

Multivariate Probability and Statistics

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Introduction

This presentation is a brief summary of some of the topics introduced in chapter 4 of the book *Natural Image Statistics* (reference at the last slide).

Since there won't be enough time to cover the whole chapter 4, I decide to go through the things that are needed for understanding the *Baye's rule*, which I think is one of the core content in this chapter.

I will also introduce *expectation* and *variance* in these slides, since they are needed in the exercises.

Random Variables

Discrete random variable is a variable which can have any value from some given range with some given probability. Example: outcome of throwing a dice. The outcome can have any value from 1 to 6, each with the probability $\frac{1}{6}$.

Continuous random variable is a variable which can have any value from some given continuous range of values, with some given probabilities for the value being in any given interval. Example
WRITE THIS!

Random Vectors

Random vector is a n -tuple of random variables. Each of these can be either similar or different types of variables. Formally:

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

where \mathbf{z} is the random vector and the random variables z_i are its components.

Probability Density Function (pdf)

Let $P(z \text{ is in } [a, b])$ be the probability that the value of a random variable z is in the interval $[a, b]$, then the probability density function $p_z(a)$ can be defined approximately as follows:

$$p_z(a) \approx \frac{P(z \text{ is in } [a, a + \nu])}{\nu}$$

for very small values of ν .

Similarly, if \mathbf{z} is a random vector:

$$p_{\mathbf{z}}(\mathbf{a}) \approx \frac{P(z_i \text{ is in } [a_i, a_i + \nu] \text{ for all } i)}{\nu^n}$$

for very small values of ν . Rigorous definitions are obtained by taking the one-sided limit $\nu \rightarrow 0^+$.

Joint and Marginal pdfs

The pdf $p(z_1, z_2, \dots, z_n)$ of a random vector \mathbf{z} is also called joint pdf, since it depends on all the components z_i .

The marginal pdfs $p_{z_i}(z_i)$ are obtained by integrating over the other variables (that is z_j , with $j \neq i$). Example in two dimensions:

$$p_{z_1}(z_1) = \int p(z_1, z_2) dz_2$$

or more rigorously:

$$p_{z_1}(\nu_1) = \int p(\nu_1, \nu_2) d\nu_2$$

Conditional Probabilities

(continuous case, discrete case)

$$p(z_2|z_1 = a) = \frac{p_z(a, z_2)}{\int p_z(a, z_2) dz_2}$$

or more concisely:

$$p(z_2|z_1 = a) = \frac{p_z(a, z_2)}{p_{z_1}(a)} = \text{e.t.c.}$$

for discrete case the integral is replaced by sum, that is

$$P_{z_1}(z_1) = \sum_{z_2} P_z(z_1, z_2)$$

Independence

definition

$$p(z_2|z_1) = p(z_2) \text{ for every } z_1 \text{ and } z_2$$

which implies that

$$\frac{p(z_1, z_2)}{p(z_1)} = p(z_2) \text{ or otherway around that } p(z_1, z_2) = p(z_1)p(z_2)$$

for all z_1 and z_2 .

Baye's Rule

Baye's rule

$$p(\mathbf{s}|\mathbf{z}) = \frac{p(\mathbf{z}|\mathbf{s})p_{\mathbf{s}}(\mathbf{s})}{\int p(\mathbf{z}|\mathbf{s})p_{\mathbf{s}}(\mathbf{s})d\mathbf{s}}$$

Or more rigorously WRITE HERE?

Appendix I, Expectation

Expectation $E\{z\}$ of a random variable z is a weighted average of the outcomes which the variable can obtain, with the weights been the individual propabilities (or values of the pdf in the continuous case).

$$\text{Discrete case:} \quad E\{z\} = \sum_z P_z(z)z$$

$$\text{Continuous case:} \quad E\{z\} = \int p_z(z)zdz$$

The expectation can be also defined for random vectors component wise:

$$E\{\mathbf{z}\} = \begin{pmatrix} E\{z_1\} \\ E\{z_2\} \\ \vdots \\ E\{z_n\} \end{pmatrix}$$

where $E\{z_i\}$ is defined as above, with z repleced by z_i for all i .

Appendix II, Variance

Variance in some sense tells how close the probability mass is concentrated near the expectation value. It is defined as follows:

$$\text{var}(z) = E\{z^2\} - (E\{z\})^2.$$

Alternatively it can be defined as

$$\text{var}(z) = E\{(z - E\{z\})^2\}.$$

WRITE THIS IN A BETTER WAY!

References



Aapo Hyvärinen, Jarmo Hurri, and Patrik O. Hoyer

Natural Image Statistics: A Probabilistic Approach to Early
Computational Vision

Springer-Verlag, 2009