Diff Eqs Study Guide

Direction Field

- @ each pt a short line is drawn whose slope = value of f @ that pt from $\frac{dy}{dt} = f(t,y)$
- gives idea of overall behavior of the solution

→ $\frac{dy}{dt} = ry - k$ often (not always)

$u(t) = e^{\int -r\,dt}$

$y = \frac{1}{u(t)} \int u(t)\,k\,dt$

Standard Form

$y'' + p(t)y' + q(t)y = g(t)$

→ $\frac{dy}{dt} + ay = g(t)$ $\times u(t)$ to get $\frac{d}{dt}(uy) = u\frac{dy}{dt} + y\frac{du}{dt}$

$u(t)\frac{dy}{dt} + au(t)y = u(t)g(t)$

product rule ↳ $u(t) = e^{\int a\,dt}$

$\int \frac{d}{dt}(uy) = \int ug(t)$

$uy = \int_{x_0}^{x} ug(t)$

→ $y = e^{-at} \int_{x_0}^{x} e^{at} g(t)\,dx + ce^{-at}$

$\frac{d}{dt}(uy) = u\frac{dy}{dt} + uay$

$u'y + uy'$

$> u'y = uay$

$u' = au$

$\int \frac{u'}{u} = \int a$

$\ln|u| = \int a$

Seperable Equations

→ $\frac{dy}{dx} = -\frac{M(x)}{N(y)}$

$\int N(y)\,dy = -\int M(x)\,dx + c$

solve for c, then y

Solution Existance

→ are $f(t,y)$ (same as $\frac{dy}{dx}$ in standard form) & $\frac{\partial f}{\partial y}$ both cont. around given initial value?

↳ yes → a solution exists on the discovered integral (& is unique)

↳ no → there is no solution or the solution is not unique

solution ϕ where $\phi'(t) = f(t, \phi(t))$

Exact Solutions

→ Form $M(x,y) + N(x,y)y' = \emptyset$

→ exact equations if $\frac{\partial}{\partial y} M = \frac{\partial}{\partial x} N$

$\psi = \int M(x,y)\,dx + c$ ← $c = h(y)$

$\psi_y + h'(y) = N(x,y)$

find $h(y)$ & sub into ψ

↳ $\psi(x,y)$ → $\int M(x,y) + h(y) = c$

→ not exact? try multiply by μ!

$M\mu_y - N\mu_x + (M_y - N_x)\mu = \emptyset$

assume $\mu(x)$ or $\mu(y)$ only

Autonomous Equations

$\frac{dy}{dt} = f(y)$ not t

① Exponential → $\frac{dy}{dt} = ry$ → $y = y_0 e^{rt}$

② Logistic → $\frac{dy}{dt} = r(1 - \frac{y}{k})y$ → $y = \frac{y_0 k}{y_0 + (k - y_0)e^{-rt}}$

→ Equilibrium points - root $f(y) = \emptyset$ if $f(c) = \emptyset$ then $y(t) = c$ is a solution to $y' = f(t)$

↳ asymptotically stable

↳ unstable

↳ semistable or

→ use 1st deriv test to find ($f(y) = \frac{dy}{dt}$) or take limit of solution

Euler Method

t_{n+1}, $y_{n+1} = y_n + f(t_n, y_n)\underbrace{(t_{n+1} - t_n)}_{h}$

$t_n = nh$ → $t_n = t_0 + nh$

backward → $y_{n+1} = y_n + f(t_{n+1}, y_{n+1})(t_{n+1} - t_n)$

Homo 2nd Linear Eqn

→ find 2 special solutions by changing $y'' \to r^2$ $y' \to r$ $y \to 1$

→ $y(t) = c_1 y_1 + c_2 y_2$

① 2 distinct & real ($b^2 - 4ac > 0$)

$y_1 = e^{r_1 t}$ $y_2 = e^{r_2 t}$

② 2 complex ($b^2 - 4ac < 0$)

$r = \lambda \pm i\mu$

$y_1 = e^{\lambda t}(\cos\mu t + i\sin\mu t)$

$y_2 = e^{\lambda t}(\cos\mu t - i\sin\mu t)$

$y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$

③ repeated root ($b^2 - 4ac = \emptyset$)

$y_1 = e^{rt}$ $y_2 = te^{rt}$ $r = \frac{-b}{2a}$

OR $y_2 = y_1 v(t)$ plug into og eqn & solve that way

Wronskian

det ↓

$W[y_1, y_2](t_0) = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$

$= y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$

→ if $W \neq \emptyset$ for y_1 & y_2, then they form a solution

→ $W[y_1, y_2](t) = ce^{-\int p(t)\,dt}$ if $p(t)$ & $q(t)$ are cont in I

→ How to check if y_1 & y_2 are solutions?

① $p(t)$ & $q(t)$ continous on I?

② plug in y_1 & y_2 ③ Wronskian $\neq \emptyset$

Exam 2

Complex Roots - see above

NonHomogenous ($g(t) \neq \emptyset$)

- $y(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t)$

① find y_1 & y_2 by treating eqn as homogenous.

② guess $Y(t)$ based on $g(t)$

↳ $g(t) = 3e^{2t}$, $Y = Ae^{2t}$

↳ $g(t) = 2\sin t\, e^t$

$Y = (A\sin t + B\cos t)e^t$

* if Y guess = multiple of y_1 or y_2, multip by $t \to Y \cdot t$

③ find Y' & Y''

④ plug into og equation & solve for unknown constants

Mechanical Vibrations

- $F_{spring} = -mg = -kL$ (static dist from rest; spring)
- $\underset{mass}{mu''} + \underset{damping}{\gamma u'} + \underset{position}{ku} = \underset{external}{F(t)}$
- Case 1 ($F(t) = \emptyset$, $\gamma = \emptyset$)

↳ $mu'' + ku = \emptyset$

↳ $u(t) = A\cos\omega_0 t + B\sin\omega_0 t$

$\omega_0 = \sqrt{k \div m}$

$u(t) = R\cos(\omega_0 t - \delta)$

$R = \sqrt{A^2 + B^2}$ $\delta = \arctan\frac{A}{B}$

- Case 2 ($F(t) = \emptyset$)

↳ $mu'' + \gamma u' + ku = \emptyset$

↳ $\gamma^2 - 4km < \emptyset$

↳ $u(t) = e^{\frac{-\gamma}{2m}t}(A\cos\mu t + B\sin\mu t)$

$\mu = \sqrt{4km - \gamma^2}/(2m)$

$u(t) = Re^{\frac{-\gamma t}{2m}}\cos(\mu t - \delta)$

- $mu'' + ku = F_0\cos\omega t$

↳ $u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega_0 - \omega}{2}t\right)\sin\left(\frac{\omega_0 + \omega}{2}t\right)$

if $\omega_0 \neq \omega$

↳ solved same as homo & nonhomo linear eqns

*only need $\lambda + \mu i$ or $\lambda - \mu i$ for V & U

Reduction of Order

- $y^{(n)} + p(t)y' + q(t)y = g(t)$

$x_1 = y$ $x_2 = y' = x_1'$ etc to $n-1$

$x_1' = x_2$

$x_2' = -q(t)x_1 - p(t)x_2 + g(t)$

- for $x_1' = F_1(t, x_1, \ldots, x_n)$ to $x_n' = F_n(t, x_1, \ldots, x_n)$

a unique solution near t_0 exists if $F_1 \ldots F_n$ & $\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial x_n} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$ are cont. near t_0

- $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$

$\vec{x}(t_0) = \vec{\xi}_0$ has a unique solution on I if all funcs. in $P(t)$ & $g(t)$ are cont. on I

Matrices

- $A_{m \times n}$ → $m\downarrow\overset{n\rightarrow}{[\quad]}$
- A^T rows ⟷ cols
- $\bar{A}$ i values $\times(-1)$
- (scalar multip) row
- $(A_{m \times n})(B_{n \times r}) = C_{m \times r}$
- A^{-1} → ① $[A \mid I]$

② row ops

③ $[I \mid B]$ $B = A^{-1}$

- det A (Mariano)
- $\frac{d}{dt}A = \frac{d}{dt}[a_{ij}(t)]$
- $\int A = [\int a_{ij}(t)\,dt]$
- $\frac{d}{dt}(A^{-1}) = -A^{-1}A'A^{-1}$
- $A\vec{x} = \vec{b}$ find $\vec{x}$

① $[A \mid \vec{b}]$

② row ops

③ $[I \mid \vec{c}]$ $\vec{c} = \vec{x}$

③ stop at RE form & find $x_1, x_2, \ldots, x_n$

Linear (In)Dependance

① form $A = [\vec{x}_1 \ldots \vec{x}_n]$

② find rref(A)

③ free variable?

↳ yes → lin dep

↳ no → lin indep

- also if $\det A \neq \emptyset$ A lin indep

Eigenvalues & vectors

- see lin alg sheet

Homo System 1st Order

- $\vec{x}' = A\vec{x}$

① $\det(A - \lambda I)$ = evalues

② $(A - \lambda I)\vec{\xi} = \vec{\emptyset}$ → evectors

③ $\vec{x}^{(i)} = \vec{\xi}_i e^{\lambda_i t}$

④ $\vec{x}(t) = \sum \vec{\xi}_i c_i e^{\lambda_i t}$

↳ if $\lambda = r \pm \mu i$

③ $\vec{x}^{(i)} = \vec{\xi}_i e^{\lambda}(\cos\mu t + i\sin\mu t)$

④ multiply into ξ for $U(t)$ & $V(t)$

⑤ $\vec{x}(t) = c_1 U(t) + c_2 V(t)$

↳ repeated roots

③ find $\vec{\xi}$ & $\vec{\eta}$ where $(A - \lambda I)\vec{\eta} = \vec{\xi}$

④ $\vec{x} = c_1\vec{\xi}e^{\lambda t} + c_2(\vec{\xi}te^{\lambda t} + \vec{\eta}e^{\lambda t})$

Fundamental Matrices

$\Psi(t) = [\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)]$

$\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$

↳ $\vec{x} = \Phi(t)\vec{x}_0$

NonHomogenous Lin Systems

$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$

① Diagonalization

1) $\det[A - \lambda I]$ → evalues

2) find evectors

3) $T = [\vec{\xi}_1, \ldots, \vec{\xi}_n]$ $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$\vec{h} = T^{-1}\vec{g}(t)$

4) $\vec{y}' = D\vec{y} + \vec{h}$

↳ $y_n' = \lambda_n y_n + h_n$

$y_n = e^{\lambda_n t}\int e^{-\lambda_n t}h_n\,dt$

5) $\vec{x}(t) = T\vec{y}$

② Variation Paramcters

1) $\det[A - \lambda I]$ → evalues

2) find evectors

3) $\Psi = [\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}]$

4) $\vec{x} = \Psi\int\Psi^{-1}\vec{g}(t)\,dt$

↳ $\Psi^{-1} = \frac{1}{\det\Psi}[\ldots]$

③ Undetermined Coeffs

1) $\det[A - \lambda I]$ → evalues

2) find evectors

3) guess solution (use $\vec{g}$)

only matching part of guess, not all

↳ $\begin{bmatrix}\cos \\ \sin\end{bmatrix}$ ⇒ $\vec{a}\sin t + \vec{b}\cos t$

$\begin{bmatrix}e^{t} \\ 0\end{bmatrix}$ ⇒ $\vec{a}e^{rt}$

$[P_n(t)]$ ⇒ $\vec{a}^{(n)}t^n + \ldots + \vec{a}^{(1)}t + \vec{a}^{(0)}$

→ if guess matches an evector → multiply by t

4) $\overrightarrow{guess}' = A\overrightarrow{guess} + \vec{\xi}_1 e^{\lambda_1 t} + \ldots$

↳ compare coeffs to find $\vec{a}, \vec{b}$, etc

5) $\vec{x}$ = guess $+ c_1\vec{\xi}_1 e^{\lambda_1 t} + \ldots$

- check solutions

↳ $\vec{x}' = A\vec{x} + \vec{g}(t)$?

* if $\vec{g}$ has e^{-t} (ie e^{-3t})

↳ guess $\vec{a}e^{-2t} + \vec{b}e^{-t} + \vec{c}\ln t + \vec{d}$

↳ so $\vec{y}' = -2\vec{a}e^{-3t} - be^{-2t} + \vec{c}t^{-1}$

Exact Solutions Cont

$\mu = \mu(x)$, $\mu_y = \emptyset$

↳ $\frac{d\mu}{dx} = \frac{1}{N}(M_y - N_x)\mu$

$\mu = \mu(y)$, $\mu_x = \emptyset$

↳ $\frac{d\mu}{dy} = \frac{1}{M}(N_x - M_y)\mu$

* cannot have 2 diff variables in μ

$\int\frac{1}{\mu} = \int\frac{(\square - \square)}{\square}$ → can drop const c