Multi-variate Functions to Lagrange Multipliers
MATH 209

Fall 2019

Contents

Notes About This Package:

(1) This package has many incomplete sections. It is recommended to use other sources alongside this one, especially sources that focus on visual graphing and animations.

1 Functions of Multiple Variables

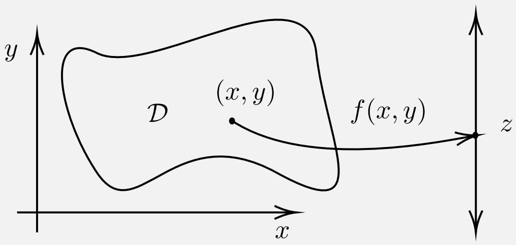
Up until now we have been concerned with functions which take one input and assign one output. Now we will be considering functions which can take up to n inputs.

1.1 Functions of Two or Three Variables

A Function of Two Variables — Multivariate Calculus

A function of two variables is a function which takes any point in a subset of the plane and assigns it to a value. Meaning $f : \mathcal{D} \to \mathbb{R}$ where $(x, y) \in \mathcal{D} \subset \mathbb{R}^2$. The subset of the plane $\mathcal{D}$ is called the **domain set** of the function, and the set of all $z = f(x, y)$ on the domain is called the **range set** of the function.

The power of these functions is they tell us the relationship between three variables of interest. For example a function of two variables can tell us the relationship between heat, space, and time.



Remarks:

(1) If we take the z axis pictured above, and assign it to be perpendicular to both x and y then $z = f(x, y)$ would be a **surface** over the domain $\mathcal{D}$.

(2) If $f(x, y) = k$, where k is some real constant. Then $f(x, y) = k$ describes **level curves** of $f(x, y)$. These functions, often implicit, define a curve of intersection between the surface described by $z = f(x, y)$ and the plane $z = k$.

Example — Domains of Functions of Two Variables

Find and plot the sets $\mathcal{D}$ which describe the domains of the following functions:

$$f_1(x, y) = \frac{1}{x^2 + y^2 - 1}$$
$$f_2(x, y) = \sqrt{2x + y}$$
$$f_3(x, y) = \ln(x^2 + y^2 - 9)$$
$$f_4(x, y) = \sqrt{\sin(\pi(x^2 + y^2))} + \ln(2 - |x|).$$

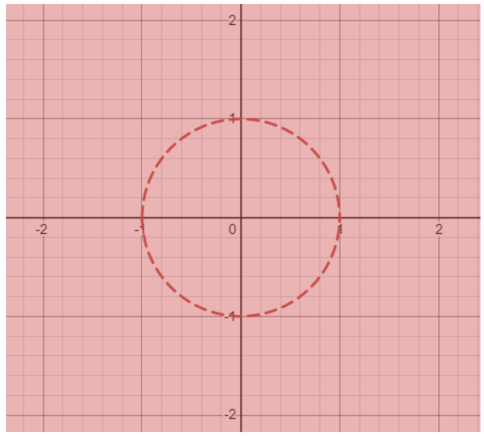
Solution: The function $f_1(x, y)$ is undefined where the denominator is zero. This means we cannot have any points (x, y) that satisfy

$$x^2 + y^2 - 1 = 0 \Longrightarrow x^2 + y^2 = 1.$$

Meaning the set $\mathcal{D}_1$ that describes the domain of f_1 is

$$\boxed{\mathcal{D}_1 := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \neq 1\right\}}$$

Graphically, this is the entire real xy plane excluding the circle of radius one centered about the origin. This is plotted below.



The red region is the domain and the dotted red line is excluded from the domain.

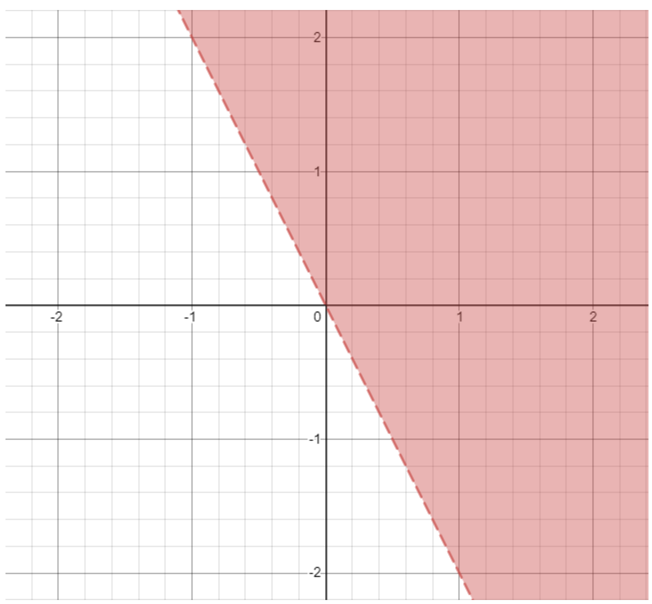
The function $f_2(x, y)$ is undefined where the function under the square root is less than zero. Meaning we cannot have any points (x, y) that satisfy

$$2x + y < 0 \implies y < -2x.$$

Thus the set $\mathcal{D}_2$ that describes the domain of f_2 is

$$\boxed{\mathcal{D}_2 := \left\{(x, y) \in \mathbb{R}^2 \,|\, 2x + y \geq 0\right\}}$$

Graphically this corresponds to the entire real plane excluding the region below the line $y = -2x$. This is plotted below.



The red region is the domain, and the dotted red line is included in the domain.

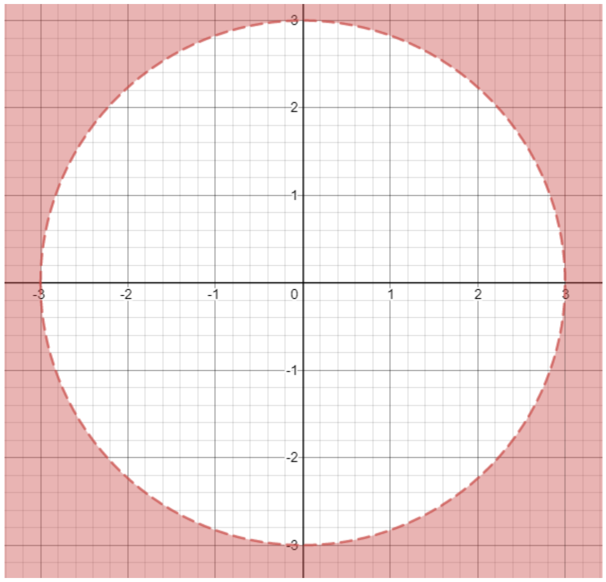
The function f_3 is undefined where the argument of the logarithm is zero. Thus the only points which satisfy the domain will be (x, y) satisfying

$$x^2 + y^2 - 9 > 0 \implies x^2 + y^2 > 9.$$

Thus the set $\mathcal{D}_3$ which describes the domain of f_3 is

$$\boxed{\mathcal{D}_3 := \left\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 > 9\right\}}$$

Graphically this corresponds to the real plane excluding all points contained within and on the circle $x^2+y^2 = 9$. This is plotted below.



The red region is the domain and the dotted red line is excluded from the domain.

Finding the domain of $f_4(x,y)$ will be the most involved process so far. We have two simultaneous restrictions on the domain of f, namely

$$\sin(\pi(x^2+y^2)) \geq 0, \quad 2 - |x| > 0.$$

The solution we seek will be the intersection of the two domains. The easiest part of the solution is finding the domain restriction from the logarithm: $|x| < 2$. This describes all points x satisfying $-2 < x < 2$. Now we consider with detail the first restriction. We let $\theta = \pi(x^2+y^2)$, then we require

$$\sin(\theta) \geq 0.$$

We know that this occurs on the top half of the unit circle, thus we expect $\theta \in [0, \pi]$. Thus we may expect

$$0 \leq \pi(x^2+y^2) \leq \pi \implies 0 \leq x^2+y^2 \leq 1,$$

to be the solution we are looking for, however we would be mistaken if we reported this as the final answer. The reason that this answer is not enough is because $\sin(\theta)$ is periodic, meaning it is also positive when $\theta \in [2\pi, 3\pi]$, when $\theta \in [4\pi, 5\pi]$ and so on. In general if $n \in \mathbb{Z}^+$, then we expect $\theta \in [2n\pi, (2n+1)\pi]$. So we can finally write

$$2n\pi \leq \pi(x^2+y^2) \leq (2n+1)n\pi$$
$$2n \leq x^2+y^2 \leq 2n+1 \ \forall n \in \mathbb{Z}^+$$

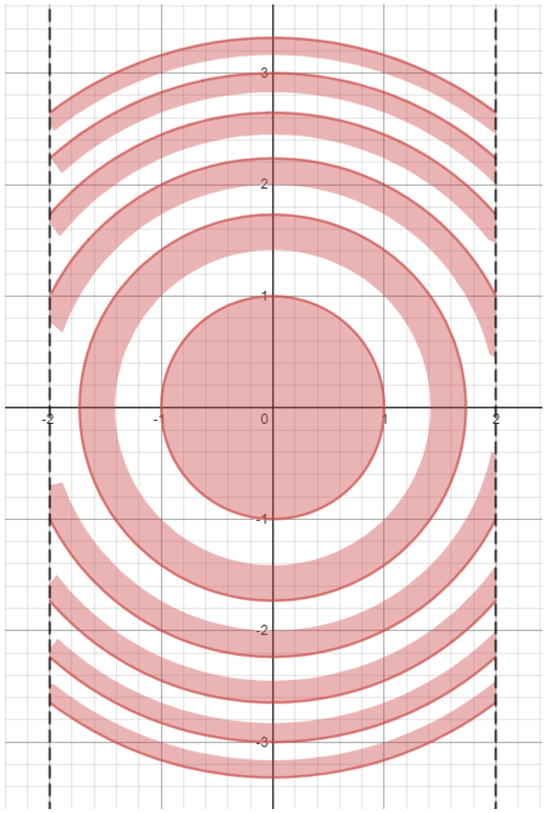
Computation for the first few values of n gives

$$n = 0 \,:\, 0 \leq x^2+y^2 \leq 1$$
$$n = 1 \,:\, 2 \leq x^2+y^2 \leq 3$$
$$n = 2 \,:\, 4 \leq x^2+y^2 \leq 5.$$

Thus the domain of the function f_4 is a disk of radius one centered about the origin and an infinite series of annular regions of decreasing width, all restricted for $-2 < x < 2$. Thus

$$\boxed{\mathcal{D}_4 := \left\{(x, y) \in \mathbb{R}^2,\ n \in \mathbb{Z}^+ \,|\, 2n \leq x^2 + y^2 \leq 2n + 1, \quad -2 < x < 2\right\}}$$

This result is plotted below.



The red regions are the domain of the function, and the dotted black lines are excluded from the domain of the function.

$\square$

Example	Level Curves

Plot the surface described by $z = f(x, y)$ given by

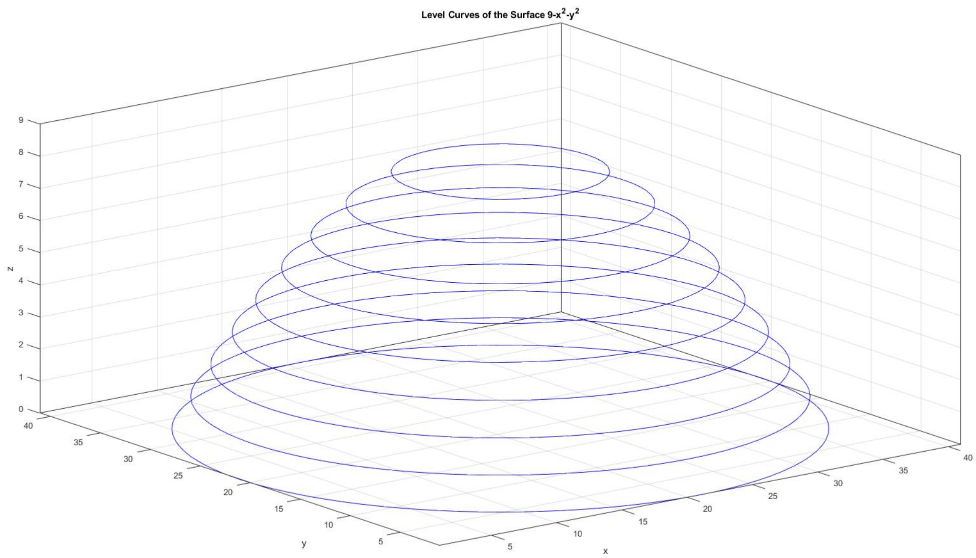
$$z = f(x, y) = 9 - x^2 - y^2$$

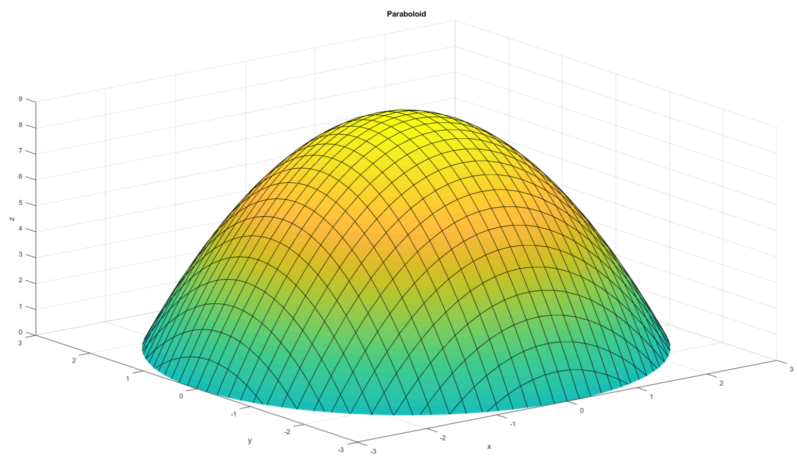
where $z \geq 0$ by considering its level curves, i.e by considering $k = f(x, y)$ where $0 \leq k \leq 9$.

Solution: The level curves for this function are

$$9 - k = x^2 + y^2, \quad 0 \leq k \leq 9$$

Thus we plot for each k, $z = k$ and on each of these planes the circle $9 - k = x^2 + y^2$. From this we see that we have circles in each plane increasing in radius until we reach the xy plane where $z = 0$. Below is a plot of integer k level curves and the surface itself.



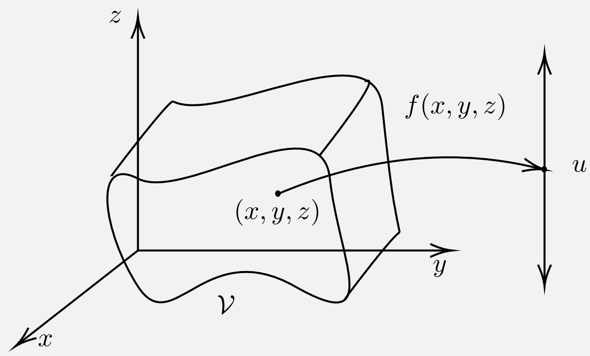


$\square$

A Function of Three Variables — Multivariate Calculus

A function of three variables is a function which takes any point in a subset of space and assigns it to a value. Meaning $f : \mathcal{V} \to \mathbb{R}$ where $(x, y, z) \in \mathcal{V} \subset \mathbb{R}^3$. The subset/subvolume of the space $\mathcal{V}$ is called the **domain set** of the function, and the set of all $z = f(x, y, z)$ acting on $\mathcal{V}$ is called the **range set** of the function.

These functions can allow us to create animations for the relationship between four variables. Take for example a function $u(x, y, t)$ that describes the relationship between heat, two spacial variables and time.



Remark:
For a function $u = f(x, y, z)$, if we set $u = k$ where k is some arbitrary constant, then we obtain **level surfaces** of f. Thus $k = f(x, y, z)$ are often implicit functions which describe level surfaces.

Example — Domains of Functions of Three Variables

Find the domain of the following function

$$f(x, y, z) = \ln(9 - x^2 - y^2 - z^2) + \sqrt{3 - x - y}.$$

Solution: We know that the argument of the logarithm must be strictly positive, and the argument of the square root must be greater than or equal to zero:

$$9 > x^2 + y^2 + z^2, \quad 3 \geq x + y.$$

The domain is therefore the union of these expressions:

$$\mathcal{V} := \left\{(x, y, z) \in \mathbb{R}^3, \,|\, x^2 + y^2 + z^2 < 9, \quad x + y \leq 3\right\}.$$

Geometrically this corresponds to all the volume enclosed by the sphere $x^2 + y^2 + z^2 = 9$, but not including the surface of the sphere. This volume is then cut by the plane $x + y = 3$. The domain is everything inside the sphere, and below the plane.

$\square$

1.2 Functions of n Variables

A function of n variables is a map from the tuple $(x_1, x_2, \ldots, x_n)$ to some number in $\mathbb{R}$. Thus $f : \mathcal{M} \to \mathbb{R}$ where $(x_1, x_2, \ldots, x_n) \in \mathcal{M} \subset \mathbb{R}^n$. The range of the function is the set $\mathcal{R} := \{f(x_1, x_2, \ldots, x_n) \,|\, (x_1, x_2, \ldots, x_n) \in \mathcal{M}\}$. We often don't consider cases above $n = 3$.

2 Limits of Multivariate Functions

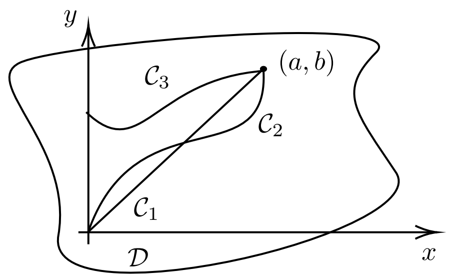
We will be focusing on functions of two and three variables for this section. Limits of multivariate functions are more complicated than their univariate counterparts. To understand why complications arise, let's consider what it intuitively means to compute something like

$$\lim_{(x,y)\to(a,b)} f(x,y).$$

Here the notation is masking the problem: It is totally ambiguous as to how one gets to the point (a, b). In a one dimensional limit we simply have to approach $x = a$ from the left and the right to see if the limit exists. Here we can take *infinitely many paths* to the point (a, b). So if the following limit exists

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

then every path in the domain of f to (a, b) must converge to L. Let $\mathcal{D}$ be the domain of $f(x, y)$, below is an illustration of 3 possible paths we can take to $(a, b) \in \mathcal{D}$.



We can take infinitely many paths to (a, b) in the domain.

The question now arises: How do we check infinitely many possible paths? One good answer is that sometimes we don't have to. Let's consider the case of showing a limit doesn't exist – this case is often the easiest. All we have to do to show a limit doesn't exist is show that the limit is different for two different paths in the domain of the function considered.

Example — Limits via Contradiction

Show that the following limit doesn't exist via contradiction:

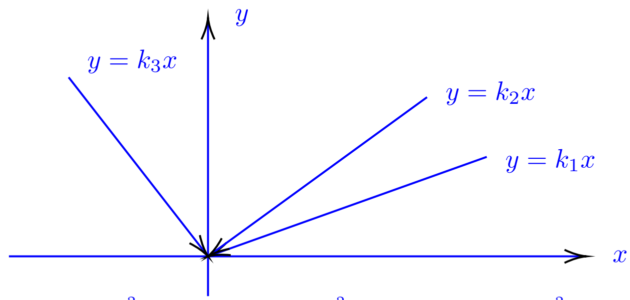
$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + 3y^4}.$$

I.e find two paths, or a family of paths, on which the value of the limit is different.

Solution: Let us choose a family of paths. Since the limit is going to the origin, let us consider the paths $y = mx$ where m is some real constant. Then we obtain

$$\lim_{(x,mx)\to(0,0)} = \frac{m^2x^4}{x^4 + 3m^4x^4} = \lim_{x\to 0} \frac{x^4}{x^4} \frac{m^2}{1 + 3m^4} = \frac{m^2}{1 + 3m^4}.$$

Therefore the limit doesn't exist since for every value of m, the above value is different. We can visualize why this is wrong by considering $y = mx$ as infinitely many lines in the domain of the $f(x, y)$ in the limit. Let $m = k_1, k_2, k_3$ where $k_1 \neq k_2 \neq k_3$ for three separate lines to the origin in the domain of f, then the resulting scenario is imaged below.



$L_1 = \frac{k_1^2}{1+3k_1^4}$ $\quad L_2 = \frac{k_2^2}{1+3k_2^4}$ $\quad L_3 = \frac{k_3^2}{1+3k_3^4}$

$$L_1 \neq L_2 \neq L_3$$

Thus we have found three paths to the origin which do not converge to the same limit, therefore the limit cannot exist.

Remark: If the expression for the limit when $y = mx$ was the same for all values of m, this **does not mean that the limit exists**. Take for example the limit above that we just showed doesn't exist. Let us check the path along the x axis ($y = 0$) and then the y axis ($x = 0$):

$$\lim_{(x,0)\to 0} \frac{x^2 \cdot 0}{x^4 + 3 \cdot 0^4} = 0$$
$$\lim_{(0,y)\to 0} \frac{y^2 \cdot 0}{0^4 + 3y^4} = 0.$$

This does not tell us that the limit exists, only that it is the same along two paths. Even if we found for example some $y = f(x, m)$ such that the limit is the same for all m this still does not mean that the limit exists! It only is the same for an infinite family of functions, but this infinite family of functions is not all possible paths.

$\square$

We have another case where we do not have to evaluate infinitely many paths. This is the easiest possible case where the point (a, b) is in the domain of $f(x, y)$, or where (a, b) causes a removable point in f, then we can easily evaluate the limit by direct substitution.

Example — **Limits via Direct Substitution**

Evaluate the following limits via direct substitution, or by manipulating them into a form where they can be evaluated via direct substitution:

$$L_1 = \lim_{(x,y)\to(5,1)} \frac{x^2 + e^y}{1 + y^2}$$
$$L_2 = \lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2}.$$

Solution: Evaluating L_1 is easy since the point $(5, 1)$ and the neighbourhood around it is in the domain of the function in the limit. Thus

$$\boxed{L_1 = \lim_{(x,y)\to(5,1)} f(x, y) = f(5, 1) = \frac{25 + e}{2}}$$

Evaluating L_2 is tricky since it at first seems that $(1,1)$ is not in the domain of the function in the limit we denote g. However, we can factor g in the following way:

$$g(x,y) = \frac{2x^2 - xy - y^2}{x^2 - y^2} = \frac{(2x+y)(x-y)}{(x-y)(x+y)} = \frac{2x+y}{x+y}$$

This expression for g allows us to evaluate the limit as

$$\boxed{L_2 = \lim_{(x,y)\to(1,1)} \frac{2x+y}{x+y} = \frac{3}{2}}$$

□

We have considered cases where the limit doesn't exist, or can be evaluated by direct substitution. The next method we can use is **squeeze theorem**, this will allow us to evaluate a class of limits that the previous two methods would not let us evaluate. This method often requires a bit of clever thinking.

Example — Limits via Squeeze Theorem

Evaluate the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{(x^2+y^2)^{3/2}}.$$

Solution: First we break the limit into two

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} - \lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2+y^2}}.$$

We consider the case in x first. Notice that addition of y^2 in the denominator always serves to make the total ratio smaller. This means we can write

$$\frac{1}{\sqrt{x^2+y^2}} \leq \frac{1}{|x|} \implies \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2}{|x|} = |x|.$$

Thus we have the following inequalities of limits:

$$-\lim_{x\to 0} x \leq \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} \leq \lim_{x\to 0} x.$$

Therefore via squeeze theorem we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0.$$

Notice that the case in y will be exactly the same as the case in x, and will also evaluate to zero via similar argumentation. Thus we have

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2+y^2}} = 0}$$

via squeeze theorem.

The second limit takes the same approach, notice

$$\frac{x^4}{(x^2+y^2)^{3/2}} \leq \frac{x^4}{(x^2)^{3/2}} = |x|$$

Thus we can use squeeze theorem:

$$-\lim_{x\to 0} x \le \lim_{(x,y)\to(0,0)} \frac{x^4}{(x^2+y^2)^{3/2}} \le \lim_{x\to 0} x \implies \lim_{(x,y)\to(0,0)} \frac{x^4}{(x^2+y^2)^{3/2}} = 0.$$

A similar argument shows

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{(x^2+y^2)^{3/2}} = 0.$$

Thus via squeeze theorem

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)^{3/2}} = 0}$$

□

Example Limits

Evaluate the following limit or show that it doesn't exist:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}.$$

Note the domain of the function in the limit is restricted such that $y \neq -x$.

Solution: Let $t = x+y$ then as $(x,y)\to(0,0)$ we have $t\to 0$. Then we have

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = \lim_{t\to 0} \frac{\sin(t)}{t} = 1}$$

The result in t is a standard result in a first course in calculus and may be derived using L'Hopitals rule or the definition of the derivative.

Note: This method isn't technically rigorous, however we can strengthen the result since $\sin(\theta)$ and the polynomial $x+y$ are both continuous everywhere, and we can apply the limits in the following manner:

$$\frac{\sin\left(\lim_{(x,y)\to(0,0)} (x+y)\right)}{\lim_{(x,y)\to(0,0)} (x+y)}$$

We know that for the domain of the function in the limit, taking $x+y$ in any point in the domain will always add to a constant. Meaning $t = x+y$ will implicitly describe all possible paths to the origin. Therefore the limit implicitly behaves exactly as the $\sin(t)/t$ case.

□

2.1 Continuity

Continuity | **Multivariate Calculus**

We primarily focus on functions of 2 and 3 variables. Thus a function of 2 variables, $f(x, y)$, is continuous at the point (a, b) if and only if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b),$$

and likewise continuity for a function of 3 variables holds if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c).$$

Example | **Continuity**

Let

$$f(x, y) = \begin{cases} \frac{x+y-1}{\sqrt{x}+\sqrt{1-y}}, & (x, y) \neq (0, 1) \\ c, & (x, y) = (0, 1). \end{cases}$$

Find the value of c, if any, so that f is continuous at $(0, 1)$.

Solution: We know that f is continuous at $(0, 1)$ if and only if

$$f(0, 1) = c = \lim_{(x,y)\to(0,1)} f(x, y).$$

Thus we compute the limit:

$$\begin{aligned} \lim_{(x,y)\to(0,1)} \frac{x+y-1}{\sqrt{x}+\sqrt{1-y}} &= \lim_{(x,y)\to(0,1)} \frac{x+y-1}{\sqrt{x}+\sqrt{1-y}} \frac{\sqrt{x}-\sqrt{1-y}}{\sqrt{x}-\sqrt{1-y}} \\ &= \lim_{(x,y)\to(0,1)} \frac{(x+y-1)(\sqrt{x}-\sqrt{1-y})}{x+y-1} \\ &= \lim_{(x,y)\to(0,1)} \sqrt{x}-\sqrt{1-y} \\ &= 0 \end{aligned}$$

Thus the value of c which gives continuity is $\boxed{c = 0}$.

$\square$

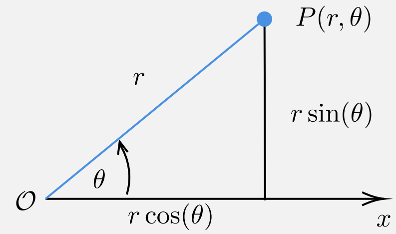
2.2 Limits via Polar and Spherical Coordinates

Often it is useful to switch to polar or spherical coordinates when evaluating limits. We start by recalling what defines these coordinate systems.

Polar/Spherical Coordinates — Multivariate Calculus

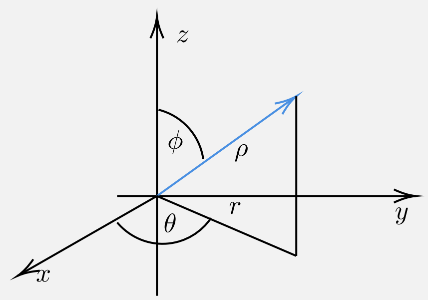
Polar coordinates are defined by $x = r\cos(\theta)$ and $y = r\sin(\theta)$ where $x^2+y^2 = r^2$ and $y/x = \tan(\theta)$.

These coordinates are sometimes used in the evaluation of 2D limits or 3D limits with cylindrical symmetry.



Spherical coordinates are defined by $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$ where $x^2 + y^2 + z^2 = \rho^2$, $y/x = \tan(\theta)$ and $z/\rho = \cos(\phi)$.

These coordinates are sometimes used in the evaluation of 3D limits, usually those with spherical symmetry.



Example — Limits via Polar Coordinates

Evaluate the following limit or show that it doesn't exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}} \ln(x^2+y^2).$$

Solution: We see that we have polar symmetry with the various $x^2 + y^2$ terms. Thus we introduce the polar transformation $x = r\cos(\theta)$ and $y = r\sin(\theta)$. We note that as $(x, y) \to (0, 0)$ we have $r \to 0^+$. This gives us the 1D limit in r:

$$\lim_{r\to 0^+} \frac{1}{r} \ln(r^2)(r^3)(\cos^3(\theta) + \sin^3(\theta)) = \lim_{r\to 0^+} r^2 \ln(r^2)(\cos^3(\theta) + \sin^3(\theta)).$$

We can re-write this limit as a ratio of indeterminate forms:

$$\lim_{r\to 0^+} \frac{\ln(r^2)}{1/r^2} \left(\cos^3(\theta) + \sin^3(\theta)\right).$$

Here we make use of L'Hopitals rule:

$$\lim_{r\to 0^+} \frac{2r}{r^2} \frac{1}{-2/r^3} \left(\cos^3(\theta) + \sin^3(\theta)\right) = \lim_{r\to 0^+} -r^2(\cos^3(\theta) + \sin^3(\theta)).$$

This limit evaluates to zero, thus

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}}\ln(x^2+y^2) = 0}$$

Remarks:
(1) We take $r \to 0^+$ since we always define r to be in the interval $[0, \infty)$.
(2) This remark is an important detail. In order to really say

$$\lim_{r\to 0^-+} -r^2(\cos^3(\theta)+\sin^3(\theta)) = 0.$$

It is of crucial importance to note that $\exists\, \beta \in \mathbb{R}$ such that

$$|\cos^3(\theta)+\sin^3(\theta)| \le \beta < \infty\ \forall\ \theta \in \mathbb{R}.$$

If there is a value of θ that causes it to diverge, then we cannot say for certain that the limit is zero, and we would have to pursue other methods of evaluation. In general if we have

$$\lim_{(x,y)\to(0,0)} f(x,y) \Longrightarrow \lim_{r\to 0^+} g(r)h(\theta)$$

then it is crucial that $\exists\beta \in \mathbb{R}$ such that $h(\theta) \le \beta$ for all θ, this required behaviour allows us to apply squeeze theorem successfully. In this way, we mostly use the polar coordinates transformation to obtain a 1D application of squeeze theorem.

□

Example — Limits via Polar Coordinates

Evaluate the following limit or show that it doesn't exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

Solution: This example was chosen to specifically illustrate remark (2) in the previous example, and to illustrate that one needs to be careful when applying polar limits. We may be tempted to apply polar coordinates here, and transform the limit into

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0^+} \frac{r^3\cos^2(\theta)\sin(\theta)}{r^2(r^2\cos^4(\theta)+\sin^2(\theta))} = \lim_{r\to 0^+} \frac{r\cos^2(\theta)\sin(\theta)}{r^2\cos^4(\theta)+\sin^2(\theta)}.$$

We may be tempted to say that this is zero, however consider the following ratio

$$\frac{\cos^2(\theta)\sin(\theta)}{r^2\cos^4(\theta)+\sin^2(\theta)} \underset{r=0}{\Longrightarrow} \frac{\cos^2(\theta)\sin(\theta)}{\sin^2(\theta)} = \frac{\cos^2(\theta)}{\sin(\theta)}$$

Since $\sin(\theta)$ and $\cos(\theta)$ are never zero at the same time, we have unbounded behaviour if $\theta = n\pi$ where $n \in \mathbb{Z}$. So in our original expression, as $r \to 0$, the ratio heads to zero, but along any line where $\theta = n\pi$, we have undetermined behaviour. In fact if we restrict $\theta = n\pi$ then under the $r \to 0^+$ limit we have

$$\lim_{r\to 0^+} \frac{r\cos^2(\theta)\sin(\theta)}{r^2\cos^4(\theta)+\sin^2(\theta)} = \frac{0}{\sin^2(n\pi)} \Longrightarrow \frac{0}{0} \text{ indeterminant form.}$$

Thus we cannot really use polar coordinates to evaluate this limit. Choosing the path $y = mx^2$ where m is a real constant gives

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,mx^2)\to(0,0)} \frac{mx^4}{x^4(1+m^2)} = \frac{m}{1+m^2}.$$

Therefore the limit in question does not exist since it is different for all values of m.

$\square$

Example — Limits via Polar Coordinates

Evaluate the following limits or show that they do not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x+y}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+3y^2}.$$

Solution: We make use of the polar transformation on the first limit to obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x+y} = \lim_{r\to0^+} \frac{r^2}{r(\cos(\theta)+\sin(\theta))} = \lim_{r\to0^+} \frac{r}{\cos(\theta)+\sin(\theta)}$$

It is tempting to say that this limit evaluates to zero. However, once again if $\theta = 3\pi/4, 7\pi/4, \ldots$, then

$$\exists\,\theta\,:\,\left|\frac{1}{\cos(\theta)+\sin(\theta)}\right| \to \infty$$

Therefore there cannot exist β such that $|h(\theta)| \leq \beta\ \forall\ \theta \in \mathbb{R}$. So we cannot say for sure if this limit exists with the polar transformation. Consider the path $y = mx^2 - x$, then we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x+y} \implies \lim_{(x,mx^2-x)\to(0,0)} \frac{x^2+(mx^2-x)^2}{x+mx^2-x} = \lim_{x\to0} \frac{x^2+m^2x^4-2mx^3+x^2}{mx^2} = \frac{2}{m}.$$

Clearly, this result changes for any $m \neq 0$. Therefore the limit does not exist.

For the second limit we again use the polar transformation to obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+3y^2} = \lim_{r\to0^+} \frac{r^3\cos^2(\theta)\sin^3(\theta)}{2+\sin^2(\theta)}.$$

Notice that $2+\sin^2(\theta) \neq 0$ for all θ. Therefore,

$$\exists\,\beta \in \mathbb{R}^+\,:\,\left|\frac{\cos^2(\theta)\sin^3(\theta)}{2+\sin^2(\theta)}\right| \leq \beta\ \forall\ \theta \in \mathbb{R}.$$

Thus we can invoke squeeze theorem and conclude

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+3y^2} = 0}$$

$\square$

Example — Limits via Spherical Coordinates

Evaluate the following limit or show that it doesn't exist

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}.$$

Solution: We use the spherical coordinate transformation which gives

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0^+} \rho^3 \frac{(\cos(\theta)\sin(\phi)\sin(\theta)\sin(\phi)\cos(\phi))}{\rho^2} = \lim_{\rho\to 0^+} \rho(\cos(\theta)\sin(\theta)\cos(\phi)\sin^2(\phi)).$$

Note that all the trig functions are bounded thus their product must be bounded as well, meaning

$$\exists\, \beta \in \mathbb{R}^+ \,:\, |\cos(\theta)\sin(\theta)\cos(\phi)\sin^2(\phi)| \leq \beta \;\forall\; \theta, \phi \in \mathbb{R}.$$

Thus we can sufficiently use squeeze theorem to obtain

$$\boxed{\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0}$$

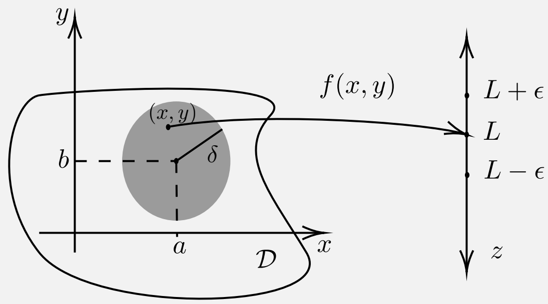
$\square$

2.3 Epsilon-Delta Definition of a Limit

The epsilon-delta definition of a limit is the rigorous method with which we prove that a limit exists. Unfortunately in two variables this method can be quite difficult to apply. We will discuss the 2D case, but the result easily generalizes to nD.

Epsilon Delta Definition of a 2D Limit — Multivariate Calculus

Let f be a function of (x, y), i.e $f : \mathcal{D} \to \mathcal{R}$. Let $\mathcal{D}$ be an open region around the point (a, b). The limit $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if and only if for each $\epsilon > 0$, $\exists\, \delta > 0$ such that for all points (x, y) in a disk with radius δ about (a, b) - except for possibly at (a, b) itself - the value of $f(x, y)$ is no more than ϵ away from L.



Mathematically, the limit exists and is L if for any choice of $\epsilon > 0$, there exists a δ such that

$$0 \le \sqrt{(x-a)^2 + (y-b)^2} \le \delta$$

whenever

$$|f(x, y) - L| \le \epsilon.$$

Note: The picture illustrates the seemingly obscure mathematics. We can choose any $\epsilon > 0$, and if we can find a corresponding disk of radius ϵ in the domain that will have all $f(x, y)$ in the disk be in the range $(L - \epsilon, L + \epsilon)$, then the limit exists and is L.

Example — Epsilon Delta Definition of a 2D Limit

Prove that the following limit holds

$$\lim_{(x,y)\to(1,2)} (x + y) = 3$$

using the $\epsilon - \delta$ definition of a limit.

Hint: It may be useful to recall the triangle inequality which states

$$|x + y| \le |x| + |y|.$$

Solution: The limit given exists and is 3 if there exists $\delta > 0$ for every choice of $\epsilon > 0$ such that

$$0 \le \sqrt{(x-1)^2 + (y-2)^2} \le \delta \text{ and } |(x + y) - 3| \le \epsilon.$$

Notice that $-3 = -2 - 1$ thus we can write the epsilon inequality as

$$|(x - 1) + (y - 2)| \le \epsilon \implies \sqrt{((x - 1) + (y - 2))^2} \le \epsilon.$$

Now we make use of the triangle inequality which gives

$$\sqrt{((x - 1) + (y - 2))^2} \le \sqrt{(x - 1)^2} + \sqrt{(y - 2)^2} \le \epsilon.$$

Further notice that

$$\sqrt{(x-1)^2} \leq \sqrt{(x-1)^2+(y-2)^2}$$
$$\sqrt{(y-2)^2} \leq \sqrt{(x-1)^2+(y-2)^2}.$$

Adding these two inequalities gives

$$\sqrt{(x-1)^2} + \sqrt{(y-2)^2} \leq 2\sqrt{(x-1)^2+(y-2)^2}$$

Thus we have finally arrived at

$$|(x+y)-3| \leq 2\sqrt{(x-1)^2+(y-2)^2} \leq \epsilon$$

Remember that we are looking for δ such that

$$0 \leq \sqrt{(x-1)^2+(y-2)^2} \leq \delta.$$

From inequality we derived for $|(x+y)-3|$ we can easily see that our choice of ϵ is

$$\epsilon = 2\delta.$$

Thus for every choice of $\epsilon > 0$ there always exists a δ, specifically $\delta = \epsilon/2$, that satisfies the definition of a limit. Thus

$$\lim_{(x,y)\to(1,2)} (x+y) = 3.$$

The amount of difficulty that went into this proof of such a trivial limit is why the epsilon-delta definition is seldom used for higher dimensional limits.

$\square$

3 Partial Differentiation and Applications

Partial Derivatives — **Multivariate Calculus**

The partial derivative of $f(x,y)$ with respect to x is defined as

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{f((x+h),y) - f(x,y)}{h}.$$

The partial derivative of $f(x,y)$ with respect to y is defined as

$$\frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \frac{f(x,(y+h)) - f(x,y)}{h}.$$

These definitions are seldom used and they generalize easily to higher dimensions.

Note: The following notations are all used for a partial derivative:

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x} = f_x(x,y) = f_x = \partial_x f(x,y) = \partial_x f.$$

There is probably more notation than this.

In practice partial derivatives are computed in the following manner. If we have $f(x,y)$, then its partial derivative with respect to x is computed by treating y as constant, and computing the regular derivative with respect to x. The process is the same for y and easily generalizes to higher dimensions.

Example — **Partial Derivatives**

Find the partial derivatives of the following functions with respect to both x and y:

$$f_1(x,y) = xy^2 + yx^2$$
$$f_2(x,y) = \ln(x^2 + y^2).$$

Then compute the following partial derivatives with respect to z for the following functions:

$$f_3(x,y,z) = e^{x^2+y^2} + xyz$$
$$f_4(x,y,z) = e^{z(x^2+y^2)} + \frac{z}{xy}$$
$$f_5(x,y,z) = ze^{z(x^2+y^2)}.$$

Solution: For all cases we simply apply the rules described above. We start with f_1:

$$\frac{\partial f_1}{\partial x} = y^2 + 2xy$$
$$\frac{\partial f_1}{\partial y} = 2xy + x^2.$$

Now onto f_2:

$$\frac{\partial f_2}{\partial x} = \frac{2x}{x^2+y^2}$$
$$\frac{\partial f_2}{\partial y} = \frac{2y}{x^2+y^2}.$$

Now onto f_3, f_4 and f_5 for the second part of the question:

$$\frac{\partial f_3}{\partial z} = xy$$
$$\frac{\partial f_4}{\partial z} = (x^2+y^2)e^{z(x^2+y^2)} + \frac{1}{xy}$$
$$\frac{\partial f_5}{\partial z} = e^{z(x^2+y^2)} + z(x^2+y^2)e^{z(x^2+y^2)}$$

□

3.1 Interpreting Partial Derivatives

Imagine we have a surface described by $z = f(x,y)$. What exactly does the partial derivatives of this function tell us? In the process of computing the partial derivative of one variable, we hold the other constant. So let's set $y = y_0$ so we have $f(x,y) = f(x,y_0)$. This is now a function of one variable, x, and it it describes the curve of intersection between the plane $y = y_0$ and the surface $f(x,y)$. What the partial derivative with respect to x, f_x in this case, will tell us what the slope of the tangent line to this curve of intersection is for all (x, y_0).

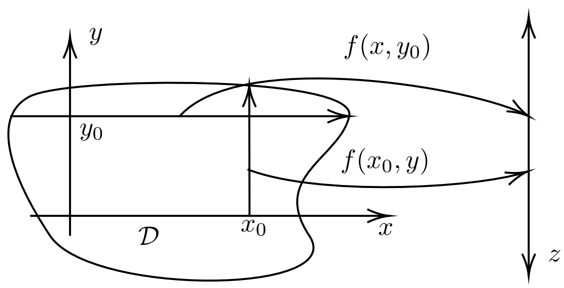


Illustration of how a function is evaluated along a line.

For animation of this process, I turn to 3Blue1Brown who has beautifully animated what this process looks like. His animation of this process occurs in the context of investigating the temperature distribution over a rod over time governed by a function $T(x,t)$. See `https://www.youtube.com/watch?v=ly4S0oi3Yz8` and watch from 4:58 to 6:53.

3.2 Tangent Planes and Normal//Tangent Lines

Suppose $z = f(x, y)$ defines a surface. The equation of tangent plane to that surface at the point $P(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Suppose we want to find the line normal to $f(x, y)$ at a point (a, b). To do this we note that $f(x, b)$ and $f(a, y)$ define two curves which pass through (a, b). The normal line through $(a, b, f(a, b))$ has direction given by the tangent plane through $(a, b, f(a, b))$. This tangent plane has normal vector

$$\mathbf{N} = \frac{\partial f}{\partial x}(a, b)\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \mp \mathbf{k}.$$

It is standard to take the $\mathbf{k}$ component to be negative, but it is arbitrary. Thus the normal line has equation

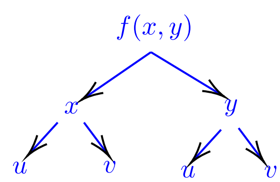
$$\mathbf{N}(t) = \langle a, b, f(a, b)\rangle + t\left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1 \right\rangle,\ \forall\, t \in \mathbb{R}.$$

3.3 The Chain Rule Revisited

Example — Chain Rule

Let f be a differentiable function of x and y and let $g(u, v) = f(2u - v, v^2 - 4u)$. Suppose that $f_x(0, 0) = 4$ and $f_y(0, 0) = 8$. Then what is $g_v(1, 2)$?

Solution: Note that the problem statement gives us that $x(u, v) = 2u - v$ and $y(u, v) = v^2 - 4u$. Thus we have the following chain rule tree:



Then

$$g_v = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Further note that $x(u, v) = 0$ when $u = 1, v = 2$ and $y(u, v)$ when $u = 1, v = 2$, thus the point corresponding to $(u, v) = (1, 2)$ is $(x, y) = (0, 0)$. Therefore

$$g_v(1, 2) = f_x(0, 0)(-1) + f_y(0, 0)(4) = -4 + 8(4) = \boxed{28}.$$

$\square$

3.4 Implicit Differentiation

Implicit differentiation of multivariate functions works entirely the same as implicit differentiation of single variable functions.

Example | **Implicit Differentiation**

Consider the following relationship between $x, y, z(x, y)$:

$$xyz = e^{zx} + y.$$

Find $\partial_x z$.

Solution: We differentiate both sides of the equation with respect to x:

$$\frac{\partial}{\partial x}(xyz) = y\frac{\partial}{\partial x}(xz) = \frac{\partial}{\partial x}(e^{zx} + y)$$

Since x and $z(x, y)$ are both changing with respect to x, we must use the product rule on the left hand side:

$$y\frac{\partial}{\partial x}(xz) = yz + xy\frac{\partial z}{\partial x}.$$

On the ride hand side we must use the chain rule, and within that chain rule we must use the product rule:

$$\frac{\partial}{\partial x}(e^{zx} + y) = ze^{zx}\frac{\partial}{\partial x}(zx) = ze^{zx}\left(x\frac{\partial z}{\partial x} + z\right).$$

Thus we have obtained the following expression:

$$yz + xy\frac{\partial z}{\partial x} = ze^{zx}\left(x\frac{\partial z}{\partial x} + z\right).$$

All that remains is solving for $\partial_x z$:

$$\frac{\partial z}{\partial x} = \frac{z(ze^{zx} - y)}{x(y - ze^{zx})} = -\frac{z}{x}.$$

□

While implicit differentiation is the same for multivariate functions, it can be the case that the computation becomes quite tedious. There exists, under certain conditions, a formula that allows one to entirely bypass the implicit differentiation process.

Theorem — Implicit Differentiation

Let $F(x,y,z)$ be a function which describes the implicit relationship between $x, y, z(x,y)$ in the form

$$F(x,y,z)=k$$

where k is some real constant. Then $\partial_x z$ and $\partial_y z$ can be found via the following formulae

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \text{ and } \frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Let $F(x,y,z,u)$ be a function which describes the implicit relationship between $x, y, z, u(x,y,z)$ in the form

$$F(x,y,z,u)=k$$

where k is some real constant. Then $\partial_x u$, $\partial_y u$ and $\partial_z u$ can be found via the following formulae

$$\frac{\partial u}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial u}},\quad \frac{\partial u}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial u}},\quad \frac{\partial u}{\partial z}=-\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial u}}$$

Proof: For the derivative with respect to x, we differentiate both sides of $F(x,y,z(x,y))=k$ with respect to x:

$$\frac{\partial}{\partial x}F(x,y,z(x,y))=\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial F}{\partial x}=0 \Longrightarrow \frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

For the derivative with respect to y, we differentiate both sides of $F(x,y,z(x,y))=k$ with respect to y:

$$\frac{\partial}{\partial y}F(x,y,z(x,y))=\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial F}{\partial y}=0 \Longrightarrow \frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

The proof of the results for the case of $F(x,y,z,u(x,y,z))$ follows from the exact same method as above. □

Example — Implicit Differentiation

Verify that for the relationship $xyz=e^{zx}+y$, the change of z with respect to x is

$$\frac{\partial z}{\partial x}=-\frac{z}{x}$$

by using the formulas derived above.

Solution: We define $F(x,y,z)=xyz-e^{zx}-y=0$. Next we use the formula for $\partial_x z$ derived above:

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{yz-ze^{zx}}{xy-xe^{zx}}=-\frac{z(y-e^{zx})}{x(y-e^{zx})}=-\frac{z}{x}.$$

This example reveals the utility of this method since F is explicitly related to the variables (x, y, z) therefore making the computation simpler.

$\square$

4 The Gradient

Cartesian Gradient in 2D/3D — Multivariate Calculus

Let f be a function of n Cartesian variables: $f : \mathbb{R}^n \to \mathbb{R}$ that has continuous partial derivatives. Then the **Gradient** of f, denoted ∇f is a vector valued map in the form

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathbf{e}_i,$$

where each $\mathbf{e}_i$ is a coordinate basis vector. For the case where $n = 3$, then $\mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$, that is

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Note: The **Gradient operator**, ∇, is given as

$$\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathbf{e}_i.$$

This operator acts on functions in various ways to produce many physical results, one of which is the gradient of f. The gradient operator will make a return later.

Example — Cartesian Gradient

Compute the gradient of the function

$$f(x, y, z) = e^z \sin(x^2 + y^2).$$

Evaluate the gradient at the point $P(1, 1, 1)$.

Solution: We compute all the partial derivatives of f:

$$\begin{aligned} f_x(x, y, z) &= 2xe^z \sin(x^2 + y^2) \\ f_y(x, y, z) &= 2ye^z \sin(x^2 + y^2) \\ f_z(x, y, z) &= e^z \sin(x^2 + y^2). \end{aligned}$$

Now we write these as the components of a vector to find the gradient:

$$\nabla f(x, y, z) = \left\langle 2xe^z \sin(x^2 + y^2), 2ye^z \sin(x^2 + y^2), e^z \sin(x^2 + y^2) \right\rangle$$

Evaluating the gradient returns the following vector:

$$\boxed{\nabla f(1, 1, 1) = \left\langle 2e \sin(2), 2e \sin(2), e \sin(2) \right\rangle}$$

$\square$

Example — Gradient

The tangent plane to the surface given by $4xyz = 1$ at $P(a, b, c)$ has equation $x + 2y + 2z = 3$. What is $a + b + c$?

Solution: Firstly, note that the tangent plane must intersect the surface at P, so we know that $4abc = 1$. We can interpret $4xyz = 1$ as a level surface of the function $f(x, y, z) = 4xyz$, then we know that

$$\nabla f(a, b, c) \parallel \mathbf{N}$$

where $\mathbf{N} = \langle 1, 2, 2 \rangle$. Thus we have the following system of equations:

$$\begin{aligned} 4bc &= 1 \\ 4ac &= 2 \\ 4ab &= 2 \\ 4abc &= 1 \end{aligned}$$

We divide the fourth equation by the third to obtain $c = 1/2$, plugging this into the second equation gives $a = 1$. Plugging c into the first equation also gives $b = 1/2$. Therefore $\boxed{a + b + c = 2}$ and it is easily verified that (a, b, c) satisfies all the above equations.

□

Theorem — Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function which has continuous partial derivatives. Then ∇f returns a vector which points normal to the level hyper-surfaces of f and the magnitude of this vector gives the maximum rate of change of f in the direction normal to the direction of the level curves of f.

Proof: The total derivative / differential of f is given by

$$\begin{aligned} df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i &= \underbrace{\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathbf{e}_i \right)}_{\nabla f} \cdot \left(\sum_{i=1}^{n} dx_i\, \mathbf{e}_i \right) \\ &= \nabla f \cdot \left(\sum_{i=1}^{n} dx_i\, \mathbf{e}_i \right). \end{aligned}$$

Now we let

$$d\mathbf{r} = \sum_{i=1}^{n} dx_i\, \mathbf{e}_i,$$

then we obtain the expression

$$df = \nabla f \cdot d\mathbf{r} = |\nabla f| \cos(\theta)\, dr \implies \frac{df}{dr} = |\nabla f| \cos(\theta).$$

where θ is the angle between ∇f and the vector $d\mathbf{r}$. This rate of change is maximized along the direction of $d\mathbf{r}$, i.e when $\theta = 0$. Meaning

$$\left.\frac{df}{dr}\right|_{\max} = |\nabla f|.$$

Now all that remains is to show that ∇f points normal to the level hyper-surfaces of f. We will show this result for $n = 3$ as the result easily generalizes. Let $z = f(x, y)$ define a surface, then $t = f(x, y)$ where $t \in \mathbb{R}$

describes the intersection between the plane $z = t$ and $z = f(x, y)$. Let $\mathcal{C}$ describe the curve of intersection between the plane and the surface, and let $\mathcal{C}'$ be the projection of this curve onto the xy plane:

$$\mathcal{C}' : \ \mathbf{r}(\psi) = x(\psi)\mathbf{i} + y(\psi)\mathbf{j}$$

for some range of ψ which traces out $\mathcal{C}'$. The differential of f along $\mathcal{C}'$ is

$$\frac{df}{d\psi} = \frac{\partial f}{\partial x}\frac{dx}{d\psi} + \frac{\partial f}{\partial y}\frac{dy}{d\psi} = \nabla f \cdot \mathbf{r}'(\psi) = 0.$$

This result must be zero since, by definition, f does not change height on a level curve. Thus $\nabla f \perp \mathbf{r}'(\psi)$. Since $\mathbf{r}'(\psi)$ is in the direction of the level curve, it must be the case that ∇f points perpendicular to the level curve.

$\square$

Example — **Extreme Values**

Consider the function $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z}$ and the point $P(4, 2, 1)$. Find the maximum and minimum rates of change of f at point P and the directions where they occur. Find all directions where the rate of change is 0 with respect to point $P(4, 2, 1)$.

Solution: The maximum rate of change will be given by $|\nabla f(P)|$ and the minimum will be given by $-|\nabla f(P)|$. The gradient of f is given by

$$\nabla f(x, y, z) = \left\langle \frac{1}{y}, -\frac{x}{y^2} + \frac{1}{z}, -\frac{y}{z^2} \right\rangle \Longrightarrow \nabla f(4, 2, 1) = \left\langle \frac{1}{2}, 0, -2 \right\rangle.$$

Thus:

$$\text{Maximum rate of change: } = |\nabla f(P)| = \sqrt{\frac{1}{4} + 4} = \boxed{\frac{\sqrt{17}}{2}}$$

$$\text{Minimum rate of change: } = -|\nabla f(P)| = -\sqrt{\frac{1}{4} + 4} = \boxed{-\frac{\sqrt{17}}{2}}$$

The direction of the maximum rate of change is in the direction of $\nabla(P)$ and the direction of the minimum rate of change is in the direction of $-\nabla(P)$. The directions where the rate of change is zero is the directions which are along the curves given by $f(x, y, z) = k$. Thus we are looking for $\nabla f(4, 2, 1) \cdot \hat{u} = 0$ where $\hat{u}$ specifies any possible direction. Let

$$\hat{u} = \frac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}},$$

then we have

$$0 = \nabla f(P) \cdot \hat{u} = \frac{a}{2\sqrt{a^2 + b^2 + c^2}} - \frac{2c}{\sqrt{a^2 + b^2 + c^2}} \Longrightarrow a = 4c.$$

Thus b can be any value and the direction is

$$\boxed{\hat{u} = \frac{\langle 4c, b, c \rangle}{\sqrt{15c^2 + b^2}}}$$

$\square$

4.1 The Total Derivative of a Function and Revisting Tangent Planes

The Total Derivative//Differential in 2D/3D — Multivariate Calculus

Let $d\mathbf{r} = \langle dx, dy \rangle$, then the total derivative of $z = f(x, y)$ is

$$\frac{df}{d\mathbf{r}} = \nabla f.$$

Sometimes this is expressed as $df = \nabla f \cdot d\mathbf{r}$:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

This result is obviously easily generalized. If $d\mathbf{r} = \langle dx, dy, dz \rangle$ and $z = f(x, y, z)$ then

$$\frac{df}{d\mathbf{r}} = \nabla f$$

and

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

The Tangent Plane — Multivariate Calculus

Let $\mathcal{S}$ be a surface defined by $z = f(x, y)$, then the tangent plane to the surface at point $P(a, b)$ is

$$z - f(a, b) = \nabla f(a, b) \cdot \langle x - a, y - b \rangle.$$

This makes sense since we know that the gradient will define a normal vector in the projection of $\mathcal{S}$, and then adding $f(a, b)$ to the RHS will give height to this vector. Hence the combination of gradient and the function at a point will define a plane which is tangent to $\mathcal{S}$ at P.

4.2 The Gradient as a Map: The Path of Steepest Ascent or Descent

Since the gradient points normal to the level curves and the magnitude of the gradient at a point gives the maximum rate of change in that normal direction, it should be the case that following the gradient at each point will trace out the path of steepest ascent or descent. We will deal with this subject for functions of two variables for visualization purposes, but the result is easily generalized. The goal of this section is to illustrate that the gradient can be used to define a curve in the domain of a function whereby evaluation of the function along the curve will give the path of steepest ascent or descent along the surface.

Theorem — **Gradient**

Let $z = f(x, y)$ describe a surface over some domain where $(x, y) \in \mathcal{D} \subset \mathbb{R}^2$. Let the projection of the path of steepest ascent or descent of the surface on $\mathcal{D}$ be denoted $\mathcal{C}$. This path is specified in in the following manner:

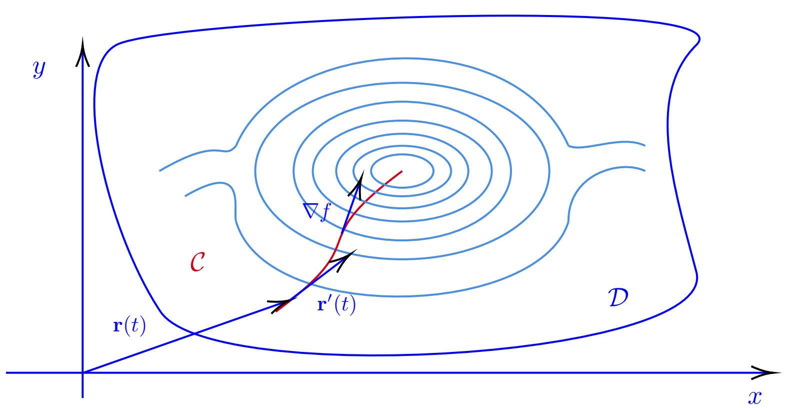
$$\mathcal{C}: \left\{ \xi = g(x, y) \;\middle|\; \frac{dy}{dx} = \frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}} \right\}$$

where g defines an implicit relationship between (x, y) and ξ is a constant. Note that it can be the case that a relationship in the form $y = y(x)$ is recovered by the ODE.

Proof: Let $\mathcal{C}$ be the projection of the path of steepest ascent/descent, i.e the path traced out in the plane by following the direction of the gradient at every point. We paramaterize $\mathcal{C}$ in the following manner

$$\mathcal{C}: \left\{ x = x(t),\ y = y(t),\ \alpha \leq t \leq \beta \;\middle|\; (x(t), y(t)) \in \mathcal{D} \right\}.$$

Let $\mathbf{r}(t)$ be the vector function which traces out $\mathcal{C}$. Then $\mathbf{r}'(t)$ is always tangent to $\mathcal{C}$, thus making it parallel to ∇f. Note that this is because we previously established that $\mathcal{C}$, by definition, is the path traced out by travelling in the normal direction to the level curves.



The thin blue curves are the level curves of $z = f(x, y)$. The red curve is the path of steepest ascent/descent. The goal of this image is to illustrate that the gradient vector and the derivative of the vector function are always parallel or anti-parallel depending on whether the path is ascent or descent.

Therefore $\exists\, \gamma \in \mathbb{R}$ such that

$$\nabla f = \gamma \mathbf{r}'(t).$$

From this we obtain the following system of equations:

$$\begin{cases} \dfrac{\partial f}{\partial x} = \gamma \dfrac{dx}{dt} \\ \dfrac{\partial f}{\partial y} = \gamma \dfrac{dy}{dt}. \end{cases}$$

Now division of the second equation by the first yields

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}.$$

If this equation has a solution in terms of elementary functions (for our purposes), then it will be in the form $\xi = g(x, y)$ where $\xi \in \mathbb{R}$. Note that technically the relationship between the gradient and the vector function is actually $\nabla f = \pm\gamma \mathbf{r}(t)$, the distinction in sign arises depending on whether the path is ascending or descending. This technicality was excluded due to the division of γ which occurs regardless.

$\square$

Example	Path of Steepest Ascent/Descent

Let $\mathcal{S}$ be the surface defined by

$$f(x, y) = e^{-(x^2+2y^2)}.$$

Find the projection of the path of steepest descent starting at the point $P(1, 1, e^{-3})$. Then write the expression which would give the path of steepest descent on the surface starting at P.

Solution: Let Γ be the path of steepest descent we wish to find. We know that the projection of the curve of steepest ascent, $\mathcal{C}$, will be the solution to the ODE

$$\frac{dy}{dx} = \frac{-4ye^{-(x^2+2y^2}}{-2xe^{-(x^2+2y^2)}} = 2\frac{y}{x}.$$

Separation and integration yields the result

$$\ln|y| = 2\ln|x| + C \implies y = \pm e^C x^2.$$

The initial data is the point $P'(1, 1)$, meaning $y(1) = 1$ and this gives $C = 0$. Note that in the definition of f, it must be the case that increasing x is the direction of descent. Thus

$$\mathcal{C} : \ \left\{y(t) = t^2, \ x(t) = t, \ t \geq 1\right\}.$$

Now that we have parameterized the curve, it is easy enough to find the actual path, Γ, which lies on $\mathcal{S}$:

$$\Gamma =: \ \left\{x(t) = t, \ y(t) = t^2, \ z(t) = f(t, t^2), \ t \geq 1\right\}.$$

$\square$

4.3 Maximizing/Minimizing Functions using the Gradient and Derivative Tests

Critical/Stationary Points | Multivariate Calculus

Let $\mathcal{D} \subset \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$. The critical points of f occur when each component of the gradient is zero, that is for each k :

$$\nabla f \cdot \mathbf{e}_k = 0.$$

The critical points could potentially be saddle points or local maxima/minima.

Example | Critical/Stationary Points

Find the extreme values of the function

$$g(x, y) = \frac{x}{4 + x^2 + y^2}$$

on $\mathcal{D} \subset \mathbb{R}^2$ where

$$\mathcal{D} := \left\{ (x, y) \in \mathcal{D} \,\middle|\, x^2 + y^2 \leq 1 \right\}.$$

Solution: We know that any extreme values of $g(x, y)$, when on a restricted domain, will occur either at the critical points of g or on the boundary of D, which is

$$\partial\mathcal{D} := \left\{ (x, y) \,\middle|\, x^2 + y^2 = 1 \right\}.$$

The critical points of g will occur at any point P where the components of the gradient will be simultaneously zero, that is:

$$\nabla f(P) \cdot \mathbf{i} = \nabla f(P) \cdot \mathbf{j} = 0.$$

Thus we compute $g_x(x, y)$ and $g_y(x, y)$:

$$g_x(x, y) = \frac{4 + y^2 - x^2}{(4 + x^2 + y^2)^2}$$
$$g_y(x, y) = \frac{2yx}{4 + x^2 + y^2}.$$

We can see from inspection that $g_y(x, y) = 0$ when $y = 0$. Plugging this into $g_x(x, y)$ will give $g_x(0, y) = 4 + y^2/(4 + y^2)^2$. Setting $g_x(0, y) = 0$ will give $y = \pm 2$. We further notice that $g_y(x, y) = 0$ when $x = 0$, the same process as previous will yield $x = \pm 2$. Thus we have critical points:

$$(0, 2), (0, -2), (2, 0), (-2, 0).$$

Note that none of these points are on $\mathcal{D}$, therefore these points are discarded and all that remains is to find the maximum value of g on the boundary of $\mathcal{D}$. The boundary of $\mathcal{D}$ is given by $y^2 = 1 - x^2$, substitution of this into $g(x, y)$ will give us an expression for g evaluated on $\partial\mathcal{D}$:

$$g(\partial\mathcal{D}) = g(x, 1 - x^2) = f(x) = \frac{x}{4 + x^2 + (1 - x^2)} = \frac{x}{5}.$$

Note that $g'(\partial\mathcal{D}) = f'(x) = 1/5$, which means that g is *always increasing on the boundary.* Thus if we start at the left most point on the boundary, $x = -1$, and move to the right most point, $x = 1$, then we will have started at the minimum value and have walked to the maximum value. Hence,

$$g_{\text{max}} = f(1) = \boxed{\frac{1}{5}}$$

$$g_{\text{min}} = f(-1) = \boxed{-\frac{1}{5}}$$

□

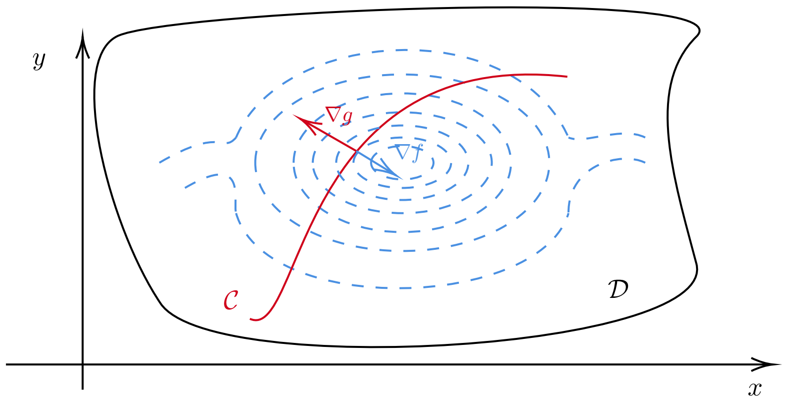
4.3.1 Second Derivative Test

If it were the case in the previous example that the critical points were in fact in the domain of the problem, then we would need a way to know if the points were maxima, minima or saddle points. We therefore need a more powerful tool to discriminate between critical points, and this motivates our investigation into the second derivative test.

5 Maximizing/Minimizing Functions using Lagrange Multipliers

5.1 Maximizing/Minimizing Functions with One Constraint

The method of of Lagrange Multipliers is used to find the extreme values of a function with imposed constraints. We start with the simple case of maximizing a function $f(x,y)$ with some constraint $g(x,y) = k$ where k is a real constant. We require that f and g have continuous partial derivatives, and we assume that along the constraint $f(x,y)$ has an extreme value, and we assume that $\mathcal{C}$ is the projection of the curve which passes through this extreme value. Further consider the projection $\mathcal{C}$ in coordination with the level curves of f as imaged below:



It must be the case that where $\mathcal{C}$ lies tangent to a level curve of f, there is an extreme value with respect to all other values $f(\mathcal{C})$. Where this occurs we have that the gradient of f is parallel to the gradient of g:

$$\nabla f = \lambda \nabla g, \ \lambda \in \mathbb{R}.$$

Thus we can define

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

and the parallel equation is equivalent to solving

$$\nabla_{x,y,\lambda}\mathcal{L}(x,y,\lambda) = 0 \Longrightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k. \end{cases}$$

The notation $\nabla_{x,y,\lambda}$ is used to emphasize that the gradient operator is being applied to the spacial variables and λ. Note that $\nabla_{x,y,\lambda}\mathcal{L} = 0$ implies that $g(x,y) = 0$, but this would simply correspond to moving the constant over, so we can safely ignore this technicality.

Example	Lagrange Multipliers//One Constraint

Maximize the function $f(x,y) = x + y$ subject to the constraint

$$g(x,y) = x^2 + y^2 - 1 = 0.$$

Solution: This problem has an easy conceptual answer. We know that the level curves of $f(x,y)$ are $k = x + y$, i.e lines. These lines will be tangent to the unit circle when $k = \pm\sqrt{2}$, this will give the critical points

$$P_1\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),\ P_2\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

Thus the max will be $f(P_1)$ and the min will be $f(P_2)$. Now we will use Lagrange multipliers to confirm this result. The method of Lagrange multipliers requires us to solve the following system

$$\nabla_{x,y}\mathcal{L}(x,y,\lambda) = \nabla f(x,y) - \lambda\nabla g(x,y) = 0 \Longrightarrow \begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ x^2 + y^2 - 1 = 0. \end{cases}$$

By direct substitution we obtain

$$2\lambda x = 2\lambda y \Longrightarrow x = y.$$

Substitution of this result into our constraint gives

$$2x^2 = 1 \Longrightarrow x = \pm\frac{\sqrt{2}}{2}$$

$$2y^2 = 1 \Longrightarrow y = \pm\frac{\sqrt{2}}{2}.$$

Therefore the critical points of the problem are

$$\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$$

and as before the maximum of the problem will be $f(\sqrt{2}/2, \sqrt{2}/2)$ and the minimum will be $f(-\sqrt{2}/2, -\sqrt{2}/2)$. Thus $f_{\max} = \sqrt{2}$ and $f_{\min} = -\sqrt{2}$.

$\square$

5.2 Maximizing/Minimizing Functions with Two Constraints

To generalize the result presented in the section involving finding the extreme values of functions on a single constraint, we have to realize that the tangent case may no longer be possible. It is entirely possible that two constraints could intersect at a point where possibly both functions are not tangent to the level curves of f. Thus the general method involves finding a point where the gradient of the function is a *linear combination of the gradients of the constraining functions.* Note that the tangency case presented in the previous section is a special case of a linear combination as $\nabla f = \lambda \nabla g$ is the expression of ∇f as a linear combination of ∇g.

Example — Lagrange Multipliers with Two Constraints

Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y, z) = yz + xy$$

subject to the constraints $xy = 1$ and $y^2 + z^2 = 1$.

6 Releasing a Particle in a Vector Field

Often it is the case that we want to see where a particle goes when it is released in a field. In MATH 209 much attention is given to line integrals around fields, which is equivalent to asking: How much work is done when moving a particle along some curve *against the field lines*. This section will be concerned with the following question: Where does a particle go when released in a vector field? Let us consider a vector field $\mathbf{F} = \langle Q(x,y), P(x,y)\rangle$. If a particle is released in a vector field with a force described by $\mathbf{F}$, then intuition tells us that $\mathbf{F}$ must be parallel to the curve which the particle follows. We assume that

$$\mathcal{C}: \ \mathbf{r}(t) = \langle x(t), y(t)\rangle, \ \alpha \leq t \leq \beta$$

is the parameterization of the curve which the particle follows. Then it must be the case that $\mathbf{F} \parallel \mathbf{r}'(t)$, i.e

$$\begin{cases} \dfrac{dx}{dt} = Q(x(t), y(t)) \\ \dfrac{dy}{dt} = P(x(t), y(t)) \end{cases}$$

Therefore the curve which the particle follows can be described as $y = y(x)$ which satisfies

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{P(x,y)}{Q(x,y)}$$

Hence, we have arrived at the following thereom.

Theorem — **Field Lines**

Let P be a particle released at point Q which interacts with vector field $\mathbf{F} = \langle Q(x,y), P(x,y)\rangle$. The path that the particle follows when acted upon by $\mathbf{F}$ is the curve $\mathcal{C}$ defined as:

$$C: \left\{ \xi = g(x,y) \,\middle|\, \frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} \right\}$$

where ξ is a constant which describes a possibly implicit relationship between x and y.

Remark: Notice that the path of steepest descent is a special case of this! If $\mathbf{F} = \nabla f$, then it is obvious that $P = \partial_y f$ and $Q = \partial_x f$, hence $\xi = g(x,y)$ describes the path of steepest descent. This observation gives some powerful results, mainly it states that if a particle is exerted on by a conservative vector field *then the path the particle takes must be the* ***fastest*** *path between highest and lowest potential*. In electromagnetics the relationship between an electric field $\mathbf{E}$ and the electric potential ϕ is given by $\mathbf{E} = -\nabla\phi$. This means that a particle released under an electric field must follow the shortest path between highest and lowest potential, i.e a path of steepest descent.