Imp. topics:

① continuity of a function :-

A function $f(x)$ is said to be continuous at $x = a$ if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

or, $LHL = RHL = f(a)$

② DIFFERENtiability of a function :-

A function $f(x)$ is said to be differentiable at $x = a$, if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = f'(a)$$

i.e. RHD = LHD = finite number

③ Higher order derivatives

* $y_n(x^m) = \frac{m!}{(m-n)!} x^{m-n}$
* $y = (ax+b)^m \Rightarrow y_n = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$
* $y = 1/x \Rightarrow y_n = \frac{(-1)^n n!}{x^{n+1}}$
* $y = \frac{1}{x+a} \Rightarrow y_n = \frac{(-1)^n n!}{(x+a)^{n+1}}$
* $y = a^x \Rightarrow y_n = a^x (\log a)^n$

* y_n (constant) $= 0$.

* $y = \sin(ax+b) \Rightarrow y_n = a^n \sin\left(ax+b+n\pi/2\right)$

* $y = \frac{1}{x^2+a^2} \Rightarrow y_n = \frac{(-1)^n n!}{a^{n+2}} \cdot \sin^{(n+1)}\theta \cdot \sin(n+1)\cdot\theta$

* $y = e^{ax}\sin bx \Rightarrow y_n = \left(a^2+b^2\right)^{n/2} e^{ax} \sin\left(bx + n\tan^{-1}\left(\frac{b}{a}\right)\right)$

(4) Leibnitz theorem :-

→ Leibnitz theorem is used to determine nth derivative of product of two function.

Statement :-

let $y = uv$

where $u = u(x)$ & $v = v(x)$; then nth derivative of their product is given by

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n$$

where, $u_n = \frac{d^n u}{dx^n}$

&

$v_n = \frac{d^n v}{dx^n}$

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Rolles theorem :-

statement :-

If $f(x)$ is real valued function such that :

(i) $f(x)$ is continuous , in closed interval $[a, b]$.

(ii) $f(x)$ is differentiable in open interval (a, b)

(iii) $f(a) = f(b)$, then

There exist a point $c \in (a, b)$ such that

$$f'(c) = 0$$

proof :-

If $f(x)$ is continuous in closed interval $[a, b]$; then there is a maximum and minimum value,

let, $f(c) = M$ be maximum value

and, $f(d) = m$ be minimum value

case (I) :-

If $M = m$, i.e maxm value & minm value are same.

Then the function is constant.

i.e. $f(x) = K$

$f'(x) = 0$

$\therefore f'(c) = 0 \quad \in (a, b)$

Hence,

theorem is verified.

case (II) :-

If $M \neq m$; i.e. maximum & minimum values are not same.

Then, at least one of them is different from $f(a)$ & $f(b)$.

we have to show $f'(c) = 0$ $c \in (a, b)$.

at $x = c$

$$RHD = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

since $f(c) = M$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - M}{h} = \frac{-ve}{+ve} \leq 0$$

and

$$LHD = \lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = \lim_{h \to 0} \frac{(c-h) - f(c)}{-h}$$

$$= \lim_{h \to 0} \frac{(c-h) - M}{-h} = \frac{-ve}{-ve} = +ve \geq 0$$

since $LHD = RHD = 0$

so, the function $f(x)$ is differenntiable at (a, b)

$\therefore \; f'(x) = 0$

$f'(c) = 0 \quad c \in (a, b)$

$\parallel^{ly}$,

$\therefore$ we can show

$f'(d) = 0 \quad d \in (a, b)$

Hence, the theorem is proved.

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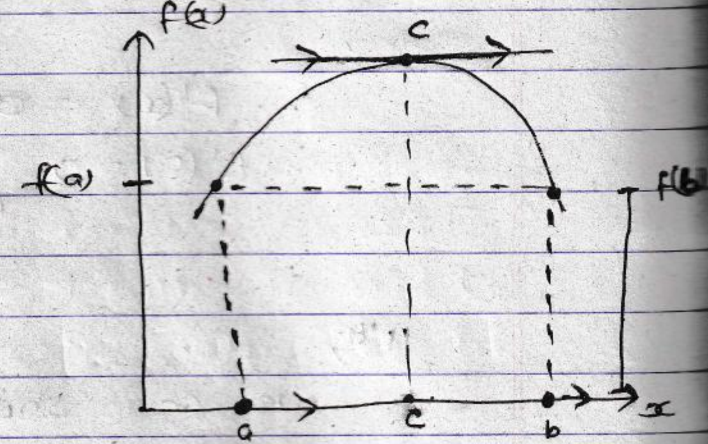
Geometrical intrepretion :-



If the function f(x) is continuous in closed interval [a, b] and differentiable in open interval (a, b) and if f(a) = f(b) then there exists a value C such that the tangent at C is parallel to the x - axis.

Basically the slope of tangent is zero (0)

or

$$f'(c) = 0$$

Lagrange's mean value theorem :-

statement :-

If $f(x)$ is a function such that

(i) $f(x)$ is continuous in $[a, b]$.

(ii) $f(x)$ is differentiable in (a, b).

Then, there exist at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

proof :-

lets define a new function

$F(x) = f(x) + Ax$ —— ①

where,

(A) is the constant to be determined ~~such that~~ :

$$F(a) = F(b) \quad \text{—— ②}$$

Since, $f(x)$ is continuous on $[a, b]$ &

$f(x)$ is differentiable (a, b).

so,

$F(x)$ is also continuous on $[a, b]$ &

$F(x)$ is also differentiable on (a, b)

such that $F(a) = F(b)$ - - - (ii)

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Therefore,

$F(x)$ satisfies all the condition of Rolle's theorem.

by Rolle's theorem

$F'(c) = 0$

$F'(c) = f'(c) + A \quad [\because F(x) = f'(a) + A]$

$0 = f'(c) + A$

$f'(c) = -A \quad$ --- (iii)

now from (i)

$F(a) = f(a) + A(a)$

$F(b) = f(b) + A(b)$

$\therefore F(a) = F(b)$

$f(a) + A(a) = f(b) + A(b)$

$A(a) - A(b) = f(b) - f(a)$

$A(a-b) = f(b) - f(a)$

$-A(b-a) = f(b) - f(a)$

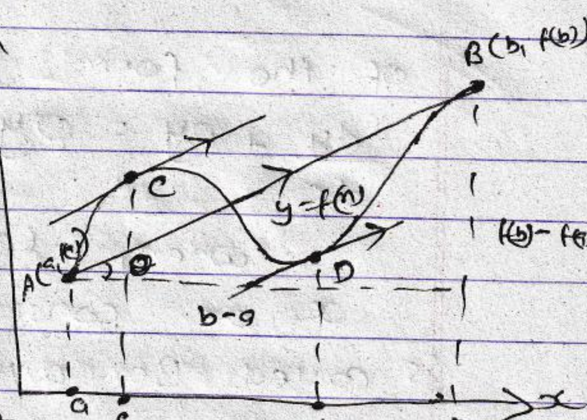
$-A = \dfrac{f(b) - f(a)}{(b-a)}$

from (iii)

$f'(c) = \dfrac{f(b) - f(a)}{b-a}$

proved.

Geometrical interpretation :-



let, f(x) be continuous on [a,b] & differentiable on (a,b) as shown in figure.

Then slope of the secan Secant line joining A and B is

$$m = \tan\theta = \frac{f(b)-f(a)}{b-a}$$

also, slope of tangent C is $f'(c)$.

we see from the graph that the tangent at c is ||el to secant line joining A & B.

i.e. $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\boxed{\frac{dy}{dx} + Py = Q}$ ← linear

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Bernoulli's equation:-

A differentiable eqn of the form

$$\frac{dy}{dx} + Py = Qy^n \quad \text{--- (1)}$$

where, (P) & (Q) are functions of x or constant & n is a real number, is called Bernoulli's eqn

Solution of Bernoulli's eqn:-

we have,

$$\frac{dy}{dx} + Py = Qy^n \quad \text{--- (1)}$$

or $$\frac{1}{y^n}\frac{dy}{dx} + \frac{Py}{y^n} = \frac{Q\cancel{y^n}}{\cancel{y^n}}$$

$$\frac{1}{y^n}\frac{dy}{dx} + y^{1-n}.P\cancel{y} = Q \quad \text{--- (ii)}$$

put $y^{1-n} = v$

diff wrt x

$$(1-n) \cdot y^{(1-n-1)} \cdot \frac{dy}{dx} = \frac{dv}{dx}$$

$$(1-n)\, y^{-n} \cdot \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{dv}{dx} \times \frac{1}{(1-n)}$$

Then eqn (11) becomes,

$$\frac{dv}{dx} \times \frac{1}{(1-n)} + v\, py = Q$$

or, $$\frac{dv}{dx} + (1-n)\, v\, py = (1-n)\, Q$$

which is the linear eqn in v.

Example

(1) $$\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3 \cos^2 y$$

Sol^n

The eqn is in the form of

$$\frac{dy}{dx} + py = Qy^n$$

So ~~it~~ ~~is~~ we can solve it by using Bernoulli's eqn.

now,

$$\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3 \cos^2 y \quad -- \text{ (1)}$$

$$\frac{dy}{dx} \times \frac{1}{\cos^2 y} + \frac{1}{x}\sin 2y \times \frac{1}{\cos^2 y} = \frac{x^3 \cancel{\cos^2 y}}{\cancel{\cos^2 y}}$$

$$\frac{dy}{dx}\sec^2 y + \frac{1}{x} \cdot \frac{2\sin y \cos y}{\cos^2 y} = x^3$$

$$\frac{dy}{dx}\sec^2 y + \frac{1}{x} \times \underline{2\tan y} = x^3 \quad -- \text{ (II)}$$

now ~~the~~ .

let $\tan y = v$

diff b.s wrt x

$$\frac{d}{dx}(\tan y) = \frac{dv}{dx}$$

$$\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

now,

eqn (II) becomes,

$$\frac{dv}{dx} + \frac{2}{x} \cdot v = x^3 \quad -- \text{ (III)}$$

comparing (iii) with

$$\frac{dy}{dx} + Py = Q$$

we get

$P = 2/x$; $Q = x^3$

now,

Integration Factor (IF) $= e^{\int P\,dx}$

$= e^{\int 2/x\,dx} \Rightarrow e^{2 \cdot \log x}$

$\Rightarrow e^{(2 \cdot \log x)}$

$\Rightarrow e^{2\log x}$

$\Rightarrow e^{\log x^2}$

$\Rightarrow x^2$

now,

$$V \times IF = \int Q \times IF\,dx + C$$

$$\tan y \times x^2 = \int x^3 \cdot x^2\,dx + C$$

$$\tan y \times x^2 = \frac{x^6}{6} + C$$

$$6\tan y = x^4 + 6\,C/x^2$$

#

Riccati's equation :-

An differentiable equation of the form

$$\frac{dy}{dx} + Py + Qy^2 = R$$

where ; P, Q, & R are the function of x or constant, is called Riccatis eqn.

* If $R = 0$; then the Equation becomes Bernoulli's equation.

* If $Q = 0$; then the equation becomes linear differentiable equation.

Case I :-

The substitution $\left[y = y_1 + \frac{1}{u} \right]$ reduces the Riccati eqn into first order linear eqn.

Case II :-

The substitution $[y = y_1 + u]$ reduces the Riccati eqn into ~~first~~ Bernoulli's eqn.

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example:

$$\frac{dy}{dx} + Py + Qy^2 = R \quad - (1) \text{ be a riccati eqn}$$

put; $y = y_1 + u \quad - (1)$

diff b.s. wrt x

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

now,

Eqn (1) becomes

$$\frac{dy_1}{dx} + \frac{du}{dx} + P(y_1 + u) + Q(y_1 + u)^2 = R$$

or $$\frac{dy_1}{dx} + \frac{du}{dx} + Py_1 + Pu + y_1^2 Q + 2y_1 u Q + Qu^2 = R$$

or $$\frac{du}{dx} + Pu + 2y_1uQ + Qu^2 + \left(\frac{dy_1}{dx} + Py_1 + Qy_1^2\right) = R$$

If ~~the~~ $(y = y_1)$ be particular soln then,

or $$\frac{du}{dx} + u(P+2y_1Q) + Qu^2 + \cancel{R} = \cancel{R}$$

$$\frac{du}{dx} + u(P+2y_1Q) + Qy^2 = 0$$

$$\frac{du}{dx} + u(P + 2y_1 Q) = -Qu^2$$

is the Bernoulli's eqn

Example

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$$2\cos x \frac{dy}{dx} = 2\cos^2 x - \sin^2 x + y^2 \quad \text{--- (1)}$$

hit & trial

put $y_1 = \sin x$

$$2\cos x \frac{d(\sin x)}{dx} = 2\cos^2 x - \cancel{\sin^2 x} + \cancel{\sin^2 x}$$

$$2\cos^2 x = 2\cos^2 x.$$

$\therefore$ The particular solution $y = \sin x$.

now,

put, $y = y_1 + \frac{1}{u}$

$y = \sin x + \frac{1}{u}$

diff b.s wrt x

$$\frac{dy}{dx} = \cos x + \frac{(-1)}{u^2} = \cos x - \frac{1}{u^2}\frac{du}{dx}$$

now, Eqn (1) becomes

$$2\cos x \cdot \left(\cos x - \frac{1}{u^2}\frac{du}{dx}\right) = 2\cos^2 x - \sin^2 x + \left(\sin x + \frac{1}{u}\right)^2$$

or, $$\cancel{2\cos^2 x} - \frac{2\cos x}{u^2}\frac{du}{dx} = \cancel{2\cos^2 x} - \cancel{\sin^2 x} + \cancel{\sin^2 x} + \frac{2\sin x}{u} + \frac{1}{u^2}$$

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$\Rightarrow \frac{-2\cos x}{u^2}\frac{du}{dx} = \frac{2\sin x}{u^2}\times u + \frac{1}{u^2}$

$\cancel{\frac{1}{u^2} + \frac{2\cos x}{u^2} = -\frac{2\sin x}{u}}$

$\Rightarrow$ or, $\frac{-2\cos x}{\cancel{u^2}}\frac{du}{dx} = 2\sin x \cdot u + 1$

$\Rightarrow \frac{du}{dx} = -\frac{2\sin x \cdot u}{2\cos x} + \frac{-1}{2\cos x}$

$\frac{du}{dx} = -\tan x \cdot u - \frac{1}{2}\sec x$

$\frac{du}{dx} + \tan x\, u = -\frac{1}{2}\sec x$ — (ii)

comp (ii) with

$\frac{dy}{dx} + Py = Q$

$\therefore P = \tan x \qquad Q = -\frac{1}{2}\sec x$

$IF = e^{\int P dx} = e^{\int \tan dx} = e^{\log(\sec x)} = \sec x$

now,

$u \times IF = \int Q \times IF\, dx + c$

$\frac{1}{y-\sin x}\times \sec x = \int -\frac{1}{2}\sec x \cdot \sec x + c$

$\frac{\sec x}{\quad} = -\frac{1}{2}\cdot \tan x + c$

$\left(1+\frac{1}{u}\right)^2$

$+\frac{2\sin x}{u}$

$\frac{1}{u^2}$

Asymptotes :- A straight line that continually approaches a given curve but doesn't meet it at any finite distance.

Types

→ Horizontal Asymptote :- → ||el to x-axis

→ Vertical Asymptote :- → ||el to y-axis

→ Oblique Asymptote :- Asymptote of type $y = mx + c$ $(\because m \neq 0)$

* If y^n is present then, there is no assymptote ||el to y-axis.

* If y^n is absent then, equating the coeff. highest available degree of y to zero. (0)

* Let $y = mx + c$ be an oblique asymptote for value of m & c.
  put $x = 1$ & $y = m$
  $\phi_n(m)$, $\phi_{(n-1)}(m)$ ---

  for value of m
  $\phi_n(m) = 0$
  --- values of m.

for non repeated values of m.

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

for repeated values of m.

$$\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

Example:

① $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0.$

Sol: Here the equation is of 3rd degree & we can clearly see that both x^3 & y^3 are present. So, there would be no asymptote $\parallel$el to x & y axis.

for oblique asymptote
let $y = mx + c$
put $x = 1$, & $y = m$ in eqn

$$\phi_3(m) = 1^3 + 2.1^2.m - 1.m^2 - 2m^3$$
$$= 1 + 2m - m^2 - 2m^3$$
$$\phi_3'(m) = 2 - 2m - 6m^2$$

$\phi_2(m) = 4m^2 + 2m$

$\phi_2'(m) = 8m + 2$

for the values of m.

$\phi_n(m) = 0$

$\therefore \phi_3(m) = 0$

$1 + 2m - m^2 - 2m^3 = 0$

or, $2m^3 + m^2 - 2m - 1 = 0$ — (1)

put $m = 1$

$2 + 1 - 2 - 1 = 0$

$\therefore (m-1)$ is factor of (1)

now,

by using scientific division.

1	2	1	-2	-1
	1	2	3	1
	2	3	1	0
	coef x^2	coef x	const	R

$\therefore \quad (2m^2 + 3m + 1)(m-1) = 0$

$(2m^2 + 2m + m + 1)(m-1) = 0$

$[2m(m+1) + 1(m+1)](m-1) = 0$

$(2m+1)(m+1)(m-1) = 0$

$\therefore \quad m = 1, -1, -1/2$

$\therefore$ since values of m are non-repeating

for $m = 1$

$$c = \frac{-\phi_{(n-1)}(m)}{\phi_n'(m)} = \frac{-(4m^2+2m)}{(2-2m-6m^2)}$$

$$= \frac{-(4m^2+2m)}{-(6m^2+2m-2)}$$

$$= \frac{-2(2m^2+m)}{-2(3m^2+m-1)}$$

$$c = \frac{2m^2+m}{3m^2+m-1} = \frac{2\cdot1+1}{3\cdot1^2+1-1} = 1$$

$\therefore \quad y = mx + c$

$y = 1x + 1 \quad$ —— (*)

for $m = -1$

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

$$= \frac{2m^2 + m}{3m^2 + m - 1} = \frac{2(-1)^2 + (-1)}{3(-1)^2 + (-1) - 1}$$

$$= \frac{2 - 1}{3 - 2} = \frac{1}{1} = 1$$

$\therefore y = -1x + 1$

$\therefore y = -x + 1$ — (*)

for $m = -1/2$

$$c = \frac{2 \times (-1/2)^2 + (-1/2)}{3(-1/2)^2 + (-1/2) - 1} = \frac{2 \times \frac{1}{4} - \frac{1}{2}}{3 \times \frac{1}{4} - \frac{1}{2} - 1}$$

$$= \frac{1/2 - 1/2}{3/4 - 2/4 - 4/4} = 0$$

$y = -1/2\, x + 0$

$2y = -x$ — (*)

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Curve tracing :-

Example (1)

$$y^2(a-x) = x^2(a+x) \quad \text{---} \; (1)$$

Sol^n

$$y^2a - y^2x = x^2a + x^3$$

(1) symmetry :- → y all power even → symm ab. x-
→ x all " " " " y-ax

The curve is symmetric about x-axis.

(2) origin and Tangent :-
The curve is passing through the origin. And equating the lowest degree term to zero.

$$y^2a = x^2a$$

or $y^2 = x^2$

or $\boxed{y = \pm x}$

ie two real & distinct tangent at origin

⇒ origin is node..

(3) point of intersection on co-ordinate axes :-

put $x=0$ in (1)	put $y=0$ in (1)
$y^2a = 0$	$x^2(a+x) = 0$
$y = 0$	$x=0, x=-a$
$(0,0)$	$(-a,0)\,(0,0)$

so, the points on axes are :-

$(0,0), (-a,0)$

(4) Asymptote :-

Asymptote parallel to y-axis

Equating the coeff. of highest degree of y to zero

$(a-x) = 0$

$a = x$

$\Rightarrow \quad x = a$

(5) Region :-

Solving the given eqn for y

$$y^2(a-x) = x^2(a+x)$$

$$y^2 = \frac{x^2(a+x)}{(a-x)}$$

$$\therefore \quad y = \pm x\sqrt{\frac{a+x}{a-x}}$$

observation :-

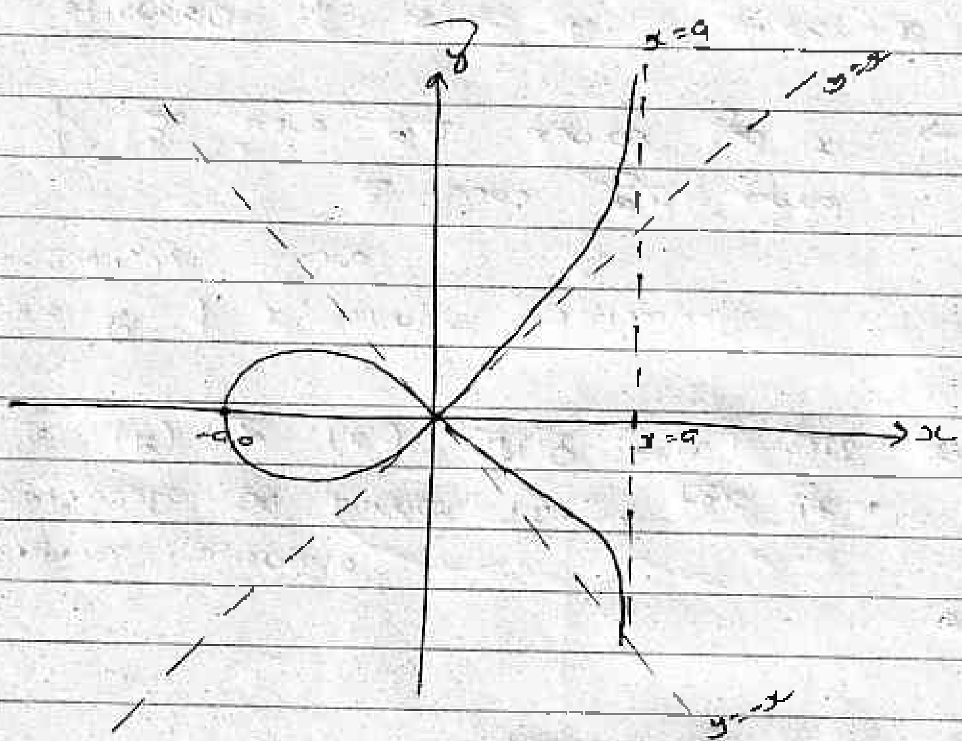
(1) y becomes imaginary when x becomes greater than a $(x > a)$

(2) y become imaginary when ~~x < (-a)~~ $x < -a$;

(3) y exist betn $-a \leq x < a$

(4) for each x $(-a \leq x < a)$, y has two equal & opposite value (sign)

Using 5 points the approximate shape of curve is



Example (2)

$$a^2y^2 = x^2(a^2 - x^2) \quad \text{— (1)}$$
$$a^2y^2 = x^2a^2 - x^4$$

(1) symmetry :-

The curve is symmetric about x-axis, y-axis & opposite quadrant

→ x की power यदि even है & y की power यदि even है

even = 2, 4, 6, 8 ——

symmetric about x & y axis.

→ यदि (x) को हटाकर $(-x)$ व (y) को हटाकर $(-y)$ रखते हैं, it would be symmetric on opposite quadrant

(2) origin & tangent :-

If $x = 0$ in (1)

$y = 0$.

∴ it passes through origin $(0, 0)$

for tangent :- equating lowest degree term to zero.

$$a^2y^2 = x^2a^2$$
$$y = \pm x$$

node

Two real & distinct tangents.

$\Rightarrow$ origion is a node.

(3) point of intersection on coordinate axes :-

put $x=0$ in (1)

$y=0$

$(0,0)$

put $y=0$ in (1)

$x^2(a^2-x^2)=0$

$x=0, \quad a^2-x^2=0$

$\pm a = x$

$x = \pm a$

$(a,0), (-a,0)$

Hence,

point of intersection $(0,0)$, $(-a,0)$ & $(a,0)$

(4) Asymptote :-

No-Asymptote

(5) Region :-

$$y = \pm \sqrt{\frac{x^2(a^2-x^2)}{a^2}}$$

$$y = \pm \frac{x}{a}\sqrt{a^2-x^2}$$

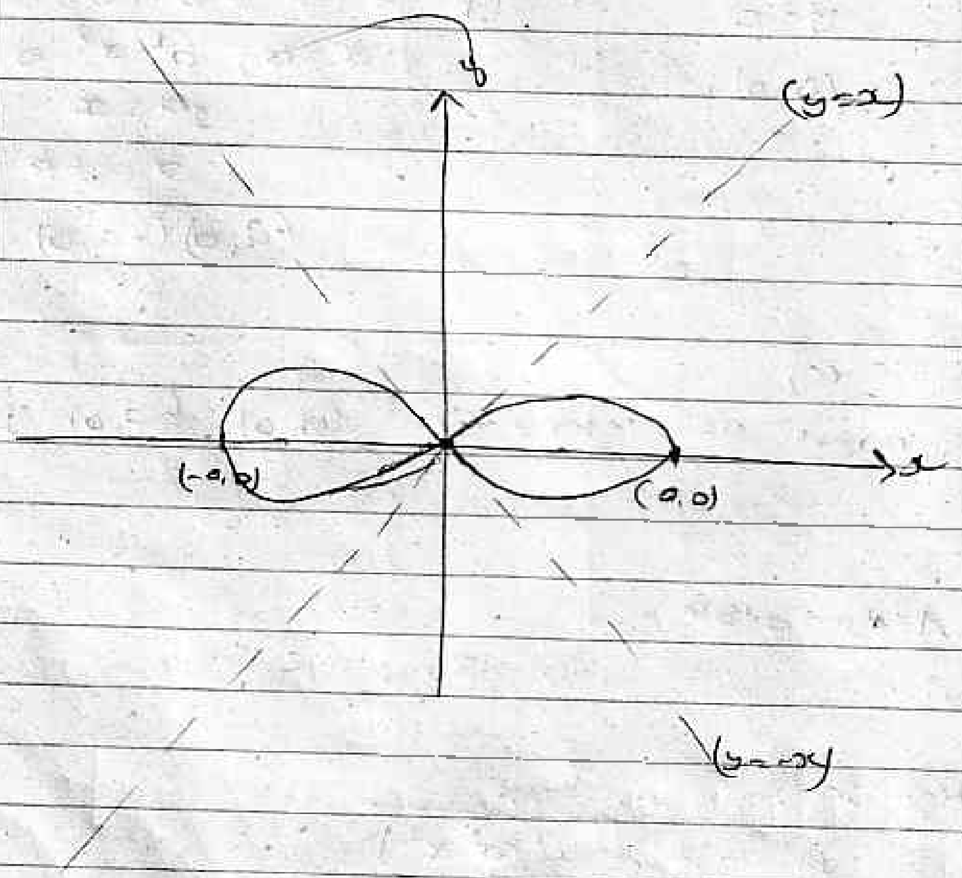
Observation :-

(i) y become imaginary when $x > a$ & $x < -a$

(ii) for each x (~~$-a<$~~ $(-a < x < a)$

y has two equal & opposite values

Approximate Curve :-



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Homogenous function :-

A function $f(x,y)$ of two independent variables x & y is said to be homogenous of degree n if it satiesfy the property $f(tx, ty) = t^n f(x,y)$. for all values of t.

Euler's theorem on homogenous function of two variables.

let $u(x,y)$ be homogenous function of two variable x & y of degree n,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

proof :-

Here $u(x,y)$ is a Homogenous function of x & y of degree n. we can write,

$$u = x^n g(y/x) \quad \text{——(1)}$$

now,

diff partially ① wrt x

$$\frac{\partial u}{\partial x} = n x^{n-1} g(y/x) + x^n \cdot g'(y/x) \cdot y \cdot \left(\frac{-1}{x^2}\right)$$

$$= n x^{n-1} g(y/x) + x^n g'(y/x) \cdot \left(\frac{-y}{x^2}\right)$$

$$= n x^{n-1} g(y/x) - x^{n-2} g'(y/x) \cdot y.$$

Again,

diff partially ① wrt y.

$$\frac{\partial u}{\partial y} = x^n \left[g'(y/x) \cdot \frac{1}{x} \cdot 1 \right]$$

$$= x^{n-1} g'(y/x)$$

now,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

$$= x\left[n x^{n-1} g(y/x) - x^{n-2} g'(y/x) \cdot y\right] + \left[x^{n-1} g'(y/x)\right] y$$

$$\Rightarrow x\, n x^{n-1} g(y/x) - \cancel{x^{n-1} g'(y/x) \cdot y} + \cancel{x^{n-1} y\, g'(y/x)}$$

$$= x\, n x^{n-1} g(y/x)$$

$$\Rightarrow n x^n g(y/x)$$

$$\Rightarrow nu$$

proved.

Euler's theorem on homogenous function of 3- variables.

let $u=f(x,y,z)$ be a homogenous function of three independent variable x, y & z of degree n.

Then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

If $u = f(x,y,z)$ be a homogenous function of $x, y,$ & z of degree n.

$$u = x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$$

let $p = y/x$ & $q = z/x$

Then,

$$u = x^n g(p, q) \quad —①$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(y/x) = y \cdot \frac{-1}{x^2} = \frac{-y}{x^2}$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}(y/x) = \frac{1}{x}$$

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}(y/x) = 0$$

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$$\frac{\partial q}{\partial x} = \frac{\partial (z/x)}{\partial x} = \frac{-z}{x^2}$$

$$\frac{\partial q}{\partial y} = \frac{\partial (z/x)}{\partial y} = 0$$

$$\frac{\partial q}{\partial z} = \frac{\partial (z/x)}{\partial z} = 1/x \cdot 1 = 1/x$$

now,

diff partially ① wrt x

$$\frac{\partial u}{\partial x} = n x^{n-1} g(p, q) + x^n \cdot g' \left[\frac{\partial g}{\partial p} \times \frac{\partial p}{\partial x} + \frac{\partial g}{\partial q} \times \frac{\partial q}{\partial x}\right]$$

$$= n x^{n-1} g(p, q) + x^n \left[\frac{\partial g}{\partial p} \times \frac{-y}{x^2} + \frac{\partial g}{\partial q} \times \frac{-z}{x^2}\right]$$

also,

diff. partially ① wrt y.

$$\frac{\partial u}{\partial y} = x^n \left[\frac{\partial g}{\partial p} \times \frac{\partial p}{\partial y} + \frac{\partial g}{\partial q} \times \frac{\partial q}{\partial y}\right]$$

$$= x^n \left[\frac{\partial g}{\partial p} \times 1/x + \frac{\partial g}{\partial q} \times 0\right]$$

also,

diff. partially (1) wrt z.

$$\frac{\partial u}{\partial z} = x^n \left[\frac{\partial g}{\partial p} \times \frac{\partial p}{\partial z} + \frac{\partial g}{\partial q} \times \frac{\partial q}{\partial z} \right]$$

$$= x^n \left[\frac{\partial g}{\partial p} \times 0 + \frac{\partial g}{\partial q} \times \frac{1}{x} \right]$$

now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= x\left[nx^{n-1} g(p,q) + x^n \frac{\partial g}{\partial p} \times \frac{-y}{x^2} + x^n \frac{\partial g}{\partial q} \times \frac{-z}{x^2} \right]$$

$$+ y\left[x^n \frac{\partial g}{\partial p} \times \frac{1}{x} \right] + z\left[x^n \frac{\partial g}{\partial q} \times \frac{1}{x} \right]$$

$$= nx^n g(p,q) + \cancel{x^{n-1} \frac{\partial g}{\partial p} \times (-y)} + \cancel{x^{n-1} \frac{\partial g}{\partial q} \times (-z)}$$

$$+ \cancel{x^{n-1} \frac{\partial g}{\partial p} \times y} + \cancel{x^{n-1} \frac{\partial g}{\partial q} \times z}$$

$$= n\, x^n g(p,q)$$

$$= n\, u$$

proved.

Taylor's series :-

Expension of Taylor's series :-

The Taylor's series expension of $f(x)$ at $x = a$.

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + (x-a)^3\frac{f'''(a)}{3!} + \cdots + (x-a)^n\frac{f^n(a)}{n!}$$

Maclaurin's series :-

~~At a=0 of~~

Taylor's infinite series of $f(x)$ at $a=0$ is known as Maclaurin's series representation of $f(x)$.

$$f(x) = f(0) + x\frac{f'(0)}{1!} + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \cdots + x^n\frac{f^n(0)}{n!}$$

example 1:- prove using maclaurin's serie

(1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

sol: The maclaurin's series is

$$f(x) = f(0) + x\frac{f'(0)}{1!} + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!}$$

$$+ x^4\frac{f^{iv}(0)}{4!} + x^5\frac{f^{v}(0)}{5!} + x^6\frac{f^{vi}(0)}{6!} + x^7\frac{f^{vii}(0)}{7!} + \cdots \quad \text{--- (1)}$$

since,
we know;

$f(x) = \sin x$ $\qquad$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\qquad$ $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\qquad$ $f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x$ $\qquad$ $f'''(0) = -1$

$f^{iv}(x) = \sin x$ $\qquad$ $f^{iv}(0) = 0$

$f^{v}(x) = \cos x$ $\qquad$ $f^{v}(0) = 1$

$f^{vi}(x) = -\sin x$ $\qquad$ $f^{vi}(0) = 0$

$f^{vii}(x) = -\cos x$ $\qquad$ $f^{vii}(0) = -1$

now Eqn (1) becomes

$$\sin x = 0 + x\frac{1}{1!} + x^2\frac{0}{2!} + x^3\frac{(-1)}{3!} + x^4\frac{0}{4!}$$

$$+ x^5\frac{1}{5!} + x^6\frac{0}{6!} + x^7\frac{(-1)}{7!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

proved

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(2) $\log \sec x = \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \cdots$

by maclaurin series

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!}$$

$$+ \frac{x^4 f^{iv}(0)}{4!} + \frac{x^5 f^{v}(0)}{5!} + \frac{x^6 f^{vi}(0)}{6!} \quad \text{---(1)}$$

$f(x) = \log \sec x$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{\sec x} \sec x \cdot \tan x$ $\qquad f'(0) = 0$

$= \tan x$

$f''(x) = \sec^2 x$ $\qquad f''(0) = 1$

$f'''(x) = 2 \sec x \cdot \sec x \cdot \tan x$

$= 2 \sec^2 x \cdot \tan x$ $\qquad f'''(0) = 0$

$f^{iv}(x) = 2 \cdot [2 \sec x \cdot \sec x \cdot \tan x \cdot \tan x + \sec^2 x \cdot \sec^2 x]$

$= 2 [2 \sec^2 x \tan^2 x + \sec^2 x \sec^2 x]$

$= 4 \sec^2 x \tan^2 x + 2 \sec^4 x$

$f^{iv}(0) = 2$

$f^{v}(x) = 4 [2 \sec x \cdot \sec x \cdot \tan x \cdot \tan^2 x + \sec^2 x \cdot 2 \tan x \cdot \sec^2 x]$

$+ 2 [4 \sec^3 x \cdot \sec x \cdot \tan x]$

$= 8 \sec^2 x \tan^3 x + 8 \sec^4 x \tan x + 8 \sec^4 x \tan x$

$= 8 \sec^2 x \tan^3 x + 16 \sec^4 x \tan x$

$f^{v}(0) = 0$

Date ______ $\frac{2}{\tan^2 x \cdot se}$

$f^{vi}(x) = 8\left[2\sec x \cdot \sec x \cdot \tan x \cdot \tan^2 x + \sec^2 x \cdot \cancel{\sec^2 x}\right.$
$+16\left[4\sec^3 x \cdot \sec x \cdot \tan x \cdot \tan x + \sec^4 x \cdot \sec^2\right.$

$= 8\left[2\sec^2 x \tan^2 x + \cancel{2}\sec^{4}x \; 3\tan^2 x \cdot \sec^2 x\right] + 16\left[4\sec^4 x \tan\right.$
$\left. + \sec^6 x \cancel{\sec^2 x}\right]$

$= 16\sec^2 x \tan^2 x + \cancel{16\sec^2 x} \; 24\sec^2 x \cdot \tan^2 x + 64\sec^4 x \tan^2$
$+16\sec^6 x$

$f^{vi}(0) = 16$

now (1) becomes

$\log \sec x = 0 + \frac{x \times 0}{1!} + \frac{x^2 \times 1}{2!} + 0 + \frac{2x^4}{4!}$
$+ 0 + \frac{16x^6}{6!}$

$=1$

$K = \frac{d\psi}{ds} = \frac{1}{\rho}$

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(K) Curvature :-

It is defined as ratio of change of direction of the curve wrt arc length.

$$\text{Curvature } (K) = \frac{d\psi}{ds}$$

(ρ) Radius of curvature :-

The reciprocal of of curvature is Radius of curvature.

i.e. $\rho = \frac{ds}{d\psi}$

(i) cartesian form

$$\rho = \frac{\left[1 + y_1^2\right]^{3/2}}{y_2} \qquad \rho = \frac{\left[1 + x_1^2\right]^{3/2}}{x_2}$$

(ii) parametric form

$$\rho = \frac{\left(x_1^2 + y_1^2\right)^{3/2}}{x_1 y_2 - y_1 x_2}$$

$x_1 = \frac{dx}{dt}$, $y_1 = \frac{dy}{dt}$

$x_2 = \frac{d^2x}{dt^2}$, $y_2 = \frac{d^2y}{dt^2}$

(iii) polar form

$$\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$r_1 = \frac{dr}{d\theta}$

$r_2 = \frac{d^2r}{d\theta^2}$

example

① $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where $y = x$ cuts it.

Sol

$\sqrt{x} + \sqrt{y} = \sqrt{a}$ — ①

at

$y = x$

now substituting $y = x$ in ①

we get,

$\sqrt{x} + \sqrt{x} = \sqrt{a}$

$2\sqrt{x} = \sqrt{a}$

$4x = a$

$x = a/4$ — ⑪

since,

$y = x$

$\therefore y = a/4$ — from ⑪

now, $P(x, y) = P(a/4, a/4)$

now,

differentiating ① wrt x

we get

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$$\frac{d}{dx}(\sqrt{x}+\sqrt{y}) = \frac{d}{dx}(\sqrt{a})$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{y}}\frac{dy}{dx}$$

$$(y_1) = \frac{dy}{dx} = \left(-\frac{\sqrt{y}}{\sqrt{x}}\right) = -\frac{\sqrt{y}}{\sqrt{x}} \quad --- \quad (*_1)$$

again,

diff $(*)$ wrt x

$$y_2 = \frac{\sqrt{x}\frac{d}{dx}(-\sqrt{y}) - (-\sqrt{y})\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}\cdot\frac{-1}{2\sqrt{y}}\frac{dy}{dx} + \sqrt{y}\times\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\cancel{\sqrt{x}}\cdot\frac{-1}{2\sqrt{y}}\times\frac{-\cancel{\sqrt{y}}}{\sqrt{x}} + \frac{\sqrt{y}}{2\sqrt{x}}}{x} \Rightarrow \frac{\frac{1}{2}+\frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$= \frac{\frac{1}{2}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}{x} \quad - (*_2)$$

substitute $(x,y) = (9/4, 9/4)$ in $(*_1)$

$$y_1 = -\frac{\sqrt{9/4}}{\sqrt{9/4}} = -1$$

&

also

substitute $(x,y) = (9/4, 9/4)$ in $(*_2)$

$$y_2 = \frac{\frac{1}{2}\left(\frac{1+\sqrt{9/4}}{\sqrt{9/4}}\right)}{9/4}$$

$$= \frac{\frac{1}{2}\left(\frac{\sqrt{9/4}+\sqrt{9/4}}{\sqrt{9/4}}\right)}{9/4} \Rightarrow \frac{\frac{1}{2}\times 2}{9/4} \Rightarrow 4/9$$

$= 9$

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Application of partial derivative :-

Critical point (stationary point) :-

let x, y be a function of two variable of pair (a, b) is a critical (stationary point) if $f_x(a, b) = 0$ & $f_y(a, b)$

ie: $\boxed{f_x = f_y = 0}$

Saddle point :-

The point (a, b) is said to be saddle point of funct $f(x, y)$ if

$f_{xx} \cdot f_{yy} - (f_{xy})^2 < 0$ at (a, b)

ie

$D < 0$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2 < 0$$

Hessian determinant :-

whose elements are partial derivative of the function

Condition for extreme values in case of 3 variable :-

If $f(x,y,z)$ be a function of 3 independent variable x, y & z then the necessary condition for function of three independent variables x, y & z to have an extreme values are as follows.

Case I : for max^m values of $f(x,y,z)$

$$f_{xx} < 0, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0 \quad \& \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} < 0$$

Case II :- for min^m value

$$f_{xx} > 0, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0 \quad \& \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} > 0$$

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⇒ equation of plane :-

The eqⁿ of plane at (x_0, y_0, z_0) on the surface of $z = f(x,y)$ is given by

$$z - z_0 = (x - x_0)\, f_x(x_0, y_0) + (y - y_0)\, f_y(x_0, y_0).$$

example

① $z = 2x^2 + y^2$ at $(1, 1, 3)$ where $x_0 = 1$, $y_0 = 1$, $z_0 = 3$

let $z = f(x,y) = 2x^2 + y^2$

$f(x) = \frac{d}{dx}(2x^2 + y^2) = 4x$, $f(1,1) = 4$

$f_y = \frac{d}{dy}(2x^2 + y^2) = 2y$, $f(1,1) = 2$

now,

Eqⁿ of tangent plane

$z - z_0 = (x - x_0)\, f_x(x_0, y_0) + (y - y_0)\, f_y(x_0, y_0)$

$z - 3 = (x - 1)\,4 + (y - 1)\,2$

$z - 3 = 4x - 4 + 2y - 2$

$4x + 2y - z - 3 = 0$

#

① find the stationary point & extreme value of following function

ⓐ $f = x^2 + xy + y^2$

Sol^n

given

$$f(x,y) = x^2 + xy + y^2$$

now,

$$f(x) = \frac{d}{dx}(x^2 + xy + y^2)$$

$$= 2x + y$$

$$f(y) = \frac{d}{dy}(x^2 + xy + y^2)$$

$$= x + 2y$$

for stationary point

we know,

$$f(x) = f(y) = 0$$

$\therefore f(x) = 0$ & $f(y) = 0$

$2x + y = 0$ — ① $x + 2y = 0$ — ⓘⓘ

~~$2x = -y$~~

from ① & ⓘⓘ

$[2x + y = 0] \times 2$

$x + 2y = 0$

$-\quad -\quad -$

$3x + 0 = 0$

$[\therefore x = 0, y = 0]$

max min

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again

$$f_{xx} = \frac{d(f_x)}{dx} = \frac{d}{dx}(2x+y) = 2$$

$$f_{xy} = \frac{d}{dy}(f_x) = \frac{d}{dy}(2x+y) = 1$$

$$f_{yy} = \frac{d}{dy}(f_y) = \frac{d}{dy}(x+2y) = 2$$

since

$$f_{xx} \cdot f_{yy} - (f_{xy})^2$$
$$= 2 \cdot 2 - (1)^2$$
$$= 4 - 1$$
$$= 3 > 0$$

$\therefore$ The function (f) is minimum at point (0,0)

and

the minimum value of f

$$f_{min} = x^2 + xy + y^2$$
$$= 0 + 0 \cdot 0 + 0$$
$$= 0$$

Lagrange's method of multiplier :-

function of two variables $f(x,y)$ under the condition $\phi(x,y) = 0$.

Let $f(x,y)$ be a function whose extreme values are to be determined under the condition $\phi(x,y) = 0$.

let us define a new function.

$$F = f + \lambda \phi$$

where,

λ = Lagrange's multiplier

for extreme value

$$\frac{\partial F}{\partial x} = 0 \quad \& \quad \frac{\partial F}{\partial y} = 0 \quad \text{at} \quad \frac{\partial F}{\partial \lambda} = 0 \Rightarrow \text{ie } \phi$$

$$f_x + \lambda \phi_x = f_y + \lambda \phi_y = 0$$

The function will be maxima if :

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} > 0$$

&

The function will be minimum if

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} < 0$$

Lagrange 3 variable problem
2 " " is just

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(1) Find the extreme values of the function $f(x,y,z) = x^2+y^2+z^2$ subjected to constraint $x+y+z=39$.

Sol^n

let, $\phi = x+y+z-39$

let define a ~~new~~ new function

$$F = f(x,y,z) + \lambda\phi$$
$$F = x^2+y^2+z^2 + \lambda(x+y+z-39)$$

$$\frac{\partial F}{\partial x} = 2x + \lambda$$

$$\frac{\partial F}{\partial y} = 2y + \lambda$$

$$\frac{\partial F}{\partial z} = 2z + \lambda$$

$$\frac{\partial F}{\partial \lambda} = \phi = 0.$$

now;

for extreme value

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

$\therefore$ $2x+\lambda=0$ $\quad$ $\therefore$ $2y+\lambda=0$ $\quad$ $\therefore$ $2z+\lambda=0$

$\lambda=-2x$ $\quad$ $\lambda=-2y$ $\quad$ $\lambda=-2z$

$x=-\lambda/2$ $\quad$ $y=-\lambda/2$ $\quad$ $z=-\lambda/2$

now,

$$x+y+z-3a=0$$

$$-\lambda/2-\lambda/2-\lambda/2-3a=0$$

$$\frac{-3\lambda}{2}-3a=0$$

$$-3\lambda-6a=0$$

$$3\lambda=-6a$$

$$\lambda=-2a$$

$$\therefore x=-\lambda/2=+\frac{2a}{2}=a$$

$$\therefore y=x=z=a$$

$$f(x,y,z)=(a,a,a)$$

now,

$\phi x=1$ $\quad$ $fxx=2$ $\quad$ $fyy=2$ $\quad$ $fzz=2$

$\phi y=1$ $\quad$ $fxy=0$ $\quad$ $fyx=0$ $\quad$ $fzx=0$

$\phi z=1$ $\quad$ $fxz=0$ $\quad$ $fyz=0$ $\quad$ $fzy=0$

now,

$$\begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{xy} & f_{xz} \\ \phi_y & f_{yx} & f_{yy} & f_{yz} \\ \phi_z & f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

$$= 0 - 1\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} + 1\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} - 1\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= 0 - 1[1 \times 4] + 1[0 - 2] - 1[-2(-2)]$$

$$= -4 - 2 - 4$$

$$= -10 < 0$$

∴ The function is minm.

The value is $f = 3a^2$

lagrange 2 - variable

(1) $f(x,y) = xy$

constraint $= 2x + y = 6$

let $\phi = 2x + y - \cancel{6} = 6$

let a new function

$$F = f(x,y) + \lambda \phi$$

$$F = xy + \lambda(2x + y - 6)$$

$\frac{dF}{dx} = y + 2\lambda$ & $\frac{dF}{dy} = x + 1\lambda$

~~since $\phi = 0$~~

~~$2x + y - 6 = 0$~~

for extreme value

$\frac{dF}{dx} = 0$ & $\frac{dF}{dy} = 0$

$y + 2\lambda = 0$ $\quad$ $x + 1\lambda = 0$

$y = -2\lambda$ $\quad$ $x = -1\lambda$

now,

$\phi = 0$

$2x + y - 6 = 0$

$2(-\lambda) + (-2\lambda) - 6 = 0$

$-2\lambda - 2\lambda - 6 = 0$

$-4\lambda - 6 = 0$

$4\lambda = -6$

$\lambda = -3/2$

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$\therefore x = -1\lambda = -1 \times \left(-\frac{3}{2}\right) = \frac{3}{2}$

$\therefore y = -2\lambda = 3$

$\therefore f(x,y) = \left(\frac{3}{2}, 3\right)$

$\phi_x = 2 \qquad f_{xx} = 0 \qquad f_{xy} = 1$

$\phi_y = 1 \qquad f_{yy} = 0 \qquad f_{yx} = 1$

$$\therefore \begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -2\left[(-1)\right] + 1\left[2\right]$$

$$= 2+2 = 4 > 0$$

$\therefore$ The function is maxm at $\left(\frac{3}{2}, 3\right)$.

The maxm value is $f(x,y)$

$= x \times y$

$= \frac{3}{2} \times 3 = \frac{9}{2}$ //

Imp

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Q find the dimension of rectangular box open at the top with volume 32 cc requiring least material for the construction.

Sol let the volume; ~~be~~ x, y, z be length, breadth & height respectively,

∴ The volume of rectangular box = $l \times b \times h$

according to qn,
volume = 32 cc

$\therefore l \times b \times h = 32$ cc

$\therefore 32 = xyz \quad \therefore z = \frac{32}{xy}$ — ①

again,
Surface (open at top)

$$S = xy + 2yz + 2zx$$
$$= xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x$$
$$= xy + \frac{64}{x} + \frac{64}{y}$$

$$S_x = y - \frac{64}{x^2}$$

$$S_y = x - \frac{64}{y^2}$$

3).

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For extreme value :-

$S_x = 0$ & $S_y = 0$

$y - \frac{64}{x^2} = 0$

$y = \frac{64}{x^2}$

$x - \frac{64}{y^2} = 0$

$x = \frac{64}{y^2}$

$x = \frac{64}{\left(\frac{64}{x^2}\right)^2}$

$x = \frac{64}{(64)^2} \times (x^2)^2$

$x = \frac{x^4}{64}$

$\therefore x^4 - 64x = 0$

$x(x^3 - 64) = 0$

either,

$x = 0$, or $x^3 = 64$

$\boxed{x = 4}$

$\therefore y = \frac{64}{x^2} = \frac{64}{16} = 4$

$\therefore z = \frac{32}{xy} = \frac{32}{4 \times 4} = 2$

now,

$$S_{xx} = 0 - 64 \times (-2) x^{-3}$$

$$= \frac{+128}{x^3} \Rightarrow = \frac{128}{64} = 2 > 0$$

$$S_{yy} = \frac{128}{y^3} = 2$$

$$S_{xy} = 1$$

$\therefore$

$$S_{xx} \cdot S_{yy} - (S_{xy})^2$$

$$= 2 \cdot 2 - (1)^2$$

$$= 4 - 1$$

$$= 3 > 0$$

which is minimum value.

en

(1) Show that the function $u(x,y) = x^3 + y^3 - 3xy$ has a saddle point at $(0,0)$.

Sol^n

$f(x,y) = x^3 + y^3 - 3xy$

$f(x) = 3x^2 - 3y$ at $(0,0) = 0$
$f(y) = 3y^2 - 3x$ at $(0,0) = 0$

$f_{xx} = 6x$ $\quad f_{yy} = 6y$
$f_{xy} = -3$

at 0,0

$f_{xx} = 0$; $f_{yy} = 0$
$f_{xy} = -3$

now,
for saddle point

$f_{xx} \cdot f_{yy} - (f_{xy})^2$
$= (0 \cdot 0) - (-3)^2$
$= -9 < 0$

$\therefore$ The function has a saddle point (-9) at $(0,0)$

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Application of integral :-

① when the function $y = f(x)$ at $a \le x \le b$ then the arc length (l)

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

② If the function is parametric function.
$x = f(t)$ & $y = g(t)$ at $ta \le x \le tb$
then

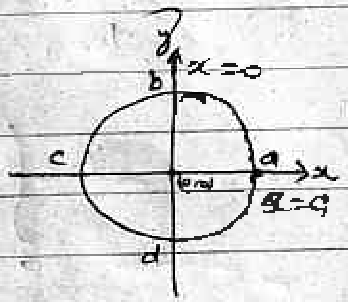
$$l = \int_{ta}^{tb} \sqrt{\left(\frac{dx}{dt}\right)^2 dt + \left(\frac{dy}{dt}\right)^2} \, dt$$

③ If the function is polar form
$r = f(\theta)$ & $\theta_1 \le \theta \le \theta_2$
then

$$l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

example 1:-

Show that the perimeter of a circle $x^2+y^2=a^2$ is $2\pi a$.



Sol

we have

$$x^2+y^2=a^2$$
$$y^2=a^2-x^2$$

diff b.s wrt x

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(a^2-x^2)$$

$$2y\frac{dy}{dx}=0-2x$$

$$\therefore \frac{dy}{dx}=-\frac{x}{y}$$

now,

The length of arc (a,b) is

$$l=\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

so for perimeter we have to find for all 4 sides

so,

$$l = 4 \times \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \cancel{4\times}$$

$$= 4 \times \int_0^a \sqrt{1+\frac{x^2}{y^2}}\,dx = 4 \times \int_0^a \sqrt{\frac{y^2+x^2}{y^2}}\,dx$$

$$= 4 \times \int_0^a \sqrt{\frac{a^2}{y^2}}\,dx = 4 \times \int_0^a \frac{a\,dx}{\sqrt{a^2-x^2}}$$

$$= 4a \int_0^a \frac{dx}{\sqrt{a^2-x^2}} \Rightarrow 4a \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

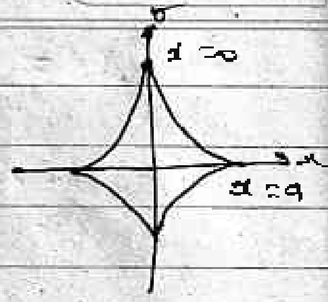
$$\Rightarrow 4a \cdot \left[\sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}\left(\frac{0}{a}\right)\right]$$

$$\Rightarrow 4a \times \frac{\pi}{2}$$

$$\Rightarrow 2\pi a$$

$\therefore$ The perimeter of a circle is $2\pi a$.

Example (2) Find the perimeter of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$.



Sol:

Here,

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$\frac{d}{dx}(x^{2/3}) + \frac{d}{dx}(y^{2/3}) = \frac{d}{dx}(a^{2/3})$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\frac{2}{3y^{1/3}} \frac{dy}{dx} = -\frac{2}{3x^{1/3}}$$

$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

$\therefore$ The perimeter is

$$l = 4 \times \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4 \times \int_0^a \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}}\, dx$$

$$= 4 \times \int_0^a \sqrt{\frac{x^{2/3} + a^{2/3}}{x^{2/3}}}\, dx \quad \Rightarrow 4 \times \int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx$$

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$\Rightarrow 4 \times \int_0^a a^{1/3} \cdot x^{-1/3} \, dx$

$\Rightarrow 4 a^{1/3} \cdot \left[\frac{x^{-1/3+1}}{-1/3+1} \right]_0^a$

$\Rightarrow 4 a^{1/3} \frac{3}{2} \times \left[x^{2/3} \right]_0^a$

$\Rightarrow 6 a^{1/3} \cdot \left[a^{2/3} - 0 \right]$

$\Rightarrow 6 a^{1/3} \cdot a^{2/3}$

$\Rightarrow 6 a^{1/3 + 2/3}$

$\Rightarrow 6 a^{\frac{3}{3}}$

$\Rightarrow 6 a^{1}$

$= 6a$

#

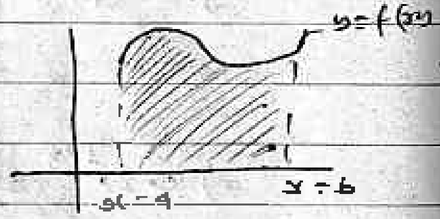
$x = a$

dx

Area under the curve

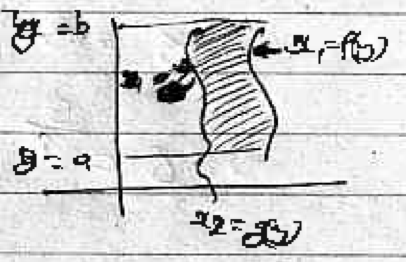
① The area under the curve $y = f(x)$, x-axis $(y = 0)$, from $x = a$ to $x = b$

$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)\,dx$$



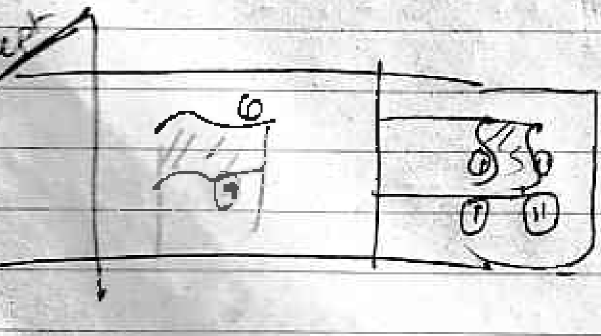
(ii) Area between two curve $x_1 = f(y)$ & $x_2 = g(y)$ y-axis from point $y = a$ to $y = b$ is given by

$$\text{Area} = \int_a^b (x_2 - x_1)\,dy$$

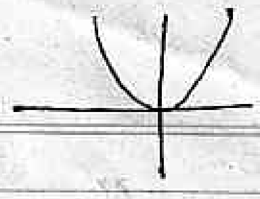


$$\Rightarrow \int_a^b (g(y) - f(y))\,dy$$

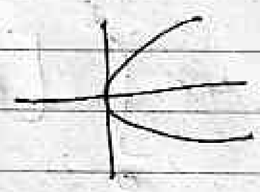
concept



$x^2 = 4by \rightarrow$



$y^2 = 4ax \rightarrow$



qn

Show that the area of Asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$.

Here,

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$$

If $x = 0$,

$$y = \pm a$$

$$\therefore \text{Area} = 4 \times \int_0^a y\,dx$$

$$= 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$$

$\Rightarrow$ put $x = a\sin^3\theta$

$$\frac{dx}{d\theta} = a\,3\sin^2\theta \cdot \cos\theta$$

$$\therefore dx = 3a\sin^2\theta\cos\theta\,d\theta$$

$x = 0,\ \theta = 0$, If $x = a,\ \theta = \pi/2$

now,

$$\text{Area} = 4 \times \int_0^{\pi/2} \left[a^{2/3} - (a\sin^3\theta)^{2/3}\right]^{3/2} \cdot 3a\sin^2\theta\cos\theta\, d\theta$$

$$= 4 \times 3 \times a \times \int_0^{\pi/2} a \cdot \left[1 - \sin^2\theta\right]^{3/2} \cdot \sin^2\theta\cos\theta\, d\theta$$

$$= 12a^2 \times \int_0^{\pi/2} \cos^3\theta \cdot \sin^2\theta\cos\theta\, d\theta$$

$$= 12a^2 \times \int_0^{\pi/2} \cos^4\theta \sin^2\theta \cdot d\theta$$

using Gamma function

$$= 12a^2 \times \frac{\Gamma\frac{4+1}{2} \cdot \Gamma\frac{2+1}{2}}{2\,\Gamma\frac{4+2+2}{2}}$$

$$= \cancel{12}a^2 \times \frac{3/2 \times 1/2 \times \Gamma 1/2 \times 1/2\, \Gamma 1/2}{\cancel{2 \times 3!}}$$

$$\Rightarrow a^2 \times \frac{3}{4} \times \pi \times \frac{1}{2} \times \pi$$

$$\Rightarrow \frac{3a^2\pi^2}{8} \quad \text{sq. unit.}$$

vertex $\Rightarrow (x-h)^2 = 4a(y-k)$

$\boxed{V = \pi \int_a^b y^2 \, dx}$

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volume of solid revolution

imp

(1) find the volume of solid in the region bonded by the curve $y = x^2 + 1$ & the line $y = -x + 3$ revolve about x-axis.

Soln

The curve is $y = x^2 + 1$
line is $y = -x + 3$

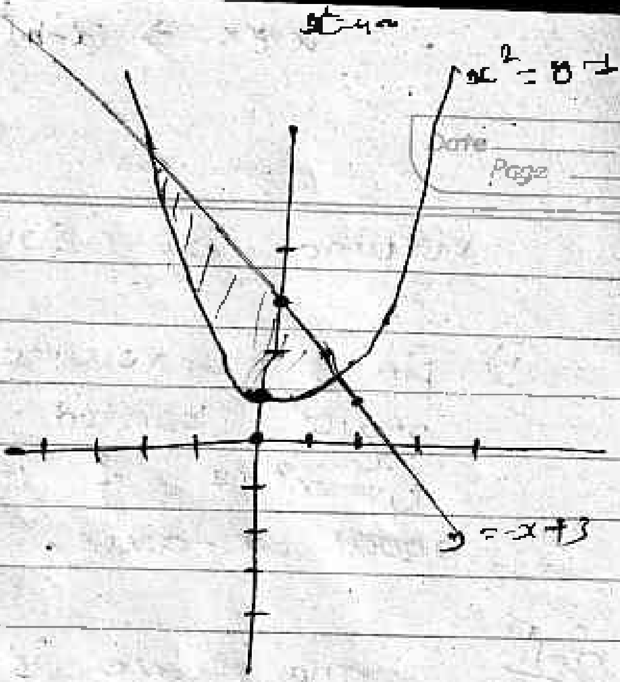
To find the vertex
$y = x^2 + 1$
$x = (y-1)^{1/2}$
$x^2 = (y-1)$
$(x-0)^2 = (y-1)$ — (1)

comparing (1) with
$(x-h)^2 = 4a(y-k)$

$\therefore x = 0, \; y = 1$
$\therefore$ The vertex of curve is $(0, 1)$

for line : $y = -x + 3$

x	0	1	2
y	3	2	1



The limit is

$y = x^2 + 1$ — (i)

$y = x + 3$ — (ii)

Equating (i) & (ii)

$x^2 + 1 = x + 3$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$x = -1, 2$

$\boxed{-1 \le x \le 2}$

$\therefore$ Area $= \pi \int_{-1}^{2} \left[(-x+3)^2 - (x^2+1)^2 \right] dx$

$$= \pi \int_{-1}^{2} \left(x^2 - 2x \cdot 3 + 9 - (x^2)^2 - 2x^2 \cdot 1 - 1\right) dx$$

$$= \pi \int_{-1}^{2} \left(x^2 - 6x + 9 - x^4 - 2x^2 - 1\right) dx$$

$$\Rightarrow \pi \left[\frac{x^3}{3} - \frac{6}{2}x^2 + 9x - \frac{x^5}{5} - \frac{2}{3}x^3 - 1x \right]_{-1}^{2}$$

$$\Rightarrow \pi \left[\frac{1}{3}(2^3+1^3) - \frac{6}{2}(2^2+1^2) + 9(2+1) - \frac{1}{5}(2^5+1) - \frac{2}{3}(2^3+1) - 1[2+1] \right]$$

$$= \pi \left[3 - 3\times5 + 9\times3 - \frac{33}{5} - 6 - 3 \right]$$

$$= \pi \left[3 + 27 - 15 - 6 - 3 - \frac{33}{5} \right]$$

$$= \pi \left[6 - \frac{33}{5} \right.$$

Amitha.

Second order ordinary differential eqn.

A differential equation of form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \text{--- (1)}$$

it is non-homogenous eqn.

where,

P, Q & R are function of x or constant.

If $R = 0$, then (1) would be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

i.e. a homogenous eqn.

Terms	Real - Logic	
Solution method for 2nd order linear homogenous eqn.	with constant coefficient	with Variable coefficient (Euler Cauchy)
The form of eqn is	$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$	$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$
Both M_1 & M_2 are real and distinct function	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$	$y = c_1 x^{m_1} + c_2 x^{m_2}$
Both m_1 & M_2 are real & equal $(m_1 = M_2 = m)$	$y = (c_1 + c_2 x) e^{mx}$	$y = (c_1 + c_2 \log x) x^{(\frac{1-a}{2})}$
If the roots (M_1, m_2) are imaginary ie: $m = \alpha \pm \beta i$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$	$y = x^{\alpha}[A\cos(\beta\log x) + B\sin(\beta\log x)]$

constent coeff. वाला
use hunxa (हुन्छ)।

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By method of variation of parameter (wronskian method)

example :

① $y'' - y' - 2y = 2e^{-x}$ — R

Sol^n

consider a homogenous eqn.

$y'' - y' - 2y = 0$

and, $R = 2e^{-x}$

~~I.e.~~ ~~Auxilo~~ ~~Axi~~

It's Auxiliary equation (A·E) is

$m^2 - m - 2 = 0$ — ①

solving the eq^n ①

$m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$(m+1)(m-2) = 0$

$m = -1, 2$ ---- ⓐ

since, the booth roots are real & dinstint.

By using constent coefficient.

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It's complementary solution is,

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\therefore y_h(x) = c_1 e^{-1x} + c_2 e^{2x} \quad \text{—} (1) \quad \left[\because m_1 = -1,\ m_2 = 2 \text{ from } (*)\right]$$

from eqn (1)

$y_1 = e^{-x}$ and $y_2 = e^{2x}$

also,

$y_1' = -1 e^{-x}$ and $y_2' = 2e^{2x}$

now,

by using wronskian method.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \Rightarrow \begin{vmatrix} e^{-x} & e^{2x} \\ -1e^{-x} & 2e^{2x} \end{vmatrix}$$

$$W \Rightarrow (e^{-x}).(2e^{2x}) - [(-1e^{-x}).(e^{2x})]$$

$$W \Rightarrow 2.e^{x} - (-1e^{x})$$

$$W \Rightarrow 2e^{x} + e^{x}$$

$$W \Rightarrow 3e^{x}$$

now,

for parametric solution

$$y_p = -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx$$

$$y_p = -y_1 \int \frac{y_2 R}{W}\, dx + y_2 \int \frac{y_1 R}{W}\, dx$$

$$y_p = -(e^{-x}) \cdot \int \frac{(e^{2x}) \times (2e^{-x})}{3e^{x}}\, dx + e^{2x} \int \frac{(e^{-x}) \times (2e^{-x})}{3e^{x}}\, dx$$

$$y_p = -e^{-x} \int \frac{2}{3} e^{2x-x-x}\, dx + e^{2x} \int \frac{2}{3} e^{-x-x-x}\, dx$$

$\boxed{e^0 = 1}$

$$y_p = -e^{-x} \times \frac{2}{3} \int 1\, dx + e^{2x} \times \frac{2}{3} \int e^{-3x}\, dx$$

$$y_p = -e^{-x} \times \frac{2}{3} \cdot (x) + e^{2x} \times \frac{2}{3} \frac{e^{-3x}}{-3}$$

$$y_p(x) = -\frac{2}{3} x e^{-x} - \frac{2}{9} e^{-x}$$

होरे x वाले terms मात्र लेखने

Finally,

The general solution of non homogenous eqn is

$$y = y_h(x) + y_p(x)$$
$$= c_1 e^{-x} + c_2 e^{2x} - \frac{2}{3} x e^{-x} - \cancel{\frac{2x}{5} e^{-x}}$$

#

Integration

का IMP Topics

(I) Gamma function (5) (Γ Γ)

(II) Definite integral (5)

(III) Purai Integration आंख 4 (5)
for (5) marks

Best of luck Buddy!

HARE krishna ♡!

♡C!