Function Representation with Orthogonal Sets
MATH 201

December 29, 2019

Contents

1 The Vector Space of Piecewise Continuous Functions

The set of all pieceiwse continuous functions form a vector subspace of all functions, and this subspace is of infinite dimension.

Definition — Piecewise Continuous Function

A function is **piecewise continuous** on an interval (a, b) if the interval (a, b) can be broken up into a finite amount of subintervals on which the function is continuous on each open subinterval which *is not* the subinterval(s) containing the end points of the function.

We denote $PWC(a, b)$ to be the set of all piecewise continuous functions on the interval (a, b).

The set of all piecewise continuous functions form a vector subspace.

Theorem — Vector Space of PWC Functions

The set of all piecewise continuous functions on some open interval I is a vector subspace of all functions $f : I \to \mathbb{R}$, i.e if $f, g \in PWC(I)$, then the following holds

1. $\alpha f + \beta g \in PWC(I), \ \forall \ \alpha, \beta \in \mathbb{R}$
2. $f \cdot g \in PWC(I)$
3. $f = 0 \in PWC(I)$

Note:
(1) If $f \in PWC(a, b)$ then it is guaranteed that f is absolutely integrable, i.e

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$$

(2) The dimension of $PWC(I)$ is infinite, where I is some open interval.
(3) This vector subspace is also a **inner product space**.

The 3rd note in the above theorem is what will eventually give the conceptual leap to Fourier series. We will now discuss what an inner product is. We know what the inner product is for regular vectors, it is the dot product. Recall that for two vectors $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\mathbf{w} = \langle w_1, w_2, \ldots, w_n \rangle$, their inner product/dot product is given by

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i \cdot w_i$$

We want to know what the analogue of the dot product is for PWC functions. In order to figure out what this analogue is, we first need to define specifically what it is when mathematicians refer to an inner product.

Definition **Inner Product**

An **inner product** on a vector space, V, is any function denoted $\langle \mathbf{u}, \mathbf{v} \rangle$ that acts on vectors in V that satisfies the following properties

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = 0$
3. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
4. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
5. $\langle \lambda \mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle$

The dot product discussed previous to this definition is an example of an inner product.

The question is: What is the inner product on the vector space of piecewise continuous functions? Upon closer examination of the inner product properties, it should be obvious that the inner product on PWC functions is integration. Let $f, g, h \in PWC(a, b)$ and let λ be a real constant. Then we verify the inner product definition with integration in the following manner:

$$(1)\ \int_a^b f(x)f(x)\,dx = \int_a^b f(x)^2\,dx \geq 0$$

$$(2)\ \int_a^b f(x)f(x)\,dx = \int_a^b f(x)^2\,dx = 0 \iff f(x) = 0.$$

$$(3)\ \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx$$

$$(4)\ \int_a^b f(x)(g(x) + h(x))\,dx = \int_a^b f(x)g(x)\,dx + \int_a^b f(x)h(x)\,dx$$

$$(5)\ \int_a^b \lambda f(x), g(x)\,dx = \lambda \int_a^b f(x)g(x)\,dx.$$

These are all simply properties of integration. We now present the result in one definition.

Definition — **Inner Product of PWC Functions**

Let $f, g, w \in PWC(a,b)$ with $w(x) \geq 0$. The **inner product** of f and g with **weight** w is defined to be

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\,dx$$

The function $w(x)$ is called a **weight function**, and in most applications it is taken to be $w(x) = 1$.

Remarks:

(1) It is possible that $f \in PWC(a,b)$ and $\langle f, f\rangle = 0$ but $f \neq 0$. In this case f is a null function of the inner product operation. This is a non-issue if we consider more properties of integration. Two PWC functions f and g are seen as identical under integration if they differ at finitely many points.

(2) Two functions $f(x), g(x)$ are said to be **orthogonal** on (a,b) if $\langle f, g\rangle = 0$, that is

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\,dx = 0.$$

(3) The **norm** of the function $f(x)$ on (a,b) is

$$\big||f(x)|\big| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f(x)^2\,dx}.$$

(4) Let $f, g \in PWC(a,b)$, then the **vector projection** of f onto g is

$$\text{proj}(f)_g = \frac{\langle f, g\rangle}{\langle g, g\rangle} g = \frac{\displaystyle\int_a^b f(x)g(x)\,dx}{\displaystyle\int_a^b g(x)^2\,dx}\, g(x).$$

The definition box above is the reason why briefly journeyed into the world of linear algebra. The main idea to be conveyed is that piecewise continuous functions behave exactly like vectors and with this knowledge we can intuitively derive the generalized Fourier series of a function.

Example — **Orthogonality**

Show that the functions

$$\begin{cases} c_n(x) = \cos\left(\dfrac{n\pi x}{L}\right) \\[2ex] s_n(x) = \sin\left(\dfrac{m\pi x}{L}\right) \end{cases}$$

are orthogonal with respect to $w(x) = 1$ on $x \in (0, 2L)$ for all $(m,n) \in \mathbb{Z}$.

Solution: Using the definition of the inner product we have

$$\langle c_n, s_n\rangle = \int_0^{2L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

If $n = m = k$, then the inner product becomes

$$\begin{aligned}
\langle c_k, s_k \rangle &= \frac{1}{2} \int_0^{2L} \sin\left(\frac{2k\pi x}{L}\right) dx \\
&= -\left(\frac{1}{2}\right)\left(\frac{L}{2k\pi}\right) \left\{ \cos\left(\frac{2k\pi}{L} x\right) \Big|_0^{2L} \right\} \\
&= \left(\frac{L}{4k\pi}\right)\Big[1 - \cos(4k\pi)\Big] = 0 \ \forall \ k \in \mathbb{Z}.
\end{aligned}$$

Now consider the case where $n \neq m$, for this we utilize the substitution $u = x - L$:

$$\begin{aligned}
\int_0^{2L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx &= \int_{-L}^{L} \cos\left(\frac{n\pi}{L}(u+L)\right) \sin\left(\frac{m\pi}{L}(u+L)\right) du \\
&= \int_{-L}^{L} \cos\left(\frac{n\pi u}{L}\right) \sin\left(\frac{m\pi u}{L}\right) du \\
&= 0
\end{aligned}$$

This integral is zero since $\cos(\alpha)\sin(\beta)$ is an odd function, and the integral is on the symmetric interval $(-L, L)$. Therefore the following is a complete set of orthogonal functions:

$$\{c_0, c_1, s_1, c_2, s_2, \dots\} = \left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots\right\}$$

The Kronecker delta is given by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Using this notation we can write that $\langle c_n, s_n \rangle = \delta_{ij} 2L$, since if we have two cosines or two sines under the given inner product with the same argument it will evaluate to $2L$, otherwise it will evaluate to zero.

$\square$

2 Representation of Functions via Orthogonal Sets

Any regular Cartesian vector can be constructed as sum of its orthogonal components. Consider the vector

$$\mathbf{v} = \langle 1, 2, 3 \rangle.$$

This vector is simply a sum of the orthogonal components $\mathbf{i}$, $2\mathbf{j}$ and $3\mathbf{k}$. Note that v is the sum of its projection onto the x axis, the y axis and the z axis. Any vector can be represented this way. This idea holds true for PWC functions as well, the only difference being that the dimension of the space is infinite so there should be infinitely many components.

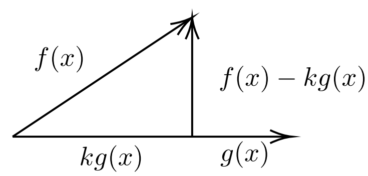


Figure 1: Simplified vector diagram of f, there should be infinitely many orthogonal components.

Lets suppose that the set of functions $(g_0(x), g_1(x), \ldots, g_N(x))$ is orthogonal on (a, b). We wish to answer the question: How can we represent some $f \in PWC(a, b)$ as a sum of orthogonal components? The answer should be the exact same as with regular vectors: the sum of the projection of f onto all of the orthogonal components. Meaning we should be able to write

$$f(x) = \frac{\langle f, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0 + \frac{\langle f, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 + \cdots + \frac{\langle f, g_N \rangle}{\langle g_N, g_N \rangle} g_N + \cdots + = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\langle f, g_n \rangle}{\langle g_n, g_n \rangle} g_n(x)$$

In the above vector diagram, each k_j would be

$$k_j = \frac{\langle f, g_j \rangle}{\langle g_j, g_j \rangle} = \frac{\displaystyle\int_a^b f(x) g_j(x)\, dx}{\displaystyle\int_a^b g_j^2(x)\, dx},$$

thus we have expressed $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} k_n g_n(x)$$

which is a **generalized Fourier series** of $f(x)$.

While the above results are exactly true, it was mostly an intuitive guess. Here is more of a proof of what is going on. We approximate f by a sum of its orthogonal components:

$$f(x) \approx \sum_{n=0}^{N} k_n g_n(x)$$

and we define the error of this approximation to be

$$e(x) = f(x) - \sum_{n=0}^{N} k_n g_n(x).$$

We wish to find each k_n which minimizes the legnth/norm of the error of the approximation. We do this by minimizing the square of the length of the error vector. In fig(1) this would correspond to making the vector $f(x) - kg(x)$ smaller. Thus we wish to minimize

$$E(k_0, k_1, \ldots, k_N) = \big|\big|e(x)\big|\big| = \int_a^b \left\{ f(x) - \sum_{n=0}^{N} k_n g_n(x) \right\}^2 dx$$

In order to do this we recall the following theorem from mathematics:

Theorem **Extreme Values**

If $f(x_1, x_2, \ldots, x_n)$ has an extreme value at the point P_0, then each component of its gradient must be zero. That is

$$\nabla f(P_0) = \frac{\partial f(P_0)}{\partial x_1}\mathbf{e}_1 + \frac{\partial f(P_0)}{\partial x_2}\mathbf{e}_2 + \cdots + \frac{\partial f(P_0)}{\partial x_n}\mathbf{e}_n = 0,$$

or more compactly, the gradients jth component satisfies $\nabla f(P_0) \cdot \mathbf{e}_j = 0$.

Let us therefore compute the jth component of the gradient of the length of the error function:

$$\begin{aligned} \nabla E(k_0, k_1, \ldots, k_n) \cdot \mathbf{e}_j &= \frac{\partial}{\partial c_j} \int_a^b \left\{ f(x) - \sum_{n=0}^{N} k_n g_n(x) \right\}^2 dx \\ &= -2 \int_a^b \Big(f(x) g_j(x) - g_j(x) \sum_{n=0}^{N} k_n g_n(x) \Big)\, dx. \end{aligned}$$

Setting this expression to zero and re-arranging yields us

$$\int_a^b f(x) g_j(x)\, dx = \int_a^b g_j(x) \sum_{n=0}^{N} k_n g_n(x)\, dx$$

In order to proceed, consider the expansion of the right hand side:

$$\begin{aligned} \int_a^b g_j(x) \sum_{n=0}^{N} k_n g_n(x)\, dx &= \int_a^b g_j(x) \Big(k_0 g_0(x) + k_1 g_1(x) + \cdots + k_j g_j(x) + \cdots + k_n g_n(x) \Big)\, dx \\ &= k_0 \int_a^b \underbrace{g_0(x) g_j(x)}_{\text{Orthogonal}}\, dx + \cdots + k_j \int_a^b (g_j(x))^2\, dx + \cdots + k_n \int_a^b \underbrace{g_n(x) g_j(x)}_{\text{Orthogonal}}\, dx \\ &= k_j \int_a^b (g_j(x))^2\, dx \end{aligned}$$

Above we made use of the fact that the set $(g_1, \ldots, g_N)$ is an orthogonal set of functions on (a, b). After plugging this back into our zero gradient expression we can see that the jth coefficient that minimizes error is given by

$$\boxed{k_j = \frac{\displaystyle\int_a^b f(x) g(x)\, dx}{\displaystyle\int_a^b (g(x))^2\, dx} = \frac{\langle f, g_j \rangle}{\langle g_j, g_j \rangle}}$$

We can conceptually justify that this is a minimum since orthogonality implies shortest distance. Further notice that $k_j g_j$, by definition, is the orthogonal projection of f onto g

$$k_j g_j = \frac{\langle f, g_j \rangle}{\langle g_j, g_j \rangle} g_j.$$

When we let $N \to \infty$ we have a complete orthogonal basis for f, and we have found that f can be represented by

$$f(x) = \lim_{N\to\infty} \sum_{n=0}^{N} k_n g_n(x), \text{ where } k_n = \frac{\langle f, g_j \rangle}{\langle g_j, g_j \rangle}.$$

We have not shown convergence of this series, but it turns out to converge for most practical applications. This series is called the **generalized Fourier series** of $f(x)$.

Also note that this representation of f will only be exact if the norm of the error function is zero, we have only shown that these coeffecients will give a minimum. However, we will discuss Bessel's Inequality and Parsevals Identity which will guarantee that the norm of the error is zero for functions which have a Fourier series.

2.1 Generalized Fourier Series

Definition — Generalized Fourier Series

The **generalized Fourier series** of $f(x)$ on $x \in (a, b)$ is

$$f(x) \sim \lim_{N\to\infty} \sum_{n=0}^{N} k_n g_n(x),$$

where, on $x \in (a, b)$, each k_j satisfies

$$k_j = \frac{\displaystyle\int_a^b f(x) g_j(x)\, dx}{\displaystyle\int_a^b (g_j(x))^2\, dx} = \frac{\langle f, g_j \rangle}{\langle g_j, g_j \rangle}.$$

Note: We can extend the class of functions which have a Fourier series by considering $f : \mathbb{C} \to \mathbb{C}$. If we have a complex signal $f(x)$, then it will have Fourier coefficients in the form

$$k_j = \frac{\displaystyle\int_a^b f(x) \bar{g}_j(x)\, dx}{\displaystyle\int_a^b |\bar{g}_j(x)|^2\, dx},$$

where $\bar{g}$ denotes the complex conjugate of g.

2.2 Trigonometric Fourier Series

One important Fourier series is the **trigonometric Fourier series**.

Definition — Trigonometric Fourier Series

A periodic function $f(x)$ has an associated Fourier series known as the trigonometric Fourier series in the form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

on $x \in (a, b)$ with period $T = b - a$.

Example — Trigonometric Fourier Series

Show that in the trigonometric Fourier series, the coefficients a_n and b_n are given by

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx} \quad \boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx}$$

on the interval $x \in (-L, L)$.

Solution: We use the definition of the Fourier coefficients:

$$k_j = \frac{\displaystyle\int_a^b f(x) g_j(x)\, dx}{\displaystyle\int_a^b (g_j(x))^2\, dx} = \frac{\langle f, g_j \rangle}{\langle g_j, g_j \rangle}.$$

We choose as our orthogonal basis the set

$$G = \{c_0, c_1, s_1, c_2, s_2, \dots\} = \left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots\right\},$$

and we divide this set into the set containing the even functions, and the set containing the odd functions $G =: \{c_0, c_1, \dots\} \cup \{s_1, s_2, \dots\}$:

$$\{c_0, c_1, \dots\} = \left\{1, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots\right\}$$
$$\{s_1, s_2, \dots\} = \left\{\sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots\right\}$$

We define the k_j coefficients for the even set to be a_n and the k_j coefficients for the odd set to be b_n and we obtain

$$a_n = \frac{\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx}{\displaystyle\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx}, \quad b_n = \frac{\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx}{\displaystyle\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx}$$

We will skip evaluating the integrals on the denominators, but it turns out that for both the sine and cosine integral, their evaluation is $1/L$. Thus we have finally arrived at:

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx}, \quad \boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)}$$

It is important to note that a_0 differs from the a_n formula, that is

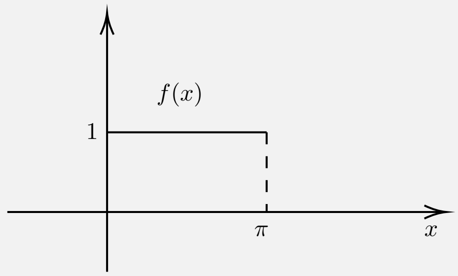
$$\boxed{a_0 = \frac{\displaystyle\int_{-L}^{L} f(x)\,dx}{\displaystyle\int_{-L}^{L} dx} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx}$$

□

Example — Trigonometric Fourier Series

Compute the trigonometric Fourier series (plotted below for one period) for the function with period $T = 2\pi$ given by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0 \\ 1 & \text{if } 0 \le x \le \pi \end{cases}$$



Solution: We start by computing a_0:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} dx = \frac{1}{2}.$$

Now we compute a_n:

$$\begin{aligned} a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx \\ &= 0. \end{aligned}$$

Likewise we compute b_n :

$$\begin{aligned} b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx \\ &= -\frac{1}{n\pi}\cos(nx)\Big|_{0}^{\pi} \\ &= \frac{1}{n\pi}(1-\cos(n\pi)) \\ &= \frac{1}{n\pi}(1-(-1)^n)\ \forall\ n \in \mathbb{Z}^+/\,\{0\} \\ &= \frac{2}{n\pi} \text{ when } n \text{ is odd.} \end{aligned}$$

Thus the trigonometric Fourier series for $f(x)$ is given by

$$\boxed{f(x) \sim \frac{1}{2} + \sum_{n \text{ odd}}^{\infty} \frac{2}{n\pi}\sin(nx)}$$

□

Example — Trigonometric Fourier Series

Find the trigonometric Fourier series associated with the functions

$$f_1(x) = 5 + \cos(4x) - \sin(5x)$$
$$f_2(x) = 6\sin(x)\cos(x)$$

over the interval $x \in (-L, L)$. Then, over the same interval, find the a_n coefficients for the function $f_3(x) = x^3$.

Solution: The function $f_1(x)$ is already a trigonometric Fourier series with $a_0 = 5$, $a_4 = 1$ and $b_5 = -1$ and all other coefficients zero.

The function $f_2(x)$ is not already a trigonometric Fourier series, but we can write $f_2(x) = 3\sin(2x)$, and this is the associated trigonometric Fourier series with $b_2 = 3$ and all other coefficients zero.

The a_n coefficients for the function $f_3(x)$ are all equal to zero, this is because $f_3(x)$ is odd, and it will not have representation by even functions on a symmetric interval like $(-L, L)$.

□

Example — Trigonometric Fourier Series

Compute the trigonometric Fourier series for the function

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ 2\sin(x), & \text{if } 0 < x < \pi. \end{cases}$$

Solution: This function has neither odd or even symmetry, thus we cannot invoke symmetry arguments to eliminate one set of Fourier coeffecients. Thus we start with a_0:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{2}{\pi}.$$

Now onto a_n:

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} \sin\left[(n+1)x\right]\,dx + \frac{1}{\pi}\int_0^{\pi} \sin\left[(1-n)x\right]\,dx \end{aligned}$$

Here we made use of the trigonometric product identity:

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta).$$

Now we finish our computation of a_n:

$$\begin{aligned} a_n &= -\frac{1}{\pi}\frac{1}{n+1}\cos[(n+1)x]\Big|_0^\pi - \frac{1}{\pi}\frac{1}{1-n}\cos[x(1-n)]\Big|_0^\pi \\ &= \frac{1}{\pi}(1-(-1)^{n+1})\left(\frac{1}{n+1} - \frac{1}{n-1}\right),\ n \neq 1. \end{aligned}$$

Note: One of the most common errors students make on exams is forgetting to check values of n for which a_n or b_n is undefined. Here $n \neq 1$ for a_n, thus we need to manually check a_1:

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(x)\,dx = \frac{1}{\pi}\int_0^\pi \sin(2x)\,dx = 0.$$

Now we calculate b_n:

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{\pi}\int_0^\pi \sin(x)\sin(nx)\,dx \\ &= \frac{1}{\pi}\int_0^\pi \cos[(1-n)x]\,dx - \frac{1}{\pi}\int_0^\pi \cos[(1+n)x]\,dx \end{aligned}$$

Here we made use of the trigonometric product identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta).$$

We proceed with calculation of b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\frac{1}{1-n}\sin[(1-n)x]\Big|_0^\pi - \frac{1}{n+1}\sin[(1+n)x]\Big|_0^\pi \\ &= 0\ n \neq 1. \end{aligned}$$

Note: Again notice that we cannot use this formula for b_1, therefore we must check it manually:

$$b_1 = \frac{2}{\pi}\int_0^\pi \sin(x)\sin(x)\,dx = \frac{2}{\pi}\int_0^\pi \sin^2(x)\,dx = \frac{1}{\pi}\int_0^\pi (1-\cos(2x))\,dx = 1.$$

Therefore we have the following for the Fourier coefficients:

$$\begin{cases} a_n = \frac{1}{\pi}(1-(-1)^{n+1})\left(\frac{1}{n+1} - \frac{1}{n-1}\right), & \text{if } n \neq 1 \\ a_n = 0, & \text{if } n = 1. \\ b_n = 0, & \text{if } n \neq 1 \\ b_n = 0, & \text{if } n = 1. \end{cases}$$

Thus the Fourier series for f is

$$f(x) \sim a_0 + a_1\cos(x) + b_1\sin(x) + \sum_{n=2}^{\infty}\frac{1}{\pi}(1-(-1)^{n+1})\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos(nx)$$

$$f(x) \sim \frac{2}{\pi} + \sin(x) + \sum_{n=2}^{\infty}\frac{1}{\pi}(1-(-1)^{n+1})\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos(nx)$$

$$f(x) \sim \frac{2}{\pi} + \sin(x) + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{1-(2k)^2}\cos(2kx).$$

$\square$

2.3 Fourier Sine and Cosine Series

Definition — Fourier Sine and Cosine Series

The Fourier sine series of f on $[0, L]$ is

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

The Fourier cosine series of f on $[0, L]$ is

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx,$$

and

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx.$$

We will only discuss how the sine series is obtained, the result for the cosine series is obtained in a nearly identical manner. Let f be defined on $[0, L]$. We extend this function to $[-L, L]$ by considering the *odd extension* of f. Since f is odd on $[-L, L]$, we know $a_n = 0\ \forall\ n \geq 0$. Thus the Fourier series is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

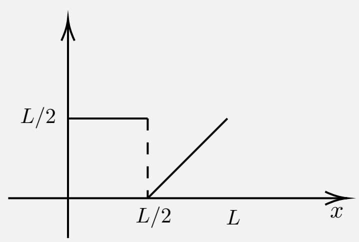
with

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{\text{odd}}(x)\sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

This result is obtained since the product of two odd functions is even. The Fourier cosine series is found by considering the even extension $f_{\text{even}}(x)$ on $[-L, L]$.

Example — Trigonometric Fourier Series

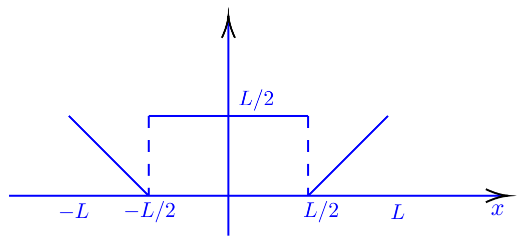
Find the Fourier cosine of the function below.



Solution: The function given is

$$f(x) = \begin{cases} \frac{L}{2} & \text{if } x \in [0, L/2) \\ x - \frac{L}{2} & \text{if } x \in [L/2, L] \end{cases}$$

To compute the Fourier cosine series, we need to find the even extension of $f(x)$:



From this image, it is easy to see that the even extension of f is

$$f_{\text{even}}(x) = \begin{cases} -(x + L/2) & \text{if } x \in [-L, L/2) \\ L/2 & \text{if } x \in [-L/2, L/2] \\ x - L/2 & \text{if } x \in (L/2, L] \end{cases}$$

All that remains is computation of a_0 and a_n:

$$a_0 = \frac{1}{L}\int_0^L f_{\text{even}}(x)\,dx = \frac{1}{L}\int_0^{L/2} L/2\,dx + \frac{1}{L}\int_{L/2}^L (x - L/2)\,dx = \frac{3L}{8}.$$

$$\begin{aligned} a_n = \frac{2}{L}\int_0^L f_{\text{even}}(x)\cos\left(\frac{n\pi x}{L}\right)dx &= \frac{2}{L}\int_0^{L/2} \frac{L}{2}\cos\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^L (x - L/2)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= L\text{sinc}\left(\frac{n\pi}{2}\right) + \frac{L}{n^2\pi^2}\left(-2\cos\left(\frac{n\pi}{2}\right) + 2\cos(n\pi)\right) \\ &= L\text{sinc}\left(\frac{n\pi}{2}\right) + \frac{2L}{n^2\pi^2}\left(-\cos\left(\frac{n\pi}{2}\right) + (-1)^n\right) \end{aligned}$$

The result was obtained by an application of integration by parts. Thus the Fourier cosine series of f is

$$f(x) \sim \frac{3L}{8} + \sum_{n=1}^{\infty}\left(L\text{sinc}\left(\frac{n\pi}{2}\right) + \frac{2L}{n^2\pi^2}\left(-\cos\left(\frac{n\pi}{2}\right) + (-1)^n\right)\right)\cos\left(\frac{n\pi x}{L}\right)$$

$\square$

3 Convergence of Fourier Series

Theorem — Convergence of Fourier Series

Let $f(x)$ be defined on $[-L, L]$ then the trigonometric Fourier series ($F(x)$) of f converges pointwise to f in the following manner:

$$F(x_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x_0}{L}\right) + b_n \sin\left(\frac{n\pi x_0}{L}\right) = \frac{f(x_0^-) + f(x_0^+)}{2}.$$

Therefore if f is continuous at some $x_0 \in (-L, L)$, then

$$F(x_0) = f(x_0)$$

and if f is discontinuous at some $x_0 \in (-L, L)$, the Fourier series converges to the average value of the discontinuity:

$$F(x_0) = \frac{f(x_0^-) + f(x_0^+)}{2}.$$

3.1 Bessels Inequality

Theorem — Bessels Inequality

Let H be any abstract vector space with an inner product and suppose that x is some element of H and that $e_0, e_1, e_2, \ldots,$ form an orthonormal sequence in H. Then for any $x \in H$, Bessels inequality states

$$\sum_{k=0}^{\infty} |\langle x, e_k \rangle|^2 \leq ||x||^2.$$

Corollary(s):
(1) If H is the space $PWC(-L, L)$ and $f \in PWC(-L, L)$, then suppose the set $1, \cos(n\pi x/L), \sin(n\pi x/L), \cdots \in PWC(-L, L)$ is an orthogonal sequence in $PWC(-L, L)$ then

$$a_0^2 + \frac{1}{2}\sum_{k=0}^{\infty}(a_k^2 + b_k^2) \leq \frac{1}{2L}\int_{-L}^{L} f(x)^2\, dx.$$

(2) If this inequality holds true, then the series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

will converge pointwise to a value.

Using the definition of Bessel's inequality, we will show corollary (1). We consider the orthonormal sequence

$$e_k =: \left\{\frac{1}{\sqrt{2L}}\right\} \cup \left\{\frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right), \frac{1}{\sqrt{L}}\cos\left(\frac{2n\pi x}{L}\right), \ldots,\right\} \cup \left\{\frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right), \frac{1}{\sqrt{L}}\sin\left(\frac{2n\pi x}{L}\right), \ldots\right\}$$

We know that this is the orthonormal sequence since:

$$||1|| = \sqrt{\int_{-L}^{L} (1)^2\, dx} = \sqrt{2L}$$

$$||\cos\left(\frac{n\pi x}{L}\right)|| = \sqrt{\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)\,dx} = \sqrt{L}$$

$$||\sin\left(\frac{n\pi x}{L}\right)|| = \sqrt{\int_{-L}^{L}\sin^2\left(\frac{n\pi x}{L}\right)\,dx} = \sqrt{L}$$

Thus the division of the orthogonal sets by these quantities give the orthonormal set. Then via the theorem above we have

$$\left|\left\langle\frac{1}{\sqrt{2L}}, f\right\rangle\right|^2 + \sum_{k=1}^{N}\left|\left\langle\frac{1}{\sqrt{L}}\cos\left(\frac{k\pi x}{L}\right), f\right\rangle\right|^2 + \left|\left\langle\frac{1}{\sqrt{L}}\sin\left(\frac{k\pi x}{L}\right), f\right\rangle\right|^2 \leq |\langle f, f\rangle|^2$$

$$2La_0^2 + \frac{L^2}{L}\sum_{k=1}^{N}(a_k^2 + b_k^2) \leq \int_{-L}^{L} f^2(x)\,dx$$

$$a_0^2 + \frac{1}{2}\sum_{k=1}^{N}(a_k^2 + b_k^2) \leq \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx$$

Letting $N \to \infty$ we obtain corollary (1):

$$a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx.$$

3.2 Parsevals Identity

Theorem — Parsevals Theorem

Let H be any abstract vector space with an inner product and suppose that x is some element of H and that $e_0, e_1, e_2, \ldots,$ form an orthonormal sequence in H. Further suppose that this orthonormal sequence is an orthonormal basis for the space H, then Parsevals theorem states

$$\sum_{k=0}^{\infty}|\langle x, e_k\rangle|^2 = \left|\left|x\right|\right|^2.$$

Corollary: If H is the space $PWC(-L, L)$ and $f \in PWC(-L, L)$, and suppose f can be written as a trigonometric Fourier series (can be represented as a complete linear combination of sine and cosine basis functions), then Parsevals theorem/identity states that Bessels inequality is strengthened to equality:

$$a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx.$$

Example **Parsevals Identity**

Show that the Fourier series of $f(x) = x$ on $[-L, L]$ is given by

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right),$$

then use Parsevals identity to show the famous result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution: We know that $f(x) = x$ is odd on the interval $[-L, L]$, thus $a_n = 0 \; \forall \; n \geq 0$. Thus all that remains is computation of b_n:

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L}\int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx \\ &= -\frac{2L}{n\pi}(-1)^n \\ &= \frac{2L}{n\pi}(-1)^{n+1}. \end{aligned}$$

The above result was obtained via integration by parts and by recognizing that for integer n we have $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$. Thus the Fourier series of $f(x)$ on $x \in (-L, L)$ is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right).$$

To show the summation result (the solution to the Bassel problem), we make use of Parsevals identity

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2L}\int_{-L}^{L} f^2(x)\, dx.$$

For this problem we have

$$\begin{aligned} \frac{1}{2}\sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2} &= \frac{1}{2L}\int_{-L}^{L} x^2\, dx \\ \frac{2L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{1}{2L}\left(\frac{L^3}{3} - \frac{(-L)^3}{3}\right) \\ \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{L^2}{3}\frac{\pi}{2L^2} = \frac{\pi^2}{6}. \end{aligned}$$

$\square$

4 Use of Fourier Series in Solving PDEs

Fourier series are used in the solution of separable PDEs. A PDE is a partial differential equation. Math 201 mostly focuses on the heat equation and sometimes the wave equation. While most of this note package has provided derivations and other information, this section will only contain examples. It is assumed that boundary conditions, linearity and superposition have been discussed in class.

4.1 Heat Equation in 1D

Example	Homogeneous Heat Equation

Solve the following heat equation IVBP on $x \in [0, \pi]$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t \geq 0 \\ u(x,0) = 2\sin(2x) + 4\sin(4x), & 0 < x < \pi. \end{cases}$$

How would the solution to the problem change if $u(x,0) = \sin\left(\dfrac{1}{2}x\right)$?

Solution: We seek a separated solution in the form $u(x,t) = X(x)T(t)$, substitution into the differential equation will give

$$X(x)T'(t) = 3X''(X)T(t).$$

Rearranging this gives the ratio

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{3T(t)}.$$

It is important to note that the only way that these two ratios are equal are if the ratios themselves are constant. Why? Imagine that we choose any particular moment in time $t_0 > 0$, then

$$\frac{X''(x)}{X(x)} = \frac{T'(t_0)}{3T(t_0)}.$$

Thus for all x in the domain of the problem, it must be the case that $X''(x)/X(x)$ is equal to a constant since $T'(t_0)/3T(t_0)$ is a a constant. A similar argument will hold for any $x_0 \in (0, \pi)$ while t is free to vary. Thus we write

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{3T(t)} = -\lambda,$$

where $\lambda \in \mathbb{R}$. Setting the ratios equal to $-\lambda$ is customary, but know that there isn't really a deep meaning as to why it is negative, only that it makes the math easier. This ratio gives us the following two differential equations:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ T'(t) + 3\lambda T(t) = 0. \end{cases}$$

For the X problem, how do we obtain the initial conditions to solve it? Consider the boundary conditions which tell us that the heat at the ends of the rod is zero. Let us write them out in separated form

$$u(0,t) = X(0)T(t) = 0 \Longrightarrow X(0) = 0.$$

Note the contradiction that arises if $T(t) = 0$. If $T(t) = 0$ then we will obtain a trivial solution for u, i.e $u(x,t) = 0$ and we do not care for this solution. Thus we take $X(0) = 0$. Likewise

$$u(\pi,t) = X(\pi)T(t) = 0 \Longrightarrow X(\pi) = 0.$$

Thus we have the following IVP for $X(x)$:

$$X''(x) + \lambda X(x) = 0, \;\; X(0) = X(\pi) = 0.$$

Technically, this ODE has three solutions. One solution is for $\lambda > 0$, another for $\lambda < 0$ and the final one for $\lambda = 0$. Let us start with the easiest case, $\lambda = 0$:

$$X''(x) = 0 \Longrightarrow X(x) = Ax + B$$

where A, B are real constants. Application of the initial conditions gives $X(x) = 0$, which gives the trivial solution $u(x,t) = 0$ to the PDE. Therefore we discard the $\lambda = 0$ case.

Next we consider the case where $\lambda < 0$. If $\lambda < 0$ then the characteristic equation for the ODE is $r = \pm\sqrt{-\lambda}$ and the general solution is

$$X_n(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

Application of the initial conditions gives us the system of equations

$$\begin{cases} A + B = 0 \\ Ae^{\pi\sqrt{-\lambda}} + Be^{-\pi\sqrt{-\lambda}} = 0. \end{cases}$$

To determine (A, B) we write this system as the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 1 \\ e^{\pi\sqrt{-\lambda}} & e^{-\pi\sqrt{-\lambda}} \end{bmatrix}}_{T} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let us take the determinant of T since the solution for (A, B) will require $\det(T)$:

$$\det(T) = e^{-\pi\sqrt{-\lambda}} - e^{\pi\sqrt{-\lambda}} \neq 0$$

Notice that $\det(T) \neq 0$ since $\lambda < 0$. Therefore the only way the above matrix equation could possibly be true is if $(A, B) = (0, 0)$. Therefore we discard the $\lambda < 0$ case since it gives the trivial solution $u(x,t) = 0$.

Finally we consider the case where $\lambda > 0$. For this case the characteristic equation of the ODE is $r = \pm i\sqrt{\lambda}$ and we have

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Application of the first initial condition gives

$$X(0) = 0 = A.$$

Application of the second initial condition gives

$$X(\pi) = 0 = B\sin(\sqrt{\lambda}\pi).$$

If $B = 0$, then we once again arive at the trivial solution $u(x,t) = 0$. Therefore our only hope of finding a non-trivial solution to the problem is if

$$\sin(\sqrt{\lambda}\pi) = 0,$$

which can only be true if

$$\sqrt{\lambda_n}\pi = n\pi \Longrightarrow \lambda_n = n^2$$

where n is a positive integer. What we just solved is called an **eigenvalue problem** for λ_n. This problem has associated **eigenfunctions**

$$X_n(x) = \sin(nx).$$

Now we return to the time ODE $T_n'(t) + 3\lambda T_n(t) = 0$, and since $\lambda_n > 0$, we know it will have nth solution

$$T_n(t) = e^{-3\lambda_n t} = e^{-3n^2 t}.$$

Thus the nth solution to the heat problem is $u_n(x,t) = X_n(x)T_n(t) = \sin(nx)e^{-3n^t}$. Why have we been ignoring constants until now? The answer to that question involves the linearity of the PDE. We know that the solution to the PDE will be an infinite *linear combinition* of this nth solution we found, therefore we are simply waiting to take care of constants until the end to make things easier on ourselves. The general solution to the heat problem is

$$u(x,t) = \sum_{n=1}^{\infty} d_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} d_n \sin(nx) e^{-3n^2 t}.$$

How do we compute d_n? We use the initial condition $u(x,0)$:

$$u(x,0) = 2\sin(2x) + 4\sin(4x) = \sum_{n=1}^{\infty} d_n \sin(nx).$$

We can see that the only non-zero terms are d_2 and d_4 by inspection:

$$\begin{cases} d_n = 0 & \text{if } n \neq 2,4 \\ d_n = 2 & \text{if } n = 2 \\ d_4 = 4 & \text{if } n = 4. \end{cases}$$

Thus the final solution to this heat problem is

$$u(x,t) = 2\sin(2x)e^{-12t} + 4\sin(4x)e^{-48t}.$$

If the initial condition was $u(x,0) = \sin(\frac{1}{2}x)$, then we would not be able to state the solution by inspection since we would have

$$u(x,0) = \sin\left(\frac{1}{2}x\right) = \sum_{n=1}^{\infty} d_n \sin(nx)$$

We have two methods of solution (they will give the same answer). The first is to recognize this is the Fourier sine series of $\sin(\frac{1}{2}x)$ on $(0,\pi)$ and we could write

$$d_n = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{1}{2}x\right)\sin(nx)\,dx,$$

and computation would give the result for d_n. Or if we did not notice this, we could use the definition of a generalized Fourier series:

$$d_n = \frac{\displaystyle\int_0^{\pi} \sin\left(\frac{1}{2}x\right)\sin(nx)\,dx}{\displaystyle\int_0^{\pi} \sin^2(nx)\,dx} = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{1}{2}x\right)\sin(nx)\,dx.$$

$\square$

Example	Heat Equation

Find the solution to the heat flow problem

$$\begin{cases} u_t = 5u_{xx}, \;\; x \in (-\pi, 0), \; t > 0 \\ u(-\pi, t) = u(0,t) = 0, \;\; t > 0 \\ u(x,0) = f(x) = x(\pi + x), \;\; x \in (-\pi, 0) \end{cases}$$

Solution: To solve we substitute the separation $u(x,t) = X(x)T(t)$ into the equation and we obtain the ODEs

$$\begin{cases} X''(x) + \lambda(x) = 0, \;\; X(-\pi) = X(0) = 0 \\ T'(t) + 5\lambda T(t) = 0 \end{cases}$$

The spacial eigenvalue problem has eigenvalues $\lambda_n = n^2$, and corresponding eigenfunctions $X_n(x) = \sin(nx)$. The nth time solution is thus given by

$$T_n(t) = e^{-5n^2 t}$$

and we can write

$$u(x,t) = \sum_{n=1}^{\infty} f_n \sin(nx) e^{-5n^2 t}.$$

Applying the initial condition we obtain

$$u(x,0) = f(x) = x(\pi + x) = \sum_{n=1}^{\infty} f_n \sin(nx).$$

Note that here we cannot simply say that the Fourier sine series specifies the coeffecients f_n. This is because the PDE is not specified on the interval $x \in (0, L)$ for which the original formulas were derived. Thus we have to derive the coeffecients for this expansion. The family of eigenfunctions $X_n(x) = \sin(nx)$ is orthogonal, and complete, thus we can write

$$\begin{aligned} f_n &= \frac{\displaystyle\int_{-\pi}^{0} f(x)\sin(nx)\,dx}{\displaystyle\int_{-\pi}^{0} \sin^2(nx)\,dx} \\ &= \frac{2}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_{-\pi}^{0} x(\pi + x)\sin(nx)\,dx \\ &= 2\int_{-\pi}^{0} x\sin(nx)\,dx + \frac{2}{\pi}\int_{-\pi}^{0} x^2\sin(nx)\,dx \\ &= \frac{4}{n^3\pi}(1 - (-1)^n) \end{aligned}$$

Thus we have

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi}(1 - (-1)^n)\sin(nx)e^{-5n^2 t},$$

we can of course simplify by using only the $n = 2k - 1$ terms:

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8}{(2k-1)^3\pi}\sin((2k-1)x)e^{-5(2k-1)^2 t}.$$

Note: The problem on $x \in (a, b)$ can also be solved by reducing it to the case of $x \in (0, L)$ through the substitution $L = b - a$, then we let $v(x,t) = u(x - \pi, t)$. Here $L = \pi$, $a = -\pi$, $b = 0$, and this will yield us the problem

$$\begin{cases} v_t = 5v_{xx}, \;\; x \in (0, \pi), \;\; t > 0 \\ v(0,t) = v(\pi, t) = 0, \;\; t > 0 \\ v(x, 0) = f(x - \pi) = (x - \pi)x, \;\; x \in (0, \pi). \end{cases}$$

Then after separation, and computation of the regular Fourier sine series, reverse the substitution: $u(x,t) = v(x - \pi, t)$. However, a simple substitution to shift the spacial position may not always work, and the first method presented is more general.

$\square$

4.2 Method of Eigenfunction Expansions

Now we consider a forcing function to the heat equation. The forcing scenario is complicated because the forcing function could be dependent on both x and t. In this case we may note assume $u(x,t) = w(x,t) + v(x,t)$. The critical idea behind this method is using the eigenfunctions of the homogeneous problem as an eigenbasis for the solution to the non-homogeneous problem. Meaning we can expand some function in terms of eigenfunctions where

$$f_n = \frac{\displaystyle\int_a^b f(x) X_n(x)\,dx}{\displaystyle\int_a^b (X_n(x))^2\,dx}$$

and these coefficients belong to the sum

$$f(x) = \sum_{n=0}^{\infty} f_n X_n(x).$$

We start with an example that appeared on an old final examination.

Example Heat Equation//Previous Final Exam Question

Find the solution to the heat equation in $\mathbb{R}$ given by

$$u_t = u_{xx} + \sin(\pi x)e^{-t}, \;\; 0 < x < 1, \;\; t > 0,$$

with the boundary/initial conditions

$$u(0,t) = u(1,t) = 0, \;\; u(x,0) = 0.$$

Solution: We will solve this problem using the method of eigenfunction expansions. Thus we consider the homogeneous problem solved by $w(x,t)$.

$$w_t = w_{xx}, \quad w(0,t) = w(1,t) = 0.$$

The eigenvalue problem will be solved with eigenvalues $\lambda_n = n^2\pi^2$ and associated eigenfunctions

$$X_n(x) = \sin(n\pi x).$$

The work to get to this point was skipped since it is not the area of the problem that I want to illustrate. We want to use the eigenfunctions of the homogeneous problem as an orthogonal basis for the solution to the non-homogenous problem. Meaning we write the separated solution to the non-homogeneous problem as:

$$u(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x).$$

Another useful aspect of this method is that we can also write the forcing term as an expansion of the eigenfunctions of the homogeneous problem.:

$$\sin(\pi x)e^{-t} = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x).$$

Substitution into the PDE, then re-arranging gives

$$\sum_{n=1}^{\infty}(w_n'(t) + n^2\pi^2 w_n(t))\sin(n\pi x) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x).$$

This gives us the family of ODEs

$$w_n'(t) + n^2\pi^2 w_n(t) = f_n(t).$$

We know that $f_n(t)$ has two cases, if $n = 1$ then $f_1(t) = e^{-t}$, for all other $n > 1$ we have $f_n(t) = 0$. The initial conditions for these ODEs will be obtained by application of initial conditions to the expression

$$u(x,t) = \sum_{n=1}^{\infty} w_n(t) X_n(x) = \sum_{n=1}^{\infty} w_n(t) \sin(n\pi x).$$

Using the initial condition we obtain:

$$u(x,0) = 0 = \sum_{n=1}^{\infty} w_n(0) \sin(n\pi x) \implies w_n(0) = 0 \;\forall\; n \geq 1.$$

We have therefore arrived at the following problem for w:

$$w_n'(t) + n^2\pi^2 w_n(t) = \begin{cases} e^{-t}, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \qquad w_n(0) = 0 \;\forall\; n \geq 1.$$

The easier problems to solve are the $n > 1$ cases, this is easily solved by separation or exponential factor:

$$w_n(t) = Ae^{-n^2\pi^2 t}, \;\; n > 1.$$

Applying the initial condition gives $A = 0 \;\forall\; n > 1$. Now we focus on the $n = 1$ case which is the IVP

$$w_1'(t) + \pi^2 w_1(t) = e^{-t}, \;\;\; w_1(t) = 0.$$

For this we introduce the exponential factor $\mu(t) = e^{\pi^2 t}$ and multiply the entire equation by it. Then applying the product rule in reverse we obtain

$$\left(w_1(t) e^{\pi^2 t}\right)' = e^{t(\pi^2 - 1)},$$

integration and solving for $w_t(t)$ gives

$$w_1(t) = \frac{1}{\pi^2 - 1} e^{-t} + ce^{-\pi^2 t}.$$

The initial condition will give

$$c = -\frac{1}{\pi^2 - 1}.$$

Thus the final expression for $w_n(t)$ is

$$w_n(t) = \begin{cases} \dfrac{1}{\pi^2 - 1}\left(e^{-t} - e^{-\pi^2 t}\right), & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Thus the only non-zero component of u is at $n = 1$, and we have finally arrived at the solution to the non-homogeneous problem:

$$u(x,t) = \frac{\sin(\pi x)}{\pi^2 - 1}\left(e^{-t} - e^{-\pi^2 t}\right).$$

$\square$

Example Eigenfunction Expansions

Solve the following IVBP for the heat equation given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < 1,\ t > 0 \\ u_x(0,t) = 1,\ u_x(1,t) = 2, & t > 0 \\ u(x,0) = x^2, & 0 \le x \le 1. \end{cases}$$

Solution: We suppose that $u(x,t)$ is in the form

$$u(x,t) = \phi(x) + w(x,t)$$

where $\phi(x)$ takes on the non-homogeneous boundary conditions, and $w(x,t)$ takes on homogeneous boundary conditions. Thus we need a function $\phi(x)$ that is twice differentiable, and that satisfies $\phi'(0) = 1$ and $\phi'(1) = 2$. One such function that satisfies these conditions

$$\phi(x) = \frac{1}{2}x^2 + x.$$

Now we focus on the homogeneous problem from w, before we do this we note

$$u(x,0) = x^2 = \phi(x) + w(x,0) = \frac{1}{2}x^2 + x + w(x,0) \Longrightarrow w(x,0) = \frac{1}{2}x^2 - x.$$

With this fact, all that remains is solving the problem in $w(x,t)$. We obtain the w problem by substitution of $u = \phi + w$ into the PDE:

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + 1 \\ w_x(0,t) = w_x(1,t) = 0 \\ w(x,0) = \dfrac{1}{2}x^2 - x. \end{cases}$$

To solve this problem we make use of the method of eigenfunction expansions. Therefore we find the eigenfunctions of the non-sourced problem. The non sourced probem ($w_t = w_{xx}$) will have eigenfunctions and associated eigenvalues

$$X_n(x) = \cos(n\pi x),\ \lambda_n = n^2\pi^2.$$

We seek to expand $w(x,t)$ and the sourcing term using the eigenbasis given by $X_n(x)$, this is possible since $X_n(x)$ form a complete orthogonal set. Thus we write

$$w(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t)\cos(n\pi x),\ \text{and}\ 1 = \beta_0(t) + \sum_{n=1}^{\infty} \beta_n(t)\cos(n\pi x)$$

Note that we can find $\beta_n(t)$ by inspection:

$$\beta_n(t) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases}$$

We can find w_t and w_{xx} directly from differentiating the sums:

$$\frac{\partial w}{\partial t} = \sum_{n=0}^{\infty} \alpha_n'(t)\cos(n\pi x)$$
$$\frac{\partial^2 w}{\partial x^2} = \sum_{n=0}^{\infty} -(n^2\pi^2)\alpha_n(t)\cos(n\pi x)$$

Substitution of all of this into our PDE and re-arranging gives

$$\sum_{n=0}^{\infty}(\alpha_n(t)' + n^2\pi^2\alpha_n(t))\cos(n\pi x) = \sum_{n=0}^{\infty}\beta_n(t)\cos(n\pi x)$$

This gives rise to the following family of ODEs:

$$\alpha_n'(t) + n^2\pi^2\alpha_n(t) = \beta_n(t) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases}$$

We need initial conditions on $\alpha_n(t)$ to solve the problem

$$w(x,0) = \sum_{n=0}^{\infty}\alpha_n(0)\cos(n\pi x)$$

This is the Fourier cosine series for $w(x,0)$ with coefficients $\alpha_n(0)$. Thus

$$\alpha_n(0) = 2\int_0^1 \left(\frac{1}{2}x^2 - x\right)\cos(n\pi x)\,dx$$

Repeated application of integration by parts gives:

$$\alpha_n(0) = \frac{2}{n^2\pi^2},\ n \neq 0$$

For the case where $n = 0$ we have

$$a_0(0) = \int_0^1 \left(\frac{1}{2}x^2 - x\right)\,dx = -\frac{1}{3}.$$

Therefore we have the following initial conditions for $n \geq 0$:

$$\alpha_n(0) = \begin{cases} -\frac{1}{3}, & \text{if } n = 0 \\ \frac{2}{n^2\pi^2}, & \text{if } n > 0. \end{cases}$$

We have arrived at two ODE's which we need to solve:

$$\begin{cases} \alpha_n'(t) + n^2\pi^2\alpha_n(t) = 0, & \alpha_n(0) = \frac{2}{n^2\pi^2} \\ \alpha_0'(t) = 1, & \alpha_0(0) = -\frac{1}{3} \end{cases}$$

The $n = 0$ case is easy as we only have to apply direct integration: $\alpha_0(t) = t + C$ and application of initial condition yields $\alpha_0(t) = t - 1/3$. For the case for $n > 0$ we introduce the integration factor $e^{n^2\pi^2 t}$ and we obtain

$$\left(\alpha_n(t)e^{n^2\pi^2 t}\right)' = 0 \Longrightarrow \alpha_n(t) = Ce^{-n^2\pi^2 t}.$$

Application of the initial condition will give C, thus we have obtained

$$\alpha_n(t) = \begin{cases} t - \frac{1}{3}, & \text{if } n = 0 \\ \frac{2}{n^2\pi^2}e^{-n^2\pi^2 t}, & \text{if } n > 0. \end{cases}$$

The solution to this problem for $w(x,t)$ is therefore

$$w(x,t) = \left(t - \frac{1}{3}\right) + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \cos(n\pi x) e^{-n^2\pi^2 t}.$$

The solution to the original problem posed is $\phi(x) + w(x,t)$ therefore we finally have arrived at

$$u(x,t) = \frac{1}{2}x^2 + x + t - \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \cos(n\pi x) e^{-n^2\pi^2 t}$$

$\square$

5 Recap

What to Know About the Fourier Series

(1) Know that any function which is absolutely integrable on $(-L, L)$ will have an associated Fourier series

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right).$$

(2) Know that when computing a Fourier series if a_n or b_n is undefined for any particular value of n, compute it directly. For example if a_n is undefined at $n = 2$, compute a_2 directly with the integral definition of a_n. Also know when computing Fourier series on an interval $x \in (-L, L)$, if $f(x)$ is even then $b_n = 0$, and if $f(x)$ is odd then $a_n = 0$.

(3) When computing Fourier series, the following trigonometric identities are often useful:

$$\begin{aligned}
\sin(\alpha)\cos(\beta) &= \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)\\
\sin(\alpha)\sin(\beta) &= \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)\\
\cos(\alpha)\cos(\beta) &= \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)
\end{aligned}$$

(4) Know Parsevals identity. If f can be represented as a Fourier series on $(-L, L)$, then

$$a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx.$$

(5) In the study of eigenvalue problems regarding linear PDEs, know that the associated eigenfunctions in MATH 201 will always be orthogonal. Meaning that the set $X_n(x)$ form a complete orthogonal set and they can be used in a Fourier series. If $X_n(x)$ are orthogonal eigenfunctions of an eigenvalue problem on (a, b) then:

$$f(x) \sim \sum_{n=0}^{\infty} k_n X_n(x) \text{ where } k_n = \frac{\displaystyle\int_a^b f(x)X_n(x)\,dx}{\displaystyle\int_a^b X_n^2(x)\,dx}.$$

(6) A sourced linear PDE (heat or wave equation in Math 201) that has a sourcing function $F(x, t)$ and **homogeneous boundary conditions** can be solved by using the eigenfunctions of the homogeneous problem $X_n(x)$ as an orthogonal basis for the solution to the non-homogeneous problem and as an orthogonal basis for $F(x, t)$. Substitute the following into the PDE:

$$u(x,t) = \sum_{n=1}^{\infty}\alpha_n(t)X_n(x) \text{ and } F(x,t) = \sum_{n=1}^{\infty}\beta_n(t)X_n(x).$$

(7) Make sure to know if a given eigenvalue problem has $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$ as possible cases. Make sure to always check if you are not sure.