Linear Algebra (English)
2023 Fall
Final Exam
2024.1.10
Exam Duration: 3 Hours

Name: ____________________

Student ID: ____________________

This exam includes 11 pages (including this page) and 5 problems. Please check to see if there is any missing page, and then write down your name and student ID number on this page and the first page of your answer sheets. Also write down the initials of your name on the top of every page of your answer sheets, in case they are scattered.

This exam is open book. You are allowed to consult your textbook and notes, and elementary calculators are fine (those that cannot instantly multiply matrices). Plagerism of all kinds are strictly forbidden and will be severly punished.

Please write down your answers to the problems in the **SEPARATE ANSWER SHEETS**, and follow the following rules:

- **Always explain your answer.** You should always explain your answers. Any problem answered with nothing but a single answer would receive no credit.
- **Write cleanly and legible.** Make sure that your writings can be read. The graders are NOT responsible to decipher illegible writings.
- **Partial credits will be given.**
- Blank spaces are provided in the exams. Feel free to use them as scratch papers. However, your formal answer has to be written in the **SEPARATE ANSWER SHEETS**, as required by the University.
- The total score of the exam is 60. If your total score exceeds 60 (there are 64 points in total), it will be recorded as 60.

Problem	Points	Score
1	14	
2	12	
3	17	
4	14	
5	7	
Total:	64	

1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$, which looks like a chinese character “田”.

   (a) (4 points) Find an orthonormal basis for the kernel of A. What property of A guarantees that its kernel and range are orthogonal complements?

   (b) (4 points) Note that the two columns of $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ form a basis for the range of A. Perform QR decomposition for B, what is the 5×2 matrix Q and what is the upper triangular R with positive diagonal entries?

   (c) (2 points) For the matrix Q above, find a matrix X such that $A = QXQ^{\mathrm{T}}$. Is such a matrix X unique? Prove it or show counter example.

   (d) (2 points) How are the eigenvalues, geometric and algebraic multiplicities, and eigenvectors of A and X related? Prove why.

   (e) (2 points) Find the orthogonal projection to $\mathrm{Ran}(A)$.

Answer:

1. $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\0\\-2\\0\\1\end{bmatrix}$. A is real and symmetric, so its kernel and range are orthogonal complements.
2. $Q = \begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0\end{bmatrix}$, $R = \begin{bmatrix}\sqrt{3} & \sqrt{3}\\ 0 & \sqrt{2}\end{bmatrix}$.
3. $X = \begin{bmatrix}3 & \sqrt{6}\\ \sqrt{6} & 0\end{bmatrix}$, and this is unique. If $QX_1Q^{\mathrm{T}} = QX_2Q^{\mathrm{T}}$, then we multiply Q^{T} on the left and multiply Q on the right. Then $X_1 = X_2$.
4. Note that $A = (QX)Q^{\mathrm{T}}$ while $X = Q^{\mathrm{T}}(QX)$. So they have the same eigenvalues and algebraic multiplicities, except that X has no zero eigenvalue and A has eigenvalue 0 with algebraic multiplicity 3. Since they are both real symmetric, they are both diagonalizable, hence they have the same geometric multiplicities as their algebraic multiplicity. Finally, if $A\boldsymbol{v} = \lambda\boldsymbol{v}$, then $QXQ^{\mathrm{T}}\boldsymbol{v} = \lambda\boldsymbol{v}$, and hence $XQ^{\mathrm{T}}\boldsymbol{v} = \lambda Q^{\mathrm{T}}\boldsymbol{v}$. And since $\mathrm{Ker}(A) = \mathrm{Ker}(Q^{\mathrm{T}})$, if $A\boldsymbol{v} \neq \mathbf{0}$, then $Q^{\mathrm{T}}\boldsymbol{v} \neq \mathbf{0}$. So for each non-zero eigenvalue λ, Q^{T} would send the corresponding eigenvector of A to some corresponding eigenvector of X. Conversely, for each non-zero eigenvalue λ, if $X\boldsymbol{v} = \lambda\boldsymbol{v}$, then $AQ\boldsymbol{v} = QXQ^{\mathrm{T}}Q\boldsymbol{v} = \lambda Q\boldsymbol{v}$, so Q would send the corresponding eigenvector of X to some corresponding eigenvector of A.
5. The range of A is the same as the range of Q. $QQ^{\mathrm{T}} = \begin{bmatrix}\frac{1}{3} & & \frac{1}{3} & & \frac{1}{3}\\ & \frac{1}{2} & & \frac{1}{2} & \\ \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3}\\ & \frac{1}{2} & & \frac{1}{2} & \\ \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3}\end{bmatrix}$.

2. Behold the earliest record of Gaussian elimination in history: the Nine Chapters of Mathematical Arts (Jiuzhang Suanshu) in ancient China, of course! In this ancient text, a linear equation $A\boldsymbol{x} = \boldsymbol{b}$ is solved by Gaussian elimination, where $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix}$.

   Now, let $A_t = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1+t \\ 1 & 2 & 3 \end{bmatrix}$.

   (a) (4 points) Find the LDU decomposition for $A_t = L_t D_t U_t$. Do L_t, D_t, U_t depend on t continuously?

   (b) (1 point) When A_t is symmetric, is it positive definite or negative definite or else? Prove your claim.

   (c) (4 points) Let us consider a variant of QR decomposition, $A_t = Q_t R_t$ where Q_t has orthogonal columns (but no need to be unit vectors), and R_t is unit upper triangular. Find Q_t, R_t. Do they depend on t continuously?

   (d) (3 points) Find all t such that A_t is NOT diagonalizable, and prove the impossibility of diagonalizations in these cases. (This means that the diagonalization of A_t does NOT depend on t continuously, as we have found some locations of discontinuity.)

Answer:

1. $L_t = \begin{bmatrix} 1 & & \\ \frac{2}{3} & 1 & \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & & \\ & \frac{5}{3} & \\ & & \frac{12}{5} - \frac{4}{5}t \end{bmatrix}$, $U_t = \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ & 1 & \frac{1}{5} + \frac{3}{5}t \\ & & 1 \end{bmatrix}$. They all depends on t continuously.
2. Only A_1 is symmetric, and the pivots are all positive in this case.
3. $R_t = \begin{bmatrix} 1 & 1 & \frac{4}{7} + \frac{1}{7}t \\ & 1 & 1 + \frac{1}{3}t \\ & & 1 \end{bmatrix}$, $Q_t = \begin{bmatrix} 3 & -1 & \frac{2}{7} - \frac{2}{21}t \\ 2 & 1 & -\frac{8}{7} + \frac{8}{21}t \\ 1 & 1 & \frac{10}{7} - \frac{10}{21}t \end{bmatrix}$. They depends on t continuously.
4. We need repeated eigenvalues. Note that 2 is always an eigenvalue. Let the eigenvalues be $\lambda_1, \lambda_2, 2$, then by trace and determinant, we have $\lambda_1 + \lambda_2 = 7, \lambda_1\lambda_2 = 6 - 2t$. If 2 is a repeated eigenvalue, then $\lambda_1 = 2, \lambda_2 = 5$, so $t = -2$. In this case, $A_{-2} - 2I$ has rank 2 because the first two columns are not parallel. Therefore, the geometric multiplicity of 2 is one, hence A_{-2} is indeed not diagonalizable. If 2 is not a repeated eigenvalue, then we must have $\lambda_1 = \lambda_2$, so both are $\frac{7}{2}$, and $t = -\frac{25}{8}$. In this case, $A_t - \frac{7}{2}I$ has rank 2 because the first two columns are not parallel. Therefore, the geometric multiplicity of $\frac{7}{2}$ is one, hence A_t is indeed not diagonalizable. For all $t \neq -2, -\frac{25}{8}$, then we must have three distinct eigenvalues, so A_t will be diagonalizable.

3. Let V be the set of all 2×2 Hermitian matrices. Note that $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \in V$, but $\begin{bmatrix}\mathrm{i} & 0\\ 0 & \mathrm{i}\end{bmatrix} \notin V$, so this is NOT a complex vector space. However, for any real $s, t \in \mathbb{R}$ and any $X, Y \in V$, then $sX + tY \in V$. So V is a REAL vector space. (Recall that a real vector space do NOT mean that all entries of elements are real. They only mean that elements can do real-coefficient linear combinations, but NOT complex-coefficient linear combinations.)

Let $X_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $X_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $X_3 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $X_4 = \begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix}$. This is a basis for the REAL vector space V.

Let $A = \begin{bmatrix}3 & \mathrm{i}\\ 1+\mathrm{i} & 2\end{bmatrix}$. Let $L_1 : V \to V$ such that $L_1(X) = \frac{1}{2}(AX + XA^*)$, and let $L_2 : V \to V$ such that $L_2(X) = \frac{1}{2\mathrm{i}}(AX - XA^*)$.

(a) (3 points) Show that for any Hermitian 2×2 matrix X, then $L_1(X), L_2(X)$ are indeed still Hermitian. Also show that $L_1L_2 = L_2L_1$.

(b) (4 points) Find the matrices for L_1, L_2 under the basis (X_1, X_2, X_3, X_4).

(c) (4 points) Let the eigenvalues of A be λ, μ (say $|\lambda| > |\mu|$), and let $\boldsymbol{v}$ be an eigenvector for λ with first coordinate 1, $\boldsymbol{w}$ be an eigenvector for μ with first coordinate 1. Let $Y_1 = \boldsymbol{v}\boldsymbol{v}^*$, $Y_2 = \boldsymbol{w}\boldsymbol{w}^*$, $Y_3 = \boldsymbol{v}\boldsymbol{w}^* + \boldsymbol{w}\boldsymbol{v}^*$, $Y_4 = \mathrm{i}(\boldsymbol{v}\boldsymbol{w}^* - \boldsymbol{w}\boldsymbol{v}^*)$. Find the change of coordinate matrix from (Y_1, Y_2, Y_3, Y_4) to (X_1, X_2, X_3, X_4).

(d) (4 points) Find the matrices for L_1, L_2 under the basis (Y_1, Y_2, Y_3, Y_4).

(e) (2 points) When A is some unknown 2×2 complex matrix, with two distinct eigenvalues λ, μ, and $A\boldsymbol{v} = \lambda\boldsymbol{v}, A\boldsymbol{w} = \mu\boldsymbol{w}$ for some non-zero complex vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^2$. What can you deduce about the eigenvalues of L_1, L_2?

Answer:

1. Direct verification shows that images of L_1, L_2 are indeed still Hermitian. $L_1L_2(X) = L_2L_1(X) = \frac{1}{4\mathrm{i}}(A^2X - X(A^*)^2)$.

2. The matrix for L_1 is $\frac{1}{2}\begin{bmatrix} 6 & 0 & 0 & -2 \\ 0 & 4 & 2 & 2 \\ 1 & 0 & 5 & 0 \\ 1 & -1 & 0 & 5 \end{bmatrix}$. The matrix for L_2 is $\frac{1}{2}\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$.

3. $A = XDX^{-1}$ where $X = \begin{bmatrix} 1 & 1 \\ 1 & -1+\mathrm{i} \end{bmatrix}$ and $D = \begin{bmatrix} 3+\mathrm{i} & \\ & 2-\mathrm{i} \end{bmatrix}$. So $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1+\mathrm{i} \end{bmatrix}$. The change of coordinate matrix is $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & -2 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

4. The matrix for L_1 is $\frac{1}{2}\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$. The matrix for L_2 is $\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

5. Under basis (Y_1, Y_2, Y_3, Y_4), L_1 will have a matrix of $\frac{1}{2}\begin{bmatrix} 2\mathfrak{Re}(\lambda) & & & \\ & 2\mathfrak{Re}(\mu) & & \\ & & \mathfrak{Re}(\lambda+\overline{\mu}) & -\mathfrak{Im}(\lambda+\overline{\mu}) \\ & & \mathfrak{Im}(\lambda+\overline{\mu}) & \mathfrak{Re}(\lambda+\overline{\mu}) \end{bmatrix}$. So its eigenvalues are $\mathfrak{Re}(\lambda), \mathfrak{Re}(\mu), \frac{1}{2}(\lambda+\overline{\mu}), \frac{1}{2}(\overline{\lambda}+\mu)$. For L_2, its matrix will be $\frac{1}{2}\begin{bmatrix} 2\mathfrak{Im}(\lambda) & & & \\ & 2\mathfrak{Im}(\mu) & & \\ & & \mathfrak{Im}(\lambda-\overline{\mu}) & \mathfrak{Re}(\lambda-\overline{\mu}) \\ & & -\mathfrak{Re}(\lambda-\overline{\mu}) & \mathfrak{Im}(\lambda-\overline{\mu}) \end{bmatrix}$. So its eigenvalues are $\mathfrak{Im}(\lambda), \mathfrak{Im}(\mu), \frac{1}{2\mathrm{i}}(\lambda-\overline{\mu}), \frac{1}{2\mathrm{i}}(\overline{\lambda}-\mu)$.

4. Benin is a country in west Africa. As far as I know, its national flag is the only national flag in the world with rank two. If we denote green as 2, yellow as $\sqrt{2}$, and red as $-\sqrt{2}$, then its flag looks like a matrix $A = \begin{bmatrix} 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \end{bmatrix}$.

   (a) (4 points) Find the limit $\lim_{n\to\infty}(\frac{1}{12}AA^{\mathrm{T}})^n$.

   (b) (3 points) Find the singular value decomposition $A = U\Sigma V^{\mathrm{T}}$.

   (c) (2 points) The matrix $I - V \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} V^{\mathrm{T}}$ is an orthogonal projection to which of these subspaces, $\mathrm{Ran}(A), \mathrm{Ker}(A), \mathrm{Ran}(A^{\mathrm{T}}), \mathrm{Ker}(A^{\mathrm{T}})$? Pick the correct subspace and prove that this is indeed the corresponding orthogonal projection.

   (d) (2 points) Find the block LDU decomposition $\begin{bmatrix} AA^{\mathrm{T}} & -A \\ -A^{\mathrm{T}} & O \end{bmatrix} = LDL^{\mathrm{T}}$ where L is block unit lower triangular and D is block diagonal. Find all eigenvalues of D.

   (e) (1 point) Is $\begin{bmatrix} AA^{\mathrm{T}} & -A \\ -A^{\mathrm{T}} & O \end{bmatrix}$ positive semidefinite, negative semidefinite or indefinite? Prove your claim.

   (f) (2 points) Pick any $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{y} \in \mathbb{R}^4$. If $||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 = 1$, find the maximum and minimum of $\boldsymbol{x}^{\mathrm{T}}AA^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} - \boldsymbol{y}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{y}$.

Answer:

1. $AA^{\mathrm{T}} = \begin{bmatrix} 10 & -2 \\ -2 & 10 \end{bmatrix}$, it has eigenvalue 8 with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and eigenvalue 12 with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So $AA^{\mathrm{T}} = UDU^{-1}$ where $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 12 & \\ & 8 \end{bmatrix}$. So the limit is $U \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.
2. $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$, where the last two columns of V are arbitrary as long as V is orthogonal.
3. Since A has rank 2, the first two columns of V form an orthonormal basis for $\mathrm{Ran}(A^{\mathrm{T}})$. So $V \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} V^{\mathrm{T}}$ is the orthogonal projection to $\mathrm{Ran}(A^{\mathrm{T}})$. Hence I minus that is the orthogonal projection to the orthogonal complement of $\mathrm{Ran}(A^{\mathrm{T}})$, i.e., $\mathrm{Ker}(A)$.
4. $\begin{bmatrix} AA^{\mathrm{T}} & -A \\ -A^{\mathrm{T}} & O \end{bmatrix} = LDL^{\mathrm{T}}$ where $L = \begin{bmatrix} I & \\ -A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1} & I \end{bmatrix}$ and $D = \begin{bmatrix} AA^{\mathrm{T}} & \\ & -A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A \end{bmatrix}$. Note that the upper left block of D have eigenvalues $12, 8$ from AA^{T}. The lower right block of D have eigenvalues $-1, -1, 0, 0, 0$ because its negation is the orthogonal projection to $\mathrm{Ran}(A^{\mathrm{T}})$.
5. So all eigenvalues of D are both positive and negative eigenvalues, hence D is indefinite. So the same is true for the original matrix, which is congruent to D.
6. This is a Rayleigh quotient for $\begin{bmatrix} AA^{\mathrm{T}} & -A \\ -A^{\mathrm{T}} & A^{\mathrm{T}}A \end{bmatrix}$. By taking out $\begin{bmatrix} U & \\ & V \end{bmatrix}$ and $\begin{bmatrix} U^{\mathrm{T}} & \\ & V^{\mathrm{T}} \end{bmatrix}$ from both sides, the original matrix is similar to $\begin{bmatrix} 12 & 0 & -\sqrt{12} & 0 & & \\ 0 & 8 & 0 & -\sqrt{8} & & \\ -\sqrt{12} & 0 & 12 & 0 & & \\ 0 & -\sqrt{8} & 0 & 8 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$. This is hidden block diagonal, with zeros and blocks $\begin{bmatrix} 12 & -\sqrt{12} \\ -\sqrt{12} & 12 \end{bmatrix}, \begin{bmatrix} 8 & -\sqrt{8} \\ -\sqrt{8} & 8 \end{bmatrix}$ on the diagonal. So the eigenvalues are $12 \pm \sqrt{12}, 8 \pm \sqrt{8}, 0, 0$. So the largest eigenvalue is $12 + \sqrt{12}$ and the smallest eigenvalue is 0.

5. In a linear algebra class, the teacher wrote a linear system on the blackboard as Homework. The students are supposed to solve it by Gaussian elimination. However, in Li Lei's notes, the problem is $A\boldsymbol{x} = \boldsymbol{a}$. In Han Meimei's notes, the problem is $B\boldsymbol{x} = \boldsymbol{b}$, which is different from what Li Lei wrote down. Oh no, one of them (or both of them) must have recorded the problem wrong!

   Li Lei solved $\boldsymbol{x}$ from $A\boldsymbol{x} = \boldsymbol{a}$, and found a unique solution $\boldsymbol{x}_1$. Han Meimei solved $\boldsymbol{x}$ from $B\boldsymbol{x} = \boldsymbol{b}$, and found a unique solution $\boldsymbol{x}_2$. Which solution should they use? Maybe neither. Maybe, we can require BOTH $A\boldsymbol{x} = \boldsymbol{a}$ and $B\boldsymbol{x} = \boldsymbol{b}$, and find the least square solution.

   (a) (2 points) If we require BOTH $A\boldsymbol{x} = \boldsymbol{a}$ and $B\boldsymbol{x} = \boldsymbol{b}$, then to find the least square solution, it is equivalent to solve $M\boldsymbol{x} = \boldsymbol{c}$ for what matrix M and what vector $\boldsymbol{c}$?

   (b) (2 points) If $A = B$, prove that the least square solution is unique, and find it in terms of a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

   (c) (1 point) Show that if A, B are invertible, then M above must be invertible.

   (d) (2 points) If A, B are invertible, find matrices S, T in terms of A, B such that $\boldsymbol{x} = S\boldsymbol{x}_1 + T\boldsymbol{x}_2$.

Answer:

1. The system is $\begin{bmatrix} A \\ B \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}$, and the least square solution is from the system $(A^{\mathrm{T}}A + B^{\mathrm{T}}B)\boldsymbol{x} = A^{\mathrm{T}}\boldsymbol{a} + B^{\mathrm{T}}\boldsymbol{b}$. So $M = A^{\mathrm{T}}A + B^{\mathrm{T}}B$ and $\boldsymbol{c} = A^{\mathrm{T}}\boldsymbol{a} + B^{\mathrm{T}}\boldsymbol{b}$. *1pt for understanding that we require to multiply $\begin{bmatrix} A^{\mathrm{T}} & B^{\mathrm{T}} \end{bmatrix}$ on the left of both sides, and 1pts for the correct M and $\boldsymbol{c}$.*
2. If $A = B$, then $2A^{\mathrm{T}}A\boldsymbol{x} = A^{\mathrm{T}}(\boldsymbol{a} + \boldsymbol{b})$. So direct computation can verify that $\frac{1}{2}\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_2$ is a solution. Futhermore, since $\boldsymbol{x}_1, \boldsymbol{x}_2$ are unique solutions, therefore A is injective, so $A^{\mathrm{T}}A$ is bijective. So the solution is unique. *1pt for correct solution, and 1pt for uniqueness.*
3. If A, B are invertible, then $A^{\mathrm{T}}A, B^{\mathrm{T}}B$ are both positive definite. So $A^{\mathrm{T}}A + B^{\mathrm{T}}B$ is positive definite, so it is invertible. *Partial credit for partial success.*
4. $S = (A^{\mathrm{T}}A + B^{\mathrm{T}}B)^{-1}A^{\mathrm{T}}A, T = (A^{\mathrm{T}}A + B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}B$. *1pt for S and 1pt for T. Partial credit for partial success.*