MIS 381N

Stochastic Control and Optimization: Homework 1

Non-graded problems

Problem 1

Use definitions to prove

$$(A^{-1})^T = (A^T)^{-1},$$

Where A is an invertible square matrix and A^T means the transpose of matrix A.

Problem 2

A bank makes four kinds of loans to its personal customers and these loans yield the following annual interest rates to the bank:

- First mortgage 14%
- Second mortgage 20%
- Home improvement 20%
- Personal overdraft 10%

We are interested in the bank's lending strategy. The information we know is as following:

1. In total $250 million is lent out.
2. First mortgages are 55% of all mortgages (i.e. first and second mortgage) issued.
3. Second mortgages are 25% of all loans issued.
4. The average interest rate on all loans is 15%.

Calculate the lending strategy using matrix inversion.

Problem 3

A company manufactures four variants of the same product and in the final part of the manufacturing process there are assembly, polishing and packing operations. For each variant the time required for these operations is shown below (in minutes) as is the profit per unit sold.

	Assembly	Polish	Pack	Profit($)
Variant 1	2	3	2	1.50
2	4	2	3	2.50
3	3	3	2	3.00
4	7	4	5	4.50

Given the current state of the labor force the company estimate that, each year, they have 100000 minutes of assembly time, 50000 minutes of polishing time and 60000 minutes of packing time available.

Add the non-negative constraints which ensure a positive number of units are manufactured for each type of variant. Write down all the constraints in the form $Ax \leq b$

Problem 4

Rankings are ubiquitous (Google's PageRank, IMDB movie ratings etc.). The backbone of these systems is Linear Algebra. Let's try building our own ranking system to rank sports teams.

In a football league where several teams play against one another, we would like ratings of teams based on the margin of victory and not just the outcomes (win/loss/draw). Say our five teams had the following results playing against each other.

	Team 1	Team 2	Team 3	Team 4	Team 5
Team 1		7-52	21-24	7-38	0-45
Team 2			34-16	25-17	27-7
Team 3				7-5	3-30
Team 4					14-52

An entry *x-y* at this (i, j) position in this table represents a match where *i* scored *x* goals and *j* scored *y* goals. We will first rate the teams and then convert the ratings to rankings. Consider a simple model for rating the teams

$$b_k = r_i - r_j$$

Here b_k is the difference in points in the match k and r_i, r_j are the ratings for teams i ,j .The first step in creating the rankings is to write down a system of linear equations in the form relating the unknown r to the known b

$$A\,r = b$$

The number of rows of A will equal the number of total matches played.

The above is not solvable since it is overdetermined - matrix A is non-invertible

However, we can solve for a rating by converting the 10 equations to 5 equations by multiplying both sides by A^T and generate a new system of equations

$$A^T A\,r = A^T b$$

Note that this is just one way of making the system determinable. The new system is an approximation in the sense that it considers all the games played by a particular team at once rather than look at each individual match.

While $A^T A$ is a square matrix, you can see that, it is still not invertible. This is because, a unique solution for this system still does not exist. For example, if r solves the system of equations, *r + 10,* will also solve it (as it should cause of our earlier definition of r). We make it invertible, by adding a condition/equation that fixes the sum of the ratings. For simplicity, we say the ratings add up to 0.

Write down the system of linear equations, with the condition, in the form $\hat{A}r = \hat{b}$ and solve for the individual ratings of the teams to find the ranking of the team.

Deliverables

This HW is due next Wednesday (31st January) at 11:59 pm.