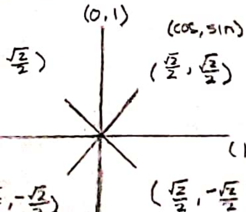
Calc 3 – Equations Sheet

- ~~Distance $P_1 \to P_2$~~
  ↳ $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
- **Unit Vector**
  ↳ $e = \frac{\vec{v}}{|\vec{v}|}$
- **Angle btwn Vectors**
  $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$
- **Dot Product**
  $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \dots$
  * if $\vec{u}\cdot\vec{v} = 0$, $\vec{u} \perp \vec{v}$
  $\vec{u}\cdot\vec{u} = |\vec{u}|^2$
  $\vec{u}\cdot\vec{v} = |u||v|\cos\theta$
- **Projection $\vec{u}$ on $\vec{v}$**
  $\text{Proj}_v u = \left(\frac{\vec{u}\cdot\vec{v}}{|v|}\right)\frac{\vec{v}}{|v|}$
  ↳ scalar component
  $|\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|v|} = \vec{u}\cdot\frac{\vec{v}}{|v|}$
- **Cross Product** unit vector
  $|\vec{u}\times\vec{v}| = (|u||v|\sin\theta)\vec{e}$
  * Right hand rule
  * If $\vec{u}\times\vec{v} = 0$, $\vec{u}\parallel\vec{v}$
  (solve as $\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{pmatrix}$)
- **Line to Vector**
  $(x-x_0)\vec{i}+(y-y_0)\vec{j}+(z-z_0)\vec{k} = t(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})$
  OR
  $r(t) = r_0 + t\vec{v}$
  ↳ $x = x_0 + tv_1$
  $y = y_0 + tv_2$
  $z = z_0 + tv_3$
- **Point → Line distance ($\parallel \vec{v}$)**
  point S to line P
  $d = \frac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}$
- **Point S → Plane w/ $\vec{n}$ @ Point P**
  $d = \left|\vec{PS}\cdot\frac{\vec{n}}{|\vec{n}|}\right| = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$
- **Distance btwn skew lines**
  $d = \frac{|\vec{PQ}\cdot(\vec{v_1}\times\vec{v_2})|}{\|\vec{v_1}\times\vec{v_2}\|}$ ← abs value
  * if $PQ\cdot(v_1\times v_2) \neq 0$ lines are skew
- **Plane vector form**
  $\vec{n}\cdot\vec{P_0P} = 0$
- **Equation of a plane**
  $A(x-x_0)+B(y-y_0)+C(z-z_0) = 0$
  OR
  $Ax+By+Cz = D$ ← $\vec{n}\cdot r_p$
  OR
  $r(t) = P + t\vec{u} + s\vec{v}$
- **Unit Circle**



$\tan = \frac{\sin}{\cos}$

~~$x^2+y^2+z^2 = R^2$ ∩ plane a~~
↳ $y^2+z^2 = R^2 - a^2$
(circle)

Ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Elliptical Paraboloid
$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{z}{c}$

Elliptical Cone
$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{z^2}{c^2}$

Hyperboloid of 2 sheets
$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$

Hyperboloid of 1 sheet
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$

Cylinder
$x^2+y^2 = z^2$

Hyperbolic Paraboloid
$\frac{y^2}{b^2}-\frac{x^2}{a^2} = \frac{z}{c}$

All vectors can be decomposed to → $u = \underbrace{\left(\frac{u\cdot v}{|v|^2}\right)v}_{\parallel} + \underbrace{\left(u - \left(\frac{u\cdot v}{|v|^2}\right)\right)v}_{\perp}$

Torque
$\tau = r\times F$

$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$
$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$

Reparametrizing
$s = \int_0^t \|\vec{v}\| du$
- solve for t
- plug into og equa

Arc Length (curve)
$L = \int_{t_0}^{t} |\vec{v}|\, dt$ $\left[\begin{matrix}\text{parameter } s\\ \int_0^t |v|\, dt\end{matrix}\right]$

Curvature
$K(t) = \frac{1}{|v(t)|}\left|\frac{d\vec{T}}{dt}\right|$

Frenet Frame
$\vec{T} = (\vec{v}(t)) \div |\vec{v}|$
$\vec{N} = \left(\frac{d\vec{T}}{dt}\right) \div \left|\frac{d\vec{T}}{dt}\right|$
$\vec{B} = \vec{T}\times\vec{N}$
(tangent, normal, binormal)

Domain
→ find curve representing inequality & graph

Limit
→ easier to prove DNE

Cross Product Rules
① $(ru)\times(sv) = (rs)u\times v$
② $u\times(v+w) = u\times v + u\times w$
③ $u\times(v\times w) = (u\cdot w)\times v - (u\cdot v)\times w$

Chain Rule
$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Level Curves
$f(x,y) = c$
"plane", constant

Particle Trajectory
$r = (v_0\cos\alpha)t\,\vec{i} + \dots + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\vec{j}$

Chain Rule
$\frac{dz}{dx} = \frac{-F_x}{F_z}$

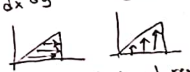
Gradient, Directional Deriv
$\nabla f = \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right)$
direct deriv $= \nabla f(P_0)$
* $\nabla f(P_0)$ is normal to tangent plane @ P_0
$D_u f(P_0) = \nabla f(P_0)\cdot\vec{e}_u = \|\nabla f(P_0)\|\cos\theta$

Linearization ↳ max
$L(x,y) = f(P_0) + f_x(P_0)(x-x_0) + \dots + f_y(P_0)(y-y_0)$
$f(x,y) \approx L(x,y)$

2nd deriv test (extremes)
cp @ $f_x = f_y = 0$ or DNE
$\det H \begin{pmatrix}f_{xx} & f_{xy}\\ f_{xy} & f_{yy}\end{pmatrix}$
↳ $= 0$ → inconclusive
↳ < 0 → saddle pt
↳ > 0 → $f_{xx} > 0$ → ∪
→ $f_{xx} < 0$ → ∩
Find abs extremes by calculating f & comparing

Lagrange multiplier
$\begin{cases}\nabla f = \lambda\nabla g\\ g(x,y,z) = 0\end{cases}$

Double Integrals
dx dy   dy dx



A = area of closed region
$A = \iint_R 1\, dA$
$f_{avg} = \frac{1}{A}\iint_R f\, dA$

Cartesian → polar
$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = dy\,dx = r\,dr\,d\theta$

Reduction Formula
$\int \sin^n x\, dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

Triple Integral
volume $= \iiint_D 1\, dV$
avg volume $= \frac{1}{\text{area of } D}\iiint_D f\, dV$
Σ mass $= \iiint_D (\text{density})\, dV$
CoM = $\bar{x} = \frac{M_{yz}}{M}$, $\bar{y} = \frac{M_{xz}}{M}$, $\bar{z} = \frac{M_{xy}}{M}$
$M_{yz} = \iiint_D x(\text{density})\, dV$
$\text{Inertia}_{x\,axis} = \iiint (y^2+z^2)\,\delta\, dV$

Cylindrical Coordinates
$x = r\cos\theta$ $r^2 = x^2+y^2$
$y = r\sin\theta$ $\tan\theta = y/x$
$z = z$ $dV = r\,dz\,dr\,d\theta$

Spherical Coordinates
$x = \rho\sin\phi\cos\theta = r\cos\theta$
$y = \rho\sin\phi\sin\theta = r\sin\theta$
$z = \rho\cos\phi$
$r = \rho\sin\phi$
$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$
$dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

	Domain	Range
asin	[-1, 1]	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
acos	[-1, 1]	$[0, \pi]$
atan	[ℝ]	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Jacobian
$\det\begin{vmatrix}\frac{dx}{du} & \frac{dx}{dv}\\ \frac{dy}{du} & \frac{dy}{dv}\end{vmatrix}$
$\iint_G f(g(u,v), h(u,v))|J|\, du\, dv$

Line Integral
"int $\lambda(x,y)$ over curve"
$\int \lambda\, ds = \int_{t_1}^{t_2} \lambda\|\vec{v}\|\, dt$
converting to t
↳ $P_0 + t(P_f - P_i) = \vec{r}$

$\vec{v} = \vec{r}\,'\, dt$
$\vec{r}(t) = P + t(\vec{PQ})$

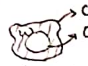
Work, Flow, Circulation
$\int_a^b F\|\vec{v}\|\, dt$ or $\int_a^b \vec{F}\cdot\vec{v}\, dt$
work → F = force field
flow → F = velocity field
start = end → circulation

Flux $\vec{n} = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$
$\oint_C F\vec{n}\, ds = \oint M\,dy - N\,dx$
$F = M(x,y)\vec{i} + N(x,y)\vec{j}$

Curl
$\begin{vmatrix}d_x & d_y & d_z\\ F_1 & F_2 & F_3\end{vmatrix} = 0$ → conserv field

Find Potential from F → use $\vec{v}$ (pot) for work
P → Q
V(Q) − V(P)
$F = F_1\,dx + F_2\,dy + F_3\,dz$
$V = \int^x F_1 + g(y,z) + h(z)$
$\frac{dF}{dy} = F_2 = \int^x F_1\, dy + dg$
$\frac{dF}{dz} = F_3 = (\int^x F_1 + g)\, dz + dh$

Greene's Theorem
Outward flux
↳ $\iint_R \frac{dM}{dx} + \frac{dN}{dy}\, dx\, dy$
Counterclockwise work
↳ $\iint_R \frac{dN}{dx} - \frac{dM}{dy}\, dx\, dy$
area R = $\oint\left(\frac{-y}{2}, \frac{x}{2}\right)\cdot\vec{v}\, dt$
if $\frac{dN}{dx} - \frac{dM}{dy} = 1$



$\oint_{C_1} - \oint_{C_2} = \omega\iint$
work = $\iint_R (\nabla\times\vec{F})\cdot\vec{k}\, dy\, dx$

$d\sigma$ → need $\vec{n}$ to translate

Area of a Surface
$\iint \left\|\frac{d\vec{r}}{du}\times\frac{d\vec{r}}{dv}\right\| dv\, du$
* be sure to parametrize 1st!

Integrate over a Surface
① parametrize $f(x,y,z)$
② $\iint f(u,v)\cdot\left\|\frac{d\vec{r}}{du}\times\frac{d\vec{r}}{dv}\right\| dv\, du$

Stokes Theorem
work = $\iint(\nabla\times\vec{F})\vec{n}\, dS$
$\nabla\times\vec{F}$ = curl
$\vec{n} = \frac{d\vec{r}}{du}\times\frac{d\vec{r}}{dv} \div \|\ \|$
$d_s = \|\ \|\, dv\, du$

Right Hand Rule
- thumb → + along C
- index → tangent + S (inward)
- middle → $\vec{n}$

Divergence Theorem
S is enclosed surface
$\iint F\cdot\vec{n}\, dS = \iiint_R (\nabla\cdot\vec{F})\, dV$
$\nabla\cdot\vec{F}$ = divergence $= \frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz}$
dV = dx dy dz, r dz dr dθ, etc
* if $\nabla\cdot\vec{F} = 1$
volume(R) $= \frac{1}{3}\iint(x,y,z)\cdot\vec{n}\, dS$

Jacobian
$J = \det\begin{pmatrix}dx/du & dy/du\\ dx/dv & dy/dv\end{pmatrix}$

$\vec{n} = \frac{\nabla g}{\|\nabla g\|}$
$\int \sin^2\theta = \frac{1}{2}(\theta - \frac{1}{2}\sin(2\theta))$
$\vec{F}\cdot\vec{n}$ $\frac{\|\nabla g\|}{|\nabla g\cdot \vec{p}|}$ $\left(\frac{d}{dx}c, \frac{d}{dy}c, \frac{d}{dz}c\right)$
Surface integral
↳ norm to shadow of object (0,0,1) ex

Some Standard Parametrization

- Cylinder radius R (center on z axis)

  $\vec{r}(\theta, z) = (R\cos\theta, R\sin\theta, z)$

  outward $\vec{n} = T_\theta \times T_z = R\langle \cos\theta, \sin\theta, 0\rangle$

  $dS = \|\vec{n}\|\, d\theta\, dz = R\, d\theta\, dz$

- Sphere radius R (center on origin)

  $\vec{r}(\theta, \phi) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$

  unit radial vector $= e_r = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$

  outward $\vec{n} = T_\phi \times T_\theta = (R^2\sin\phi)\, e_r$

  $dS = \|\vec{n}\|\, d\phi\, d\theta = R^2\sin\phi\, d\phi\, d\theta$

- Graph of $z = g(x,y)$

  $G(x,y) = (x, y, g(x,y))$

  $\vec{n} = T_x \times T_y = \langle -g_x, -g_y, 1\rangle$

  $dS = \|\vec{n}\|\, dx\, dy = \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy$

Directional derivative = unit $\cdot$ gradient
(AKA rate of change)