Monte Carlo Simulation Assessment 6

Q1. Using Monte Carlo estimator to approximate the expected value,

$I = E[\exp(\sqrt{U})]$ where, $U \sim U(0, 1)$

for values of M = 100, 1000, 10000, 100000.

Also finding the exact value of I .

Finding Confidence Interval using Central Limit Theorem,

An asymptotic confidence interval for μ can be computed using central limit theorem. The **central limit theorem (CLT)** states the following: Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables with mean μ and finite variance $\sigma^2 > 0$. Then

$$\sqrt{n}\,\frac{\widehat{\mu}_n - \mu}{\sigma} \xrightarrow{\mathcal{D}} Z \sim N(0,1),$$

i.e., for all $z \in \mathbb{R}$

$$P\left(\sqrt{n}\,\frac{\widehat{\mu}_n - \mu}{\sigma} \le z\right) \to \Phi(z),$$

as $n \to \infty$, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

CLT can be used to get asymptotic confidence interval for μ, but it requires that we know σ. However, if σ is unknown, we can proceed as follows. Note that using weak law of large numbers, $s_n^2 \xrightarrow{\mathcal{P}} \sigma^2$. Now, using Slutsky's Theorem,

$$\sqrt{n}\,\frac{\widehat{\mu}_n - \mu}{s_n} \xrightarrow{\mathcal{D}} Z \sim N(0,1).$$

In other words, for all $z \in \mathbb{R}$,

$$\mathbb{P}\left(\sqrt{n}\,\frac{\widehat{\mu}_n - \mu}{s_n} \le z\right) \to \Phi(z),$$

as $n \to \infty$. Hence, for $\Delta > 0$

$$\begin{aligned}\mathbb{P}\left(\widehat{\mu}_n - \frac{\Delta s_n}{\sqrt{n}} \le \mu \le \widehat{\mu}_n + \frac{\Delta s_n}{\sqrt{n}}\right) &= \mathbb{P}\left(-\Delta \le \sqrt{n}\,\frac{\widehat{\mu}_n - \mu}{s_n} \le \Delta\right)\\ &\to \Phi(\Delta) - \Phi(-\Delta)\\ &= 2\Phi(\Delta) - 1.\end{aligned}$$

For a 95% confidence interval, set $2\Phi(\Delta) - 1 = 0.95$. Then $\Delta = \Phi^{-1}(0.975) = 1.96$, yielding the **familiar 95% confidence interval** $\left(\widehat{\mu}_n - 1.96\frac{s_n}{\sqrt{n}},\, \widehat{\mu}_n + 1.96\frac{s_n}{\sqrt{n}}\right)$. Similarly 99% confidence interval can be found as $\left(\widehat{\mu}_n - 2.58\frac{s_n}{\sqrt{n}},\, \widehat{\mu}_n + 2.58\frac{s_n}{\sqrt{n}}\right)$.

Theoretical Estimation:

Consider U ~ U(0,1), a uniform Random Variable,

Define g(u) = e^u

If f is the density function of the distribution of a random variable U, then

E(g(X)) = $\int_{-\infty}^{\infty} f(x)g(x)\,dx$

and there's no need to find the probability distribution, including the density, of the random variable X

We have,

f(x) = 1 for $0 \leq x \leq 1$, and 0 otherwise

hence,

$$
\begin{aligned}
E(g(U)) &= \int_0^1 g(u)du \\
&= \int_0^1 e^{\sqrt{u}}\, du \\
&= 2\left(\sqrt{u} - 1\right) . e^{\sqrt{u}} \text{ evaluated at 1 and 0} \\
&= 2.
\end{aligned}
$$

Readings

For M = 100,

Observed Mean = 1.98359009881 35189

Observed Variance = 0.18680106247806333

95 % Confidence Interval = [1.8988794979909895, 2.068300699636048]

For M = 1000,

Observed Mean = 1.9991818488601112

Observed Variance = 0.19234938114701589

95 % Confidence Interval = [1.971999092802966, 2.0263646049172563]

For M = 10000,

Observed Mean = 2.0025151461660986

Observed Variance = 0.19408194926198863

95 % Confidence Interval = [1.993880577178998, 2.011149715153199]

For M = 100000,

Observed Mean = 2.00087890059453

Observed Variance = 0.1949999714388782

95 % Confidence Interval = [1.9981419600392405, 2.00361584114982]

M	Estimated Value	Exact Value	Variance	95% Confidence Interval
100	1.9835900988135189	2.00000000	0.18680106247806333	[1.8988794979909895, 2.068300699636048]
1000	1.9991818488601112	2.00000000	0.19234938114701589	[1.971999092802966, 2.0263646049172563]
10000	2.0025151461660986	2.00000000	0.19408194926198863	[1.993880577178998, 2.011149715153199]
100000	2.00087890059453	2.00000000	0.1949999714388782	[1.9981419600392405, 2.00361584114982]

Comparison between Exact Value and Estimated Values:

M	Estimated Value	Exact Value
100	1.9835900988135189	2.000000000
1000	1.9991818488601112	2.000000000
10000	2.0025151461660986	2.000000000
100000	2.00087890059453	2.000000000

OBSERVATIONS:

- The estimated value of I converges to the value of 2 which we observed to be the exact value of I, as the sample size(M) increases.
- As the sample size (M) increases, the confidence interval (CI) decreases in width or becomes narrower.

- Ayush Kumar
(210123010)