4. 因 $(X_i-\mu)^2$ i.i.d, $E(X_i-\mu)^2=\sigma^2$

$\therefore \dfrac{\sum_{i=1}^{n}(X_i-\mu)^2}{n} \xrightarrow{P} \sigma^2$

5. ~~不妨假设 EX_i 都为0，否则用 X_i-EX_i 代替 X_i 不影响题目~~

~~則 $r(X_i,X_j)=\dfrac{EXY}{\sqrt{DX}\sqrt{DY}}=\dfrac{EXY}{\sqrt{EX^2}\sqrt{EY^2}}$~~

由 Chebyshev 不等式，$P\left(\left|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}EX_k\right|\geq\varepsilon\right)\leq\dfrac{D\frac{\sum_{k=1}^{n}X_k}{n}}{\varepsilon^2}=\dfrac{D\sum_{k=1}^{n}X_k}{n^2\varepsilon^2}$

$\therefore$ 只需证 $\lim_{n\to\infty}\dfrac{D\sum_{k=1}^{n}X_k}{n^2}=0$

$Cov(X_i,X_j)=\sqrt{DX}\sqrt{DY}\,r(X_i,X_j)\leq C$　　$\lim_{|i-j|\to\infty}Cov(X_i,X_j)=0$

$\frac{1}{n^2}D\sum_{k=1}^{n}X_k=\frac{1}{n^2}\left(\sum_{k=1}^{n}DX_k+\sum_{i\neq j}Cov(X_i,X_j)\right)$　　$\frac{1}{n^2}\sum_{k=1}^{n}DX_k\leq\frac{Cn}{n^2}\longrightarrow 0$

对 $\forall\delta>0$，$\exists N$，当 $|i-j|\geq N$ 时 $|Cov(X_i,X_j)|<\delta$

$\therefore$ 对 $n\geq N$，　　$\frac{1}{n^2}\sum_{i\neq j}Cov(X_i,X_j)=\frac{1}{n^2}\left(\sum_{|i-j|<N}Cov(X_i,X_j)+\sum_{|i-j|\geq N}Cov(X_i,X_j)\right)$

满足 $|i-j|\leq N$ 的数对有 $2nN-N^2$ 个

$\leq\frac{1}{n^2}\left((2nN-N^2)C+(n(n-1)-(2nN-N^2))\delta\right)$

令 $n\to+\infty$，由 δ 的任意性，得该式趋近于0

$\therefore$ 原式得证

9. 记 $X_i=\cos(iX)$

则 $EX_i=0$, $DX_i=\frac{1}{2}$ $Cov(X_i,X_j)=0$

由第5题结论 可知 $\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}EX_k\xrightarrow{P}0$

$$\therefore \frac{S_n}{n}\xrightarrow{P}0$$

11. 设 $X_i=\begin{cases}1 & \text{i号信放在第i个盒子里}\\ 0 & \text{其他}\end{cases}$

则 ~~$P(X_i$~~ 则 $X_i\sim\begin{pmatrix}1 & 0\\ \frac{1}{n} & 1-\frac{1}{n}\end{pmatrix}$ $\qquad S_n=\sum_{i=1}^{n}X_i$

$$EX_i=\frac{1}{n}\quad DX_i=\frac{n-1}{n^2}$$

$$Cov(X_i,X_j)=\frac{1}{n^2(n-1)}$$

$$\therefore DS_n=D\sum_{i=1}^{n}X_i=\sum_{i=1}^{n}DX_i+2\sum_{1\le i<j\le n}Cov(X_i,X_j)=1$$

$$\therefore P\left(\left|\frac{S_n-ES_n}{n}\right|\ge\varepsilon\right)\le\frac{D\left(\frac{S_n}{n}\right)}{\varepsilon^2}=\frac{1}{n^2\varepsilon^2}\to0$$

$$\therefore \frac{S_n-ES_n}{n}\xrightarrow{P}0$$

8. $E\bar{X} = E\frac{X_1+\cdots+X_n}{n} = EX_1 = 0$

$Var(\bar{X}) = Var\frac{X_1+X_2+\cdots+X_n}{n} = \frac{1}{n^2}\cdot n\, Var X_1 = \frac{1}{3n}$

10.

$$\sum_{i<j}(x_i-x_j)^2 = (n-1)\sum_{i=1}^{n}x_i^2 - 2\sum_{i<j}x_ix_j$$

$$= n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2$$

$$= n\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$\therefore \frac{1}{n(n-1)}\sum_{i<j}(x_i-x_j)^2 = s^2$$

24. ~~PE~~

(1) $P(X_{(16)}>10) = 1-P(X_{(16)}\le 10) = 1-(P(X_1\le 10))^{16} = 1-(\Phi(1))^{16}$

(2) $P(X_{(1)}>5) = (P(X_1>5))^{16} = (1-\Phi(-1.5))^{16} = (\Phi(1.5))^{16}$

29. (1) $f_{F(X)}(x) = \begin{cases} 1 & 0\le x\le 1 \\ 0 & 其他 \end{cases}$ $\therefore F(X)\sim U(0,1)$

$\therefore F(X_{(6)})$为来自$U(0,1)$的次序统计量，有 $F(X_{(6)})\sim Be(6,5)$

$\therefore E(F(X_{(6)})) = \frac{6}{11}$ $Var(F(X_{(6)})) = \frac{5}{242}$

(2) 在$x=0.15$函数值为 $\int_0^{0.15} \frac{1}{B(6,5)} x^5 (1-x)^4 = 0.0014$

32. $f_{X_{(2)},X_{(4)}}(x,y) = \frac{5!}{1!1!1!} x^3 (y^3-x^3)(1-y^3)\cdot 3x^2 \cdot 3y^2$

$= 1080 x^5 y^2 (y^3-x^3)(1-y^3) \qquad 0<x\le y<1$

则 $f_{\frac{X_{(2)}}{X_{(4)}},X_{(4)}}(u,v) = f(uv,v)v = 1080u^5(1-u^3)v^{11}(1-v^3)$

可分离成u的函数与v的函数

$\therefore \frac{X_{(2)}}{X_{(4)}}$ 与 $X_{(4)}$ 相互独立