35. $E(X|\eta=n)=\frac{n+a}{2}$ 而 $\eta\sim U[0,a]$

$\therefore E(X|\eta)\sim U[\frac{a}{2},a]$

41. $E(X_1|X_1<X_2)=\frac{E(X_1 I_{X_1<X_2})}{P(X_1<X_2)}$

$=\frac{\iint_{0<x<y} x f(x,y)dxdy}{\iint_{0<x<y} f(x,y)dxdy}$

由于 ~~对于~~ X_1,X_2 独立，$\therefore f(x,y)=f_{X_1}(x)f_{X_2}(y)=\lambda_1\lambda_2 e^{-\lambda_1 x-\lambda_2 y}$

$\therefore$ 上式 $=\frac{\frac{1}{(\lambda_1+\lambda_2)^2}}{\frac{1}{\lambda_1+\lambda_2}}=\frac{1}{\lambda_1+\lambda_2}$

49. $f_{X,Y}(x,y)=f_X(x)f_Y(y)=e^{-x-y}$ $(x>0,y>0)$

$f_{U,V}(u,v)=f_{X,Y}(\frac{uv}{v+1},\frac{u}{v+1})|J(u,v)|$

$=\begin{cases} ue^{-u}\frac{1}{(1+v)^2} & u>0 \quad v>0 \\ 0 & \text{其他} \end{cases}$

$f_{U,V}(u,v)=ue^{-u}\cdot\frac{1}{(1+v)^2}$ $\therefore$ U V 独立

50. $f_{U,V}(u,v) = f_{X,Y}(u,v)\,|J_1(u,v)| + f_{X,Y}(v,u)\,|J_1(u,v)|$

$$= \begin{cases} \frac{12}{7}(u+v)^2 & u<v \\ 0 & 其他 \end{cases}$$

51. $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(e^{-\frac{y_1^2+y_2^2}{2}}, \arccos\frac{y_1}{\sqrt{y_1^2+y_2^2}}\right)|J_1(y_1,y_2)|$

$$+ f_{X_1,X_2}\left(e^{-\frac{y_1^2+y_2^2}{2}}, 2\pi - \arccos\frac{y_1}{\sqrt{y_1^2+y_2^2}}\right)|J_1(y_1,y_2)|$$

$$= \frac{1}{2\pi}e^{-\frac{y_1^2+y_2^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{y_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{y_2^2}{2}}$$

$\therefore Y_1\ Y_2$ 独立，且满足 $N(0,1)$ 分布

53. $F_{Z_n}(x) = P(Z_n \le x) = P(\max(X_1, X_2 \ldots X_n) \ge 1-\frac{x}{n})$

$x \le 0$ 时，$F_n(x) = 0$

$x > 0$ 时，上式 $= 1-(P(X_1) < 1-\frac{x}{n})^n$

$= 1-(1-\frac{x}{n})^{2n}$

当 $n \to \infty$ 时，$F_{Z_n}(x) \to \begin{cases} 1-e^{-2x} & x>0 \\ 0 & x\le 0 \end{cases}$

1. (a) $\varphi(\theta) = \int_a^b \frac{e^{i\theta x}}{b-a}\,dx = \int_a^b \frac{\cos\theta x}{b-a}\,dx + i\int_a^b \frac{\sin\theta x}{b-a}\,dx$

$$= \cancel{\frac{\sin}{b-a}} \; \frac{\sin(b\theta) - \sin(a\theta) \cancel{-} i\cos(\theta b) + i\cos(a\theta)}{\theta(b-a)}$$

$$= \frac{e^{ib\theta} - e^{ia\theta}}{i\theta(b-a)}$$

∴ $[-a, a]$ 上均匀分布

$\varphi(\theta) = \frac{\sin a\theta}{a\theta}$

(b) $I(\theta) = \frac{1}{a\pi}\int_{-\infty}^{+\infty} \frac{e^{i\theta x}}{x^2+1}\,dx$

$$= \frac{2}{a\pi}\int_0^{+\infty} \frac{\cos\theta x}{x^2+1}\,dx$$

$\theta>0$ 时

$I'(\theta) = \frac{2}{a\pi}\int_0^{+\infty} \frac{x\sin\theta x}{x^2+1}\,dx$

$$= -\frac{2}{a\pi}\int_0^{+\infty} \frac{\sin\theta x}{x}\,dx + \frac{2}{a\pi}\int_0^{+\infty} \frac{\sin\theta x}{x(x^2+1)}\,dx$$

$$= -\frac{1}{a} + \frac{2}{a\pi}\int_0^{+\infty} \frac{\sin\theta x}{x(x^2+1)}\,dx$$

$I''(\theta) = \frac{2}{a\pi}\int_0^{+\infty} \frac{\cos\theta x}{x^2+1}\,dx = I(\theta)$

∴ $I(\theta) = C_1 e^{\theta} + C_2 e^{-\theta}$ $\quad I(0) = \frac{1}{a}$ $\quad I'(0) = -\frac{1}{a}$

∴ $I(\theta) = \frac{1}{a} e^{-\theta}$ $\quad \theta > 0$ $\quad$ ∴ $I(\theta) = \frac{1}{a} e^{-|\theta|}$

∴ $\varphi_X(\theta) = e^{ib\theta - a|\theta|}$

(c) $\varphi(\theta) = Ee^{i\theta X} = \int_0^{+\infty} e^{i\theta x} \frac{\beta^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} dx$

$$= \int_0^{+\infty} \frac{\beta^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-(\beta - i\theta)x} dx$$

$$= \frac{1}{\Gamma(\alpha)} \left(1 - \frac{i\theta}{\beta}\right)^{-\alpha} \int_0^{+\infty} y^{\alpha-1} e^{-y} dy \qquad (y = (\beta - i\theta)x)$$

$$= \left(1 - \frac{i\theta}{\beta}\right)^{-\alpha}$$

6.

设 $\varphi(\theta)$ 为 X 的特征函数

取 $Y \sim U(-a,a)$ 则 $\varphi_Y(\theta) = \frac{\sin a\theta}{a\theta}$

$\therefore$ $\varphi_a(\theta)$ 为 $X+Y$ 的特征函数

7.

(1) $\varphi_X(\theta) = \int_0^{\infty} e^{i\theta x} e^{-x} dx = \int_0^{\infty} \cos(\theta x) e^{-x} dx + i \int_0^{\infty} \sin(\theta x) e^{-x} dx$

$$= \frac{1}{1 - i\theta}$$

(2) $\varphi_{X-Y}(\theta) = \varphi_X(\theta) \cdot \varphi_Y(-\theta) = \frac{1}{1+\theta^2}$

(3) $f_{X-Y}(x) = \int_{-\infty}^{+\infty} f_X(u) f_Y(u-x) du = \frac{1}{2} e^{-|x|}$

$$\varphi_{X-Y}(\theta) = \int_{-\infty}^{+\infty} e^{i\theta x} \frac{e^{-|x|}}{2} dx = \frac{1}{1+\theta^2}$$

$$f_{X-Y}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \varphi_{X-Y}(\theta) d\theta = \frac{1}{2} e^{-|x|}$$

10、设 $X_1 \sim Cauchy(m, a)$

则 $\varphi_{X_1}(\theta) = e^{im\theta - a|\theta|}$

则 $\varphi_{\frac{X_1+X_2+\cdots+X_n}{n}} = \left(e^{im\frac{\theta}{n} - a|\frac{\theta}{n}|}\right)^n = e^{im\theta - a|\theta|}$

$\therefore \frac{X_1+X_2+\cdots+X_n}{n} \sim Cauchy(m, a)$