

Natural Image and Video Statistics
Homework for Chapter 4
Model solutions (Jussi Martin)

1. Math exercise 1 from chapter 4.

Solution: If we write the conditional probability as

$$p(z_2|z_1 = a) = \frac{p_{\mathbf{z}}(a, z_2)}{\int p_{\mathbf{z}}(a, z_2) dz_2}$$

we see that integration over z_2 gives us

$$\int p(z_2|z_1 = a) dz_2 = \int \frac{p_{\mathbf{z}}(a, z_2)}{\int p_{\mathbf{z}}(a, z_2) dz_2} dz_2 = \frac{\int p_{\mathbf{z}}(a, z_2) dz_2}{\int p_{\mathbf{z}}(a, z_2) dz_2} = 1$$

since the denominator is just a constant.

2. Math exercise 2 from chapter 4.

Solution: Lets denote the random variable by z (in the sequel I will denote also its values by z). Since it is distributed uniformly on the interval $[a, b]$ we have

$$p(z) = \frac{1}{b-a} \quad \text{for every } z \text{ in } [a, b].$$

Its mean is

$$E\{z\} = \int_a^b p(z)z dz = \frac{1}{b-a} \int_a^b z dz = \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2} \right).$$

Since

$$b^2 - a^2 = (b+a)(b-a)$$

we get

$$E\{z\} = \frac{1}{2} \frac{(b+a)(b-a)}{b-a} = \frac{b+a}{2}.$$

Its variance is

$$\text{var}(z) = E\{z^2\} - (E\{z\})^2 = \int_a^b p(z)z^2 dz - (E\{z\})^2.$$

Using the previous result for the mean we get

$$\text{var}(z) = \int_a^b p(z)z^2 dz - \frac{(b+a)^2}{4} = \frac{1}{b-a} \int_a^b z^2 dz - \frac{(b+a)^2}{4}.$$

Solving the integral we get

$$\text{var}(z) = \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{a^3}{3} \right) - \frac{(b+a)^2}{4}$$

and since

$$b^3 - a^3 = (b-a)(b^2 + ab + a^2)$$

we see that

$$\text{var}(z) = \frac{(b^2 + ab + a^2)}{3} - \frac{(b+a)^2}{4} = \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3b^2 + 6ab + 3a^2}{12}.$$

Which reduces to:

$$\text{var}(z) = \frac{b^2 - 2ab - a^2}{12} = \frac{(b-a)^2}{12}.$$

3. Math exercise 6 from chapter 4.

Solution: Now the likelihood is

$$p(z(1), \dots, z(n)|\alpha) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \sum_{i=0}^n (z(i) - \alpha)^2\right)$$

and the log-likelihood is

$$\log p(z(1), \dots, z(n)|\alpha) = -\frac{1}{2} \sum_{i=0}^n (z(i) - \alpha)^2 - \text{const.}$$

Since the logarithm is an increasing function, the value which maximizes the log-likelihood also maximizes the likelihood. Furthermore, the log-likelihood function is a down opening parabola and hence the maximum is achieved in the unique point $\hat{\alpha}$ on top of the parabola. This point is also the point where derivative of the function

$$f(\alpha) = -\frac{1}{2} \sum_{i=0}^n (z(i) - \alpha)^2$$

is zero. The maximum value of the likelihood is thus found by setting $f'(\hat{\alpha}) = 0$, which yields:

$$\left(-\sum_{i=0}^n z(i)\right) + n\hat{\alpha} = 0 \quad \Leftrightarrow \quad \hat{\alpha} = \frac{1}{n} \sum_{i=0}^n z(i),$$

since

$$\frac{d}{d\alpha} (z(i) - \alpha)^2 = -2(z(i) - \alpha) \text{ for every } i.$$