Linear Algebra (English) **Name:** ______________________

2021 Fall

Final Exam

2021.12.22

Exam Duration: 3 Hours **Student ID:** ______________________

This exam includes 11 pages (including this page) and 5 problems. Please check to see if there is any missing page, and then write down your name and student ID number on this page and the first page of your answer sheets. Also write down the initials of your name on the top of every page of your answer sheets, in case they are scattered.

This exam is open book. You are allowed to consult your textbook and notes, but no calculator. Plagerism of all kinds are strictly forbidden and will be severly punished.

Please write down your answers to the problems in the provided **SEPARATE ANSWER SHEETS**, and follow the following rules:

- **Always explain your answer.** You should always explain your answers. Any problem answered with nothing but a single answer would receive no credit.
- **Write cleanly and legible.** Make sure that your writings can be read. The graders are NOT responsible to decipher illegible writings.
- **Partial credits will be given.**
- Blank spaces are provided in the exams. Feel free to use them as scratch papers. However, your formal answer has to be written in the **SEPARATE ANSWER SHEETS**, as required by the University.
- The total score of the exam is 50. If your total score exceeds 50 (there are 53 points in total), it will be recorded as 50.

Problem	Points	Score
1	13	
2	10	
3	11	
4	10	
5	9	
Total:	53	

1. We have points $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ in the space $\mathbb{R}^3$. Together they form a data matrix $A = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix}$. (Note that the points are already centered.)

   (a) (3 points) Find a singular value decomposition of A.

   (b) (3 points) Find two mutually orthogonal lines of best fit.

   (c) (2 points) Find the plane of best fit.

   (d) (2 points) Find the maximum and minimum Rayleigh quotient $\frac{\boldsymbol{v}^{\mathrm{T}} S \boldsymbol{v}}{\boldsymbol{v}^{\mathrm{T}} \boldsymbol{v}}$ for $S = (A^{\mathrm{T}}A)^2 + 2A^{\mathrm{T}}A + 3I$.

   (e) (3 points) Find a polynomial $p(x)$ such that $(A^{\mathrm{T}}A + 2I)^{-1} = p(A^{\mathrm{T}}A)$.

Answer:

1. $A^{\mathrm{T}}A = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 9 & & \\ & 9 & \\ & & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^{\mathrm{T}}$.
   So the singular value decomposition has $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 0 \end{bmatrix}$. Now $A\boldsymbol{v}_1, A\boldsymbol{v}_2$ gives us $\boldsymbol{u}_1, \boldsymbol{u}_2$, and $\boldsymbol{u}_3$ is any non-zero vector in the kernel of A^{T}. So we have $U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$. So we are done.
2. The two lines of best fit is along $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.
3. The plane of best fit is the span of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.
4. Note that $A^{\mathrm{T}}A$ is a 3×3 matrix with eigenvalues $9, 9, 0$. So S has eigenvalues $102, 102, 3$. So the maximum value of the Rayleigh quotient is 102 and the minimum value is 3.
5. $A^{\mathrm{T}}A = 9I - 3B$ where B is the all one matrix, and $B^2 = 3B$. So the inverse is $aI + bB$ for some B. Then $I = (A^{\mathrm{T}}A + 2I)(aI + bB) = (11I - 3B)(aI + bB) = 11aI + (2b - 3a)B$. So $a = \frac{1}{11}$ and $b = \frac{3a}{2}$. Hence the inverse is $\frac{1}{11}I + \frac{3}{22}B = \frac{1}{2}I - \frac{1}{22}A^{\mathrm{T}}A$.

2. In the xy-plane, we have a line $ax + y = 1$ for some unknown constant $a \in \mathbb{R}$. Suppose we also know that the line must go through the points $\begin{bmatrix} 1 \\ b \end{bmatrix}, \begin{bmatrix} 2 \\ -b \end{bmatrix}, \begin{bmatrix} -2 \\ 4b \end{bmatrix}$ for some unknown constant $b \in \mathbb{R}$. We wish to find all possible a, b.

   (a) (2 points) Find a 3×2 matrix A and a vector $\boldsymbol{u}$ such that a, b is a possible solution to the problem above if and only if $A \begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{u}$.

   (b) (2 points) Show that $A \begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{u}$ above has no solution.

   (c) (4 points) Find the QR decomposition of A, where Q is 3×2 with orthonormal columns and R is 2×2 and upper triangular with positive diagonal entries.

   (d) (2 points) Find the least square solution to $A \begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{u}$.

Answer:

1. $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
2. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -2 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ Oops.
3. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = (\frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}) \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$.
4. $\begin{bmatrix} A^{\mathrm{T}}A & A^{\mathrm{T}}\boldsymbol{u} \end{bmatrix} = \begin{bmatrix} 9 & -9 & 1 \\ -9 & 18 & 4 \end{bmatrix} \to \begin{bmatrix} 9 & 0 & 6 \\ 0 & 9 & 5 \end{bmatrix}$. So the least square solution is $a = \frac{6}{9}$ and $b = \frac{5}{9}$.

3. Consider $F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{bmatrix}$.

   (a) (2 points) Is F Hermitian or skew-Hermitian or Unitary or none of these? Can you tell without calculation if F is diagonalizable or not?

   (b) (4 points) Calculate F^2 and find a basis for each eigenspace.

   (c) (3 points) Find all eigenvalues of F and their algebraic multiplicity, and find a basis for the eigenspaces of non-real eigenvalues.

   (d) (2 points) Let $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the matrix X such that $2F = X \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} P$. (This is the fundation of the famous Fast Fourier Transform algorithm, ranked as one of the top 10 algorithms of 20-th century.)

Answer:

1. F is unitary, so it is diagonalizable.
2. $F^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. So 1 is an eigenvalue and its eigenspace has a basis $\boldsymbol{e}_1, \boldsymbol{e}_2 + \boldsymbol{e}_4, \boldsymbol{e}_3$. And -1 is another eigenvalue whose eigenspace is spanned by $\boldsymbol{e}_2 - \boldsymbol{e}_4$.
3. F can only have eigenvalues $\pm 1, \pm 1, \pm 1, \pm \mathrm{i}$. Since the trace is $1 + \mathrm{i}$, we must have eigenvalues $1, 1, -1, \mathrm{i}$.
4. $X = \begin{bmatrix} I & D \\ I & -D \end{bmatrix}$ where $D = \begin{bmatrix} 1 & \\ & \mathrm{i} \end{bmatrix}$.

4. Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix}$. We aim to find the orthogonal projection matrix to $\mathrm{Ran}(A)$.

   (a) (3 points) Find the reduced row echelon form of A^{T}, and find a basis for $\mathrm{Ker}(A^{\mathrm{T}})$.

   (b) (2 points) Find the orthogonal projection matrix P_1 to $\mathrm{Ker}(A^{\mathrm{T}})$.

   (c) (2 points) Find the orthogonal projection matrix P_2 to $\mathrm{Ran}(A)$.

   (d) (3 points) Let $B = \begin{bmatrix} 5 & 5 & 6 \\ 5 & 8 & 9 \\ 6 & 9 & 13 \end{bmatrix}$. Show that $\begin{bmatrix} I & A^{\mathrm{T}} \\ A & B \end{bmatrix}$ is positive definite. Please explicitly states the criteria of positive-definiteness that you are using. (Hint: Block $\mathrm{LDL}^{\mathrm{T}}$ decomposition.)

Answer:

1. A^{T} has RREF of $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $\mathrm{Ker}(A^{\mathrm{T}})$ is spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

2. $\frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

3. $I - \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$.

4. $\begin{bmatrix} I & A^{\mathrm{T}} \\ A & B \end{bmatrix} = \begin{bmatrix} I & \\ A & I \end{bmatrix} \begin{bmatrix} I & \\ & B - AA^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & A^{\mathrm{T}} \\ & I \end{bmatrix}$ and $B - AA^{\mathrm{T}} = I$. So our matrix is LL^{T} for an invertible matrix $L = \begin{bmatrix} I & \\ A & I \end{bmatrix}$. (Or say that pivots are positive.)

5. Let V be the space of homogeneous polynomials in x, y with degree two. (Homogeneous means we only have degree two terms. For example, elements of V could be $x^2 + 2xy + 3y^2$, $4x^2 - 3xy - y^2$ and so on. There cannot be degree one or degree zero terms.)

   Let W be the space of polynomials in x with degree at most two. Now, given any polynomial in V, say $x^2 + 2xy + 3y^2$, we can substitute y by $x + 2$, and therefore get $x^2 + 2x(x+2) + 3(x+2)^2$. This would be an element of W. Hence "substitute y by $x + 2$" is an abstract map $S : V \to W$. This is a linear map.

   (a) (1 point) Verify that $S(2x^2 + 3xy) = 2S(x^2) + 3S(xy)$.

   (b) (2 points) Pick basis x^2, xy, y^2 for V and basis $x^2, x, 1$ for W, find the matrix A for S under these basis.

   (c) (2 points) Find all possible $p \in V$ such that $S(p) = 3x^2 + 6x + 4$.

   (d) (2 points) Pick basis $x^2, x(y - x), (y - x)^2$ for V and basis $x^2, x, 1$ for W, find the matrix B for S under these basis.

   (e) (2 points) Let X the change of coordinate matrix in V from the basis x^2, xy, y^2 to the basis $x^2, x(y-x), (y-x)^2$. How are A, B, X related? Calculate X from this relation.

Answer:

1. $S(2x^2+3xy) = 2x^2+3x(x+2) = 2S(x^2)+3S(xy)$.
2. $S(x^2) = x^2, S(xy) = x^2+2x, S(y^2) = x^2+4x+4$. So the matrix is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.
3. S is invertible so the solution is unique. Gaussian elimination (or simply staring at it) gives x^2+xy+y^2.
4. $y-x$ is sent to 2. So the matrix is $\begin{bmatrix} 1 & & \\ & 2 & \\ & & 4 \end{bmatrix}$.
5. $X = B^{-1}A = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix}$.