University of Alberta

ECE 370: Engineering Electromagnetics I

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Contents

1 A Review of Prerequisite Mathematics and Physics

The most relevant mathematics to a first course in classical electromagnetics is vector calculus. Thus the first part of ECE 370 consists of a fairly comprehensive review and expansion of ideas discussed towards the end of MATH 209. One needs not only a strong ability to perform computations with vector calculus, but also a strong conceptual grasp of what the mathematics is saying to have a truly immersive understanding of introductory electromagnetics. In this note package we will often use shorthand notation for integrals in multiple dimensions, and the dimensionality of the integral will be primarily denoted by the space subscript and differential of the integral. For example an integral over V denotes a volume integral, i.e a triple integral and this integral will have a dV in the integrand.

1.1 Basic Vector Operations

1.2 Differentiating Vectors

1.3 The Gradient

The gradient is the result of a vector operation on a multi-dimensional function which returns a vector in the domain space of that function which points towards directions of maximal rates of change, either positive or negative. For example in two dimensions, the gradient is the vector returned from a function $f : \mathcal{D} \to \mathbb{R}$ where $(x, y) \in \mathcal{D} \subset \mathbb{R}^2$ which will always lay in $\mathcal{D}$ and at any point (x, y) it will point to some (x', y') which will greatly increase the value of f. In 2D Cartesian coordinates we denote the gradient as follows:

$$\text{grad}(f) := \nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_2 \tag{1.3.1}$$

Where ∇ is the **gradient operator**, i.e the vector operation which returns the vector discussed above:

$$\nabla_{x,y} := \frac{\partial}{\partial x}\mathbf{e}_1 + \frac{\partial}{\partial y}\mathbf{e}_2 \tag{1.3.2}$$

The defining property of the operator (1.3.2) is that when it acts on f it returns (1.3.1). When the gradient acts on itself it returns the **Laplacian**:

$$\nabla_{x,y}\left\{\nabla_{x,y}\right\} := \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.3.3}$$

The operator (1.3.3) will be discussed thoroughly in the coming sections.

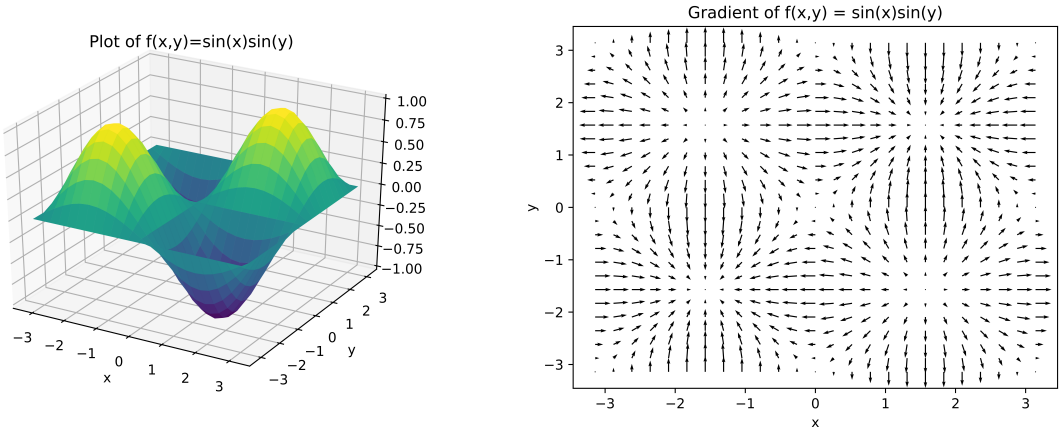
What does the gradient tell us? Let us begin with an example considering the simple function:

$$f(x, y) = \sin(x)\sin(y) \tag{1.3.4}$$

Application of (1.3.1) returns the following vector field (known as a scalar field since it is the result of the gradient operator):

$$\nabla_{x,y} f(x, y) = \cos(x)\sin(y)\mathbf{e}_1 + \sin(x)\cos(y)\mathbf{e}_2 \tag{1.3.5}$$

At this point, professors like to *tell* students that the gradient returns a set of vectors in the domain space of the function which acts as a map whereby every point on the map contains a vector which points to the direction of fastest increase or decrease – and further that paths which take one from any location on the surface to the highest region on the surface are specified by the gradient. I can *show* that this is true with the following python generated plots:

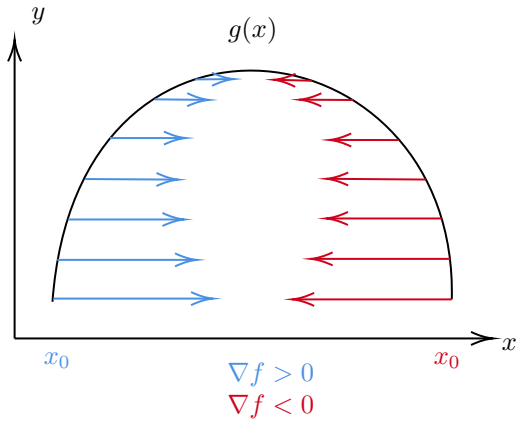


We can see that the vectors on the vector plot of (1.3.5) point towards the positive peaks of (1.3.4) and they point away from the lowest peaks of (1.3.4). From this example there seems to be some merit to the idea that the map from any $f(x, y)$ to its gradient seems to return a set of vectors which map out paths of extreme ascent/descent.

To start understanding the gradient, let's start by re-thinking the ordinary derivative for a function of one variable in a slightly different way. If we accept that (1.3.1) is the multivariate case of the ordinary derivative then the ideas presented above will easily reveal themselves. If we consider a simple smooth function of one variable, $f(x)$, and start at any point on the curve and follow the direction of the derivative, we will head towards the nearest maxima or minima. To see this let $f(x, y) = g(x)$, then applying (1.3.1) to $f(x, y)$ will give:

$$\nabla_{x,y} f(x, y) = \frac{d}{dx} g(x)\mathbf{e}_1 = g'(x)\mathbf{e}_1 \tag{1.3.6}$$

Let's proceed as follows: We start at some point x_0 and we evaluate (1.3.6) at that particular x_0, then if the derivative is positive we move in the positive x direction along the curve (in proportion to the magnitude of that derivative). Likewise if (1.3.6) is negative, we move in the negative x direction along the curve. Below is a (rough plot) of the vectors:



Note: The gradient (1.3.6) strictly returns a vector which is on the x axis, however they were placed on the curve for conceptual understanding and (obviously) to reduce clutter.

In a first course in calculus, we know that to find the maxima or minima of a function, we simply find where the derivative is zero since the slope of the tangent line to the curve will be zero there. The process we just

went through is *slightly* different, and is a more iterative process. We can see that if we start at the blue x_0 in the figure above, we have to go right until the vector returned is zero, and once that occurs we will have found the maxima of the function. Likewise if we start at the red x_0 we will have to go left until the vector returned is zero and we will have also found the maxima of the function. Note that in the one dimensional case there is only one direction of maximum increase from a given point, this property will be subtly used in the following extension to multiple dimensions.

To extend this idea to multiple dimensions we can use the idea of projections. Consider a general function $f(x, y)$ with continuous, smooth first order partial derivatives. We can consider two planes through a point on which to *project* the function $f(x, y)$. Let these planes be $x = x_0$ and $y = y_0$ respectively. On these planes we have:

$$h(x) = f(x, y_0) \tag{1.3.7}$$

and

$$g(y) = f(x_0, y) \tag{1.3.8}$$

Next we compute the gradients of these functions:

$$\nabla_{x,y} f(x, y_0) = h'(x)\mathbf{e}_1 \quad y = y_0 \tag{1.3.9}$$

$$\nabla_{x,y} f(x_0, y) = g'(y)\mathbf{e}_2 \quad x = x_0 \tag{1.3.10}$$

Now for (1.3.9) we ask ourselves: Where do we have to travel in the domain space to increase the value of the function $f(x, y)$ starting from the point (x_0, y_0)? The answer: Follow the direction where the derivative is positive. When we do this, we will be following the curve $h(x)$ starting at the point (x_0, y_0) *contained within the plane* $y = y_0$. We likewise ask the same question for (1.3.10) but instead we will be following the curve $g(y)$ starting at the point (x_0, y_0) *contained within the plane* $x = x_0$. If we simply answer the question at hand *independently* for the two functions we will end at the nearest maxima of the function $f(x, y)$ contained within the respective planes the function is projected on.

It seems like we're stuck since – strictly speaking – we are asking the question: What is the maximum value of f when restricted to the plane $x = x_0$ and $y = y_0$ respectively? In general the answer to this question will have us arrive at two different points – if this isn't obvious think about how the directions of travel demanded by (1.3.9) and (1.3.10) are in orthogonal directions. However, we are well equipped to investigate the following question: What is the vector rate of change of the function $f(x, y)$ if we first walk towards the direction of steepest increase in the x direction, *and then* walk towards the direction of steepest increase in the y direction? Let the small step in the x direction be denoted δx and let the small step in the y direction be denoted δy, further let these steps be small enough that $x_0 + \delta x_0 \approx x_0$ (note that these steps will be in proportion to the relative rates of change in each direction), then we have:

$$h(x_0)\Big|_{y=y_0} \implies h(x + \delta x)\Big|_{y=y_0} \tag{1.3.11}$$

and then in the y direction (at the new point):

$$g(y_0)\Big|_{x=x_0} \implies g(y_0 + \delta y)\Big|_{x \approx x_0} \tag{1.3.12}$$

In the x direction, we know that the direction of change was strictly in the x direction, thus we can write the vector rate of change (in terms of $f(x, y)$) as follows:

$$\Delta_x = \frac{f(x_0 + \delta x, y_0) - f(x_0, y_0)}{\delta x}\mathbf{e}_1 \tag{1.3.13}$$

and likewise in the y direction:

$$\Delta_y = \frac{f(x_0, y_0 + \delta y) - f(x_0, y_0)}{\delta y}\mathbf{e}_2 \tag{1.3.14}$$

Consider again what we have done up until this point. Firstly, we walked up the surface $f(x,y)$ in the fastest manner along the x direction (in a small amount) and then in the y direction in the fastest manner (in a small amount). Then the direction of the net fastest change is the vector sum between them, and in the limit as $\delta x, \delta y$ goes to zero we obtain the direction to go from the point (x_0, y_0) to obtain the fastest change:

$$\lim_{\delta x \to 0, \delta y \to 0} (\Delta_x + \Delta_y) = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x, y_0) - f(x_0, y_0)}{\delta x} \mathbf{e}_1 + \lim_{\delta y \to 0} \frac{f(x_0, y_0 + \delta y) - f(x_0, y_0)}{\delta y} \mathbf{e}_2 \tag{1.3.15}$$

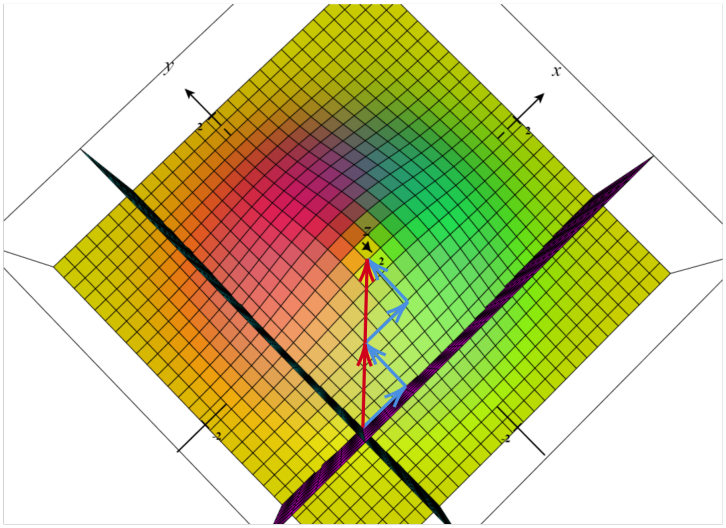
Of course this is just the gradient of $f(x,y)$ evaluated at the point (x_0, y_0) :

$$\lim_{\delta x \to 0, \delta y \to 0} (\Delta_x + \Delta_y) = \nabla_{x,y} f(x_0, y_0) = \frac{\partial f}{\partial x} \mathbf{e}_1 \Big|_{(x_0, y_0)} + \frac{\partial f}{\partial y} \mathbf{e}_2 \Big|_{(x_0, y_0)} \tag{1.3.16}$$

This process may have seemed tedious, however it does reveal the fastest increase property of the gradient. To animate the process slightly, let's consider the particular example of

$$f(x,y) = 2e^{-(x^2+y^2)} \tag{1.3.17}$$

The following diagram is a heuristic diagram of what is occurring:



In red: The gradient vector. In blue: The vector components which make up the gradient as described above. Note that the two planes are moving each step of the way, but this wasn't shown to reduce clutter.

The image speaks for itself – we are simultaneously increasing f most optimally in both the x and y direction. There are, of course, easier and more elegant mathematical proofs containing more rigor than the approach described above that will directly show that the gradient points in the direction of fastest increase on the surface, but in those methods the visualization is often lost.

The following question comes naturally at this point: How can we arrive at a mathematical expression for the path of steepest ascent on a surface from any given starting point? We can easily arrive at this answer via the following approach, let us denote the path of steepest ascent – by path of steepest ascent I specifically mean the path one takes *in the domain space of the function* to ascend the surface in the most timely fashion – Γ and let's parameterize it by the vector function $\mathbf{r}(t)$ given by

$$\mathbf{r}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2, \quad \alpha \leq t \leq \beta \tag{1.3.18}$$

where α, β are chosen such that the path is traversed once. Recall that the gradient is guiding us each step of the way, meaning that the gradient is always parallel (i.e is tangent) to this path. Then there exists a constant k such that

$$\mathbf{r}'(t) = k\nabla_{x,y} f \tag{1.3.19}$$

Therefore we have the following system of equations:

$$x'(t) = k\frac{\partial f}{\partial x} \tag{1.3.20}$$

$$y'(t) = k\frac{\partial f}{\partial y} \tag{1.3.21}$$

Taking the ratio of (1.3.21) and (1.3.20) and applying an identity from differential geometry (content towards the end of MATH 101) we obtain

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}} \tag{1.3.22}$$

Then the curve Γ is specified in its entirety by the following initial value problem:

$$\Gamma := \left\{ (x,y) \;\middle|\; \frac{dy}{dx} = \frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}}, \quad y(x_0) = y_0 \right\} \tag{1.3.23}$$

To find the path on the surface we simply evaluate $f(x,y)$ on the curve Γ as follows:

$$f(\Gamma) = f(x(t), y(t)) \tag{1.3.24}$$

Later in the course, we will undergo a nearly identical process to find the path a particle takes when released in a vector field.

The last property of the gradient to discuss is that the gradient of a multivariate function always points orthogonal to the level curves of that function everywhere on those curves. A level curve of a function of two variables is always specified by:

$$k = f(x,y), \quad k \in \mathbb{R} \tag{1.3.25}$$

The curve is the intersection of the plane $z = k$ and the function $z = f(x,y)$, thus on a level curve the value of the function $f(x,y)$ is not changing. There are various ways to prove that the level curves of a surface are always orthogonal to the gradient, but using the fact that the function is "level" on the curve is the easiest way to do it. Let $\mathbf{r}(t)$ parameterize the projection of the level curve on the xy plane of a function $f(x,y)$ and let it be in the same form as (1.3.18). The derivative of this vector function will always be tangent to the level curve specified by (1.3.25). On the curve we have:

$$f(x,y) = f(x(t), y(t)), \quad (x,y) \in \text{Level curves} \tag{1.3.26}$$

Using the multivariable chain rule we obtain:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla_{x,y} f \cdot \mathbf{r}'(t) \tag{1.3.27}$$

However, the function is level on the curve meaning that it does not change with respect to the parameterization variable t thus (1.3.27) is always zero regardless of the value of t. Therefore the gradient always points orthogonal to the projection of the surfaces level curves.

1.4 Line Integrals Over Vector Fields

Line integrals aren't utilized directly too much in this course, however some review will still be covered. Firstly let's start with the fundamental theorem of line integrals. Let Γ be a smooth contour parameterized by some vector function $\mathbf{r}(t)$ over the interval $t \in [\alpha, \beta]$ where α and β are chosen to traverse the curve once, then let f be a surface with a gradient ∇f. The fundamental theorem of line integrals states:

$$\int_{\Gamma} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(\beta)) - f(\mathbf{r}(\alpha)) \tag{1.4.1}$$

Now at this point usually a proof using the chain rule is utilized, but I think the following proof is much more elegant. We know that $d\mathbf{r}$ is directly proportional to the tangent vector of the curve Γ.

1.5 Flux

1.6 Divergence Theorem

The traditional definition of divergence theorem is the following: Let V be a volume enclosed by the oriented surface $\mathcal{S}$ and let $\mathbf{F}$ be a vector field with continuous first order partial derivatives, then

$$\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_{V} \operatorname{div} \mathbf{F}\, dV = \int_{V} \nabla \cdot \mathbf{F}\, dV \tag{1.6.1}$$

Now the definition presented in (1.6.1) and (1.6.2) is useful in terms of computation but this definition and often partial proofs of this definition leave much lacking in terms of conceptual information. In order to expand upon the knowledge that (1.6.1) provides we start with an investigation of divergence itself. Let $\mathbf{F} = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z) \rangle$, in a first course in vector calculus divergence is presented as the following in Cartesian coordinates:

$$\operatorname{div} \mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.6.2}$$

And then professors will usually *tell* students that somehow this gives information about *sources* or *sinks* of a vector field. However just by looking at (1.6.2) one would wonder how one arrives to such a powerful conceptual conclusion just by looking at that definition. The following definition of divergence provides a wealth of conceptual information: The divergence of a vector field at a point P is as follows:

$$\operatorname{div} \mathbf{F}\Big|_{P} = \lim_{V \to 0} \frac{1}{V} \oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \tag{1.6.3}$$

The reason why (1.6.3) is a more useful conceptual tool than (1.6.2) is simply because thinking about flux is, in general, quite easy and (1.6.3) is the true definition of divergence whereas (1.6.2) is simply an equivalent definition that provides an ease of calculation. It is important to re-iterate that (1.6.3) is a definition, *there is no derivation for it*, when we say divergence we really mean the point-wise application of (1.6.3) in order to investigate the property we call divergence for a given vector field.

So what does (1.6.3) tell us about what divergence is? Let's start by investigating the process that lead mathematicians, engineers and physicists to consider (1.6.3) as a useful definition in the first place. Consider a vector field $\mathbf{F}$ with continuous first order partial derivatives and a closed oriented surface $\mathcal{S}$ which encloses a volume V. The surface has a flux ϕ through it given by

$$\phi = \oint_{\mathcal{S}} \mathbf{F} \cdot\, d\mathbf{S} \tag{1.6.4}$$

Now imagine we split $\mathcal{S}$ into two pieces through the middle somewhere such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and we keep these two pieces very close to each other. At the region of the cut the surfaces are oppositely oriented so that their contribution to the flux adds to zero, and the remaining net flux through both surfaces is the same, meaning we can write:

$$\phi = \phi_1 + \phi_2 = \oint_{\mathcal{S}_1} \mathbf{F} \cdot \, d\mathbf{S}_1 + \oint_{\mathcal{S}_2} \mathbf{F} \cdot \, d\mathbf{S}_2 \tag{1.6.5}$$

We can do this N times since cutting the remaining surfaces apart and applying the same process above will not change the net flux through all the surfaces:

$$\phi = \sum_{n=1}^{N} \phi_n = \sum_{n=1}^{N} \left\{ \oint_{\mathcal{S}_n} \mathbf{F} \cdot \, d\mathbf{S}_n \right\} \tag{1.6.6}$$

As we partition the surface more and more the volume of each partition decreases, this indicates that the net flux will also decrease since the relevant area of the surface will also decrease. **Therefore flux is directly proportional to the volume of the enclosed surface.** In many areas of physics, engineering and math we give new names to quantities with proportionalities divided out. Like force $\mapsto$ acceleration when the proportionality of mass is divided out. When the number of partitions approaches infinity, the net flux remains ϕ and the volume approaches zero. Thus we call **divergence** the quantity that arises when the volume is shrunk to zero about a point P while measuring flux:

$$\operatorname{div} \mathbf{F}\Big|_P := \frac{1}{V} \lim_{N\to\infty} \sum_{n=1}^{N} \left\{ \oint_{\mathcal{S}_n} \mathbf{F} \cdot \, d\mathbf{S}_n \right\} = \lim_{V\to 0} \frac{1}{V} \oint_{\mathcal{S}} \mathbf{F} \cdot \, d\mathbf{S} \tag{1.6.7}$$

Thus divergence is really the *point wise measurement of the volumemetric density of flux*. This statement is powerful, it directly leads to the terminology of sources and sinks (which we will discuss more in a moment) and it even is a conceptual statement of divergence theorem. To show how (1.6.7) implies divergence theorem consider that we can write (1.6.6) in terms of infinitely many partitions and the flux will remain the same:

$$\oint_{\mathcal{S}} \mathbf{F} \cdot \, d\mathbf{S} = \lim_{N\to\infty} \sum_{n=1}^{N} \left\{ \oint_{\mathcal{S}_n} \mathbf{F} \cdot \, d\mathbf{S}_n \right\} \tag{1.6.8}$$

As discussed above, we know that each of those volume partitions have a flux which is proportional to their volume. Let the nth volume be denoted V_n, we can write the following equivalent statement of (1.6.8):

$$\oint_{\mathcal{S}} \mathbf{F} \cdot \, d\mathbf{S} = \lim_{N\to\infty} \sum_{n=1}^{N} \left\{ \frac{1}{V_n} \oint_{\mathcal{S}_n} \mathbf{F} \cdot \, d\mathbf{S}_n \right\} V_n \tag{1.6.9}$$

As $N \to \infty$ as under the limit in (1.6.9) we know that each of the flux contributions will converge to the divergence at the point P_n that the nth partition is closing in towards, meaning:

$$\oint_{\mathcal{S}} \mathbf{F} \cdot \, d\mathbf{S} = \lim_{N\to\infty} \sum_{n=1}^{N} \operatorname{div} \mathbf{F}\Big|_{P_n} V_n \tag{1.6.10}$$

This is simply a Riemann volume integral of the divergence function and thus we have arrived at (1.5.1):

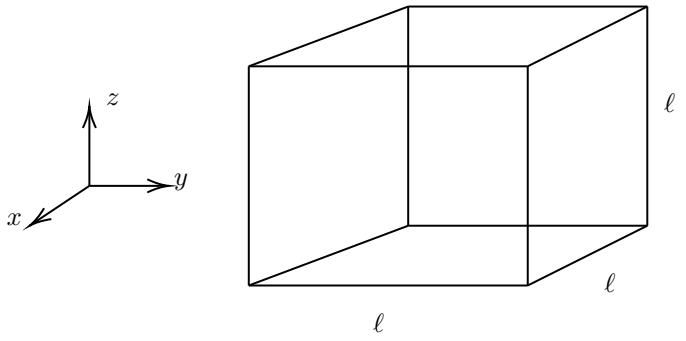
$$\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{F} \, dV \tag{1.6.1}$$

This derivation makes the divergence theorem seem completely intuitive, if divergence is simply the pointwise measurement of volumemetric flux density all throughout the volume enclosed by $\mathcal{S}$, then the volume weighted sum of those contributions should return the flux through $\mathcal{S}$.

What can we say about divergence after all of this math? Well first of all the divergence theorem (1.6.1) is a quite powerful result as it essentially tells us that divergence tells us about local flux and information about one will yield information about net flux. With this local flux information, physicists and engineers (and reluctantly mathematicians) started using the terminology **sources** and **sinks** to describe local behaviour of vector fields. Also (1.6.7) indicates to us that the value of divergence is **coordinate free** since the value of surface integrals are independent of paramaterization.

What is a **source** of a vector field?

The only remaining question lingering in the air is the following: How does (1.6.2) contain the same information as (1.6.3)? The remainder of this section will focus on the derivation of (1.6.2) and derivations of (1.6.2) in different coordinate systems follow from similar arguments. It is not necessary to the course, but continue reading if you are interested. Let $\mathbf{F}$ have continuous first order partial derivatives and be of the form $\mathbf{F} = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\rangle$. Consider a small box in $\mathbb{R}^3$ with side length of ℓ whose center is point P as shown below:



Let ℓ be small enough that the flux through any face of the cube is approximately constant throughout each face. Let point P (the center of the cube) be located at (x', y', z'). The shortest distance to each face from point P is always $\ell/2$. Let the surfaces whose normal vectors are parallel to the $\mathbf{e}_1$ or x direction be denoted $\mathcal{S}_{1,x}$ and $\mathcal{S}_{2,x}$ respectively. The net flux, ϕ_x, through these surfaces is:

$$\phi_x = \int\limits_{\mathcal{S}_{1,x}} \mathbf{F} \cdot d\mathbf{S}_{1,x} + \int\limits_{\mathcal{S}_{2,x}} \mathbf{F} \cdot d\mathbf{S}_{2,x} \tag{1.6.11}$$

When ℓ is small enough (1.6.11) can be approximated as follows:

$$\phi_x \approx \ell^2 \left[F_x\left(x' + \frac{\ell}{2}, y', z'\right) - F_x\left(x' - \frac{\ell}{2}, y', z'\right)\right] \tag{1.6.12}$$

Likewise for ϕ_y:

$$\phi_y \approx \ell^2 \left[F_y\left(x', y' + \frac{\ell}{2}, z'\right) - F_y\left(x', y' - \frac{\ell}{2}, z'\right)\right] \tag{1.6.13}$$

And also for ϕ_z:

$$\phi_z \approx \ell^2 \left[F_z\left(x', y', z' + \frac{\ell}{2}\right) - F_z\left(x', y', z' - \frac{\ell}{2}\right)\right] \tag{1.6.14}$$

Thus the divergence of $\mathbf{F}$ via (1.6.7) is as follows:

$$\operatorname{div} \mathbf{F}\Big|_{(x',y',z')} = \lim_{\ell \to 0} \frac{1}{\ell^3}\Big(\phi_x + \phi_y + \phi_z\Big) \tag{1.6.15}$$

Plugging in (1.6.12) through (1.6.14) into (1.6.15) yields:

$$\operatorname{div}\mathbf{F}\Big|_{x',y',z'} = \begin{cases} \lim\limits_{\ell\to 0}\dfrac{F_x(x'+\ell/2,y',z')-F_x(x'-\ell/2,y',z')}{\ell} \\ +\lim\limits_{\ell\to 0}\dfrac{F_y(x',y'+\ell/2,z')-F_y(x',y'-\ell/2,z')}{\ell} \\ +\lim\limits_{\ell\to 0}\dfrac{F_z(x',y',z'+\ell/2)-F_z(x',y',z'-\ell/2)}{\ell} \end{cases} \tag{1.6.16}$$

This is simply a linear combination of partial derivatives as follows:

$$\operatorname{div}\mathbf{F}\Big|_P = \frac{\partial F_x}{\partial x}\Big|_P + \frac{\partial F_y}{\partial y}\Big|_P + \frac{\partial F_z}{\partial z}\Big|_P \tag{1.6.17}$$

Thus we have arrived at (1.6.2) by letting P be anywhere within a relevant domain.

1.7 Stokes Theorem

Stokes Theorem is presented as follows in a first course in vector calculus: Let $\mathcal{S}$ be an oriented smooth surface that is bounded by a simple, smooth, closed boundary curve Γ. Let $\mathbf{F}$ be a vector field with continuous first order partial derivatives, then:

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot d\mathbf{S} \tag{1.7.1}$$

In 3D Cartesian coordinates, the curl of the vector field $\mathbf{F}=\langle F_x,F_y,F_z\rangle$ is usually presented as:

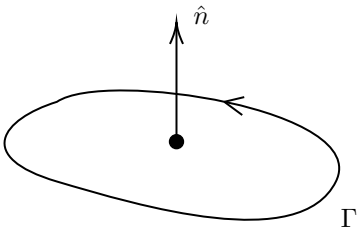
$$\operatorname{curl}\mathbf{F} := \nabla\times\mathbf{F} = \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{e}_2 + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{e}_3 \tag{1.7.2}$$

However, much like the usual definition of divergence, this definition of curl is simply a tool for computational ease and doesn't reflect much of the true nature of curl. If I simply stated now that (1.7.2) evaluated at a point gives the area density of the fields tendency to cause rotation about an orthogonal axis, you would likely have no idea how it is even possible to know that information from (1.7.2).

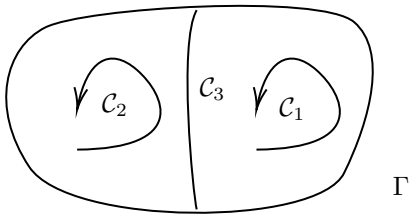
Therefore just as we did in the divergence section, we are going to investigate the original definition of curl. Let P be a point inside the surface. Let us intersect three planes through this point P such that each plane is defined by the coordinate vectors at point P, i.e the first plane is defined by point P and $\hat{n}=\mathbf{e}_1$, the second plane is defined by point P and $\hat{n}=\mathbf{e}_2$, and the third plane is defined by point P and $\hat{n}=\mathbf{e}_3$. Let Γ_1,Γ_2 and Γ_3 be the curves of intersection between the planes 1,2 and 3, and the surface $\mathcal{S}$. Each of these curves bound an area A_1,A_2 and A_3. The curl at point P is defined as:

$$(\nabla\times\mathbf{F})(P) = \sum_{k=1}^{3}\left(\lim_{A_k\to 0}\frac{1}{A_k}\oint_{\Gamma_k}\mathbf{F}\cdot d\mathbf{r}\right)\mathbf{e}_k \tag{1.7.3}$$

Now (1.7.3) is a bit harder to break down than divergence, however it is still possible to gain an understanding of curl from this definition. Just like the divergence definition, we are going to undergo the process that made mathematicians/physicists and engineers decide that this was a useful property to measure in the first place. In order to do this we are only going to consider one plane, i.e the projection of the curl onto any of the three planes. Consider the following image below:



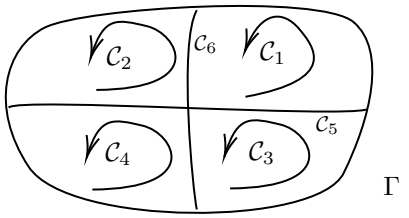
We can split the contour Γ into three contours as follows:



The curve $\mathcal{C}_3$ has no net contribution to the line integral since it is traversed twice in opposite orientations by $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus we can write:

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} \tag{1.7.4}$$

Now imagine we add a fourth curve $\mathcal{C}_4$ in the following manner:



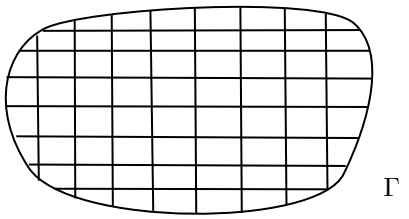
We can see again that the contributions from the curves $\mathcal{C}_5$ and $\mathcal{C}_6$ are negligible since they are always traversed on opposite orientations. Thus we can write:

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} \tag{1.7.5}$$

We can see that if we partition the domain into N sections the result will always be the same:

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \sum_{k=1}^{N} \left\{ \oint_{\mathcal{C}_k} \mathbf{F} \cdot d\mathbf{r} \right\} \tag{1.7.6}$$

As N gets large we will obtain a grid that looks something like this:



As N gets larger, the area of each grid section gets smaller and eventually the sum of the small line integrals will cover the entire area of the region bounded by Γ. Therefore (1.7.6) is proportional to the area. We can break this area up into the area of each smaller region and we define the following:

$$\operatorname{curl}\mathbf{F}\cdot\hat{n}=\lim_{N\to\infty}\sum_{k=1}^{N}\left\{\frac{1}{A_k}\oint_{\mathcal{C}_k}\mathbf{F}\cdot d\mathbf{r}\right\}=\sum_{k=1}^{\infty}\left\{\lim_{A_k\to 0}\frac{1}{A_k}\oint_{\mathcal{C}_k}\mathbf{F}\cdot d\mathbf{r}\right\} \tag{1.7.7}$$

This returns the curl at all points in the domain. We can see that it is the sum of the area density of the work required to circulate around a small circle about each point. Thus we can say the following:

The curl at a point within a plane is a measurement of a given vector fields tendency to cause rotation about an axis through that point and orthogonal to that plane.

The curl at a single point is defined as follows:

$$\operatorname{curl}\mathbf{F}\cdot\hat{n}(P)=\lim_{A\to 0}\frac{1}{A}\oint_{\Gamma}\mathbf{F}\cdot d\mathbf{r} \tag{1.7.8}$$

With this definition in mind it is easy to arrive at (1.7.1). To do this we will take a step back to (1.7.6) except in a slightly different form:

$$\oint_{\Gamma}\mathbf{F}\cdot d\mathbf{r}=\sum_{k=1}^{N}\left\{\frac{1}{A_k}\oint_{\mathcal{C}_k}\mathbf{F}\cdot d\mathbf{r}\right\}A_k \tag{1.7.9}$$

As the number of partitions becomes infinity we obtain:

$$\oint_{\Gamma}\mathbf{F}\cdot d\mathbf{r}=\lim_{N\to\infty}\sum_{k=1}^{N}\operatorname{curl}\mathbf{F}\cdot\hat{n}A_k \tag{1.7.10}$$

This is simply a Riemann integral for a surface:

$$\oint_{\Gamma}\mathbf{F}\cdot d\mathbf{r}=\int_{\mathcal{S}}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \tag{1.7.1}$$

The only remaining thing to discuss is how (1.7.8) encapsulates the same information as (1.7.2). We will only show the derivation of the $\mathbf{e}_3$ component, the rest of the components easily follow via identical arguments. Consider the following square centered about the point $P=(x',y',z')$ in the xy plane:

1.8 Taylor's Expansion in 2D

Later in the course it will be required to somehow expand a function on $(x,y)\in\mathcal{D}\subset\mathbb{R}^2$ in order to examine limiting behaviour. One way to do this is with multi-dimensional Taylor expansions. Let $f(x,y)$ be defined on an open neighbourhood about the point (a,b), then $f(x,y)$ about the point (a,b) can be expanded as

$$f(x,y)=\begin{cases}f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+\\ \dfrac{1}{2!}[f_{xx}(a,b)(x-a)^2+2f_{xy}(a,b)(x-a)(y-b)+f_{yy}(a,b)(y-b)^2]+\dots\end{cases} \tag{1.8.1}$$

Thus the curl of F over the whole region is the sum of the area density of the work required to circulate around a very small circle about each point. The expansion (1.8.1) is only written to second order as that will be all that is required in ECE 370.

1.9 Gauss' Law

2 How are Field Lines Obtained?

Up until now one may have been exposed to many subtle partial falsehoods about field lines: They aren't physically realizable, they are conceptual aids, etc. The only verification one may have had about these field lines are an experiment or two which seems to verify the idea of field lines like a bar magnet placed in iron filings, or a more advanced experiment where potential is calculated over a discrete array between two metal plates with a potential difference. However, it is quite interesting to note that these field lines are *not firstly experimental* and by this I mean: The "lines of force" or the *geometry* of the field lines can be completely determined mathematically before experimentation is ever observed so long as the underlying mathematics describing the physical phenomena is understood. To understand this one must have comprehensive understanding of the material covered in 1.3.

Often in introductory multi-variable calculus courses, students spend time on expanding properties of random vector fields and performing operations on these fields to extract information. However, often a very important question is glossed over or not even addressed:

"**If a particle which interacts with a vector field is released in that field, what path does it take?**"

Before we get into the nitty gritty of it, let's consider a simple "toy example". Let us investigate the following simple vector function of two variables:

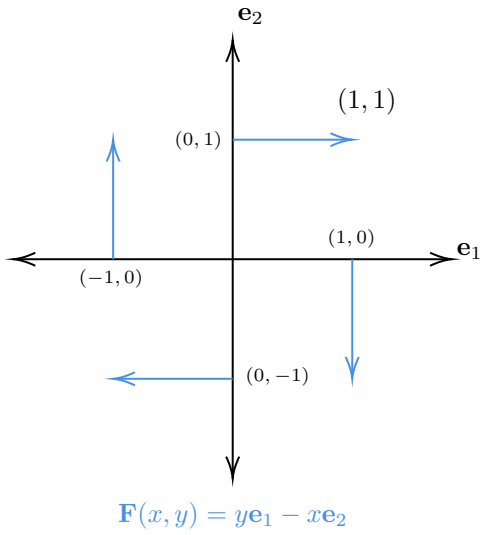
$$\mathbf{F}(x,y) = y\mathbf{e}_1 - x\mathbf{e}_2 \tag{2.1}$$

This vector function could be easily plotted by taking simple sample points like $(0,1)$, $(1,0)$ and etc. But we can get a bigger conceptual idea of what this field would do to a particle which interacts with it by considering the following fact about (2.1): The vector generated at a point (x',y') is **orthogonal** to the position vector, $\mathbf{r}(x',y')$, to (x',y'). To see this consider

$$\mathbf{F}(x',y')\cdot\mathbf{r}(x',y') = \langle y',-x'\rangle\cdot\langle x',y'\rangle = x'y' - x'y' = 0 \tag{2.2}$$

Thus (2.2) confirms that at any point the vector (2.1) is directed orthogonal to the position vector to that point. If a particle that interacts with (2.1) is released into the field, then we should intuitively expect that the particle moves in a circle as the motion described exactly is that of circular motion.

Consider the left most plot which is a partial plot of (2.1). We can see that at any point, the field points orthogonal to the point in a clockwise fashion. From this plot alone we can expect that a particle released at a point will either move in a circular fashion or a circular spiral fashion about that point. However, we do not have enough information about the problem to tell exactly what will occur. It is easy enough to verify that (2.1) is not curl-less, and in fact it has a curl of $\langle 0,0,-2\rangle$ which means we expect a particle released in the field to rotate about the z axis in a clockwise fashion, i.e the right hand rule holds here.



How can we know for sure the motion of a particle released in the field described by (2.1)? The answer to this question requires some intuitive thinking. If a particle that interacts with (2.1) is released in the field, then we should expect that the force acting on that particle is tangent to the path of travel that the particle takes. Meaning if Γ is the curve traced out by a particle under (2.1), then the vector (2.1) should be parallel to Γ at any point. Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ describe Γ for some range of t which traces out Γ only once. Then the above statement corresponds to

$$\mathbf{F}(x, y) \parallel \mathbf{r}'(t) \tag{2.3}$$

Therefore we expect there exists some non-zero constant k such that the following system of equations holds:

$$y = k\frac{dx}{dt} \tag{2.4}$$

$$-x = k\frac{dy}{dt} \tag{2.5}$$

We take the ratio of (2.5) and (2.4) to obtain:

$$\frac{-x}{y} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{dy}{dx} \tag{2.6}$$

The equation (2.6) indicates that the (possibly implicit) curve the particle follows has a slope everywhere described by (2.6). The differential equation (2.6) can be solved via separation:

$$C - \frac{1}{2}x^2 = \frac{1}{2}y^2 \tag{2.7}$$

Letting $K = 2C$, this equation can easily be rearranged to give the form of a circle:

$$K = x^2 + y^2 \tag{2.8}$$

If the particle was initially released in the field described by (2.1) at the point $(0, 1)$ for example, then the constant can be solved:

$$K = (0)^2 + (1)^2 = 1$$

and the particle should travel in the field described by

$$x^2 + y^2 = 1 \tag{2.9}$$

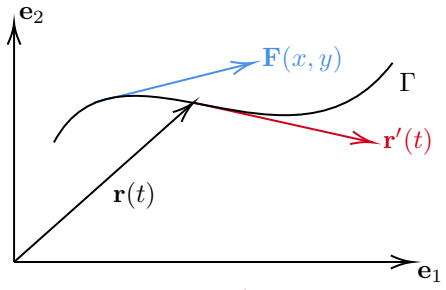
Let us now generalize this result. Let $\mathbf{F}$ be described by the following vector function:

$$\mathbf{F}(x, y) = \gamma_1(x, y)\mathbf{e}_1 + \gamma_2(x, y)\mathbf{e}_2 \tag{2.10}$$

When a particle is released under this field, we expect that this field always remains tangent to the curve it traces out Γ. Let $\mathbf{r}(t)$ be a vector function which traces out Γ:

$$\mathbf{r}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2, \quad \alpha \leq t \leq \beta \tag{2.11}$$

where α and β are chosen such that Γ is traced out only once. The following visualization should provide great aid in conceptually understanding why the following approach must be valid:



$\mathbf{F}(x, y) = k\mathbf{r}'(t)$

In the figure above the vector (2.10) was drawn at a previous location on the curve to avoid clutter, but it still illustrates that (2.10) is always tangent to Γ. Since the two vectors must be parallel we may write the following system of equations:

$$\gamma_1(x,y) = k\frac{dx}{dt} \tag{2.12}$$

$$\gamma_2(x,y) = k\frac{dy}{dt} \tag{2.13}$$

Once again taking the ratio of (2.13) and (2.12) we obtain:

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{\gamma_2(x,y)}{\gamma_1(x,y)} \tag{2.14}$$

Thus Γ is the family of curves described by

$$\Gamma := \left\{ (x,y) \,\middle|\, \frac{dy}{dx} = \frac{\gamma_2(x,y)}{\gamma_1(x,y)} \right\} \tag{2.15}$$

If (2.10) is conservative, then a very important result occurs which is traditionally predicted by a more advanced mathematical formalism in Lagrangian mechanics or Hamiltonian mechanics. Let (2.10) be conservative, i.e $\exists f(x,y)$ such that

$$\mathbf{F}(x,y) = \nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_2 \tag{2.16}$$

Then the curve traced out under this conservative field will be described by

$$\Gamma := \left\{ (x,y) \,\middle|\, \frac{dy}{dx} = \frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}} \right\} \tag{2.17}$$

By definition (2.17) describes a path of steepest descent. Usually (2.17) indicates to us the following statement (or a equivalent one) about conservative fields/systems:

Particles which interact with conservative fields, when released in such a field, will travel along the path which most quickly minimizes potential.

Consider the example of electric field, which is related to the gradient of potential: $\mathbf{E} = -\nabla\Phi$. If a positive charge is placed in this field, the path described by (2.17) will be one which seeks to minimize potential. Other fields like gravity also obey this principle. This will be discussed more thoroughly later.

3 Potential: Laplace and Poisson Equations

In general coordinates, the Poisson equation is given as

$$\nabla^2\Phi(\mathbf{r}) = f(\mathbf{r}) \tag{3.1}$$

where $\mathbf{r}$ would denote are coordinate set like (x, y, z) or (r, θ, ϕ). In 2D Cartesian coordinates, when $f(\mathbf{r}) = 0$ in (3.1), we obtain Laplace's equation in 2D Cartesian coordinates:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{3.2}$$

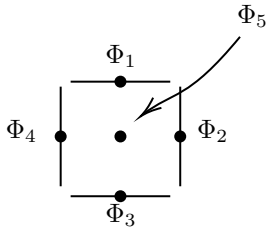
Notice that (3.2) is equivalent to the following expression:

$$\text{div}(\text{grad}\,\Phi) = 0 \tag{3.3}$$

Since $-\text{grad}(\Phi) = \mathbf{E}$, we can say that (3.2) applies in regions where there is no source or sink of electric field, i.e there is no charge density present in the region where (3.2) is to be applied. A big theme in this note package so far is an emphasis on understanding what's going on under the hood of all this vector calculus, and I'm going to take a similar approach for (3.2). Firstly, we start by stating something which will not seem apparent from (3.2) (but is more present in (3.3):

The potential at a point converges to the average value of the potential around that point.

The easiest way to show this is to consider a discrete approach to (3.2). Consider a scenario where (3.2) governs a potential function Φ over a small square region, where the boundary of the square region is made up of four metal plates which are held at 4 different constant potentials $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ as shown:



The bold statement above states:

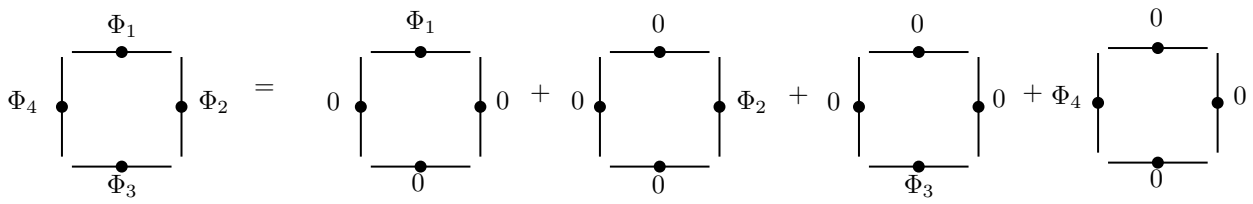
$$\Phi_{avg} - \Phi_5 = 0 \tag{3.4}$$

Or equivalently:

$$\Phi_5 = \frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4}{4} \tag{3.5}$$

This is a discrete case of the mean value property for Laplace's equation, and a full treatment of this property in the continuous case would require advanced mathematics out of the scope of ECE 370. For example in the 2D continuous case, a more accurate statement would be: The value of potential at a point is obtained by the average value of the potential on a circle (in 3D a spherical shell) about that point, integrated over the whole domain on which Laplace's equation is defined. However, as we will see later in the course, (3.5) will easily converge to the continuous case where $\phi_{1,2,3,4}$ are allowed to be functions under iterations. So all we need to keep in mind is the general bold statement above.

Now (3.5) reveals an interesting property of systems governed by Laplace's equation, but I haven't shown anything yet. There are a number of ways to arrive at (3.5), for example one way to do it would be to use the Taylor expansion (1.8.1) on (3.2). However, if we simply use the linearity of (3.2) there is a much more elegant approach. Since (3.2) is a linear PDE, we know that the response of Φ to the four boundary conditions is the same as the linear combination of the response to each of the boundary conditions independently. This is shown below:



In order to proceed we have to consider the uniqueness property of Laplace's equation. The property is as follows:

The potential (solutions to Laplace's equation) within a domain whose boundary is held at given potential is unique.

This means that if a specific solution to Laplace's equation is known, it must have come from a single unique potential on the boundary. Thus the potential at the center of the left most square, Φ_5, is unique and corresponds only to the potential given on the four boundaries.