Feynman's Trick: Differentiation Under the Integral Sign

The Leibniz rule for integration allows one to evaluate complicated integrals by turning them into solutions of differential equations. This rule was popularized by Physicist Richard Feynman by using it frequently, to the surprise of others, since it is not conventionally taught in formal mathematics education. The statement of the rule is as follows: For an integral in the form of

$$\int_{a(x)}^{b(x)} f(x,t)\,dt$$

where a, b are finite, the derivative of this integral with respect to x is

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\cdot\frac{d}{dx}b(x) - f(x,a(x))\cdot\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt$$

Often this expression is reduced into a simpler form when $a(x)$ and $b(x)$ are constants, when this occurs we have the much simpler form

$$\boxed{\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x}f(x,t)\,dt}$$

The following is a set of examples and exercises.

Example: Evaluate the following integral:

$$\int_0^1 \frac{x^{2019}-1}{\ln(x)}\,dx.$$

Solution: The solution to this problem requires us to become clever with our knowledge of derivatives. Firstly, we know

$$\frac{\partial}{\partial t}x^t = x^t\ln(x)$$

since this is a standard derivative rule. Notice that if we differentiated under the integral sign in this scenario, the $\ln(x)$ on the denominator would be canceled. Therefore, let

$$I(t) = \int_0^1 \frac{x^t-1}{\ln(x)}\,dx,\ t\geq 0$$

The value we wish to find is $I(2019)$, and notice that $I(0)=0$ since

$$I(0) = \int_0^1 \frac{1-1}{\ln(x)}\,dx = 0$$

We proceed by differentiating this function with respect to t:

$$I'(t) = \frac{d}{dt}\int_0^1 \frac{x^t-1}{\ln(x)}\,dx = \int_0^1 \frac{\partial}{\partial t}\frac{x^t-1}{\ln(x)}\,dx = \int_0^1 \frac{x^t\ln(x)}{\ln(x)}\,dx = \int_0^1 x^t\,dx$$

The value of this integral is trivial since it is an integration of a polynomial:

$$I'(t) = \int_0^1 x^t\,dx = \frac{1}{t+1}x^{t+1}\Big|_0^1 = \frac{1}{t+1}$$

Therefore we have the following differential equation whose solution is the family of integrals we wish to solve:

$$I'(t) = \frac{1}{t+1},\ I(0)=0.$$

We solve this simple differential equation by directly integrating:

$$I(t) = \int \frac{1}{t+1}\, dx = \ln|t+1| + C.$$

To find the value of the constant we simply use the initial condition $I'(0) = 0$, this yields $C = 0$ and the general result is $I(t) = \ln|t+1|$. The value of the integral is given when $t = 2019$ therefore

$$\boxed{I(2019) = \int_0^1 \frac{x^{2019}-1}{\ln(x)}\, dx = \ln(2020)}$$

Exercise 1: Evaluate the integral

$$\int_0^1 \frac{x^{2019}-x^{2018}}{\ln(x)}\, dx.$$

Hint: The previous result may be used as an intermediate step in this problem, i.e the formula

$$\int_0^1 \frac{x^t-1}{\ln(x)}\, dx = \ln|t+1|.$$

ANS: $\ln\left(\frac{2020}{2019}\right)$.

Exercise 2: Evaluate the following integral:

$$I = \int_0^1 x^{2019}\ln(x)^2\, dx$$

Then generalize the result to evaluate

$$I_n = \int_0^1 x^{2019}\ln(x)^n\, dx,\ n \geq 0$$

For fun show the following result (consider a certain Taylor series):

$$\sum_{n=0}^{\infty} \frac{1}{I_n} = 2020e^{-2020}.$$

Hint: Consider the function

$$G(t) = \int_0^1 x^t\, dx$$

ANS: The answer for the first integral is $\frac{2}{2020^3}$. The answer for the second integral is $\frac{(-1)^n n!}{2020^{n+1}}$.

Example: Evaluate the following integral:

$$\int_0^\infty \frac{\sin(x)}{x}\, dx.$$

Solution: This one is tricky since there is no obvious approach to the problem via just the integrand alone. However, we could make this an integration by parts problem by using the function e^x, and this would perfectly cancel out the x in the denominator when differentiated. Therefore consider

$$I(t) = \int_0^\infty \frac{\sin(x)}{x}e^{-tx}\, dx,\ t \geq 0$$

Notice that $I(0)$ is the value we seek and that

$$\lim_{t\to\infty}\int_0^\infty \frac{\sin(x)}{x}e^{-tx}\,dx = 0$$

This limit will serve to reduce the constant of integration which occurs when using this method. We proceed by computing $I'(t)$:

$$I'(t) = \frac{d}{dt}\int_0^\infty \frac{\sin(x)}{x}e^{-tx}\,dx = -\int_0^\infty e^{-tx}\sin(x)\,dx$$

The standard method is to use integration by parts, this method is left to the reader to verify the presented result. We proceed with a more efficient method which uses Euler's formula:

$$e^{ix} = \cos(x) + i\sin(x) \Longrightarrow \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

which we substitute into the integral to obtain

$$\begin{aligned}
\int_0^\infty e^{-tx}\sin(x)\,dx &= \frac{1}{2i}\int_0^\infty e^{-(t-i)x} - e^{-(t+i)x}\,dx \\
&= \frac{1}{2i}\left(-\frac{1}{t-i}e^{-(t-i)x}\Big|_{x=0}^{\infty} + \frac{1}{t+i}e^{-(t+i)x}\Big|_{x=0}^{\infty}\right) \\
&= \frac{1}{2i}\left(\frac{1}{t-i} - \frac{1}{t+i}\right) \\
&= \frac{1}{2i}\left(\frac{(t+i)-(t-i)}{(t-i)(t+i)}\right) \\
&= \frac{1}{t^2+1}
\end{aligned}$$

Therefore we have

$$I'(t) = -\frac{1}{t^2+1} \Longrightarrow I(t) = -\arctan(t) + C$$

To find the constant we use the limit condition:

$$\lim_{t\to\infty} I(t) = 0 \iff \lim_{t\to\infty} -\arctan(t) + C = 0 \Longrightarrow \lim_{t\to\infty} -\arctan(t) + C = -\frac{\pi}{2} + C$$

Therefore we have $C = \frac{\pi}{2}$ and we may write

$$I(t) = \int_0^\infty \frac{\sin(x)}{x}e^{-tx}\,dx = \frac{\pi}{2} - \arctan(t)$$

The requested integral is $I(0)$:

$$\boxed{I(0) = \int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}}$$

Exercise 3: Evaluate the following integral:

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx.$$

Hint: Do not use the exponential approach presented previous, think about a different way to eliminate the x on the denominator. You are given that

$$\int_0^\infty \frac{\sin(\beta x)}{x}\,dx = \frac{\pi}{2}.$$

ANS: $\frac{\pi}{2}$.

Exercise 4: Evaluate the following integral

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$$

Hint: Consider the numerator to be $\ln(tx+1)$, then after differentiation consider partial fraction expansions.

ANS: $\frac{\pi \ln(2)}{8}$.

This method is not only powerful for evaluating fun integrals, but it often finds use in derivations. The Gamma function is a function which extends the factorial function to include fractional and negative values. For example using the Gamma function we easily explain perplexing results like $(1/2)! = \sqrt{\pi}/2$. The Gamma function is given by

$$\Gamma(z+1) = \int_0^\infty x^z e^{-x}\,dx.$$

Exercise 5: By considering the function

$$F(t) = \int_0^\infty e^{-tx}\,dx,\ t > 0$$

arrive at the Gamma function and explicitly show

$$z! = \int_0^\infty x^z e^{-x}\,dx$$

thereby establishing the famous identity $\Gamma(z) = (z-1)!$

Hint: Start by directly evaluating $F(t)$, then set the result equal to the original integral $F(t)$ and proceed on both sides of the equality.

Example: Evaluate the integral

$$\int_0^\infty e^{-x^2}\,dx.$$

Solution:

Example: Given that $\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\frac{\pi}{2}}$, evaluate

$$\int_0^\infty \cos(x) e^{-x^2/2}\,dx$$

Solution: We start by considering the function

$$I(t) = \int_0^\infty \cos(tx) e^{-x^2/2}\,dx,\ t \geq 0.$$

We then differentiate this function under the integral sign:

$$I'(t) = \int_0^\infty -x \sin(tx) e^{-x^2/2}\,dx.$$

We proceed with integration by parts, letting $dv = -xe^{-x^2/2}$ and $u = \sin(tx)$. This gives

$$I'(t) = \int_0^\infty -x\sin(tx)e^{-x^2/2}\,dx = \sin(tx)e^{-x^2/2}\Big|_{x=0}^{\infty} - t\int_0^\infty \cos(tx)e^{-x^2/2}\,dx = -tI(t)$$

Therefore we have arrived at the following differential equation

$$I'(t) + tI(t) = 0.$$

How can we solve this for $I(t)$? Consider that this equation almost looks like a product rule, however it is not quite there. We can multiply this equation by a factor which will make the equation the result of application of the product rule. Consider that

$$\frac{d}{dt}e^{t^2/2} = te^{t^2/2}.$$

Multiply this to both sides of the equation:

$$I'(t)e^{t^2/2} + te^{t^2/2}I(t) = 0 \Longrightarrow \frac{d}{dt}\left(I(t)e^{t^2/2}\right) = 0 \Longrightarrow I(t)e^{t^2/2} = C \Longrightarrow I(t) = Ce^{-t^2/2}$$

To find C we use the information provided in the problem statement:

$$I(0) = C = \int_0^\infty e^{-x^2/2}\,dx = \sqrt{\frac{\pi}{2}} \Longrightarrow C = \sqrt{\frac{\pi}{2}}.$$

Therefore the requested integral is

$$\boxed{\int_0^\infty \cos(x)e^{-x^2/2}\,dx = I(1) = \sqrt{\frac{\pi}{2e}}}$$

The Weierstrass Substitution

The Weierstrass substitution has been called the sneakiest substitution in mathematics. It is used to turn trigonometric integrals into rational integrals, and all rational integrals may be computed by various Euler substitutions. Let's say we have an integral in the form

$$\int f(\sin(x), \cos(x))\, dx$$

Then we introduce the substitution $t = \tan(x/2)$. We want to find out what $\sin(x)$, $\cos(x)$ and dx are in terms of t. In order to do this consider

$$t = \tan\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

We may write $\sin(x)$ as $\sin(x) = 2\sin(x/2)\cos(x/2)$ and we note

$$\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = t = \frac{\sin(x)}{2\cos^2\left(\frac{x}{2}\right)} \Longrightarrow 2t\cos^2\left(\frac{x}{2}\right) = \sin(x)$$

To proceed we need to know what $\cos^2(x/2)$ is, and to do this we use the definition of the substitution:

$$t^2\cos^2\left(\frac{x}{2}\right) = \sin^2\left(\frac{x}{2}\right)$$

We add $\cos^2(x/2)$ to both sides and apply the Pythagorean identity:

$$\cos^2\left(\frac{x}{2}\right)(t^2+1) = 1 \Longrightarrow 2\cos^2\left(\frac{x}{2}\right) = \frac{2}{1+t^2}$$

Therefore we finally have an expression for $\sin(x)$:

$$\sin(x) = \frac{2t}{1+t^2}$$

We know that $\sin^2(x) + \cos^2(x) = 1$ therefore we solve for $\cos(x)$:

$$\cos^2(x) = 1 - \sin^2(x) = 1 - \frac{4t^2}{(1+t^2)^2} = \frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2} = \frac{1+2t^2+t^4-4t^2}{(1+t^2)^2}$$

We note that $t^4 - 2t^2 + 1 = (t^2-1)^2$ and we write

$$\cos^2(x) = \frac{(1-t^2)^2}{(1+t^2)^2} \Longrightarrow \cos(x) = \frac{1-t^2}{1+t^2}$$

All that remains is finding dx. To do this we arctan both sides of the definition of the substitution:

$$\frac{x}{2} = \arctan(t) \Longrightarrow x = 2\arctan(t) \Longrightarrow dx = \frac{2}{1+t^2}\, dt$$

Therefore via the substitution $t = \tan(x/2)$ we transformed our integral in the following manner:

$$\boxed{\int f(\sin(x), \cos(x))\, dx \overset{t=\tan(x/2)}{\Longrightarrow} \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\, dt}$$

Example: Evaluate the integral

$$\int \frac{1}{\sin(x)}\,dx$$

Solution: This integral would be difficult to manage using standard methods, however with the Weierstrass substitution it is trivial. Letting $t = \tan(x/2)$ we obtain

$$\begin{aligned}
\int \frac{1}{\sin(x)}\,dx &= \int \frac{1}{\frac{2t}{1+t^2}} \frac{2}{1+t^2}\,dt \\
&= \int \frac{2(1+t^2)}{2t(1+t^2)}\,dt \\
&= \int \frac{1}{t}\,dt \\
&= \ln|t| + C \\
&= \ln\left|\tan\left(\frac{x}{2}\right)\right| + C
\end{aligned}$$

Exercise 6: Evaluate the integral

$$I = \int \frac{1}{\cos(x)}\,dx.$$

ANS: $I = \ln|\tan(x/2)| + C$.

Exercise 7: Evaluate the integral

$$I = \int_0^{\pi/4} \frac{1}{1+\cos(x)-\sin(x)}\,dx$$

ANS: $I = \ln\left|1 + \frac{1}{\sqrt{2}}\right|$.

More Clever Substitutions

Sometimes in competition math monstrous integrals appear which do not yield to standard techniques of evaluation. However, they often have simple solutions. One very powerful substitution exploits a fundamental integral symmetry, and it is the result of a simple substitution:

$$\boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$$

Example: Evaluate the integral

$$\int_0^{\pi/2} \ln(\sin(x))\,dx$$

Solution: We use the substitution boxed to obtain

$$I = \int_0^{\pi/2} \ln(\sin(x))\,dx = \int_0^{\pi/2} \ln\left[\underbrace{\sin\left(\frac{\pi}{2} - x\right)}_{\cos(x)}\right] = \int_0^{\pi/2} \ln(\cos(x))\,dx$$

To proceed we add the two integrals:

$$2I = \int_0^{\pi/2} \ln(\sin(x)) + \ln(\cos(x))\,dx = \int_0^{\pi/2} \ln\left(\frac{1}{2}\sin(2x)\right)\,dx = \int_0^{\pi/2} -\ln(2) + \ln(\sin(2x))\,dx$$

Therefore we have

$$2I = -\frac{\pi}{2}\ln(2) + \int_0^{\pi/2} \ln(\sin(2x))\,dx$$

To proceed we let $u = 2x$ in the integral above to obtain

$$\begin{aligned}
\int_0^{\pi/2} \ln(\sin(2x))\,dx &= \frac{1}{2}\int_0^{\pi} \ln(\sin(u))\,du \\
&= \frac{1}{2}\int_0^{\pi/2} \ln(\sin(u))\,du + \frac{1}{2}\int_{\pi/2}^{\pi} \ln(\sin(u))\,du \\
&= \frac{1}{2}I + \frac{1}{2}\int_0^{\pi/2} \ln(\cos(k))\,dk \text{ via } k = u - \frac{\pi}{2} \\
&= \frac{1}{2}I + \frac{1}{2}I \\
&= I
\end{aligned}$$

Therefore we have

$$2I = -\frac{\pi}{2}\ln(2) + I \Longrightarrow I = -\frac{\pi}{2}\ln(2)$$

Thus the final result

$$\boxed{\int_0^{\pi/2} \ln(\sin(x))\,dx = -\frac{\pi}{2}\ln(2)}$$

Example: Find the value of the integral

$$I=\int_0^{\pi/2}\frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\,dx,\ n\geq 0$$

Using the substitution $x\to a+b-x$ we obtain

$$I=\int_0^{\pi/2}\frac{\sin^n\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)+\cos^n\left(\frac{\pi}{2}-x\right)}\,dx=\int_0^{\pi/2}\frac{\cos^n(x)}{\sin^n(x)+\cos^n(x)}\,dx$$

Therefore adding the two integrals together yields

$$2I=\int_0^{\pi/2}\frac{\sin^n(x)+\cos^n(x)}{\sin^n(x)+\cos^n(x)}\,dx=\int_0^{\pi/2}dx=\frac{\pi}{2}$$

Therefore

$$\boxed{\int_0^{\pi/2}\frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\,dx=\frac{\pi}{4},\ n\geq 0}$$

Exercise 8: Evaluate the integral

$$I=\int_2^4\frac{\sin\left(e^{9-x}\right)}{\sin\left(e^{9-x}\right)+\sin\left(e^{x+3}\right)}\,dx$$

ANS: $I=1$.

Exercise 9: Evaluate the following integral

$$I=\frac{\pi}{e}\int_0^1\ln\left(\frac{\pi}{e}(1-x)\right)-\ln\left(\frac{\pi^2}{e^2}x(1-x)\right)\,dx$$

ANS: $I=\dfrac{\pi}{2e}$.

For integrals on $[0,\infty)$ we may make use of the following substitution:

$$\boxed{\int_0^\infty f(x)\,dx=\int_0^\infty\frac{f\left(\frac{1}{x}\right)}{x^2}\,dx}$$

Example: Evaluate the following integral:

$$\int_{2/\pi}^\infty\frac{\ln\left(\sin\left(\frac{1}{x}\right)\right)}{x^2}\,dx$$

Solution: We replace $x\to 1/x$ to obtain the following:

$$\int_{2/\pi}^\infty\frac{\ln\left(\sin\left(\frac{1}{x}\right)\right)}{x^2}\,dx=\int_{\pi/2}^0 x^2\ln(\sin(x))\frac{dx}{-x^2}=\int_0^{\pi/2}\ln(\sin(x))\,dx=\boxed{-\frac{\pi}{2}\ln(2)}$$

The integral this was reduced to was evaluated in a previous section.

Exercise 10: Evaluate the following integral:

$$I = \int_0^\infty \frac{\ln(2x)}{x^2+1}\,dx$$

ANS: $I = \frac{\pi}{2}\ln(2)$.

Exercise 11: Evaluate the following integral:

$$I = \int_0^\infty \frac{1}{\left(x+\frac{1}{x}\right)^2}\,dx$$

ANS: $I = \frac{\pi}{4}$.