Eigenfunction of the Harmonic Oscillator and its Eigenvalues

Consider the following LTIC system which describes the mass of a block which is oscillating via a spring:

$$y''(t) + \frac{b}{m}y'(t) + \frac{k}{m}y(t) = x(t) \tag{1}$$

The eigenfunction of any LTIC system is the exponential function $y(t) = e^{\lambda t}$. Where λ is a constant. To see this we simply substitute this solution into (1) to see:

$$\left(\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}\right)e^{\lambda t} = x(t) \tag{2}$$

I.e the output of the system when given the input $y(t) = e^{st}$ is a rescaling of the function itself! We know that if we let $x(t) = 0$ to investigate the natural behaviour of the system (the zero input response) we require the following restriction on λ to satisfy (2):

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0 \tag{3}$$

What exactly is equation (3)? From an introductory ODE class one knows that equation (3) gives all possible solutions to the equation (1) when $x(t) = 0$. However, this equation is much more fundamental to the system at hand, and those who have investigated ODEs under Laplace transforms may find the following results quite familiar.

To further investigate equation (3), let us try to find it under different circumstances. Firstly, let $x''(t) = v'(t)$ and $x'(t) = v(t)$, then we obtain the following system of equations:

$$\begin{cases} v'(t) = -\frac{b}{m}v(t) - \frac{k}{m}x(t) \\ x'(t) = v(t) \end{cases}$$

We may write this system as the following matrix equation:

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \tag{4}$$

This equation effectively describes the **state space** or **phase space** of the system. This is because given any initial $x(t_0), v(t_0)$ we may find $x'(t_0), v'(t_0)$ and the plot of all such points shows how the system evolves over time. This is easily done via recursion, that is we may write equation (4) as

$$\begin{bmatrix} x'(t_{n+1}) \\ v'(t_{n+1}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x(t_n) + x'(t_n) \\ v(t_n) + v'(t_n) \end{bmatrix} \tag{5}$$

where the time steps for each n are very small. Consider a system with no damping, i.e $b = 0$. To make numbers simple we further take $k = m = 1$. The following is a plot of the phase space of this system via (5).

It may seem as if we are off track with regards to rediscovering equation (3), however we are still on the case. Consider the position vector to some point in the phase space $(x(t_k), v(t_k))$, and let L be the line which is the span of this position vector. The question we wish to ask is: "Which sets of initial data, when acted on by the system, remain on the line L defined by this initial data?" Or phrased differently: "What are the eigenvalues of the matrix equation (4)?" To find the eigenvalues all we need is to set the determinant $\det(A - \lambda Id_n)$ for the matrix in (4) to zero :

$$P(\lambda) = \det = \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} - \lambda \end{bmatrix} = \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0 \tag{6}$$

Equations (4) and (6) are identical. So we have determined that when substitution the *eigenfunction* into the system we obtain the equation for the *eigenvalues* for the matrix which describes the system. Further we know can state the following:

The linear combination of eigenfunctions for the harmonic oscillator ODE, when evaluated at the eigenvalues for the matrix which describes the harmonic oscillator system, is the general solution to the ODE at hand.

We know this to be true since two solutions λ_1, λ_2 to equation (3) give rise to the general solution

$$x(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_1 t}$$

for the system described by equation (1) with $y(t) = 0$.

There is more to be uncovered by considering the Laplace transform of equation (1) where $x(t) = 0$ and initial conditions $y(t_0) = \xi,\ y'(t_0) = \eta$:

$$\mathcal{L}\left\{y''(t) + \frac{b}{m}y'(t) + \frac{k}{m}y(t)\right\}(s) = \mathcal{L}\left\{y(t)\right\}(s)\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right) = \left(s + \frac{b}{m}\right)\xi + \eta \tag{7}$$

We may rewrite equation (7) as

$$\mathcal{L}\left\{y(t)\right\}(s) = \frac{\left(s + \frac{b}{m}\right)\xi + \eta}{s^2 + \frac{b}{m}s + \frac{k}{m}} \tag{8}$$

Those with signal processing experience know that (8) describes the frequency content of (1). The so-called resonant frequencies of the system would occur when the denominator of (8) is zero, however the denominator of (8) being zero would simply correspond to the frequencies of the system being eigenvalues of the matrix which describes the system!