1 Generation from Univariate Normal Distribution

Recall that the PDF and CDF of univariate normal distribution with mean $\mu\,(\mu \in \mathbb{R})$ and variance $\sigma^2\,(\sigma > 0)$ are

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma}) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2} \quad \text{for} \quad x \in \mathbb{R}$$

and

$$\Phi_{\mu,\sigma}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2}dt \quad \text{for} \quad x \in \mathbb{R},$$

respectively, where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \quad \text{for} \quad x \in \mathbb{R}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt \quad \text{for} \quad x \in \mathbb{R}.$$

Note that $\phi(\cdot)$ and $\Phi(\cdot)$ are PDF and CDF of **standard normal distribution**. The word **"standard" indicates** that the mean is 0 and the variance is 1. If $Z \sim N(0,1)$, then $\mu + \sigma Z \sim N(\mu, \sigma^2)$. Thus, given a method for generating samples $Z_1, Z_2, \ldots$ from the standard normal distribution, we can generate samples $X_1, X_2, \ldots$ from $N(\mu, \sigma^2)$ by setting $X_i = \mu + \sigma Z_i$. It, therefore, suffices to consider methods for sampling from $N(0,1)$.

We now discuss algorithms for generating univariate normal distribution. We assume the availability of a sequence $U_1, U_2, \ldots$ of independent random variables uniformly distributed on the unit interval $(0,1)$ and consider methods for transforming these uniform random variables to normally distributed random variables.

1.1 Using Acceptance Rejection Method

To obtain an **upper bound** of $\phi(x)$, we need to find an **lower bound** of $\frac{x^2}{2}$. Notice that

$$\frac{1}{2}\left(|x|-1\right)^2 = \frac{x^2}{2} - |x| + \frac{1}{2} \geq 0.$$

Thus, $\dfrac{x^2}{2} \geq |x| - \dfrac{1}{2}$ for all $x \in \mathbb{R}$. This shows that, for all $x \in \mathbb{R}$,

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \leq \frac{1}{\sqrt{2\pi}}e^{-|x|+\frac{1}{2}} = cg(x),$$

where $c = \sqrt{\frac{2e}{\pi}}$ and $g(x) = \frac{1}{2}e^{-|x|}$ for all $x \in \mathbb{R}$. Therefore, we can use acceptance rejection method if we can generate from g easily. To generate from $g(\cdot)$ we can use the following result. The PDF of $X = ZY$ is $g(\cdot)$ if Y and Z are independent random variables such that $Y \sim Exp(1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. To see it, notice that the CDF of X is

$$\begin{aligned}
P\left(X \le x\right) &= P\left(X \le x|Z = 1\right)P\left(Z = 1\right) + P\left(X \le x|Z = -1\right)P\left(Z = -1\right) \\
&= \frac{1}{2}P\left(ZY \le x|Z = 1\right) + \frac{1}{2}P\left(ZY \le x|Z = -2\right) \\
&= \frac{1}{2}P\left(Y \le x\right) + \frac{1}{2}P\left(Y \ge -x\right) \\
&= \begin{cases} \frac{1}{2}e^{x} & \text{if } x < 0 \\ 1 - \frac{1}{2}e^{-x} & \text{if } x \ge 0. \end{cases}
\end{aligned}$$

This shows that the PDF of X is same as $g(\cdot)$. This result tells us the method of generation from g. **First, generate** Y from **exponential distribution** with mean 1 and **then assign a random sign** to it. Therefore, we have the Algorithm 1.

Algorithm 1 Generation of $N(0, 1)$ by rejecting from Laplace PDF

1: **repeat**
2: generate U from $U(0, 1)$ ▷ Will be used to generate from $Exp(1)$
3: $X \leftarrow -\ln(U)$ ▷ $X \sim Exp(1)$
4: generate V from $U(0, 1)$ ▷ Will be used to generate random sign
5: **if** $V < \frac{1}{2}$ **then** ▷ Assigning signs with equal probability
6: $X \leftarrow -X$
7: **end if**
8: generate W from $U(0, 1)$ ▷ Will be used to implement acceptance rejection step
9: **until** $We^{\frac{1}{2}-|X|} \le e^{-\frac{X^2}{2}}$
10: **return** X

Note that the condition at Line 9 of the Algorithm 1 **does not depend on the sign** of X. Thus, we may assign the sign if the candidate is accepted. Moreover, $We^{\frac{1}{2}-|X|} \le e^{-\frac{X^2}{2}}$ is equivalent to $-2\ln W \ge (|X| - 1)^2$. Thus, Algorithm 1 can be modified to obtain a better algorithm given below.

Algorithm 2 Generation of $N(0, 1)$ by rejecting from Laplace PDF

repeat
generate U from $U(0, 1)$ ▷ Will be used to generate from $Exp(1)$
$X \leftarrow -\ln(U)$ ▷ $X \sim Exp(1)$
generate W from $U(0, 1)$ ▷ Needed to implement acceptance rejection step and assign sign
until $-2\ln(W) \ge (X - 1)^2$
if $W < \frac{1}{2}$ **then**
$X \leftarrow -X$
end if
return X

2 Box-Muller Method

Perhaps the simplest method to implement (though **not the fastest or necessarily the most convenient**) is the one by Box-Muller. This algorithm generates a sample from a bivariate standard normal distribution, each component of which is thus a univariate standard normal. The algorithm is based on the following result.

Theorem 1. *Let* $U_1, U_2 \overset{i.i.d.}{\sim} U(0, 1)$*. Define*

$$Z_1 = \sqrt{-2\ln U_1}\cos\left(2\pi U_2\right) \quad \text{and} \quad Z_2 = \sqrt{-2\ln U_1}\sin\left(2\pi U_2\right).$$

Then $Z_1, Z_2 \overset{i.i.d.}{\sim} N(0, 1)$*.*

Proof. Note that

$$\begin{aligned}
\frac{\partial z_1}{\partial u_1} &= -\frac{\cos\left(2\pi u_2\right)}{u_1\sqrt{-2\ln u_1}}, \\
\frac{\partial z_2}{\partial u_1} &= -\frac{\sin\left(2\pi u_2\right)}{u_1\sqrt{-2\ln u_1}}, \\
\frac{\partial z_1}{\partial u_2} &= 2\pi\sin\left(2\pi u_2\right)\sqrt{-2\ln u_1}, \\
\frac{\partial z_2}{\partial u_2} &= 2\pi\cos\left(2\pi u_2\right)\sqrt{-2\ln u_1}.
\end{aligned}$$

Thus, the Jacobian of the transformation is

$$J = \det\begin{pmatrix} \frac{\partial z_1}{\partial u_1} & \frac{\partial z_2}{\partial u_1} \\ \frac{\partial z_1}{\partial u_2} & \frac{\partial z_2}{\partial u_2} \end{pmatrix} = -\frac{2\pi}{u_1}.$$

Therefore, the absolute value of the Jacobian of inverse transformation is

$$\frac{1}{|J|} = \frac{u_1}{2\pi} = \frac{1}{2\pi}e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)}.$$

and hence the JPDF of (Z_1, Z_2) is

$$f_{Z_1, Z_2}(z_1, z_2) = \frac{1}{2\pi}e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)} = \phi(z_1)\phi(z_2) \quad \text{for} \quad z_1 \in \mathbb{R},\ z_2 \in \mathbb{R}.$$

This proves the result. □

Now, based on the theorem, we have the Algorithm 3 to generate a pair of independent standard normal random numbers from a pair of independent $U(0, 1)$ random numbers.

Algorithm 3 Box-Muller Method to generate $N(0, 1)$ random numbers

1: generate U_1 and U_2 from $U(0, 1)$
2: $R \leftarrow \sqrt{-2\ln U_1}$
3: $\theta \leftarrow 2\pi U_2$
4: $Z_1 \leftarrow R\cos(\theta)$
5: $Z_2 \leftarrow R\sin(\theta)$
6: **return** (Z_1, Z_2).

3 Marsaglia and Bray Method

Marsaglia and Bray developed a modification of the Box-Muller method that reduces computing time by **avoiding evaluation** of the "cos" and "sin" functions. The Marsaglia and Bray method instead uses **acceptance rejection method** to sample points **uniformly in the unit disc** and then transforms these points to normal variables. The algorithm is as follows:

Algorithm 4 Marsaglia and Bray Method to generate $N(0,\, 1)$ random numbers

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repeat
    generate U_1 and U_2 from $U(0,\, 1)$
    $U_1 \leftarrow 2U_1 - 1$ and $U_2 \leftarrow 2U_2 - 1$ $\qquad \triangleright\ U_i \sim U(-1,\, 1)$
until $U_1^2 + U_2^2 \leq 1$ $\quad \triangleright\ (U_1,\, U_2)$ is uniformly distributed on the disc of radius 1 centered at the origin
$Z_1 \leftarrow U_1 \left[\frac{-2\ln(U_1^2+U_2^2)}{U_1^2+U_2^2}\right]^{\frac{1}{2}}$ and $Z_2 \leftarrow U_2 \left[\frac{-2\ln(U_1^2+U_2^2)}{U_1^2+U_2^2}\right]^{\frac{1}{2}}$ $\quad \triangleright\ Z_i \sim N(0,\, 1)$ and they are independent
return $(Z_1,\, Z_2)$.
