Linear Algebra (in English)
2020 Fall
Final Exam
2020.12.23
Exam Duration: 3 Hours

Name: ____________________

Student ID: ____________________

This exam includes 15 pages (including this page) and 7 problems. Please check to see if there is any missing page, and then write down your name and student ID number on this page and the first page of your answer sheets. Also write down the initials of your name on the top of every page of your answer sheets, in case they are scattered.

This exam is open book. You are allowed to consult your textbook and notes, but no calculator. Plagerism of all kinds are strictly forbidden and will be severly punished.

Please write down your answers to the problems in the provided **SEPARATE ANSWER SHEETS**, and follow the following rules:

- **Always explain your answer.** You should always explain your answers. Any problem answered with nothing but a single answer would receive no credit.
- **Write cleanly and legible.** Make sure that your writings can be read. The graders are NOT responsible to decipher illegible writings.
- **Partial credits will be given.**
- Blank spaces are provided in the exams. Feel free to use them as scratch papers. However, your formal answer has to be written in the **SEPARATE ANSWER SHEETS**, as required by the University.
- The total score of the exam is 80. If your total score exceeds 80 (there are 85 points in total), it will be recorded as 80.

Problem	Points	Score
1	11	
2	13	
3	16	
4	13	
5	9	
6	14	
7	9	
Total:	85	

1. Consider the matrix $A = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

   (a) (2 points) How many terms does the big formula for $\det A$ has? Also find this determinant.

   (b) (5 points) Find the characteristic polynomial of A, and find all eigenvalues and eigenvectors of A.

   (c) (2 points) Find all vectors $\boldsymbol{v} \in \mathbb{R}^4$ such that $\lim_{n\to\infty}(\frac{1}{3}A)^n\boldsymbol{v}$ exists.

   (d) (2 points) Find a polynomial $p(x)$ such that the eigenvalues of $p(A)$ are $1, 0, 0, 0$.

Answer:

1. Only one term, $1 \times 3 \times 3 \times 1 = 9$ with positive sign. So the determinant is 9.
2. The characteristic polynomial is $x^4 - 10x^2 + 9$. The eigenvalues are $3, 1, -1, -3$, with eigenvectros $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ respectively.
3. Note that $\frac{1}{3}A$ has eigenvalues $1, \frac{1}{3}, -\frac{1}{3}, -1$. So $(\frac{1}{3}A)^n \boldsymbol{v}$ would converge if and only if $\boldsymbol{v}$ is in the span of eigenvectors of the eigenvalues $1, \frac{1}{3}, -\frac{1}{3}$, i.e., $\boldsymbol{v} \in \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix})$.
4. We just need to find a polynomial $p(x)$ such that it maps three eigenvalues of A to zero, and one eigenvalue of A to 1. Some possible answers are $-\frac{1}{48}(x-3)(x-1)(x+1)$, $\frac{1}{16}(x-3)(x-1)(x+3)$, $-\frac{1}{16}(x-3)(x+1)(x+3)$, $\frac{1}{48}(x-1)(x+1)(x+3)$.

2. We have points $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}$ on the plane $\mathbb{R}^2$. Together they form a data matrix $A = \begin{bmatrix}1 & -2 & 1\\1 & 1 & -2\end{bmatrix}$. (Note that the points are already centered.)

   (a) (4 points) Find the spectral decomposition of AA^{T}. (Note that the eigenvector for the largest eigenvalue is the direction of the best fit line.)

   (b) (3 points) Find the $\mathrm{LDL}^{\mathrm{T}}$ decomposition of AA^{T}. (Note that the bottom left entry of L is the slope for the line from linear regression.)

   (c) (4 points) Find all singular values and all left and right singular vectors of A. (Note that there might be singular vectors for the singular value zero.)

   (d) (2 points) Find the maximum and minimum Rayleigh quotient $\frac{\boldsymbol{v}^{\mathrm{T}}S\boldsymbol{v}}{\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}}$ for $S = (A^{\mathrm{T}}A)^2 - 2A^{\mathrm{T}}A + 2I$.

Answer:

1. $AA^{\mathrm{T}} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} = (\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}) \begin{bmatrix} 3 & \\ & 9 \end{bmatrix} (\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix})^{\mathrm{T}}$
2. $AA^{\mathrm{T}} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 & \\ & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ & 1 \end{bmatrix}$
3. Singular values of A are $3, \sqrt{3}$ (it is also OK to count the zero singular value or not count the zero singular value), and the corresponding left singular vectors (unit vectors) are $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The corresponding right singular vectors (unit vectors) are $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. The other right singular vectors of A are in the kernel, and the unit vector is $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The full SVD of A is $A = (\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}) \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} (\frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 2 & \sqrt{2} \\ -\sqrt{3} & -1 & \sqrt{2} \\ \sqrt{3} & -1 & \sqrt{2} \end{bmatrix})^{\mathrm{T}}$.
4. Note that $A^{\mathrm{T}}A$ is a 3×3 matrix with eigenvalues $9, 3, 0$. So S has eigenvalues $65, 5, 2$. So the maximum value of the Rayleigh quotient is 65 and the minimum value is 2.

3. We have vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 7 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 7 \\ 6 \\ 11 \end{bmatrix}$ in the space $\mathbb{R}^4$. We wish to find all vectors $\boldsymbol{x}$ perpendicular to $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

   (a) (4 points) Find a 3×4 matrix A such that $\boldsymbol{x}$ is perpendicular to $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ if and only if $A\boldsymbol{x} = \mathbf{0}$.

   (b) (4 points) Find a basis for $\mathrm{Ker}(A)$.

   (c) (4 points) Find all solutions of $A\boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$.

   (d) (4 points) Let $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find all points perpendicular to $B\boldsymbol{v}_1, B\boldsymbol{v}_2, B\boldsymbol{v}_3$.

Answer:

1. $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 4 & 4 & 7 \\ 2 & 7 & 6 & 11 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 4 & 4 & 7 \\ 2 & 7 & 6 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So the kernel is spanned by the linearly independent vectors $\begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

3. $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ for all real numbers s, t.

4. Note that $\boldsymbol{v}_i \perp \boldsymbol{x}$ if and only if $\boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{x} = 0$ if and only if $\boldsymbol{v}_i^{\mathrm{T}} B^{\mathrm{T}} (B^{\mathrm{T}})^{-1} \boldsymbol{x} = 0$. So we only need to apply $(B^{\mathrm{T}})^{-1} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix}$ to the answers in the second sub-problem. So this subspace is spanned by $\begin{bmatrix} 4 \\ -6 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 3 \\ 1 \end{bmatrix}$.

4. Consider $A = \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 2 & 8 \end{bmatrix}$. We aim to find the orthogonal projection matrix to $\mathrm{Ran}(A)$.

(a) (4 points) Find the LL^{T} decomposition of $A^{\mathrm{T}}A$.

(b) (4 points) Find the QR decomposition of A. (For simplicity, we want R to be upper triangular, while Q can be non-square but has orthonormal columns.)

(c) (3 points) Find the 3×3 matrix of orthogonal projection to $\mathrm{Ran}(A)$.

(d) (2 points) Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be the left singular vectors of A for its two singular values σ_1, σ_2. Can you find $\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^{\mathrm{T}} \\ \boldsymbol{u}_2^{\mathrm{T}} \end{bmatrix}$? (Hint: You may not need to calculate anything. But in that case you would need to show your arguments still.)

Answer:

1. $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 27 \\ 27 & 90 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 9 & \\ & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ & 1 \end{bmatrix} = \begin{bmatrix} 3 & \\ 9 & 3 \end{bmatrix} \begin{bmatrix} 3 & 9 \\ & 3 \end{bmatrix}$.
2. Note that R is the L^{T} in the problem above. So we have $A = (\frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}) \begin{bmatrix} 3 & 9 \\ & 3 \end{bmatrix}$.
3. This is $QQ^{\mathrm{T}} = \frac{1}{9} \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}$.
4. This is the same matrix as above. Since $\boldsymbol{u}_1, \boldsymbol{u}_2$ form an orthonormal basis of A, we have $\mathrm{Ran}(\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}) = \mathrm{Ran}(A)$. Furthermore, the matrix $\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}$ has orthonormal columns, so $\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}^{\mathrm{T}}$ is the orthogonal projection to $\mathrm{Ran}(A)$.

5. Consider the real matrix $M = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}$ where A, C are symmetric, and A, B, C are all 2×2 real matrices.

   (a) (3 points) If B is invertible, find a formula for $\det(M)$ in terms of determinants of 2×2 matrices.

   (b) (4 points) Suppose $B = 2A$ and $C = 4A$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find all eigenvalues and eigenvectors of M.

   (c) (2 points) (Hard) Suppose $A = C = 0$, and B has singular values $2, 1$. Find all eigenvalues of M.

Answer:

1. $\det(M) = \det \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix} = \det \begin{bmatrix} B & A \\ C & B^{\mathrm{T}} \end{bmatrix} = \det \begin{bmatrix} B & A \\ O & B^{\mathrm{T}} - CB^{-1}A \end{bmatrix} = \det(B)\det(B^{\mathrm{T}} - CB^{-1}A)$.
2. Note that M has rank 1, so it has three zero eigenvalues, and by $\mathrm{trace}(M) = 25$, we see that the last eigenvalue is 25. The eigenvector for 25 is clearly $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$, and the eigenvectors for the eigenvalue zero are all vectors orthogonal to this, because M is symmetric. A basis for this could be $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.
3. $M = \begin{bmatrix} O & UDV^{\mathrm{T}} \\ VDU^{\mathrm{T}} & O \end{bmatrix} = \begin{bmatrix} U & \\ & V \end{bmatrix} \begin{bmatrix} & D \\ D & \end{bmatrix} \begin{bmatrix} U^{\mathrm{T}} & \\ & V^{\mathrm{T}} \end{bmatrix}$. Where $D = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}$. Since U, V are orthogonal, M is similar to $\begin{bmatrix} & D \\ D & \end{bmatrix}$, and it is also similar to $P_{23} \begin{bmatrix} & D \\ D & \end{bmatrix} P_{23}^{-1} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, where P_{23} is the swapping matrix of the second and third coordinates. This is now block diagonal, and the two diagonal blocks have eigenvalues $2, -2$ and $1, -1$. so these are the eigenvalues of M.

6. Consider the space V whose vectors are real functions of the form $(ax^2+bx+c)\mathrm{e}^{2x}$ for constants $a, b, c \in \mathbb{R}$, and vector additions and scalar multiplications are defined in the obvious mannor.

   (a) (2 points) Show that if $f \in V$, then its derivative f' is also in V. (So in particular, taking derivative is a linear map $D : V \to V$)

   (b) (4 points) Using basis $\mathrm{e}^{2x}, x\mathrm{e}^{2x}, \frac{1}{2}x^2\mathrm{e}^{2x}$, write out the corresponding matrix for D. Is this matrix in the above subproblem diagonalizable? Why?

   (c) (4 points) Write out the change of coordinate matrix from basis $\mathrm{e}^{2x}, x\mathrm{e}^{2x}, \frac{1}{2}x^2\mathrm{e}^{2x}$ to basis $x^2\mathrm{e}^{2x}, (2x^2+2x)\mathrm{e}^{2x}, (4x^2+8x+2)\mathrm{e}^{2x}$.

   (d) (4 points) Using basis $x^2\mathrm{e}^{2x}, (2x^2+2x)\mathrm{e}^{2x}, (4x^2+8x+2)\mathrm{e}^{2x}$, write out the corresponding matrix for D. What is the characteristic polynomial of this matrix?

Answer:

1. The derivative of $(ax^2+bx+c)e^{2x}$ is $(2ax^2+(2a+2b)x+(2c+b))e^{2x}$.

2. $D(e^{2x}, xe^{2x}, \frac{1}{2}x^2e^{2x}) = (2e^{2x}, (2x+1)e^{2x}, (x^2+x)e^{2x}) = (e^{2x}, xe^{2x}, \frac{1}{2}x^2e^{2x}) \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$.

3. Note that $(x^2e^{2x}, (2x^2+2x)e^{2x}, (4x^2+8x+2)e^{2x}) = (e^{2x}, xe^{2x}, \frac{1}{2}x^2e^{2x}) \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{bmatrix}$.
   This is the basis transition matrix from the old basis to the new basis, and thus the change of coordinate matrix is its inverse, $\begin{bmatrix} 2 & -1 & \frac{1}{2} \\ -2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$.

4. We have $D(x^2e^{2x}, (2x^2+2x)e^{2x}, (4x^2+8x+2)e^{2x}) = ((2x^2+2x)e^{2x}, (4x^2+8x+2)e^{2x}, (8x^2+24x+12)e^{2x}) = (x^2e^{2x}, (2x^2+2x)e^{2x}, (4x^2+8x+2)e^{2x}) \begin{bmatrix} 0 & 0 & 8 \\ 1 & 0 & -12 \\ 0 & 1 & 6 \end{bmatrix}$.
   Note that this is a companion matrix, so the characteristic polynomial is $x^3-6x^2+12x-8$. Alternatively, one can also see the characteristic polynomial by the matrix for D under the old basis, which is $(x-2)^3$.

7. (Proof-intensiven problem) Suppose $A^{\mathrm{T}} = A^2$ for a real matrix A. Let us investigate the possibility of such A. Note that A is a normal matrix.

   (a) (2 points) For each eigenvalue λ of A, show that $\lambda^2 = \overline{\lambda}$, and find all possible $\lambda \in \mathbb{C}$ that satisfy this condition. (Hint: Spectral decomposition of A.)

   (b) (2 points) Show that $A^4 = A$.

   (c) (2 points) Show that $A^2\boldsymbol{x} = \boldsymbol{x}$ implies that $A\boldsymbol{x} = \boldsymbol{x}$.

   (d) (2 points) Show that A^3 is an orthogonal projection.

   (e) (1 point) Show that $I + A - A^3$ is an orthogonal and $(I + A - A^3)^3 = I$. Also prove that $\det(I + A - A^3) = 1$, so this is a rotation with period 3.

Answer:

1. Suppose $A = QDQ^*$, then $A^2 = A^{\mathrm{T}} = A^*$ implies that $D^2 = D^*$, so $\lambda^2 = \overline{\lambda}$ for each eigenvalue λ. Note that this means $\lambda^3 = |\lambda|^2$, and thus $|\lambda|^3 = |\lambda|^2$. This means $|\lambda| = 1$ or $|\lambda| = 0$. If $|\lambda| = 1$, then $\lambda^3 = 1$ implies that $\lambda = 1, \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}, \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$. If $|\lambda| = 0$, then $\lambda = 0$. All in all, we have four possibilities.
2. Since $\lambda^3 = 1$ for each eigenvalue λ, we see that $\lambda^4 = \lambda$. So $D^4 = D$ and $A^4 = QD^4Q^* = QDQ^* = A$.
3. If $A^2\boldsymbol{v} = \boldsymbol{v}$, then $A\boldsymbol{v} = A^4\boldsymbol{v} = A^2(A^2\boldsymbol{v}) = A^2\boldsymbol{v} = \boldsymbol{v}$.
4. $(A^3)^2 = A^6 = A^2A^4 = A^2A = A^3$, so this is a projection matrix. $(A^3)^{\mathrm{T}} = (A^{\mathrm{T}})^3 = (A^2)^3 = A^6 = A^3$, so this is an orthogonal projection.
5. $(I + A - A^3)^{\mathrm{T}} = I + A^{\mathrm{T}} - (A^3)^{\mathrm{T}} = I + A^2 - A^3$. Now we have $(I + A - A^3)^{\mathrm{T}}(I + A - A^3) = (I + A^2 - A^3)(I + A - A^3) = I + A + A^2 - A^3 - A^4 - A^5 + A^6 = I + A + A^2 - A^3 - A - A^2 + A^3 = I$, where we use the identity $A^4 = A$ to lower the degrees. So this matrix is orthogonal. We also have $(I+A-A^3)^2 = I+2A+A^2-2A^3-2A^4+A^6 = I+2A+A^2-2A^3-2A+A^3 = I + A^2 - A^3 = (I + A - A^3)^{\mathrm{T}} = (I + A - A^3)^{-1}$, so $(I + A - A^3)^3 = I$. Finally, $(\det(I + A - A^3))^3 = \det(I + A - A^3)^3 = \det(I) = 1$, but since $I + A - A^3$ is real, it has real determinant, so $\det(I + A - A^3) = 1$. So this is a rotation with period 3.