Midterm for Linear Algebra

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Generic Rules:

1. The due date is Nov 15th.
2. Most sub-problems worth 2 points each, but some are worth 3 points each. So you can get as much as 54 pts in total. However, we use 50 pts as full credit. If you get more than 50 pts, then your score is 50. (So you can safely skip one subproblem, or did one subproblem completely wrong.) We will give partial credits for partial solutions.
3. Feel free to use calculators or softwares to help your computations, or collaborate with your classmates. But in your submitted midterm, you should always express your ideas in your own words. **Write down the names of your collaborators if you collaborated with someone.** (Also, if you collaborated with someone, then each of you should hand in your own midterm answer in your own words. Do NOT give me a joint answer or copied answer, as it would be treated as plagiarism.)
4. If I suspect plagiarism, I might call you into my office, and ask you to do the problem in front of me to see if you actually understand what you have written down. You should be able to explain your own written answers. Failure to do so will be a confirmation of plagiarism, and will be punished accordingly.
5. Write your answers in English. You should always explain any statement you made, and always show some process of calculation (which may help you earn some partial credit). This is a mid-term after all, so please be more formal and careful in your answers.

Problem 1 (Solving Sylvester's equation)**.** *Sylvester's equation refers to the following situation: for constant matrices A, B, C, find the unknown matrix X from the equation $AX - XB = C$. This type of equations is very important in many applications, such as the next problem on electrical circuits.*

In this problem, let V be the space of all 2×2 matrices. We fix some θ that is not a multiple of π, and set $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B = A^{-1}$. Then we have a map $L : V \to V$ such that $L(X) = AX - XB$.

1. *(2pts) Verify that L is a linear map.*
2. *(2pts) We pick basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for V, then find the matrix for L, then find its RREF.*
3. *(2pts) Find a basis for the kernel of L and the range of L. (Note that the basis vectors should be elements of V, so they should be matrices. DO NOT put them in coordinates.) Geometrically, what are the elements of these subspaces?*
4. *(2pts) We pick basis $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for V. Find the change of coordinate matrix P from the old basis to this new basis, and find the new matrix for L. How are the new matrix L_{new} and the old matrix L_{old} related through P?*

5. *(3pts) Say $\theta = \frac{\pi}{2}$. Find all solutions to $AX - XB = C$ when $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and find all solutions to $AX - XB = C$ when $C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.*

Problem 2 (Circuits, Commutator and Sylvester's equation)**.** *Suppose I have n nodes, with non-zero electrical potentials $x_1, ..., x_n$. Each pair of nodes is connected by a wire of non-zero resistance r_{ij}. Then for the wire between the i-th node and the j-th node, the electrical current would flow from i-th node to j-th node with a magnitude of $y_{ij} = \frac{x_i - x_j}{r_{ij}}$. (Here if y_{ij} is negative, then the electricity is actually flowing from the j-th node to the i-th node. Furthermore, if $i = j$, then we simply set $y_{ii} = 0$.)*

We can talk about the total outflow from a node i as the value $y_i = \sum_j y_{ij}$. We also have the following matrices here that might or might not be useful. We let R be the resistance matrix whose (i,j) entry is r_{ij} when $i \neq j$, and zero when $i = j$. Let R' be the "inverse resistance matrix" whose (i,j) entry is $\frac{1}{r_{ij}}$ when $i \neq j$, and zero when $i = j$. And we have a voltage matrix V which is diagonal with x_i as the i-th diagonal entry. Let C be the electrical current matrix whose (i,j) entry is y_{ij}.

1. *(2pts) Express C in terms of R, R', V. Not all matrices have to be used.*
2. *(2pts) Show that C is skew symmetric.*
3. *(2pts) Show that $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = (VR'V^{-1} - R') \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.*
4. *(2pts) Suppose $x_1, ..., x_n$ are not all the same. Show that $VR'V^{-1} - R'$ has NO LU decomposition. BEWARE that non-invertible matrices might STILL have LU decompositions.*
5. *(2pts) For this subproblem, we allow electrical potentials to be potentially zero. Suppose we know C and R and R', but V is unknown. We are trying to solve for the diagonal matrix V from the expression of C above. Show that the set of complete solutions is an affine subspace of $M_{n\times n}$, the vector space of n by n matrices. Furthermore, show that this affine subspace has dimension one. (Hint: it is easier here to think about the physics.)*
6. *(3pts) For this subproblem, we allow electrical potentials to be potentially zero. Suppose $R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. For any 3×3 matrix C, suppose $XR - RX = C$ has a solution, then show that the collection of all matrices X that solves $XR - RX = C$ is an affine subspace, and find its dimension. (Hint: consider the range of the map $X \mapsto XR - RX$.)*

Problem 3 (Transpose)**.** *Suppose I want to invest in the stock market. There are two companies A and B selling umbrellas, and two companies C and D selling sunscreens. Now, a single company's stock might rise or fall for many factors, so I do not want to buy stocks of single companies. Rather, I want to buy a portfolio (combination of stocks) to reflect the umbrella industry as a whole (therefore immune to the risks special to some particular company), or a portfolio to reflect the sunscreen industry as a whole.*

Suppose each share of Portfolio X is made of 2 shares of A and 3 shares of B. Each share of portfolio Y is made of 1 share of C and 4 shares of D. The share price of a portfolio is the sum of the prices of all shares it contains.

Now, how many shares of portfolio X and portfolio Y should I buy? Suppose I invented a formula to optimize my investment. When the shares of X, Y have prices x, y, then I would hold $2x - y, 3y - x$ shares of X, Y respectively.

1. *(2pts) Find a matrix M such that if the prices of shares of X, Y are a vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, then I would hold $M \begin{bmatrix} x \\ y \end{bmatrix}$ shares of X, Y respectively. Then find the LDU decomposition of M.*

2. *(2pts) Find a matrix P such that if I hold $\boldsymbol{v}$ shares of X, Y, then I would be equivalently hold $P\boldsymbol{v}$ shares of A, B, C, D respectively. Also find a matrix P' such that if the prices of shares of A, B, C, D are* $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$, *then* $P'\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ *gives the prices of shares of X and Y. What is the relation between P and P'?*

3. *(2pts) Suppose we have a matrix Q such that if the prices of shares of A, B, C, D are* $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$, *then I would equivalently be holding* $Q\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ *shares of A, B, C, D respectively. Express Q in terms of M and P, and find a full rank decomposition of Q.*

4. *(2pts) Find the RREF of Q and P^{T}.*

5. *(2pts) Find bases for* $\mathrm{Ran}(Q), \mathrm{Ker}(Q), \mathrm{Ran}(Q^{\mathrm{T}}), \mathrm{Ker}(Q^{\mathrm{T}})$, *and find their dimensions.*

6. *(2pts) If the prices of shares of A, B, C, D are $\boldsymbol{v} \in \mathbb{R}^4$, and the prices of the shares of X, Y are $\boldsymbol{w} \in \mathbb{R}^2$, prove that $\boldsymbol{v}^{\mathrm{T}}Q\boldsymbol{v} = \boldsymbol{w}^{\mathrm{T}}M\boldsymbol{w}$ always. What is the meaning of this quantity?*

Problem 4 (Sudoku Matrices)**.** *The 4×4 Sudoku matrix is a block form matrix such as* $\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ \hline 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{array}\right]$, *where each row is made of* $1, 2, 3, 4$, *each column is made of* $1, 2, 3, 4$, *and each block is made of* $1, 2, 3, 4$.

1. *(2pts) For the matrix in the example, show that it is invertible. Also find a Sudoku matrix that is not invertible.*

2. *(3pts) Find the block LDU of* $\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ \hline 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{array}\right]$. *The block form should be something like* $\begin{bmatrix} I & \\ X & I \end{bmatrix}\begin{bmatrix} A & \\ & B \end{bmatrix}\begin{bmatrix} I & Y \\ & I \end{bmatrix}$ *where all blocks are 2×2. Then express the inverse of this Sudoku matrix in terms of the blocks A, B, X, Y.*

3. *(2pts) Can you write the identity matrix as a linear combination of 4×4 Sudoku matrices? How or why not?*

4. *(2pts) Consider the span of all 4×4 Sudoku matrices. What is its dimension? And what could be a basis? (This problem is not easy, but try some row/column swaps, block row/column swaps, etc..)*

Problem 5 (Chinese Chess Matrix)**.** *The Chinese chess (Zhong Guo Xiang Qi) is a very popular game. The game board is 10×9. Consider the Chinese chess matrix, where c, m, x, s, j, p, b are all distinct non-zero* *real numbers.*

$$A = \begin{bmatrix} c & m & x & s & j & s & x & m & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 & p & 0 \\ b & 0 & b & 0 & b & 0 & b & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & b & 0 & b & 0 & b & 0 & b \\ 0 & p & 0 & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & m & x & s & j & s & x & m & c \end{bmatrix}.$$

1. *(2pts) Find the RREF of A.*
2. *(2pts) Find a permutation matrix P such that PA would have the fourth row of A as the first row, the third row of A as the second row, the first row of A as the third row, and the second row of A as the fourth row, and all the other rows are identical.*
3. *(2pts) Use the permuation matrix P above, find a unit lower triangular matrix L and an REF U, such that $PA = LU$.*
4. *(3pts) Find the rank of A, and find bases for the four subspaces* $\mathrm{Ran}(A), \mathrm{Ran}(A^{\mathrm{T}}), \mathrm{Ker}(A), \mathrm{Ker}(A^{\mathrm{T}})$