清华大学本科生考试试题专用纸

考试课程　　　　　　　　　　　　　　　　　　　年　月　日

Linear Algebra (in English) **Name:** ____________
2020 Fall
Final Exam
2020.12.23
Exam Duration: 3 Hours **Student ID:** ____________

This exam includes 8 pages (including this page) and 7 problems. Please check to see if there is any missing page, and then write down your name and student ID number on this page and the first page of your answer sheets. Also write down the initials of your name on the top of every page of your answer sheets, in case they are scattered.

This exam is open book. You are allowed to consult your textbook and notes, but no calculator. Plagerism of all kinds are strictly forbidden and will be severly punished.

Please write down your answers to the problems in the provided **SEPARATE ANSWER SHEETS**, and follow the following rules:

- **Always explain your answer.** You should always explain your answers. Any problem answered with nothing but a single answer would receive no credit.
- **Write cleanly and legible.** Make sure that your writings can be read. The graders are NOT responsible to decipher illegible writings.
- **Partial credits will be given.**
- Blank spaces are provided in the exams. Feel free to use them as scratch papers. However, your formal answer has to be written in the **SEPARATE ANSWER SHEETS**, as required by the University.
- The total score of the exam is 80. If your total score exceeds 80 (there are 85 points in total), it will be recorded as 80.

Problem	Points	Score
1	11	
2	13	
3	16	
4	13	
5	9	
6	14	
7	9	
Total:	85	

1. Consider the matrix $A = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

   (a) (2 points) How many terms does the big formula for $\det A$ has? Also find this determinant.

   (b) (5 points) Find the characteristic polynomial of A, and find all eigenvalues and eigenvectors of A.

   (c) (2 points) Find all vectors $\boldsymbol{v} \in \mathbb{R}^4$ such that $\lim_{n\to\infty}(\frac{1}{3}A)^n\boldsymbol{v}$ exists.

   (d) (2 points) Find a polynomial $p(x)$ such that the eigenvalues of $p(A)$ are $1, 0, 0, 0$.

2. We have points $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ on the plane $\mathbb{R}^2$. Together they form a data matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$. (Note that the points are already centered.)

   (a) (4 points) Find the spectral decomposition of AA^{T}. (Note that the eigenvector for the largest eigenvalue is the direction of the best fit line.)

   (b) (3 points) Find the $\mathrm{LDL}^{\mathrm{T}}$ decomposition of AA^{T}. (Note that the bottom left entry of L is the slope for the line from linear regression.)

   (c) (4 points) Find all singular values and all left and right singular vectors of A. (Note that there might be singular vectors for the singular value zero.)

   (d) (2 points) Find the maximum and minimum Rayleigh quotient $\frac{\boldsymbol{v}^{\mathrm{T}} S \boldsymbol{v}}{\boldsymbol{v}^{\mathrm{T}} \boldsymbol{v}}$ for $S = (A^{\mathrm{T}} A)^2 - 2A^{\mathrm{T}} A + 2I$.

3. We have vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 7 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 7 \\ 6 \\ 11 \end{bmatrix}$ in the space $\mathbb{R}^4$. We wish to find all vectors $\boldsymbol{x}$ perpendicular to $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

   (a) (4 points) Find a 3×4 matrix A such that $\boldsymbol{x}$ is perpendicular to $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ if and only if $A\boldsymbol{x} = \mathbf{0}$.

   (b) (4 points) Find a basis for $\mathrm{Ker}(A)$.

   (c) (4 points) Find all solutions of $A\boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$.

   (d) (4 points) Let $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find all points perpendicular to $B\boldsymbol{v}_1, B\boldsymbol{v}_2, B\boldsymbol{v}_3$.

4. Consider $A = \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 2 & 8 \end{bmatrix}$. We aim to find the orthogonal projection matrix to $\mathrm{Ran}(A)$.

   (a) (4 points) Find the LL^{T} decomposition of $A^{\mathrm{T}}A$.

   (b) (4 points) Find the QR decomposition of A. (For simplicity, we want R to be upper triangular, while Q can be non-square but has orthonormal columns.)

   (c) (3 points) Find the 3×3 matrix of orthogonal projection to $\mathrm{Ran}(A)$.

   (d) (2 points) Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be the left singular vectors of A for its two singular values σ_1, σ_2. Can you find $\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^{\mathrm{T}} \\ \boldsymbol{u}_2^{\mathrm{T}} \end{bmatrix}$? (Hint: You may not need to calculate anything. But in that case you would need to show your arguments still.)

5. Consider the real matrix $M = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}$ where A, C are symmetric, and A, B, C are all 2×2 real matrices.

   (a) (3 points) If B is invertible, find a formula for $\det(M)$ in terms of determinants of 2×2 matrices.

   (b) (4 points) Suppose $B = 2A$ and $C = 4A$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find all eigenvalues and eigenvectors of M.

   (c) (2 points) (Hard) Suppose $A = C = 0$, and B has singular values $2, 1$. Find all eigenvalues of M.

6. Consider the space V whose vectors are real functions of the form $(ax^2+bx+c)\mathrm{e}^{2x}$ for constants $a, b, c \in \mathbb{R}$, and vector additions and scalar multiplications are defined in the obvious mannor.

   (a) (2 points) Show that if $f \in V$, then its derivative f' is also in V. (So in particular, taking derivative is a linear map $D : V \to V$)

   (b) (4 points) Using basis $\mathrm{e}^{2x}, x\mathrm{e}^{2x}, \frac{1}{2}x^2\mathrm{e}^{2x}$, write out the corresponding matrix for D. Is this matrix in the above subproblem diagonalizable? Why?

   (c) (4 points) Write out the change of coordinate matrix from basis $\mathrm{e}^{2x}, x\mathrm{e}^{2x}, \frac{1}{2}x^2\mathrm{e}^{2x}$ to basis $x^2\mathrm{e}^{2x}, (2x^2+2x)\mathrm{e}^{2x}, (4x^2+8x+2)\mathrm{e}^{2x}$.

   (d) (4 points) Using basis $x^2\mathrm{e}^{2x}, (2x^2+2x)\mathrm{e}^{2x}, (4x^2+8x+2)\mathrm{e}^{2x}$, write out the corresponding matrix for D. What is the characteristic polynomial of this matrix?

7. (Proof-intensiven problem) Suppose $A^{\mathrm{T}} = A^2$ for a real matrix A. Let us investigate the possibility of such A. Note that A is a normal matrix.

   (a) (2 points) For each eigenvalue λ of A, show that $\lambda^2 = \overline{\lambda}$, and find all possible $\lambda \in \mathbb{C}$ that satisfy this condition. (Hint: Spectral decomposition of A.)

   (b) (2 points) Show that $A^4 = A$.

   (c) (2 points) Show that $A^2\boldsymbol{x} = \boldsymbol{x}$ implies that $A\boldsymbol{x} = \boldsymbol{x}$.

   (d) (2 points) Show that A^3 is an orthogonal projection.

   (e) (1 point) Show that $I + A - A^3$ is an orthogonal and $(I + A - A^3)^3 = I$. Also prove that $\det(I + A - A^3) = 1$, so this is a rotation with period 3.