Graph Theory

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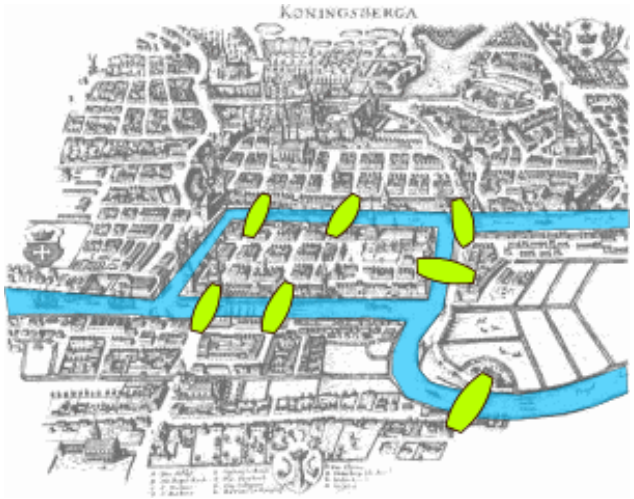
Introduction

Graph Theory

- Invented by Leonard Euler 1735
- A set V of vertices
- E (edges connecting pairs of vertices)
- First Problem: Seven Bridges of Konigsberg

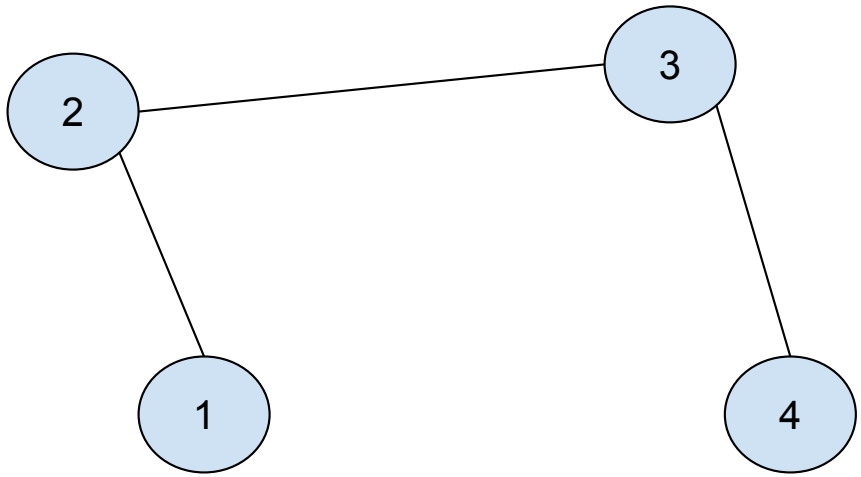
Introduction



Terminology

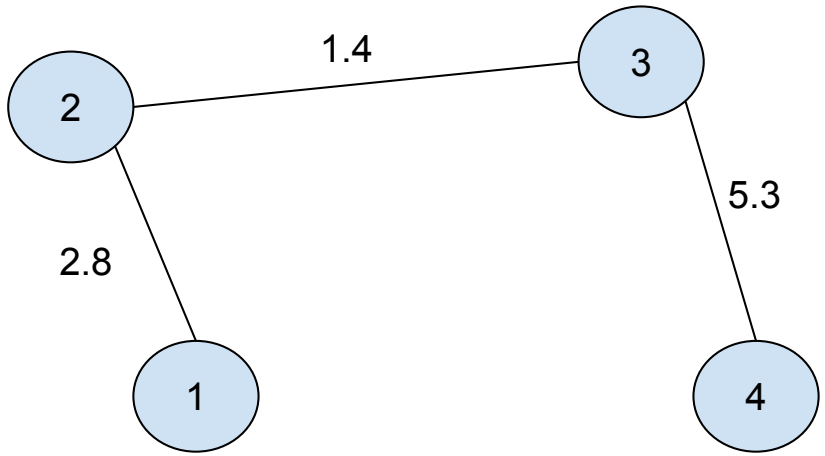
A Graph has a set V of vertices, and E edges connecting pairs of vertices

Edges are by default not directional (travel can go in both directions)



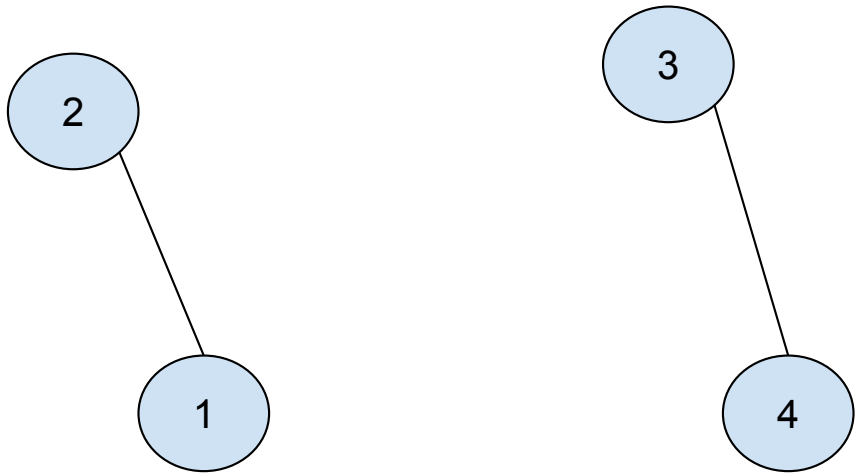
Weights

Edges may also have weights representing the cost to traverse an edge



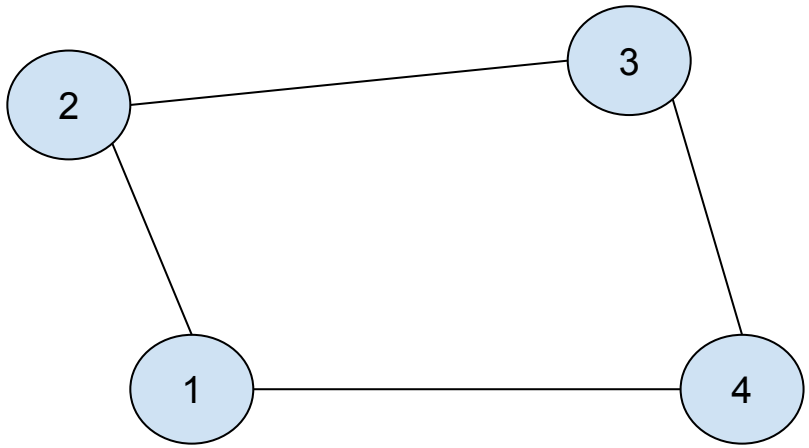
Connected Graph

A graph is said to be connected iff from each vertex it is possible to reach all other vertices



Biconnected

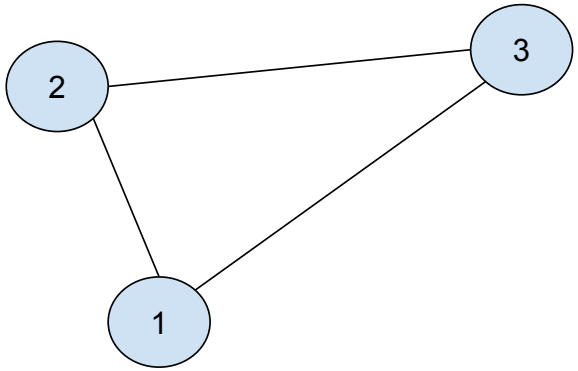
A graph is biconnected if removing any single edge leaves it still connected

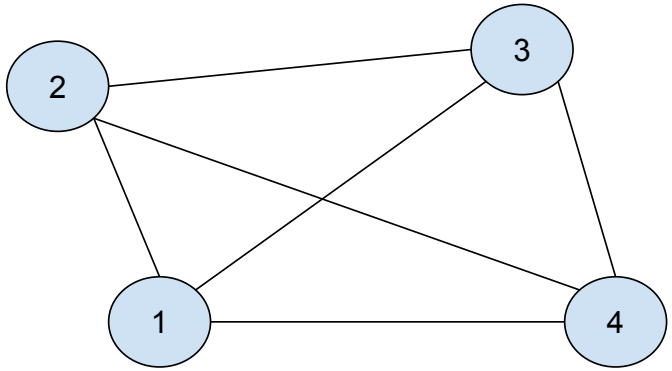


Complete Graph

A complete graph has all edges connecting every vertex to every other

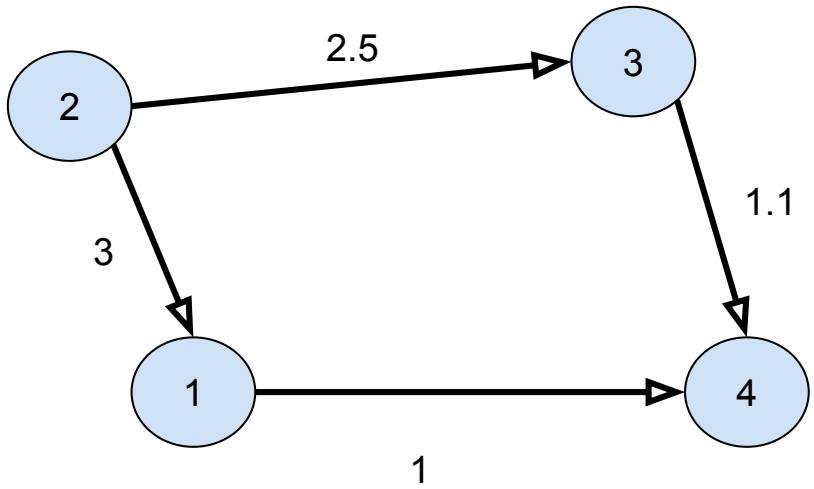
What is the number of edges to make a complete graph as a function of V?





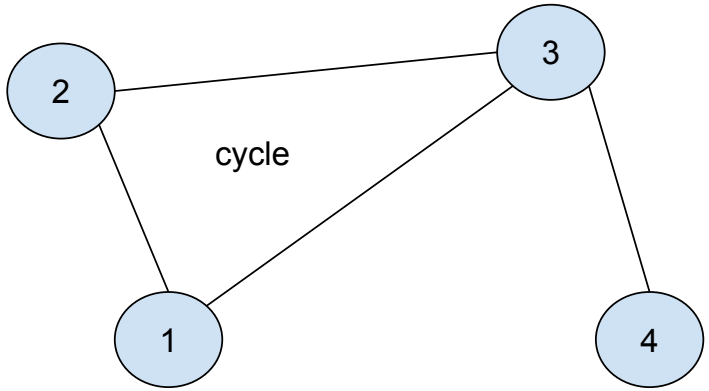
Directed Graph (DiGraph)

A Directed Graph has edges that can only be traversed in one direction



Cycles

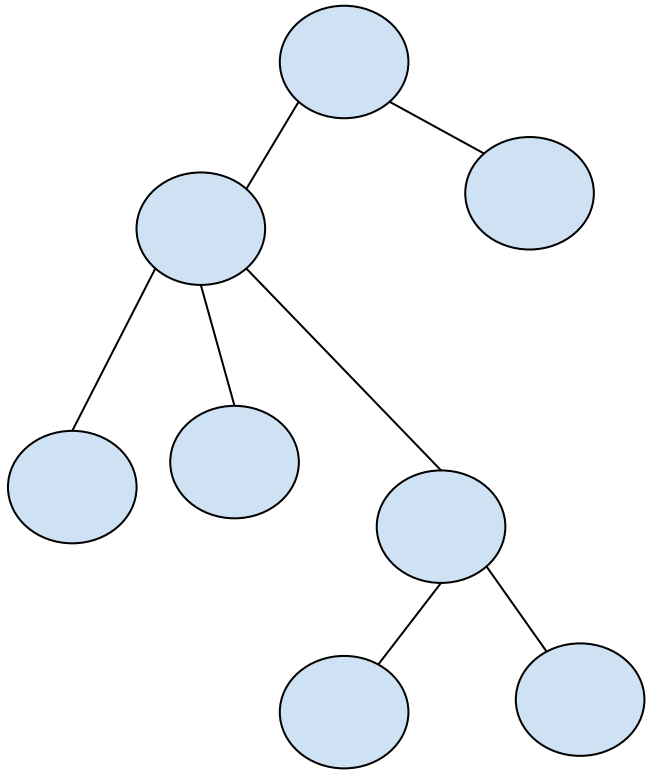
A cycle in a graph is a loop



Tree: A Special Case of Graph

A Tree is a graph

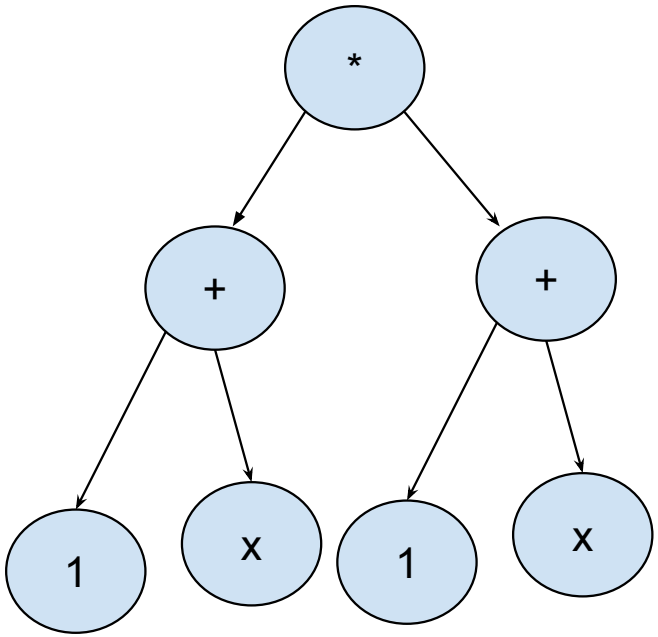
- With a designated root
- Once branches diverge they never rejoin
- In other words, a tree has no cycles



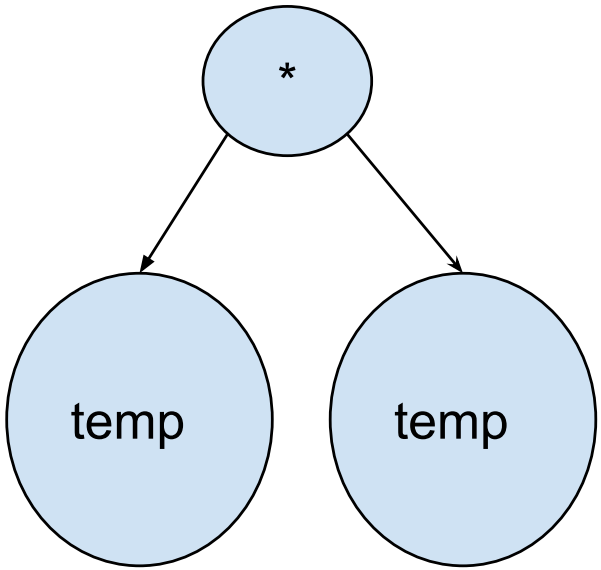
Notice that any node could be the root

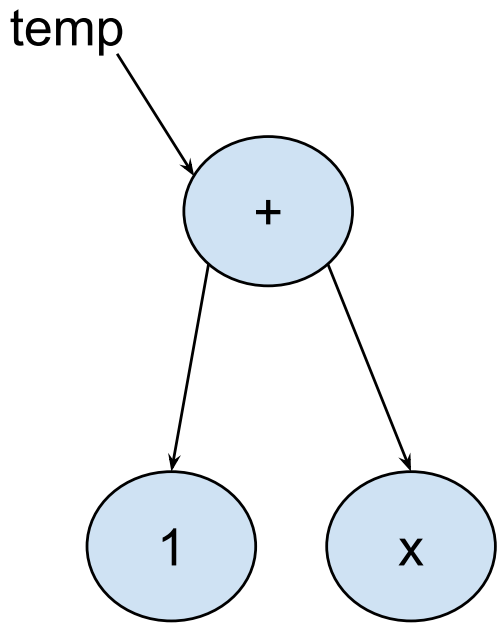
Directed Acyclic Graph (DAG)

A DAG is like a tree except the branches can rejoin. There are no cycles. Common in expressions where common terms repeat
This diagram shows x used twice



DAG Referring to Common Subexpression

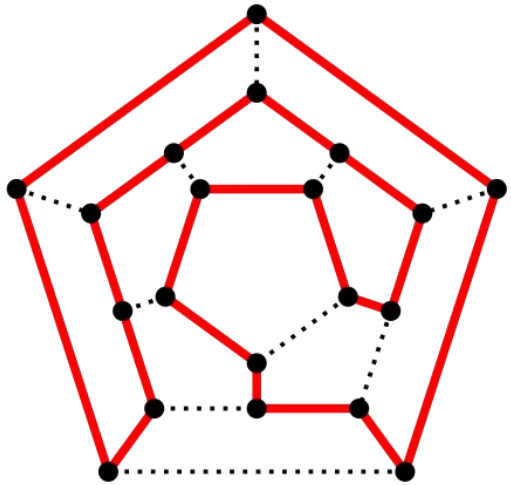




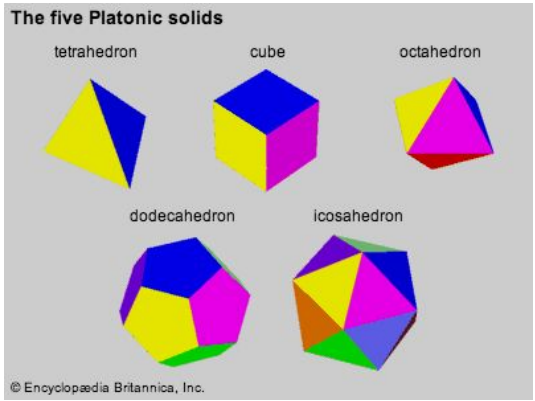
Hamiltonian Path

A Hamiltonian path

- Begins at one vertex
- Visits every vertex
- Returns to the original vertex
- All without visiting any edge twice

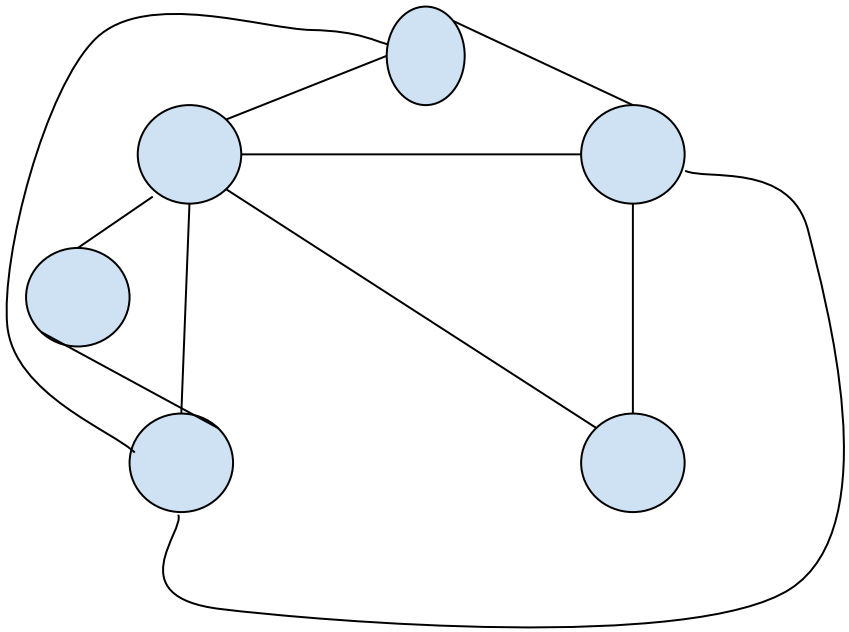


Hamilton proved that the vertices of platonic solids can be traversed with a Hamiltonian path



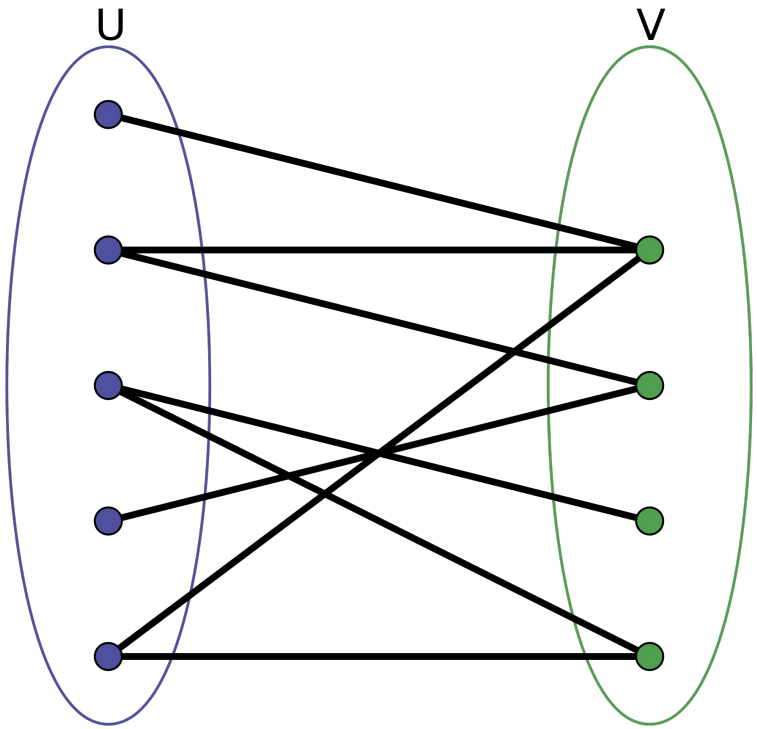
Planar

A planar graph can be drawn on a 2D surface (like paper) without any lines crossing



Bipartite Graph

A bipartite graph can be broken down into two regions, each with a corresponding value in the other



Graph Representations

There are three good graph representations, and one bad one

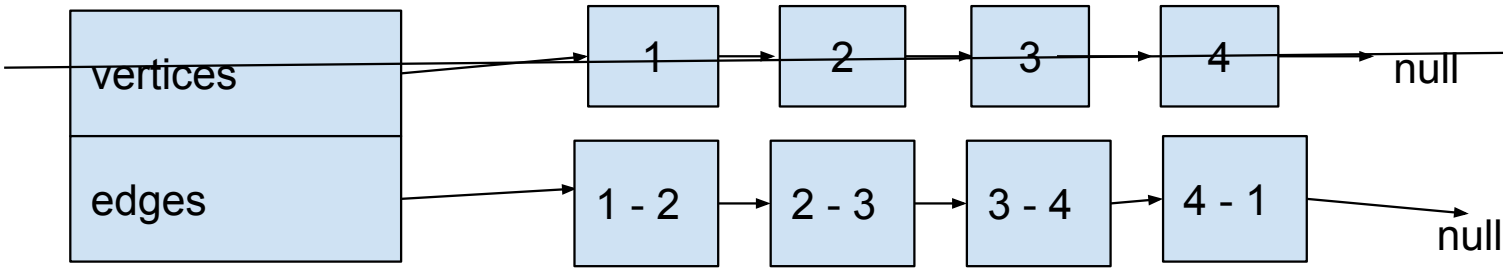
The bad one will be shown first so you know what not to use

- Global List of Edge (bad)
- Per-vertex Edge List
- Matrix
- Compressed Sparse Row (CSR)

Global Edge List

Having a single list of all the edges is bad because $E = O(V^2)$

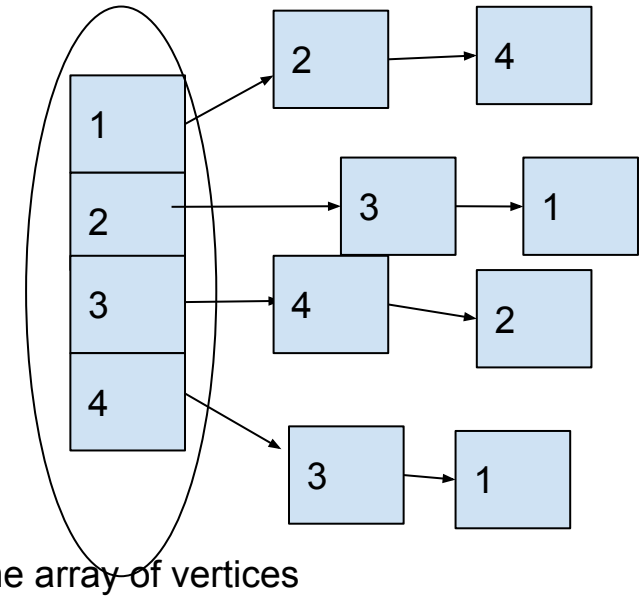
- Too much data to look through (slow)
- Other representations will have the number of edges from any vertex $= O(V)$
- By splitting edges into categories (which vertex they come from) speed can be drastically improved
- Notice, even this bad implementation does not need a list of vertices



Per-Vertex Edge List

For each vertex, there is a list of edges

- The max edges per vertex is $V - 1$ therefore $O(V)$
- The min edges per vertex is 0 therefore (1)
- Pro: Finding if vertex i adjacent to j is (1)
- Con: but it is also $O(V)$



the array of vertices must be there so best case is V

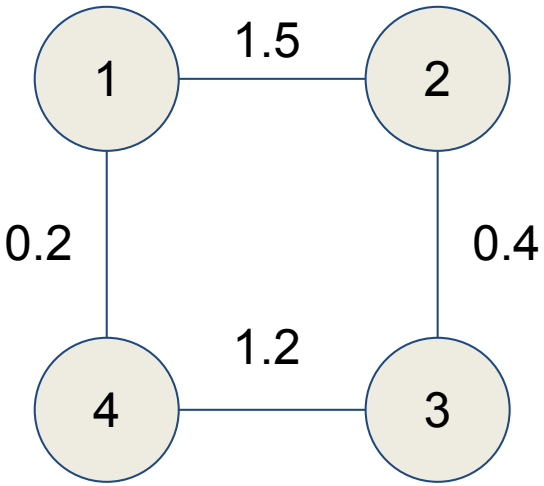
Matrix Representation

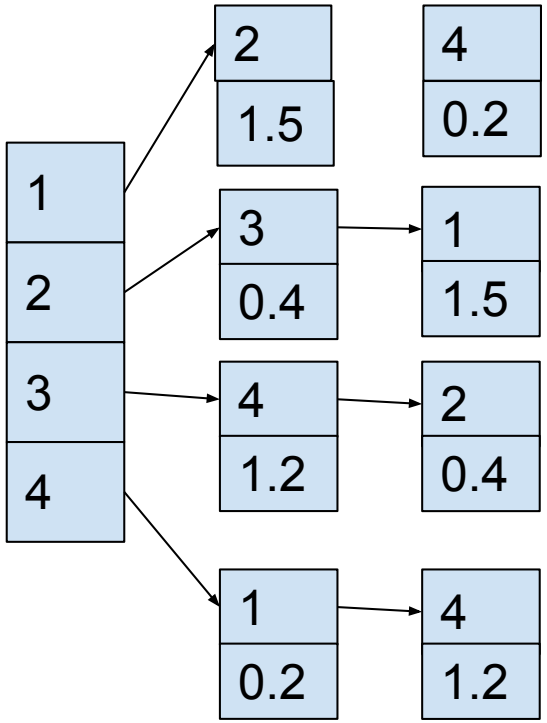
The Matrix Representation of a graph

- Pro: finding whether vertex i is adjacent to j is $O(1)$
- Con
  - Listing all vertices adjacent to i is $\theta(V)$
  - Space is $\theta(V^2)$

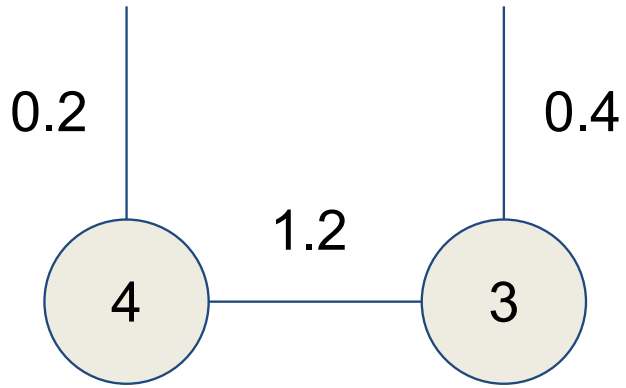
from \ to				
	1	1	0	1
	1	1	1	0
	0	1	1	1
	1	0	1	1

Per-Vertex Edge List with Weights





Matrix Representation with Weights

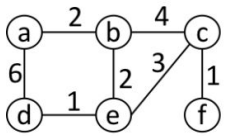


from \ to				
	∞	1.5	∞	0.2
	1.5	∞	0.4	∞
	∞	0.4	∞	1.2
	0.2	∞	1.2	∞

CSR Representation

Compressed Sparse Row is optimized for GPUs

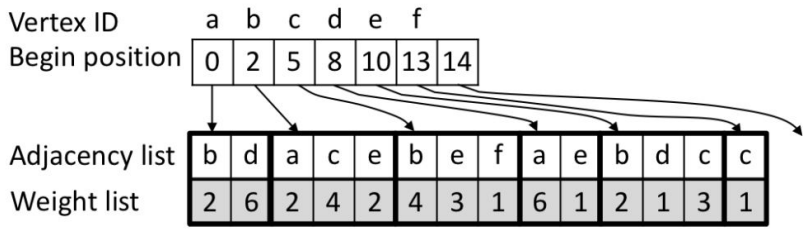
- Sequential access is much faster
- Two Lists: adjacencies and weights
- Third list is the starting position of each vertex within the lists



src	dest	weight
a	b	2
a	d	6
e	b	2
b	c	4
e	d	1
d	e	1
c	e	3

src	dest	weight
c	f	1
d	a	6
c	b	4
b	a	2
b	e	2
e	c	3
f	c	1

(a) Sample graph (b) Edge (tuple) list format (Source, destination, weight)

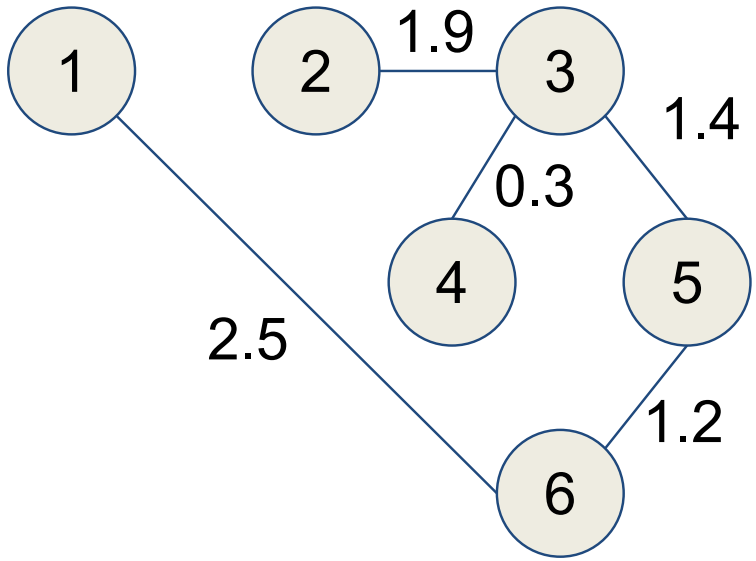


(c) CSR format

Image courtesy Professor Hang Liu

Test Yourself

Represent the graph in Edge list, matrix, and CSR Representations



Pseudocode for low-level Methods CSR

```
double isAdjacent(int from, int to) {
  int start = startPos[from]; // starting index into main arrays
  for (int i = start; i < startPos[from+1]; i++) {
    if (adjacent[i] == to)
      return weight[i];
  }
  return INFINITY;
}
```

DFS Recursive

Depth-First Search is an algorithm for

- Visiting all connected vertices in a graph
- In a definite order

Can be written easily recursively

- With slightly more work, iteratively

There is a significant advantage to avoiding recursion

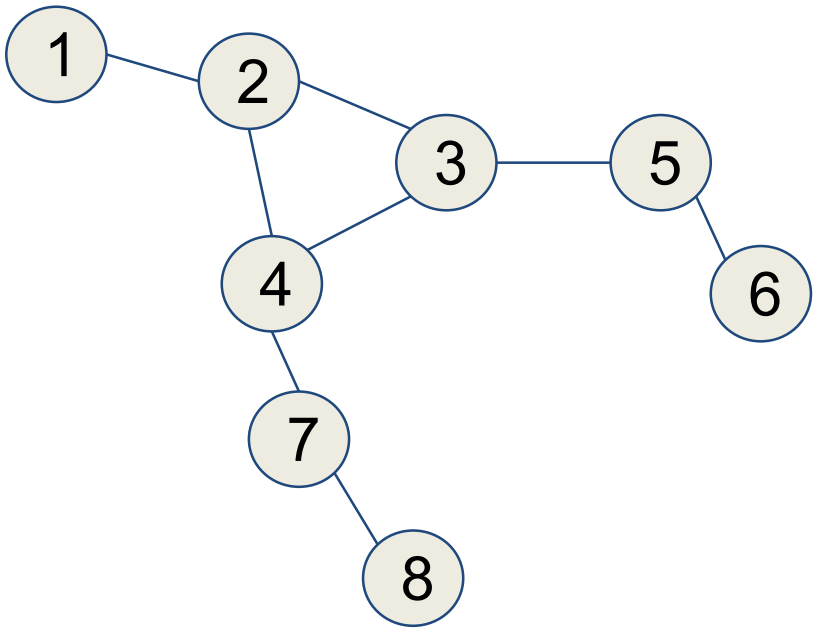
- Performance penalty for recursion
- Stack limitations on modern operating systems to defend against stack smashing attacks

DFS Visually

Start with a known vertex

For each neighbor, if it has never been visited

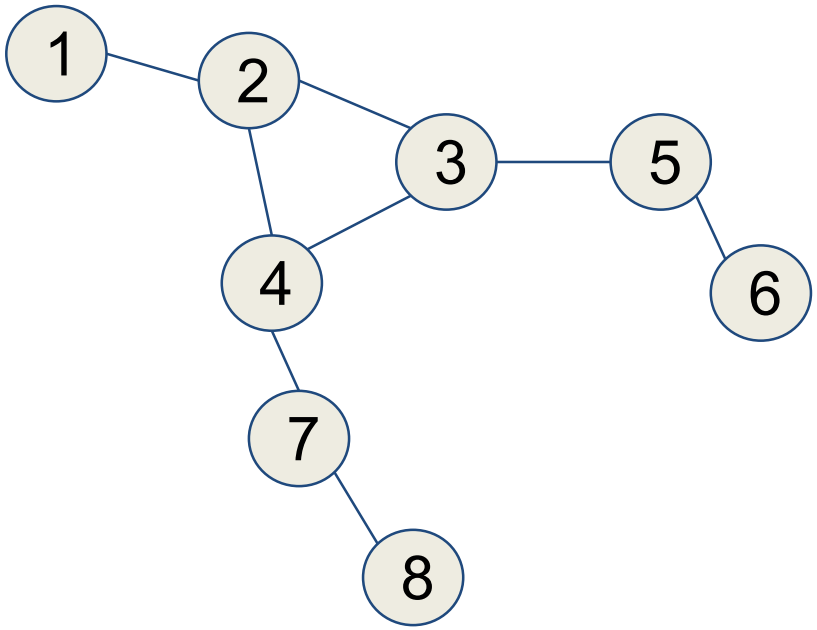
- Go there
- Mark it visited
- Example: start at 4: 4 2 1 3 5 6 7 8



DFS: Different Answers are Possible

DFS solution depends on the order of traversing neighbors For this discussion, always consider that neighbors are visited in ascending order

- Start with 1: 1 2 3 4 7 8 5 6
- Start with 5: 5 3 2 1 4 7 8 6



Depth First Search (DFS) Iterative

```
g.DFS(v)
   scheduled[*] ← false //O(V)
   todo.push(v)
   scheduled[v] ← true
   while (NOT stack.empty())
      v       todo.pop()
      print v
      foreach n ← neighbor(v)
         if NOT   scheduled[n]
            todo.push(n)
            scheduled[n] ← true
         end
      end
   end
end
```

Breadth First Search, Iterative

BFS is the same as DFS if solved iteratively

But instead of a stack, **use a queue**

```
g.BFS(v)
   visited[*] ← false
   queue.enqueue(v)
   visited[v] ← true
   while NOT queue.isEmpty() // O(V)
      v ← queue.dequeue()
          foreach n ← neighbor(v) // O(V) omega(1) i
            if NOT visited[n]
               queue.enqueue(n)
               visited[n] ← true
           end
      end
   end
end
```

Dijkstra

Dijkstra finds the lowest cost method of getting from start to end

- Space complexity $O(V)$
- Start with infinity for cost to all locations
- set cost to get to start $= 0$ (already there)
- Build a priority queue of vertices sorted on edge cost
- Add start vertex to priority queue
- While vertex is not empty remove the vertex with the cheapest cost
- For each adjacency try to find cheaper way to get there
- Stop when the Queue is empty

Dijkstra

```
Graph.Dijkstra(Graph, source)
  for v in Graph.Vertices:
    dist[v] ← ∞
    prev[v] ← null
    add v to Q
    dist[source] ← 0
    while Q is not empty
      u ← vertex in Q with min dist[u]
      remove u from Q
      foreach adjacent v of u still in Q
        alt ← dist[u] + Graph.Edges(u, v)
        if alt < dist[v]:
          dist[v] ← alt
          prev[v] ← u
```

Floyd-Warshall

Floyd-Warshall solves the minimum cost from all points to all points
Example: find the cheapest airline ticket between any two cities (EWR=Newark Airport, LHR=London Heathrow, PEK=Beijing, BOM=Mumbai)
Note: it costs nothing to go from anywhere to itself, you are already there

	EWR	LHR	PEK	BOM
EWR	0			
LHR		0		
PEK			0	
BOM				0

Website of 19 year old that finds best deals utilizing all flights:
`https://skiplagged.com/`
Article about passenger who was arrested in Germany for "fraud"
`https://www.npr.org/2019/02/13/694352593/lufthansa-airlines-sues-customer-who-skipped-part-of-his-return-flight`

Floyd-Warshall Pseudocode

```
dist ← |V| x |V| array, all infinity
for each edge (u, v)
  dist[u][v] ← w(u, v) // The weight of the edge (u
for each vertex v
  dist[v][v] ← 0 // going from v to itself costs n
for k ← 1 to V
  for i ← 1 to V
    for j ← 1 to V
      if dist[i][j] > dist[i][k] + dist[k][j]
        dist[i][j] ← dist[i][k] + dist[k][j]
      end if
```

Floyd-Warshall With Path Reconstruction

```
dist ← V x V array, all ∞
next ← V x V array, all null
for each edge (u, v)
  dist[u][v] ← w(u, v) // The weight of the edge (u
  next[u][v] ← v
for each vertex v
  dist[v][v] ← 0 // going from v to itself costs n
for k ← 1 to V
  for i ← 1 to V
    for j ← 1 to V
      if dist[i][j] > dist[i][k] + dist[k][j]
        dist[i][j] ← dist[i][k] + dist[k][j]
        next[i][j] ← next[i][k]
      end if
```

Spanning Tree

A spanning tree is a minimal graph sufficient to connect all vertices
Used in designing efficient networks

- Power
- Internet
- Water
- Sewer

Conversely, can be used by the military

- Maximum disruption per munition

Unfortunately competing goals

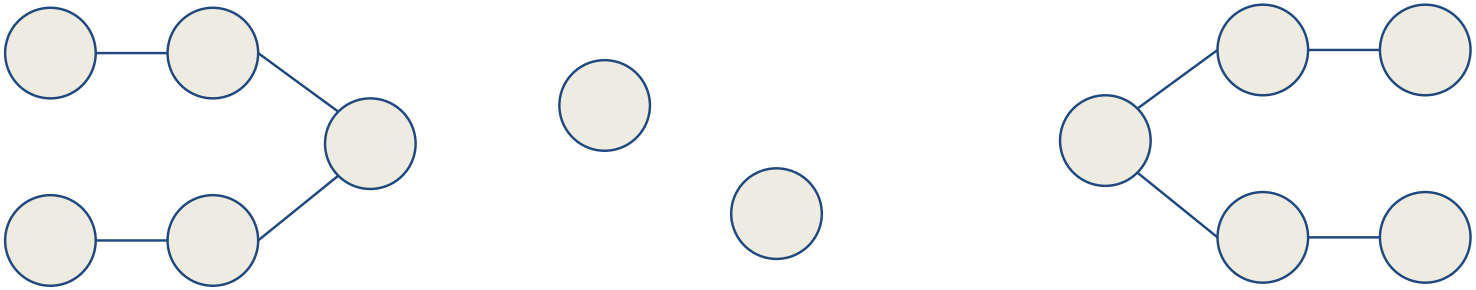
- Minimizing cost
- Reliability

Efficiency vs. Redundancy

Redundancy is expensive

Most networks are trees at the customer side

Internally there can be some redundancy

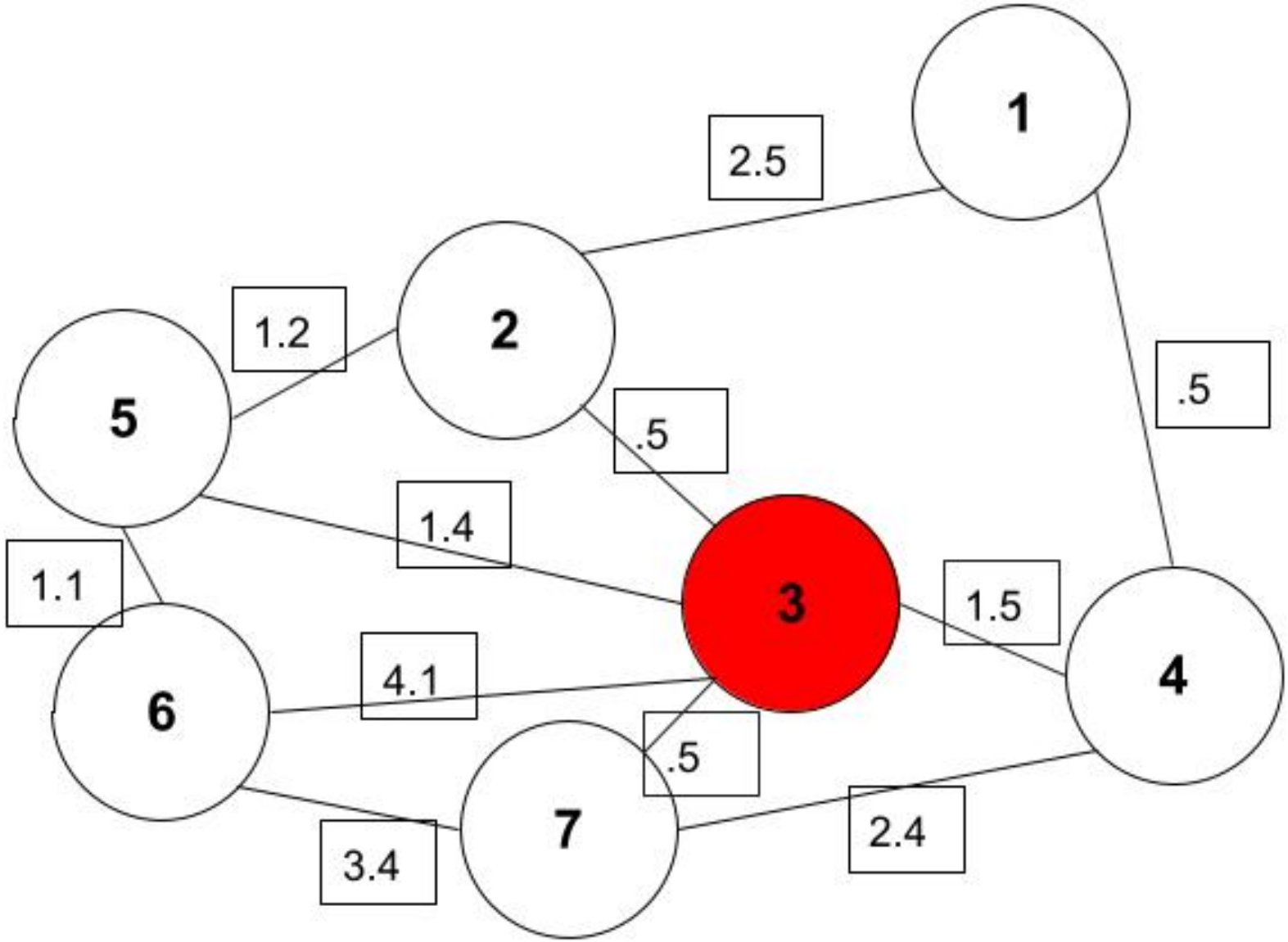


Prim

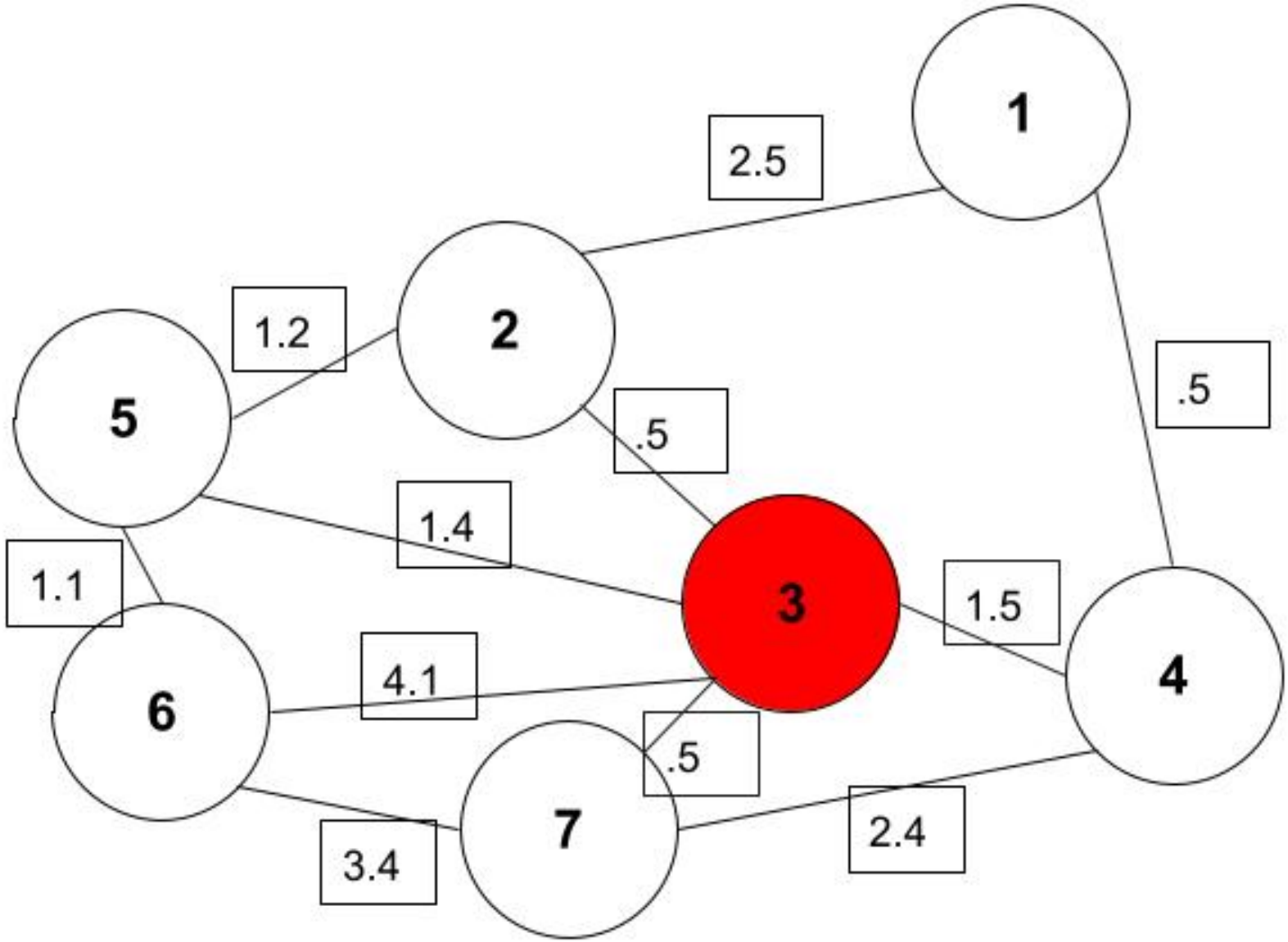
Prim's method for finding a spanning tree

- Greedy
- Start with any vertex
- Find the cheapest edge that extends the current set

Prim-1



Prim-2

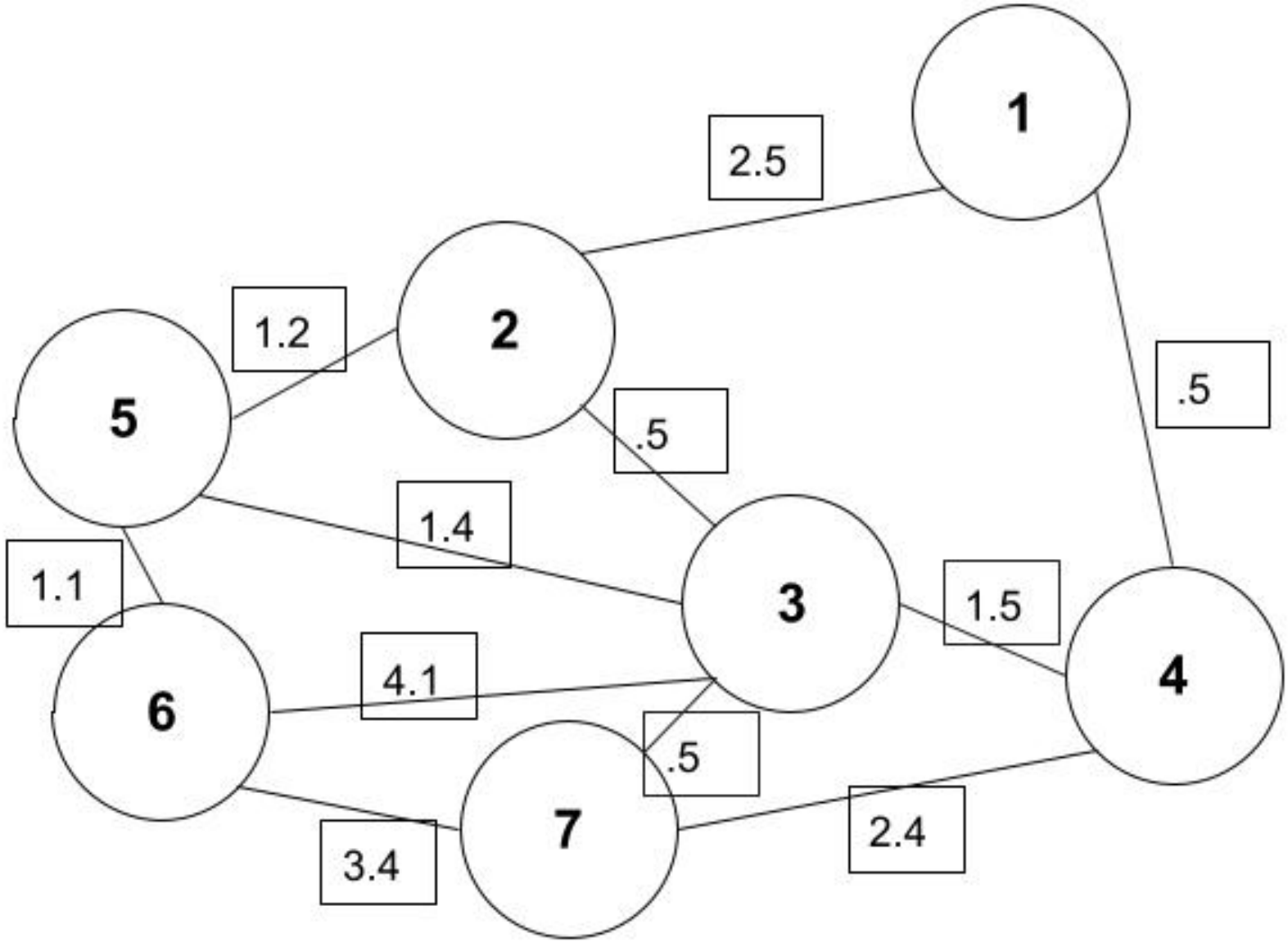


Kruskal

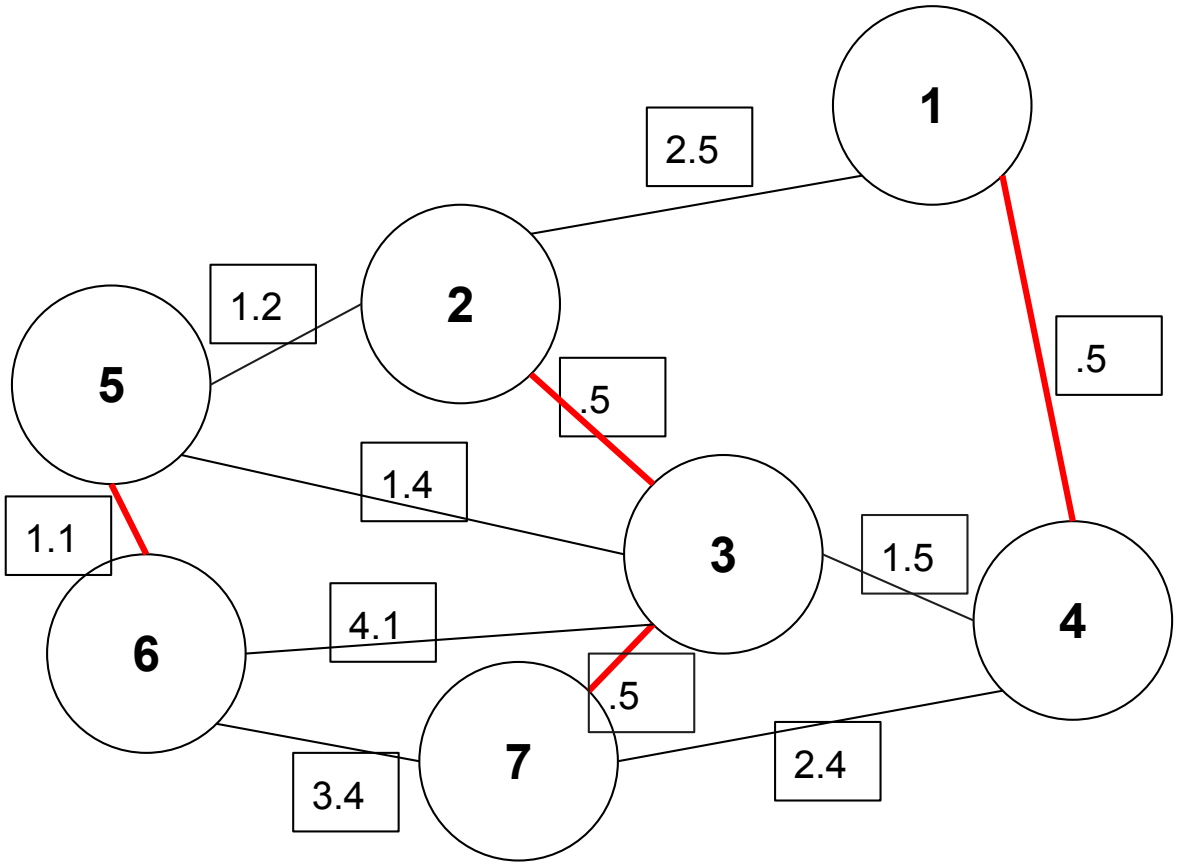
Kruskal's method for finding a spanning tree

- Greedy
- Sort edges ascending in cost
- Connect the cheapest edge that connects two vertices not yet connected
- Forest of trees is created, eventually all connecting

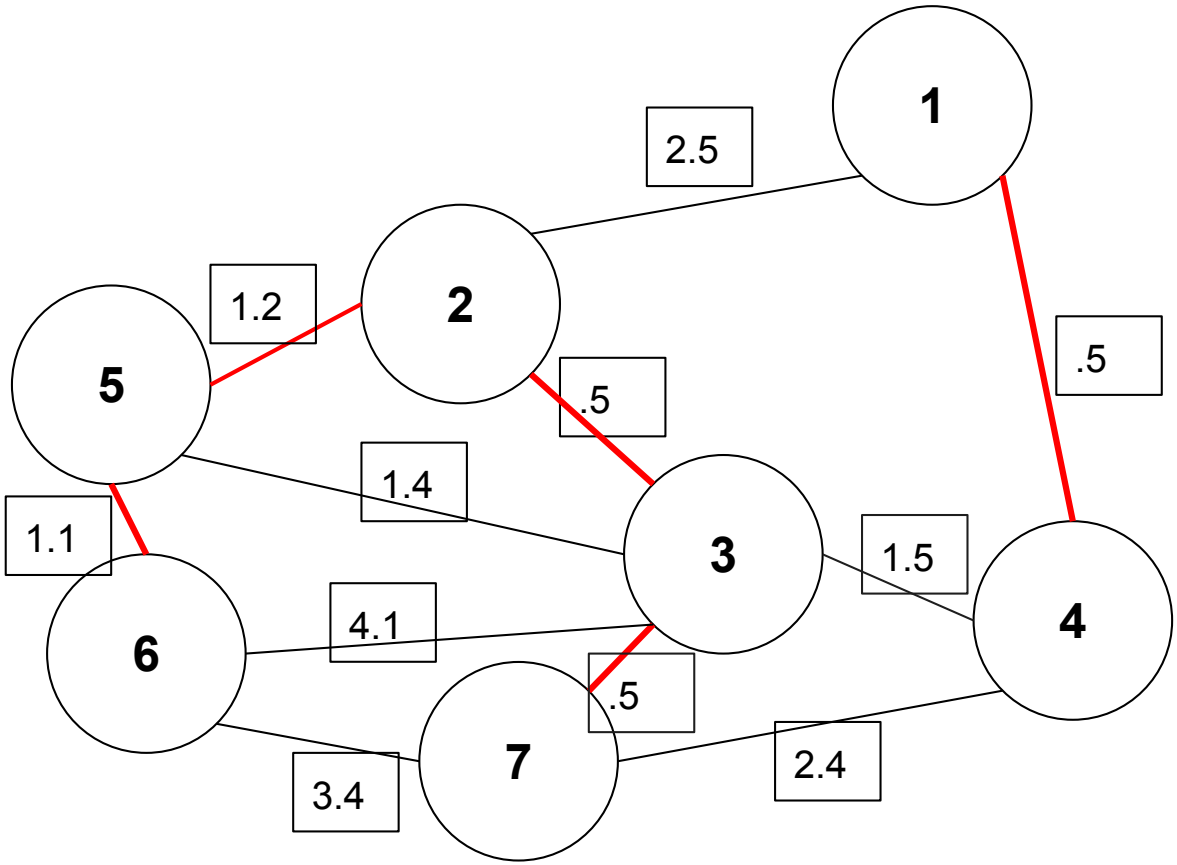
Kruskal



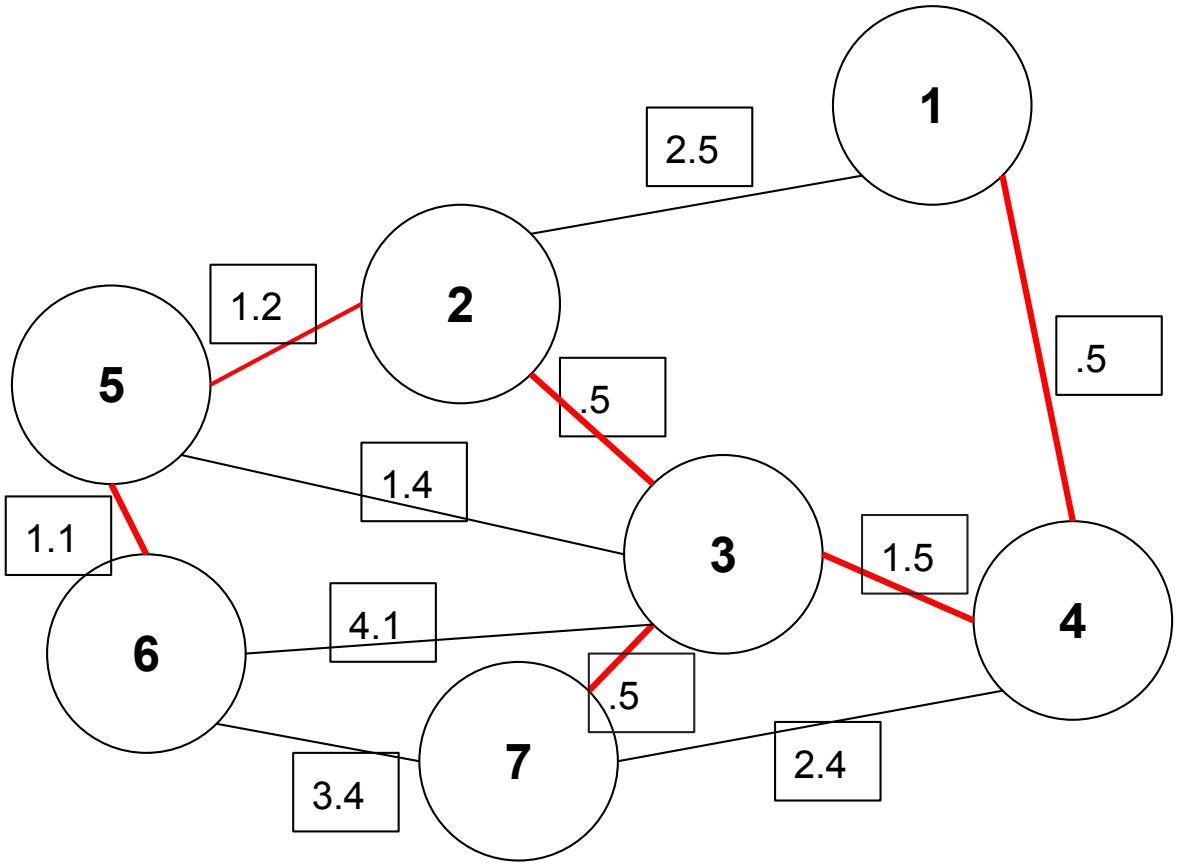
Kruskal-1



Kruskal-2



Kruskal-3



Travelling Salesman Problem (TSP)

The Travelling Salesman Problem

- Start at a vertex
- Visit all other vertices once, returning to start (tour)
- Find the minimal cost
- Cost of any edge not present considered infinite

Approaches to TSP

- Brute Force: Try all permutations
- Heuristic: Apply rules that don't guarantee the optimal solution but do allow a good one
- Incremental optimization

Applications of TSP

- Original: Travelling salesman
- Delivery Trucks
- Optimization of 3d printer motion planning

TSP Brute Force

- Start with list 1 .. v
- If any edge does not exist, cost is infinite
- Add up all costs. Stop if one is infinite
- Repeat for all permutations and choose minimum cost
- Complexity $O(n!)$

Euclidean TSP Heuristic

Euclidean TSP has weights proportional to distance

- Choose nearest neighbor
- Try swapping adjacent pairs to see if that reduces cost