Midterm for Linear Algebra

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Generic Rules:

1. The due date is Nov 12th 2AM. As this is your midterm, NO LATE WORK will be accepted.
2. Each sub-problem is worth 2 points. So you can get as much as 64 pts in total. However, we use 60 pts as full credit. If you get more than 60 pts, then your score is 60. (So you can safely skip two subproblems if you want.) We will give partial credits for partial solutions.
3. Feel free to use calculators or softwares to help your computations, or collaborate with your classmates. But in your submitted midterm, you should always express your ideas in your own words. **Write down the names of your collaborators if you collaborated with someone.** (Also, if you collaborated with someone, then each of you should hand in your own midterm answer in your own words. Do NOT give me a joint answer or copied answer, as it would be treated as plagiarism.)
4. If I suspect plagiarism, I might call you into my office, and ask you to do the problem in front of me to see if you actually understand what you have written down. You should be able to explain your own written answers. Failure to do so will be a confirmation of plagiarism, and will be punished accordingly.
5. Write your answers in English. You should always explain any statement you made, and always show some process of calculation (which may help you earn some partial credit in case you got it wrong). This is a mid-term after all, so please be more formal and careful in your answers.

Problem 1 (Affine maps)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be any affine map, i.e., $f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ for some constant $n \times n$ matrix A and some constant vector $\boldsymbol{b} \in \mathbb{R}^n$. Recall that the matrix for f is $M_f = \begin{bmatrix} A & \boldsymbol{b} \\ & 1 \end{bmatrix}$.

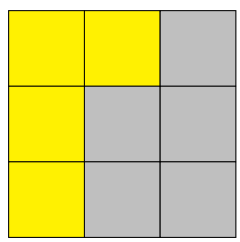
1. Suppose $n = 2$. Let f be the rotation counter clockwise by θ around some point $\boldsymbol{p} \in \mathbb{R}^2$. Find M_f.
2. Suppose $n = 2$. If $M_f = \begin{bmatrix} R_{\frac{\pi}{2}} & \boldsymbol{e}_1 \\ & 1 \end{bmatrix}$, find the center of the rotation by f.
3. Suppose some hyperplane in $\mathbb{R}^n$ has unit normal vector $\boldsymbol{n}$ and goes through some point $\boldsymbol{p} \in \mathbb{R}^n$. If f is the projection to this hyperplane, find M_f, and verify that $M_f^2 = M_f$ by block matrix multiplication.
4. Suppose some hyperplane in $\mathbb{R}^n$ has unit normal vector $\boldsymbol{n}$ and goes through some point $\boldsymbol{p} \in \mathbb{R}^n$. If f is the reflection about this hyperplane, find M_f, and verify that $M_f^2 = I$ by block matrix multiplication.
5. Say we have two distinct points $\boldsymbol{a} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$. Let f be the reflection that sends $\boldsymbol{a}$ to $\boldsymbol{b}$, find M_f.

Problem 2 (Soccer Team Prediction)**.** Suppose the recent three matches of a soccer team resulted in the scores $0:1$, then $1:2$, then $1:3$. (They lose all three matches....) Suppose there is a mysterious linear map A that we can use to predict future scores. If the previous match has a score $a:b$, then the scores for the next match are the coordinates of $A\begin{bmatrix} a \\ b \end{bmatrix}$.

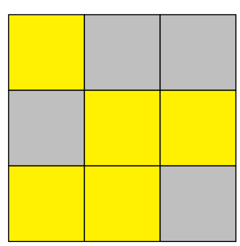
Let $\boldsymbol{v}_i$ be the scores of the i-th match, e.g., $\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and so on. By our construction, we should have $A\boldsymbol{v}_k = \boldsymbol{v}_{k+1}$. We further define $\boldsymbol{v}_0 = A^{-1}\boldsymbol{v}_1, \boldsymbol{v}_{-1} = A^{-1}\boldsymbol{v}_0$ and so on.

1. Find A and A^{-1}.
2. Show that this soccer team would lose forever.
3. Find the largest k such that $\boldsymbol{v}_k$ has first coordinate larger than the second coordinate. (We should have $k \leq 0$.)
4. Show that for all integer k, $A^k = \begin{bmatrix} -\boldsymbol{v}_m & \boldsymbol{v}_n \end{bmatrix}$ for some m, n depending on k.
5. Find all linear relations among $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. Can you generalize this relation to $\boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}$? Can you write the coordinates of $\boldsymbol{v}_k$ according the the Fibonacci sequence?
6. Find a degree two polynomial $p(x)$ such that $p(A) = 0$. (Hint: look at last subproblem.)

Problem 3 (Light out)**.** The Light out game is a very fun traditional puzzle. We start with some configuration like this.



Each tile is a light, and it might be on or off. Our goal is to turn all the lights off. Now, if we press a tile, then this tile and all adjacent tiles will change status. For example, if I press the middle tile in the example here, we will obtain



To better study this, let us give names to the tiles in the game:

1	2	3
4	5	6
7	8	9

For example, if I say the third tile, it will mean the tile on the upper right corner.

All linear algebra here is over $\mathbb{F}_2 = \{0, 1\}$, where we define $0+0 = 1+1 = 0$, $0+1 = 1+0 = 1$, $0 \times 0 = 0 \times 1 = 1 \times 0 = 0$, $1 \times 1 = 1$. (You may think of 0 as "even", and 1 as "odd".)

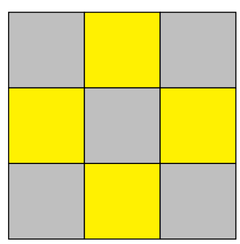
Given a vector $\boldsymbol{x} \in \mathbb{F}_2^9$, say $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then there are two interpretations. If we say we press $\boldsymbol{x}$, then it means we press the second and third tile. If we say the status of the light is $\boldsymbol{x}$, then it means tiles at location 2 and location 3 have lights on, and the other tiles are off.

Say if the status of the light is currently some vector $\boldsymbol{b}_0$. If I press some tiles by some vector $\boldsymbol{x}$, then the status of the lights will change to $\boldsymbol{b}_0 + \boldsymbol{y}$, where $\boldsymbol{y}$ is the change of status induced by $\boldsymbol{x}$. The process $\boldsymbol{x} \mapsto \boldsymbol{y}$ is a linear map f.

Let $B = \begin{bmatrix} 1 & 1 & \\ 1 & 1 & 1 \\ & 1 & 1 \end{bmatrix}$. $A = \begin{bmatrix} B & I & \\ I & B & I \\ & I & B \end{bmatrix}$.

1. We defined the linear map f to be the one that sends the vector we press to the vector of status change. Show that the matrix for f is A.

2. Suppose the current status is like the following. Which buttons should I press to turn all the lights off?



3. Prove that A is invertible, and find its block LDU decomposition or show that this does not exist.

4. Show that I, B, B^2 are linearly independent, but I, B, B^2, B^3 are not. (Reminder: we are over $\mathbb{F}_2$.)

5. Suppose my light out tiles are in an m by n array. Again let f be the linear map sending the vector we press to the vector of status change. Can you find the matrix for f?

Problem 4 (Waterflow in the ocean)**.** Suppose we want to study water flow deep under the ocean. Say the ocean is $\mathbb{R}^3$. Suppose at location $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and at time t, the velocity of the water flow here is $\begin{bmatrix} -y+z-t \\ x+z+2t \\ -x-y-3t \end{bmatrix}$. Note that sometimes some location will have zero velocity, and this is a swirl center.

1. Find a matrix A such that $A \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$ is the velocity of the water flow at location $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and at time t.

2. Find a basis for $\mathrm{Ran}(A)$. What dimension is this? (This describes the directions for the layers of swirls. At least in our model here, no water can escape their own layer.)

3. Find a basis for $\mathrm{Ker}(A^{\mathrm{T}})$. Also find equation/equations whose solution space is exactly $\mathrm{Ran}(A)$.

4. If a water molecule is in location $\begin{bmatrix}1\\1\\1\end{bmatrix}$, will it ever flow to $\begin{bmatrix}2\\2\\2\end{bmatrix}$? Why or why not?

5. Find a basis for $\mathrm{Ker}(A)$. (This means when and where will the water be stationary inside the ocean.)

6. At each time, the stationary locations inside the ocean (i.e., locations with zero velocity) would form a line. These are the "swirl centers" As time goes by, this line will gradually move around. Please describe the direction of this line at time t, and describe how the line changes as t changes.

7. (This is not linear algebra, but you might be curious about this.) Verify that

$$\boldsymbol{w}(t) = \frac{1}{3}\begin{bmatrix} 2\cos(\sqrt{3}t) - 5t + 4 \\ \sqrt{3}\sin(\sqrt{3}t) + \cos(\sqrt{3}t) - 4t - 3 \\ -\sqrt{3}\sin(\sqrt{3}t) + \cos(\sqrt{3}t) - t - 8 \end{bmatrix}$$

is a valid trajectory according to our model, i.e., at time t, $\boldsymbol{w}'(t) = A\begin{bmatrix}\boldsymbol{w}(t)\\ t\end{bmatrix}$.

Problem 5 (Chemical Reaction)**.** A box contains carbon dioxide (CO_2), water (H_2O) and oxygen (O_2). To describe the content of this box, we may say it contains x carbon dioxide molecules, y water molecules, and z oxygen molecules, and write it as $\begin{bmatrix}x\\y\\z\end{bmatrix}$. On the other hand, we may also say it contains r carbon atoms, s hydrogen atoms, and t oxygen atoms, and write it as $\begin{bmatrix}r\\s\\t\end{bmatrix}$

In short, we have a basis $\mathcal{B}$ in terms of molecules (carbon dioxide molecule, water molecule, oxygen molecule). We have another basis $\mathcal{C}$ in terms of atoms (carbon atom, hydrogen atom, oxygen atom).

1. If the content of my box is $\begin{bmatrix}x\\y\\z\end{bmatrix}$ under basis $\mathcal{B}$, what will it be under basis $\mathcal{C}$? What is the change of coordinate matrix from basis $\mathcal{B}$ to basis $\mathcal{C}$? And what is the change of coordinate matrix from basis $\mathcal{C}$ to basis $\mathcal{B}$?

2. I sell the contents of this box for money. Say each carbon dioxide molecule is 3 yuan, water molecule 5 yuan, oxygen molecule 4 yuan. If the content of my box is $\begin{bmatrix}x\\y\\z\end{bmatrix}$ under basis $\mathcal{C}$, how much money would I get?

3. Say I used some super chemical reaction, and all the molecules in my box turned into glucose ($C_6H_{12}O_6$), and some left over carbon dioxide (CO_2) and oxygen (O_2). There is no water left. This process corresponds to some linear map f. In the codomain of f, we use the basis in terms of molecules, i.e., (glucose molecule, carbon dioxide molecule, oxygen molecule). In the domain, we use basis $\mathcal{B}$. Then what is the matrix for f?

4. For the map above, if the domain and codomain both use bases in terms of atoms, i.e., (carbon atom, hydrogen atom, oxygen atom), what is the matrix for f?

Problem 6 (How to Lie Down (Rú Hé Tăng Píng)). (This problem is an example of linear optimization problems, which have wide applications and technically not part of our class. But it is still a part of linear algebra, and it is fun to do.)

Suppose you have three classes: calculus, linear algebra and probability. Say the semester has three months in total, and after the first month, linear algebra and probability will have a midterm. After the second month, calculus, linear algebra and probability will all have a midterm. Finally, after all three months, all three classes will have a final.

The score for each class is calculated as the following:

$$\begin{aligned}
\text{Calculus Score} =& 0 \times \text{Midterm One} + 0.2 \times \text{Midterm Two} + 0.8 \times \text{Final}.\\
\text{Linear Algebra Score} =& 0.3 \times \text{Midterm One} + 0.3 \times \text{Midterm Two} + 0.4 \times \text{Final}.\\
\text{Probability Score} =& 0.4 \times \text{Midterm One} + 0.3 \times \text{Midterm Two} + 0.3 \times \text{Final}.
\end{aligned}$$

Here you see a matrix $A = \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$, recording the grading patterns of the three classes.

Now, a student will spend efforts in order to get good grades. Let's say a student gives 100% effort in the first month, second month and the third month. Then the effort vector of this student is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and the scores for this student would be $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so this student would get 100% score in everything. In general, a student might give efforts a, b, c in the first month, second month and the third month, where $0 \leq a, b, c \leq 1$. Then the effort vector is $\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and the scores for this student for these classes would be $A\boldsymbol{v}$.

Say all classes needs a score at least 0.6 to pass, and a score at least 0.9 to ace. Your goal is to pass or ace these classes with the minimal amount of total efforts, i.e., you want to minimize $a + b + c$.

1. If you want to pass all three classes, what is the minimal amount of total effort you need? Give me your effort vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Keep in mind that $0 \leq a, b, c \leq 1$. (Surely $\begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}$ would pass all three classes. Can you get $a + b + c < 1.8$?)

2. If you want to ace all three classes, what is the minimal amount of total effort you need? Give me your effort vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Keep in mind that $0 \leq a, b, c \leq 1$. (Can you get $a + b + c < 2.7$?)

3. If you want to ace at least one class and pass all classes, what is the minimal amount of total effort you need? Give me your effort vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Keep in mind that $0 \leq a, b, c \leq 1$.

4. Suppose the university redesigned the scoring system for these classes, and now choose A as the identity matrix. Show that if a student wants to pass all classes or ace all classes, then this student must spend efforts evenly throughout the semester. (Of course, this A is horrible for other reasons. . . .)

5. If you are the university and you are in charge of the design of A. Suppose all students want to ace at least one class and passes all classes. How would you design A? There is no wrong answer, just give me your design and your argument for it.