17. Σ 为对称矩阵，可作正交对角化

∴ ∃ A 正交，使得 $A\Sigma A^T = D$　　D 为对角矩阵，$= diag(d_1, d_2, \dots, d_n)$

∴ 由此，$Y \sim N(0, D)$，$Cov(Y_i, Y_j) = 0$　$(i \neq j)$

∴ Y_i 互相独立，且 $Y_i \sim N(0, d_i)$

18.

(1) $\begin{pmatrix}3&2\\2&3\end{pmatrix}\begin{pmatrix}1&\frac{4}{5}\\\frac{4}{5}&1\end{pmatrix}\begin{pmatrix}3&2\\2&3\end{pmatrix}=\begin{pmatrix}\frac{113}{5}&\frac{112}{5}\\\frac{112}{5}&\frac{113}{5}\end{pmatrix}$

(2) $\begin{pmatrix}Y_1\\Y_2\end{pmatrix}\sim N\left(\begin{pmatrix}7\\8\end{pmatrix},\begin{pmatrix}\frac{113}{5}&\frac{112}{5}\\\frac{112}{5}&\frac{113}{5}\end{pmatrix}\right)$

$Y_2|Y_1$ 也为正态分布，其期望为：$8+\frac{112}{5}\times\frac{5}{113}\times(Y_1-7)=8+\frac{112}{113}(Y_1-7)$

$Y_1+Y_2=(1\ 1)\begin{pmatrix}Y_1\\Y_2\end{pmatrix}$　∴ $E(Y_1+Y_2)=(1,1)\begin{pmatrix}7\\8\end{pmatrix}=15$

(3) $\cancel{E(X_2|X_1)=}$ $X_2|X_1 \sim N\left(2+\frac{4}{5}\times1\times(X_1-1),\ 1-\left(\frac{4}{5}\right)^2\times1\right)$

∴ $E(X_2|X_1)=2+\frac{4}{5}(X_1-1)$　　$D(X_2|X_1)=\frac{9}{25}$

(4) 显然二者为高斯分布，故只需证明二者协方差为0

$E(X_2-E(X_2|X_1))=EX_2-E(E(X_2|X_1))=EX_2-EX_2=0$

$E(X_1X_2-X_1E(X_2|X_1))=EX_1X_2-E(X_1(2+\frac{4}{5}(X_1-1)))=Cov(X_1,X_2)+EX_1EX_2-E(\frac{4}{5}X_1^2)-2EX_1+\frac{4}{5}\dots$

$=0$

∴ $Cov(X_1, X_2-E(X_2|X_1))=0$　∴ 独立

20.(a) 必要性 $(X_{t_1}, X_{t_2}\cdots X_{t_n})^T$ 服从高斯分布

∴ ~~Cov(~~ 因与 $(X_{t_1+a}, X_{t_2+a}\cdots X_{t_n+a})^T$ 相同

∴ $EX_{t_1} = EX_{t_1+a}$ 由 t_1, a的任意性 ∴ EX_i 为常数

$Cov(X_{t_1}, X_{t_2}) = Cov(X_{t_1+a}, X_{t_2+a})$

由a的任意性 $Cov(X_{t_1}, X_{t_2})$ 只与 t_1-t_2 有关.

充分性: 考虑 $(X_{t_1+a}, X_{t_2+a}, \cdots X_{t_n+a})^T$ 联合分布的特征函数.

$\varphi(\vec{\theta}) = \exp\{i\vec{\theta}^T\vec{\mu} - \frac{1}{2}\vec{\theta}^T\Sigma\vec{\theta}\}$

而 $\vec{\mu}$ 与a无关. Σ 中元素为 $Cov(X_{t_i+a}, X_{t_j+a})$ 只与 $|t_i-t_j|$ 有关

∴ $\varphi(\vec{\theta})$ 与a无关.

∴ 对任意a, $\varphi_{X_{t_1}, X_{t_2}\cdots X_{t_n}}(\vec{\theta}) = \varphi_{X_{t_1+a}, X_{t_2+a}, \cdots}(\vec{\theta})$

∴ 二者有相同的联合分布.

(b) $EU_t = e^{-\frac{\alpha t}{2}} EX_{e^{\alpha t}} = 0$

~~$Cov(X_s, X_t) = EX_sX_t$~~

$$Cov(U_s, U_t) = EU_sU_t = e^{-\frac{\alpha s}{2}-\frac{\alpha t}{2}} E(X_{e^{\alpha s}}X_{e^{\alpha t}})$$

$$= e^{-\frac{\alpha s}{2}-\frac{\alpha t}{2}} Cov(X_{e^{\alpha s}}, X_{e^{\alpha t}})$$

$$= e^{-\alpha\frac{s+t}{2}} \cdot e^{\min(\alpha s, \alpha t)}$$

$$= e^{-\frac{\alpha}{2}|t-s|}$$

∴ $\{U_t, t\geq 0\}$ 为平稳高斯过程

21. (1) $\begin{pmatrix}B_1\\B_2\\\vdots\\B_n\end{pmatrix} \sim N\left(\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}1&1&\cdots&1\\1&2&2\cdots&2\\\vdots&\vdots&3\ddots&\\1&2&&n\end{pmatrix}\right)$

$B_1+B_2+\cdots+B_n=(1,1\ 1\cdots 1)\begin{pmatrix}B_1\\B_2\\\vdots\\B_n\end{pmatrix} \sim N\left(0,\ \frac{n(n+1)(2n+1)}{6}\right)$

(2) $E(B_{t+1}-B_t)=0$

$Cov(Y_s,Y_t)=EY_sY_t = E((B_{s+1}-B_s)(B_{t+1}-B_t))$

$= \min(s+1,t+1)+\min(s,t)-\min(s,t+1)$
$\quad -\min(t,s+1)$

$= \frac{1}{2}(|s-t-1|+|s-t+1|-2|s-t|)$

~~只与s,t~~ 只依赖于 $|s-t|$

∴ 是平稳过程.

23. $\begin{pmatrix}B_s\\B_t\end{pmatrix} \sim N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}s&s\\s&t\end{pmatrix}\right)$ $\quad\therefore B_s|B_t \sim N\left(\frac{s}{t}(B_t-0),\ s-\frac{s^2}{t}\right)$

$\therefore E(B_s|B_t=x)=\frac{xs}{t}$

12. $F_{M_n}(x)=P(M_n\le x)=1-P(M_n>x)$

当 $x<a$ 时, 上式 $=0$

当 $x\ge a$ 时, 上式 $=1-(1-e^{-(x-a)})^n \longrightarrow 1$

$\therefore M_n \xrightarrow{D} F_a \qquad \therefore M_n \xrightarrow{P} a$

(4) 概念:

一: $\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} X + \begin{pmatrix} B \\ D \end{pmatrix}$

$\therefore \begin{pmatrix} Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} A \\ C \end{pmatrix} \mu + \begin{pmatrix} B \\ D \end{pmatrix}, \begin{pmatrix} A\Sigma A^T & A\Sigma C^T \\ C\Sigma A^T & C\Sigma C^T \end{pmatrix} \right)$

$\because$ Y、Z 相互独立 $\Leftrightarrow$ Y的分量与Z的分量独立

$\Leftrightarrow C\Sigma A^T = 0$

二. (1) ~~$EX=0$~~ $X = (3 \quad 2 \quad -1)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \sim N(0, 18)$

$\therefore EX = 0 \quad DX = 18 \qquad \varphi_X(\theta) = e^{-9\theta^2}$

(2) X,Y 相互独立 $\Leftrightarrow \mathrm{Cov}(X,Y) = 0 \Leftrightarrow EXY = 0$

$EXY = E(3B_1 + 2B_2 - B_3)(cB_1 + B_2) = 2c + 5 = 0$

$c = -\frac{5}{2}$

(3) $E(X \mid B_2 = 1) = 3E(B_1 \mid B_2 = 1) + 2 - E(B_3 \mid B_2 = 1)$

$= \frac{5}{2}$

三、$EX^2=1 \quad EY^2=1$

$EX^4=3 \quad EY^4=3$

$\therefore DX^2=DY^2=2$

$EX^2Y^2=E(E(X^2Y^2|Y))$

$=E(Y^2E(X^2|Y))$

而 $X|Y\sim N(\rho Y,1-\rho^2)$ $\quad\therefore E(X^2|Y)=(1-\rho^2)+\rho^2Y^2$

$\therefore$ 此上式 $=\cancel{E}(1-\rho^2)EY^2+\rho^2EY^4$

$=1+2\rho^2$

$$\therefore r_{X^2,Y^2}=\frac{EX^2Y^2-EX^2EY^2}{\sqrt{DX^2DY^2}}=\rho^2$$