6.3

6. $\hat{\mu} = \frac{1}{20}\sum_{i=1}^{20} \ln x_i = 3.089$

$\hat{\sigma}^2 = \frac{1}{20}\sum_{i=1}^{20} (\ln x_i - 3.089)^2 = 0.508$

$\therefore \hat{E}(X) = e^{\hat{\mu}+\frac{\hat{\sigma}^2}{2}} = 28.305$

7. (1) $EX = \frac{3\theta}{2}$ $\quad DX = \frac{\theta^2}{12}$

$\therefore E\bar{X} = \frac{3\theta}{2}$ $\quad D\bar{X} = \frac{\theta^2}{12n}$

$\therefore E\hat{\theta} = \theta$

$D\hat{\theta} = \frac{\theta^2}{27n} \to 0$ $\quad$ ∴ 无偏且相合

(2) $L(\theta) = \left(\frac{1}{\theta}\right)^n I_{\{\theta < x_{(1)} < x_{(n)} < 2\theta\}}$

当 $\hat{\theta} = \frac{x_{(n)}}{2}$ 时, $L(\theta)$ 有最大值 ∴ 最大似然估计 $\hat{\theta} = \frac{x_{(n)}}{2}$

$E\hat{\theta} = \frac{2n+1}{2(n+1)}\theta$. $\quad$ ∴ 不为无偏估计

$D\hat{\theta} = E\hat{\theta}^2 - (E\hat{\theta})^2 = \frac{n\theta^2}{2(n+2)(n+1)^2} \to 0$ $\quad$ ∴ 为相合估计

8. (1) $L(\theta)=e^{n\theta-\sum_{i=1}^{n}x_i} I_{\{x_{(1)}>\theta\}}$

当 $\hat{\theta}_1=x_{(1)}$ 时，有最大值.

$E\hat{\theta}_1=\frac{1}{n}+\theta\neq\theta$ $\therefore\hat{\theta}_1$ 不为 θ 的无偏估计

$E\hat{\theta}_1^2=\frac{2}{n^2}+\frac{2}{n}\theta+\theta^2$

$D\hat{\theta}_1=\frac{1}{n^2}\to 0$ $\therefore\hat{\theta}_1$ 为相合估计

(2) $EX=\theta+1$ $\therefore\hat{\theta}_2=\bar{x}-1$，必为 θ 的无偏估计，

$D\hat{\theta}_2=\frac{1}{n}\to 0$

$\therefore\hat{\theta}_2$ 为 θ 的相合估计

6.4

2、对任意 ϕ 满足 $E\phi=0$，有 $Cov(T_i,\phi)=0$

$E(aT_1+bT_2)=a\theta_1+b\theta_2$

$Cov(aT_1+bT_2,\phi)=0$

$\therefore aT_1+bT_2$ 为 $a\theta_1+b\theta_2$ 的 UMVUE

6. (1) $L(\theta)=\prod_{i=1}^{n} x_i^{\theta-1}\cdot\theta^n$

$$\frac{\partial \ln L(\theta)}{\partial\theta}=\cancel{ng(\theta)}-\frac{n}{g(\theta)}-\frac{\sum_{i=1}^{n}\ln x_i}{g(\theta)^2}=0$$

得: $\hat{g}(\theta)=-\frac{1}{n}\sum_{i=1}^{n}\ln x_i$

(2) $I(\theta)=-E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x,\theta)\right)=-E\left(-\frac{1}{\theta^2}\right)=\frac{1}{\theta^2}$

$\therefore g(\theta)$的无偏估计 C-R下界为 $\frac{1}{n\theta^2}$

~~而~~ ~~ta~~ 记 $Y_i=-\ln X_i$

则 $f_{Y_i}(y)=\theta e^{-\theta y}I_{\{y>0\}}$ $\therefore Y_i\sim Ga(1,\theta)$

$\therefore \hat{g}(\theta)\sim Ga(n,n\theta)$

$\therefore E\hat{g}(\theta)=\frac{1}{\theta}$ $D\hat{g}(\theta)=\frac{1}{n\theta^2}$

$\therefore \hat{g}(\theta)$为~~g~~ $g(\theta)$的有效估计

2,

3)

12. $T=\bar{X}\sim N(\mu,1)$ 是一个充分完备统计量

且 $\bar{X}^2-\frac{1}{n}$ 为 μ^2 的一个无偏估计.

$$E(\bar{X}^2-\frac{1}{n}\mid \bar{X})=\bar{X}^2-\frac{1}{n}.$$

由 Lehman-Sheff 定理: $\bar{X}^2-\frac{1}{n}$ 为 μ 的 UMVUE

$$\text{而 } D(\bar{X}^2-\frac{1}{n})=D(\bar{X}^2)=\frac{2}{n^2}+\frac{4}{n}\mu^2$$

$$\text{C-R下界}=\frac{(g'(\mu))^2}{nI(\mu)}=\frac{4\mu^2}{n}$$

即此 UMVUE 的方差未达到 C-R 下界

14

(1) $$L(\theta)=\left(\frac{1-\theta}{2}\right)^{\frac{1}{2}\sum_{i=1}^{n}(x_i^2-x_i)}\left(\frac{1}{2}\right)^{n-\sum_{i=1}^{n}x_i^2}\left(\frac{\theta}{2}\right)^{\frac{\sum(x_i^2+x_i)}{2}}$$

$$\therefore \frac{\partial\ln L(\theta)}{\partial\theta}=-\frac{1}{2}\frac{\sum_{i=1}^{n}(x_i^2-x_i)}{1-\theta}+\frac{1}{2\theta}\sum_{i=1}^{n}(x_i^2+x_i)=0$$

$$\text{得: } \hat{\theta}_1=\frac{1}{2}+\frac{\sum_{i=1}^{n}x_i}{2\sum_{i=1}^{n}x_i^2}$$

$$E\hat{\theta}_1=\frac{1}{2}+\frac{1}{2}(\theta-\frac{1}{2})(1-\frac{1}{2n})\neq\theta$$ ∴ 不为无偏估计

(2) $EX=\theta-\frac{1}{2}$ $\quad\therefore \hat{\theta}_2=\bar{X}+\frac{1}{2}$

(3) $$I(\theta)=E\left(\frac{\partial\ln f(x,\theta)}{\partial\theta}\right)^2=\frac{1}{2\theta(1-\theta)}$$

∴ θ 的无偏估计的 C-R 下界为 $\frac{2\theta(1-\theta)}{n}$

6.5

2. $\cancel{h(x,\theta)=\pi(\theta|x)m(x)}$

$$h(x,\theta)=f(x|\theta)\pi(\theta)=\frac{1}{6},\quad 11.1<\theta<11.7$$

$$\pi(\theta|x)=\frac{h(x,\theta)}{\int_{-\infty}^{+\infty}h(x,\theta)d\theta}=\frac{5}{3}$$

∴ θ的后验分布为 $U(11.1, 11.7)$

4、若λ的先验分布为 $\cancel{\pi(\lambda)=}$ $\lambda\sim\Gamma(\alpha,\beta)$，$\pi(\lambda)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$

$$\pi(\lambda|x)=\frac{f(x_1,x_2\cdots x_n|\lambda)\pi(\lambda)}{\int_0^{+\infty}f(x_1,x_2\cdots x_n|\lambda)\pi(\lambda)d\lambda}$$

$$=\frac{(\beta+n)^{\sum_{i=1}^{n}x_i+\alpha}}{\Gamma(\sum_{i=1}^{n}x_i+\alpha)}\lambda^{\sum_{i=1}^{n}x_i+\alpha-1}e^{-(\beta+n)\lambda}\sim\Gamma(\sum_{i=1}^{n}x_i+\alpha,\beta+n)$$

∴伽玛分布为共轭先验分布

7. $$\pi(\theta|x)=\frac{f(x_1,x_2,\cdots x_n|\theta)\pi(\theta)}{\int_0^{+\infty}f(x_1,x_2\cdots x_n|\theta)\pi(\theta)d\theta}=\frac{(\lambda-\sum_{i=1}^{n}\ln x_i)^{n+\alpha}}{\Gamma(\lambda-\sum_{i=1}^{n}\ln x_i)}\theta^{n+\alpha-1}e^{-\theta(\lambda-\sum_{i=1}^{n}\ln x_i)}$$

∴ θ的后验分布为 $\Gamma(n+\alpha,\lambda-\sum_{i=1}^{n}\ln x_i)$

$$\hat{\theta}_B=\frac{n+\alpha}{\lambda-\sum_{i=1}^{n}\ln x_i}$$

8、(1) $\pi(\theta|x)=\dfrac{f(x_1,x_2,\cdots x_n|\theta)\ \pi(\theta)}{\int_{\max\{x_{(n)},\theta_0\}}^{+\infty} f(x_1,x_2\cdots x_n|\theta)\ \pi(\theta)\,d\theta}=\dfrac{(n+\beta)\cancel{\theta}(\max(x_{(n)},\theta_0))^{n+\beta}}{\theta^{n+\alpha+1}}$

这此为参数为 $n+\beta+1$ 和 $\max(x_{(n)},\theta_0)$ 的帕雷托分布.

∴帕雷托分布为共轭先验分布

(2) $\hat{\theta}_B=\int_{\max(x_{(n)},\theta_0)}^{+\infty}\theta\cdot\pi(\theta|x)\,d\theta=\dfrac{(n+\beta)\max\{x_{(n)},\theta_0\}}{n+\beta-1}$