Linear Algebra (English)
2022 Fall
Final Exam
2023.1.5
Exam Duration: 3 Hours

Name: ____________________

Student ID: ____________________

This exam includes 11 pages (including this page) and 5 problems. Please check to see if there is any missing page, and then write down your name and student ID number on this page and the first page of your answer sheets. Also write down the initials of your name on the top of every page of your answer sheets, in case they are scattered.

This exam is open book. You are allowed to consult your textbook and notes, and elementary calculators are fine (those that cannot instantly multiply matrices). Plagerism of all kinds are strictly forbidden and will be severly punished.

Please write down your answers to the problems in the **SEPARATE ANSWER SHEETS**, and follow the following rules:

- **Always explain your answer.** You should always explain your answers. Any problem answered with nothing but a single answer would receive no credit.
- **Write cleanly and legible.** Make sure that your writings can be read. The graders are NOT responsible to decipher illegible writings.
- **Partial credits will be given.**
- Blank spaces are provided in the exams. Feel free to use them as scratch papers. However, your formal answer has to be written in the **SEPARATE ANSWER SHEETS**, as required by the University.
- The total score of the exam is 50. If your total score exceeds 50 (there are 54 points in total), it will be recorded as 50.

Problem	Points	Score
1	13	
2	11	
3	10	
4	11	
5	9	
Total:	54	

1. (I am really proud of myself for constructing this problem....)

   Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$, which looks like a chinese character "囧".

   (a) (2 points) How many non-zero terms does the big formula for $\det A$ has? Also find this determinant.

   (b) (3 points) Find a basis for the kernel of A.

   (c) (3 points) Suppose $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$ is an eigenvector for a non-zero eigenvalue $\lambda \neq 0$. Show that we must have $c = 0$ and $a = d = e$.

   (d) (5 points) Find all eigenvalues of A, and find their geometric and algebraic multiplicities, and find all eigenvectors. Can A be diagonalized?

Answer:

1. None. All terms are zero, and the determinant is zero.
2. $\begin{bmatrix}1\\1\\0\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\\-1\end{bmatrix}$. Basically the kernel is $\begin{bmatrix}a\\b\\0\\-b\\-a\end{bmatrix}$ for any a, b.
3. $\lambda \begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix} = A \begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix} = \begin{bmatrix}b+d\\a+e\\0\\b+c+d\\b+d\end{bmatrix}$. Since $\lambda \neq 0$, we have $c = 0$. Then $\begin{bmatrix}b+d\\a+e\\0\\b+c+d\\b+d\end{bmatrix}$ have the same first, fourth and fifth coordinate, so $a = d = e$.
4. We solve $\lambda \begin{bmatrix}a\\b\\0\\a\\a\end{bmatrix} = A \begin{bmatrix}a\\b\\0\\a\\a\end{bmatrix} = \begin{bmatrix}a+b\\2a\\0\\a+b\\a+b\end{bmatrix}$. This gives $\lambda a = a + b$ and $\lambda b = 2a$, which shows that $\lambda(\lambda - 1) = 2$, and it shows that $\lambda = -1$ or $\lambda = 2$, with corresponding eigenvectors $\begin{bmatrix}1\\-2\\0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix}$. So both are eigenvalues with geometric multiplicity 1. We also know that 0 is an eigenvalue with geometric multiplicity 2. Hence the eigenvalues are $0, 0, -1, 2, \lambda$ for some unknown λ. Since the trace is one, the final eigenvalue is $\lambda = 0$. So the eigenvalues counting algebraic multiplicity are $0, 0, 0, -1, 2$, where $-1, 2$ can only have algebraic and geometric multiplicity one, but 0 has algebraic multiplicity three and geometric multiplicity two. Hence A cannot be diagonalized.

2. Suppose we are trying to solve a linear equation $A\boldsymbol{x} = \mathbf{0}$, where $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 1 & 1 \\ -1 & 1 & 3 & 3 \end{bmatrix}$.

   Surprisingly, for some rotation matrix R (real orthogonal with determinant one), $A\boldsymbol{x} = \mathbf{0}$ and $AR\boldsymbol{x} = \mathbf{0}$ have the same solution set! We also know that $R^2 - I$ is invertible, but $R^4 - I = 0$. What does it mean?

   (a) (2 points) Find a basis for the subspaces $\mathrm{Ran}(A), \mathrm{Ker}(A), \mathrm{Ran}(A^{\mathrm{T}}), \mathrm{Ker}(A^{\mathrm{T}})$.

   (b) (3 points) Among the subspaces $\mathrm{Ran}(A), \mathrm{Ker}(A), \mathrm{Ran}(A^{\mathrm{T}}), \mathrm{Ker}(A^{\mathrm{T}})$, which of them are R-invariant? Show why and why not. (We say W is R-invariant if for all $\boldsymbol{w} \in W$, we have $R\boldsymbol{w} \in W$.)

   (c) (2 points) For each R-invariant subspace in the last sub-problem, find the orthogonal projection to it. Find the sum of all these orthogonal projections.

   (d) (4 points) Find all eigenvalues of R counting algebraic multiplicities, and find all possible R.

Answer:

1. $\mathrm{Ran}(A)$ has basis $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}$, and $\mathrm{Ker}(A^{\mathrm{T}})$ has basis $\begin{bmatrix}-7\\4\\1\end{bmatrix}$. $\mathrm{Ran}(A^{\mathrm{T}})$ has basis $\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, and $\mathrm{Ker}(A)$ has basis $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$.
2. Since $\mathrm{Ker}(A) = \mathrm{Ker}(AR)$, therefore if $A\boldsymbol{w} = \mathbf{0}$, then $AR\boldsymbol{w} = \mathbf{0}$, so $\mathrm{Ker}(A)$ is R-invariant. Furthermore, due to spectral theorem, $\mathrm{Ran}(A^{\mathrm{T}})$ is also R-invariant. Finally, $\mathrm{Ran}(A), \mathrm{Ker}(A^{\mathrm{T}})$ are not subspaces of the domain of R, so they cannot be invariant.
3. The orthogonal projection to $\mathrm{Ran}(A^{\mathrm{T}})$ is $\frac{1}{2}\begin{bmatrix}1 & -1 & 0 & 0\\-1 & 1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix}$, and the orthogonal projection to $\mathrm{Ker}(A)$ is $\frac{1}{2}\begin{bmatrix}1 & 1 & 0 & 0\\1 & 1 & 0 & 0\\0 & 0 & 1 & -1\\0 & 0 & -1 & 1\end{bmatrix}$. Their sum is I.
4. Eigenvalues of R are $\pm\mathrm{i}$, and since trace is real, they are $\mathrm{i}, \mathrm{i}, -\mathrm{i}, -\mathrm{i}$. Note that then spectral theorem impies that R is skew symmetric, and commutativity with previous projections implies a nice block structure. But of course direct calculation would also do. Let $X = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, Y = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$, then the four possible R are $\pm\begin{bmatrix} & X\\X & \end{bmatrix}, \pm\begin{bmatrix} & Y\\-Y & \end{bmatrix}$.

3. (Imagine we want to solve the differential equation $f'(x) = xf'(x+1)$. Not easy, right? But maybe we can study it by finding some eigenvectors.)

   Let V be the space of polynomials of degree at most 3. Let $L : V \to V$ such that $f(x)$ is sent to $xf'(x+1)$. We shall see that L is diagonalizable. Let polynomials p_1, p_2, p_3, p_4 be the eigenvectors of L for eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$. Say we scale them so that they are all polynomials with leading coefficient 1. (So the coefficient for the largest degree term is 1.)

   (a) (2 points) Let $\mathcal{B} = (1, x, x^2, x^3)$ be the basis. Find the matrix for L under this basis.

   (b) (3 points) Find the characteristic polynomial of L, and all its eigenvalues. Find the eigen-polynomials p_1, p_2, p_3, p_4. (I want to see the polynomials, not the coordinate vectors.)

   (c) (2 points) We have a basis $\mathcal{C} = (p_1, p_2, p_3, p_4)$. If a polynomial has coordinates $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ under $\mathcal{C}$, what are its coordinates under $\mathcal{B}$.

   (d) (2 points) Suppose a linear transformation T has matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 6 & 24 \\ 0 & 0 & 5 & 30 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ under basis $\mathcal{B}$. Find the matrix for the same linear transformation under basis $\mathcal{C}$.

   (e) (1 point) Can you write out an abstract description of T in terms of polynomials? (E.g., an abstract description of L is that it sends $f(x)$ to $xf'(x+1)$.)

Answer:

1. The matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.
2. The char poly is $x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$, either format is correct. The eigenvalues are $0, 1, 2, 3$, and the eigenvectors are $1, x, x^2 + 2x, x^3 + 6x^2 + 7.5x$. (Note that multiples of this is also correct.)
3. Let $X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 7.5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Answer is $X \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 10.5 \\ 7 \\ 1 \end{bmatrix}$.
4. $X^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4.5 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then direct multiplication gives answer $\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 5 & \\ & & & 10 \end{bmatrix}$.
5. See that $T = L^2 + I$. So T sends $f(x)$ to $xf'(x+2) + x(x+1)f''(x+2) + f(x)$.

4. Let $A = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix}$. Let $G = A^T A$.

(a) (2 points) Find a formula for G^n. (Hint: find $(AA^T)^{n-1}$ first. Also, it is OK to leave the answer as a product of three explicit matrices.)

(b) (3 points) Find the determinant of $\begin{bmatrix} I & A \\ A^{\mathrm{T}} & 2G - I \end{bmatrix}$. Is this positive semi-definite and why?

(c) (4 points) Find the maximum and minimum of $\frac{\boldsymbol{u}^{\mathrm{T}}(G-10I)\boldsymbol{u}}{\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}}$ for any non-zero vector $\boldsymbol{u}$, and find some $\boldsymbol{u}$ that make the extremum happen.

(d) (2 points) Let $\boldsymbol{v}$ be a unit right singular vector of A for the largest singular value, and let $\boldsymbol{w}$ be a unit vector in $\mathrm{Ker}(A)$. Find $\frac{(\boldsymbol{v}+\boldsymbol{w})^{\mathrm{T}}(G-10I)(\boldsymbol{v}+\boldsymbol{w})}{(\boldsymbol{v}+\boldsymbol{w})^{\mathrm{T}}(\boldsymbol{v}+\boldsymbol{w})}$.

Answer:

1. $AA^{\mathrm{T}} = \begin{bmatrix} 10 & \\ & 34 \end{bmatrix}$. So $G^n = A^{\mathrm{T}} \begin{bmatrix} 10^{n-1} & \\ & 34^{n-1} \end{bmatrix} A$.

2. Block elimination gives $\det \begin{bmatrix} I & A \\ A^{\mathrm{T}} & 2G - I \end{bmatrix} = \det \begin{bmatrix} I & A \\ & G - I \end{bmatrix} = \det(G - I)$. Since G has eigenvalues $10, 34, 0, 0, 0$, therefore the answer is -297. Note that we must have negative eigenvalues, so the matrix is not positive semi-definite.

3. $G - 10I$ has largest eigenvalue 24 at $\begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}$ and smallest eigenvalue -10 at any non-zero vector in the span of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$, i.e., anything non-zero of the form $\begin{bmatrix} a \\ b \\ c \\ b + 4a \\ -a \end{bmatrix}$.

4. $\frac{24+(-10)}{1+1} = 7$.

5. For each real number x, we define a corresponding matrix $A_x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & x \\ 1 & 1 & x & x \\ 1 & x & x & 1 \end{bmatrix}$.

   (a) (3 points) Find the LDL^{T} decomposition of A_x and the determinant of A_x. For what x is A_x invertible?

   (b) (2 points) When $x < 1$, find the QR-decomposition of A_x. (Note that R need to have positive diagonal entries.)

   (c) (2 points) How many eigenvalues of A_x are positive, depending on x?

   (d) (2 points) Find two linearly independent eigenvectors of A_x independent of x. What are their eigenvalues? (The eigenvalues depend on x.) (Maybe try some specific values of x first? Or stare at A really hard.)

Answer:

1. When $x \neq 1$, $L = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & & & \\ & x-1 & & \\ & & x-1 & \\ & & & 2(1-x) \end{bmatrix}$. $\det(A_x) = 2(1-x)^3$ Therefore A_x is invertible if and only if $x \neq 1$.
2. $Q = \frac{1}{2}A_{-1}$, and $R = \begin{bmatrix} 2 & 1+x & 1+x & 1+x \\ & 1-x & & \\ & & 1-x & \\ & & & 1-x \end{bmatrix}$.
3. Using law of inertia, and the fact that A is congruence to D, therefore when $x > 1$, we have 3 positive eigenvalues. When $x < 1$, we have two positive eigenvalues. When $x = 1$, we have one positive eigenvalues.
4. We have eigenvalue $x - 1$ and eigenvector $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, and eigenvalue $1 - x$ and eigenvector $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}$. FYI, the other two eigenvalues are $\lambda = x + 1 \pm \sqrt{x^2 + 3}$, with corresponding eigenvector $\begin{bmatrix} 3 \\ \lambda - 1 \\ \lambda - 1 \\ \lambda - 1 \end{bmatrix}$.