Linear Alg Study Guide (row #, col #)

Transpose
$(A_{ij})^T = A_{ji}$
↳ $(A+B)^T = A^T + B^T$
↳ $(AC)^T = C^T A^T$

Vectors
$\mathbb{R}^n \rightarrow n \times 1$ matrix

Linear combo
↳ row → Σcol_1, Σcol_2, Σcol_3
↳ col → Σrow_1, Σrow_2, Σrow_3
↳ becomes 1 $n \times 1$ col

Rotation Matrix
$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ↻
multiply by A_θ w/ $\theta = \measuredangle$ cclock

Row Echelon
① $\emptyset$ rows @ bottom
② leading entries to right of prev
③ entries under leading → $\emptyset$

Reduced RE
① in RE form
② leading entries = $\emptyset$
③ entries above & below leading are $\emptyset$

Free Variables
- Found in nonpivot cols
- ∞ solutions
- Linear Dependant → $n>m$
- $A\vec{x}=\emptyset$ non $\emptyset$ solution → ∞ solutions

Linear Independance
- $A\vec{x} = \emptyset$ $\emptyset$ solution → 1 solution for each $\vec{B} \in \mathbb{R}^n$
- nullity $= \emptyset$, rank $= n$ (nullity = #col − rank; rank = # pivot col, # non$\emptyset$ rows)
- 1 vector is a multple of another

Span
↳ all possible vectors which can be achieved as linear combos of a set of vectors
- ie is $\vec{v}$ in the span of S? ⇒ can $\vec{v}$ be made by the vectors in S?
- $\vec{v}$ belongs to S if $A\vec{x}=\vec{v}$ is consistent
  ① augment $[A|\vec{v}]$
  ② find RE of $[A|\vec{v}]$

{ → cols of A span $\mathbb{R}^m$
→ every $\vec{b} \in \mathbb{R}^m$, $A\vec{x}=\vec{b}$ has solution
→ RREF(A) has pivot in every row }

Consistancy
{ → $A\vec{x}=\vec{B}$ is consistant
→ $\vec{B}$ linear combo of col vectors in A
→ RRE of $[A|\vec{B}]$ has no $[\emptyset \dots \emptyset \mid d]$ }

Multiplication
- $(A_{m\times r})(B_{r\times n}) = AB_{m\times n}$
- entry$_{ij} = \Sigma(A_{row\,i} \cdot B_{col\,j})$

Inverse
- $AB = BA = I_n$ if $RRE(A) = I_n$
- $[A \; B] \xrightarrow{\text{row ops}} [I_n \; A^{-1}B]$
- $A\begin{bmatrix} a & b \\ c & d\end{bmatrix} \rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- elementary row ops are invertible & reversible
  - $P = (E_1)(E_2)\dots(E_k)$
    ↳ $PA = R$ & $P^{-1} = A$

① Set up $[A|I_n]$
② Find RRE(A) so $[R|B]$
③ if $R = I_n$, $A^{-1} = B$

Column Correspondance
- if $RRE(A) = R$, if j^{th} col R is lin combo of other col in R, then j^{th} col A is also lin combo of same col in A as R w/ same coeff
- used often in lin dep spans / sets of vectors

LU Decomposition
- $U = RE(A)$ no row exch
- $L = I_n$ w/ $r_i + cr_j \rightarrow r_i$ on U means $L_{ij} = -c$
$[\,\dots\,] = U$ & $[\,\dots\,] = L$
- $A\vec{x}=\vec{B} \rightarrow LU\vec{x}=\vec{b}$ where $U\vec{x}=\vec{y} \rightarrow L\vec{y}=\vec{B}$
① Find L & U
② Solve for $\vec{y}$ w/ $[L|\vec{b}]$
③ Solve for $\vec{x}$ w/ $[U|\vec{y}]$

− Exam 2 (3.1−5.3) −

Determinant n×n
- Cofactor Expansion
$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
↳ along row 1
$a(|\begin{smallmatrix} e & f \\ h & i\end{smallmatrix}|) - b(|\begin{smallmatrix} d & f \\ g & i\end{smallmatrix}|) + c(|\begin{smallmatrix} d & e \\ g & h\end{smallmatrix}|)$
↳ along col 1
$a(|\begin{smallmatrix} e & f \\ h & i\end{smallmatrix}|) - d(|\begin{smallmatrix} b & c \\ h & i\end{smallmatrix}|) + g(|\begin{smallmatrix} b & c \\ e & f\end{smallmatrix}|)$
- det Δ or diag = product of diagonal
- if det $= \emptyset$, lin indep & invertible
- $\det(AB) = (\det A)(\det B)$
- $\det(A^{-1}) = \frac{1}{\det A}$
{ - row swap → $\det B = -\det A$
- row mult → $\det B = k\det A$ for each row multiplied }
- det using row ops
① Find RE form
② det = product diag
③ apply

Cramer's Rule ($A\vec{x}=\vec{B}$)
- $\vec{x}_i = \frac{\det M_i}{\det A}$
$M_1 = [\vec{B}\,\vec{A}_2\,\vec{A}_3]$ $M_2 = [\vec{A}_1\,\vec{B}\,\vec{A}_3]$
① Find det A
② Find det of M_i's
③ $\vec{x} = [\frac{\det M_1}{\det A} \; \frac{\det M_2}{\det A} \dots]^T$

Subspace of $\mathbb{R}^n$
① includes $\{\vec{\emptyset}\}$
② closed under addition
$\vec{u}$ & $\vec{w}$ in, so is $\vec{u}+\vec{w}$
③ closed under scalar mult
$\vec{u}$ in, so is $c\vec{u}$
- subspace = span of vector set
- generating set = vectors in the set of the span

Basis (exclude $\{\vec{\emptyset}\}$)
↳ lin indep generating set
- null space − all vectors who are solutions to $A\vec{x} = \vec{\emptyset}$
① Find RE or RRE of A
② null A = span of coefficients of free variables (set other free vars to $\emptyset$)
- column space − subspace generated by col vectors
↳ $\text{Col}A = \text{span}\{\vec{A}_1, \vec{A}_2, \dots\}$
↳ in $\mathbb{R}^m$
↳ $\vec{w}$ in Col A?
① $[A|\vec{w}]$
② rref ↑ consistant?
↳ basis of Col A
① rref A
② basis = pivot cols of og A (not pivot of rref A)
- row space − subspace generated by rows of A
↳ m1: rows A = cols A^T
↳ m2: non $\emptyset$ rows of rref(A)

Dimension
↳ # vectors in basis of V
- prove set B basis of V
① show B contained in V (solution)
② B lin indep
③ dim V = # vectors in B
- dim(Null) = # free vars (nullity)
[dim(Row) = rank =
dim(Col) = dim(Row^T)
rank^T]
- any basis of V has dim(V) # of vectors (gen sets ≥ dim # vectors)

Eigenvalues & Evectors
- $A\vec{v} = \lambda\vec{v}$ (multiplying A by evector only stretches vector)
- evector has 1 evalue
- evalue can have ≥1 evector
① $\det(A-\lambda I)$ for char poly
② λ = roots of char poly
③ null $(A-\lambda I)$ or rref $(A-\lambda I)$ gives evectors for λ & eigen span
- if $\det A = \emptyset$, $\emptyset$ is an evalue
- geo multiplicity (dim(espace)) ≤ alg mult (λ^n)
- diag values in Δ matrix are evalues

Diagonalization
- $A = PDP^{-1}$
- A is similar to D → same charcter poly, same evalues & # independant evectors
- P = [evectors]
- $D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$
- A diagonalizable if
  ↳ evectors of A form a basis for $\mathbb{R}^n$
  ↳ Σ multiplicites $= n$ & dim(espace) = multiplicity of corresponding λ
- $A^n = PD^nP^{-1}$

Norm & Dot Product
- norm $= \|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$
- unit vector $\vec{v} \div \|\vec{v}\|$
- dist $\vec{u} \leftrightarrow \vec{v}$ is $\|\vec{u} - \vec{v}\|$
- dot product $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \dots$
  ↳ $\vec{u} \perp \vec{v}$, if $\vec{u}\cdot\vec{v} = \emptyset$
  ↳ $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}$
- $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$

- $\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\cdot\vec{v} = \vec{w}$
$\vec{z} = \|\vec{u} - \vec{w}\|$ → dist pt to line
- if $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$
$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\cdot\|\vec{v}\|$
$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

Orthogonal
- set of vectors ortho if each pair is ortho
  ↳ will be lin indep
  ↳ if dim = n, will be a basis for $\mathbb{R}^n$
- Gramm Schmidt
$\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$
① $\vec{v}_1 = \vec{u}_1$
② $\vec{v}_2 = \vec{u}_2 - \text{proj}_{v_1}u_2$
③ $\vec{v}_k = \vec{u}_k - \text{proj}_{v_1}u_k - \text{proj}_{v_2}u_k \dots$
$\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis of $\mathbb{R}^n$
- orthogonal compliment $S^\perp$ st S in $\mathbb{R}^n$
  ↳ ⊥ subspace
  - $(\text{Row}\,A)^\perp = \text{Null}\,A$
  - $(\text{Col}A)^\perp = (\text{Row}\,A^T)^\perp = \text{Null}\,A^T$
  - $\dim S + \dim S^\perp = n$
  - basis of $S^\perp$ = basis Null A
- Ortho decomp
$\vec{w} = (\vec{u}\cdot\vec{v}_1)\vec{v}_1 + \dots + (\vec{u}\cdot\vec{v}_k)\vec{v}_k$
$\vec{z} = \vec{u} - \vec{w}$
↳ $\vec{w}$ is vector in subspace closest to $\vec{u}$
Projection Matrix
$P_W = C(C^TC)^{-1}C^T$
↳ cols of C for basis of W
$\vec{w} = P_W\vec{u}$
$\vec{z} = \vec{u} - \vec{w}$
$\|\vec{z}\|$ = dist $\vec{u}$ to W

Least Square Approx

- line of best fit w/ min $\Sigma dist^2$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = (C^TC)^{-1}C^T\vec{y}$$

$C = [\vec{v_1}, \vec{v_2}]$ $\vec{v_1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$

line → $y = a_0 + a_1x$

Orthogonal Matrices

- $n \times n$ matrix is orthogonal if cols are an orthonormal basis of $\mathbb{R}^n$
- equivalents
  - Q orthogonal
  - $Q^TQ = I_n$ → can use to test orthogonality
  - $Q^{-1} = Q^T$
  - $\vec{u} \cdot \vec{v} = Q\vec{u} \cdot Q\vec{v}$
  - $\|\vec{u}\| = \|Q\vec{u}\|$
- properties
  - $\det Q = \pm 1$
  - product of 2 ortho matrices is ortho
  - Q^T & Q^{-1} ortho

Symmetric Matrices ($A^T = A$)

- A_{symm} w/ evectors $\vec{u}$ & $\vec{v}$ who have $\boxed{\text{distinct evalues}}$ λ_u & λ_v, → $\vec{u} \perp \vec{v}$
  (but if $\vec{u} \perp \vec{v}$ & distinct → A not always symm)
- A symm iff evectors of A are orthonorm basis of $\mathbb{R}^n$
- if A symm, A is diagonalizable & P_{ortho} → $P^TAP = D$
- Find ortho diag
  1. Find evalues & evector
  2. Make evectors into orthonorm w/ GrammSchmidt
  3. $P = [\vec{z_1}, \vec{z_2} \dots \vec{z_n}]$ $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$ same
- Spectral decomposition (if A symm)

$$A = PDP^T, \quad P = [\vec{u_1}, \dots, \vec{u_n}] \quad D = \begin{bmatrix} \lambda_1 & \\ & \ddots \end{bmatrix}$$

$$= \lambda_1 \vec{u_1}\vec{u_1}^T + \dots + \lambda_n \vec{u_n}\vec{u_n}^T$$

$$= \lambda_1 [P_1] + \dots + \lambda_n [P_n]$$