**Natural Image Statistics**
**Homework for chapter 4**
**Model solutions (Jussi Martin)**

**1.** Math exercise 1 from chapter 4.

**Solution:** If we write the conditional probability as

$$p(z_2|z_1 = a) = \frac{p_{\mathbf{z}}(a, z_2)}{\int p_{\mathbf{z}}(a, z_2)dz_2}$$

we see that integration over $z_2$ gives us

$$\int p(z_2|z_1 = a)dz_2 = \int \frac{p_{\mathbf{z}}(a, z_2)}{\int p_{\mathbf{z}}(a, z_2)dz_2} dz_2 = \frac{\int p_{\mathbf{z}}(a, z_2)dz_2}{\int p_{\mathbf{z}}(a, z_2)dz_2} = 1$$

since the denominator is just a constant.

**2.** Math exercise 2 from chapter 4.

**Solution:** Lets denote the random variable by $z$ (in the sequel I will denote also its values by $z$). Since it is distributed uniformly on the interval $[a, b]$ we have

$$p(z) = \frac{1}{b-a} \quad \text{for every } z \text{ in } [a, b].$$

It's mean is

$$E\{z\} = \int_a^b p(z)zdz = \frac{1}{b-a}\int_a^b zdz = \frac{1}{b-a}\Big(\frac{b^2}{2} - \frac{b^2}{2}\Big).$$

Since

$$b^2 - a^2 = (b+a)(b-a)$$

we get

$$E\{z\} = \frac{1}{2}\frac{(b+a)(b-a)}{b-a} = \frac{b+a}{2}.$$

It's variance is

$$\mathrm{var}(z) = E\{z^2\} - (E\{z\})^2 = \int_a^b p(z)z^2dz - (E\{z\})^2.$$

Using the previous result for the mean we get

$$\mathrm{var}(z) = \int_a^b p(z)z^2dz - \frac{(b+a)^2}{4} = \frac{1}{b-a}\int_a^b z^2dz - \frac{(b+a)^2}{4}.$$

Solving the integral we get

$$\mathrm{var}(z) = \frac{1}{b-a}\Big(\frac{b^3}{3} - \frac{a^3}{3}\Big) - \frac{(b+a)^2}{4}$$

and since

$$b^3 - a^3 = (b-a)(b^2 + ab + a^2)$$

we see that

$$\mathrm{var}(z) = \frac{(b^2 + ab + a^2)}{3} - \frac{(b+a)^2}{4} = \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3b^2 + 6ab + 3b^2}{12}.$$

Which reducies to:

$$\mathrm{var}(z) = \frac{b^2 - 2ab - a^2}{12} = \frac{(b-a)^2}{12}.$$

**3.** Math exercise 6 from chapter 4.

**Solution:** Now the likelihood is

$$p(z(1), \ldots, z(n)|\alpha) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} \sum_{i=0}^{n} (z(i) - \alpha)^2)$$

and the log-likelihood is

$$\log p(z(1), \ldots, z(n)|\alpha) = -\frac{1}{2} \sum_{i=0}^{n} (z(i) - \alpha)^2 - \text{const.}$$

Since the logarithm is incresing function, the value which maximizes the log-likelihood also maximizes the likelihood. Furthermore, the log-likelihood function is a down opening parabola and hence the maximum is achieved in the unique point $\hat{\alpha}$ on top of the parabola. This point is also a point where derivative of function

$$f(\alpha) = -\frac{1}{2} \sum_{i=0}^{n} (z(i) - \alpha)^2$$

vanishes. The maximum value of the likelihood is thus found by seting $f'(\hat{\alpha}) = 0$, which yields:

$$\big( - \sum_{i=0}^{n} z(i) \big) + n\hat{\alpha} = 0 \quad \Leftrightarrow \quad \hat{\alpha} = \frac{1}{n} \sum_{i=0}^{n} z(i),$$

since

$$\frac{d}{d\alpha} (z(i) - \alpha)^2 = -2(z(i) - \alpha) \text{ for every } i.$$