Natural Image Statistics
Homework for chapter 4
Model solutions (Jussi Martin)

1. Math exercise 1 from chapter 4.

   Solution: If we write the conditional probability as

   $$p(z_2|z_1 = a) = \frac{p_\mathbf{z}(a, z_2)}{\int p_\mathbf{z}(a, z_2)dz_2}$$

   we see that integration over z_2 gives us

   $$\int p(z_2|z_1 = a)dz_2 = \int \frac{p_\mathbf{z}(a, z_2)}{\int p_\mathbf{z}(a, z_2)dz_2}dz_2 = \frac{\int p_\mathbf{z}(a, z_2)dz_2}{\int p_\mathbf{z}(a, z_2)dz_2} = 1$$

   since the denominator is just a constant.

2. Math exercise 2 from chapter 4.

   Solution: Lets denote the random variable by z (in the sequel I will denote also it's values by z). Since it is distributed uniformly on the interval $[a, b]$ we have

   $$p(z) = \frac{1}{b - a} \quad \text{for every } z \text{ in } [a, b].$$

   It's mean is

   $$E\{z\} = \int_a^b p(z)zdz = \frac{1}{b - a} \int_a^b zdz = \frac{1}{b - a}\Big(\frac{b^2}{2} - \frac{b^2}{2}\Big).$$

   Since

   $$b^2 - a^2 = (b + a)(b - a)$$

   we get

   $$E\{z\} = \frac{1}{2}\frac{(b + a)(b - a)}{b - a} = \frac{b + a}{2}.$$

   It's variance is

   $$\text{var}(z) = E\{z^2\} - (E\{z\})^2 = \int_a^b p(z)z^2dz - (E\{z\})^2.$$

   Using the previous result for the mean we get

   $$\text{var}(z) = \int_a^b p(z)z^2dz - \frac{(b + a)^2}{4} = \frac{1}{b - a}\int_a^b z^2dz - \frac{(b + a)^2}{4}.$$

   Solving the integral we get

   $$\text{var}(z) = \frac{1}{b - a}\Big(\frac{b^3}{3} - \frac{a^3}{3}\Big) - \frac{(b + a)^2}{4}$$

   and since

   $$b^3 - a^3 = (b - a)(b^2 + ab + a^2)$$

   we see that

   $$\text{var}(z) = \frac{(b^2 + ab + a^2)}{3} - \frac{(b + a)^2}{4} = \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3b^2 + 6ab + 3b^2}{12}.$$

   Which reducies to:

   $$\text{var}(z) = \frac{b^2 - 2ab - a^2}{12} = \frac{(b - a)^2}{12}.$$

3. Math exercise 6 from chapter 4.

Solution: Now the likelihood is

$$p(z(1),\ldots,z(n)|\alpha) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\sum_{i=0}^{n}(z(i) - \alpha)^2)$$

and the log-likelihood is

$$\log p(z1),\ldots,z(n)|\alpha) = -\frac{1}{2}\sum_{i=0}^{n}(z(i) - \alpha)^2 - \text{const.}$$

Since the logarithm is incresing function, the value which maximizes the log-likelihood also maximizes the likelihood. Furthermore, the log-likelihood function is a down opening parabola and hence the maximum is achieved in the uniq point $\hat{\alpha}$ on top of the parabola. This point is also a point where derivative of function

$$f(\alpha) = -\frac{1}{2}\sum_{i=0}^{n}(z(i) - \alpha)^2$$

vanishes. The maximum value of the likelihood is thus found by seting $f'(\hat{\alpha}) = 0$, which yields:

$$\left(-\sum_{i=0}^{n}z(i)\right) + n\hat{\alpha} = 0 \quad \Leftrightarrow \quad \hat{\alpha} = \frac{1}{n}\sum_{i=0}^{n}z(i),$$

since

$$\frac{d}{d\alpha}(z(i) - \alpha)^2 = -2(z(i) - \alpha) \text{ for every } i.$$