Introduction to the Boundary Element Method, and Examples

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I. Introduction and Review

The Boundary Element Method is another computational method used to solve linear PDEs – in the case of this notes package PDEs which arise in electromagnetics. This method (when applicable) is often preferred over a FEM approach for a few reasons: Firstly a standard FEM approach is $\mathcal{O}(n^3)$ whereas a standard BEM approach is $\mathcal{O}(n^2)$, secondly in linear problems only the boundary of the problem needs to be discretized in a BEM approach, whereas the entire problem must be discretized in an FEM approach, thirdly BEM approaches converge faster and with more accuracy than FEM approaches. The BEM method can be used for problems with infinite domain without the need for implementation of things like radiating boundary conditions, this is useful in antenna problems for example. The simple dipole antenna will be modelled using the BEM in this notes package. Note that this notes package assumes at least an introductory course in PDEs which covers Green's functions as prerequisite and some familiarity with numerical methods in PDEs and assumes a working knowledge of vector calculus. The goal of this notes package is not to provide the most rigorous introduction to the boundary element method, the goal is instead to provide a more problem oriented guide to solving simple problems which do not require curvilinear discretization and implementation in a programming language that isn't as old as computers, i.e Python.

At the level of heuristics, the method works on the "inverse" problem of a PDE, meaning the PDE modelling the scenario under question is converted to an integral equation (IE) and the boundary is descretized instead of the entire domain of the problem. The integral equation is then discretized to obtain a numerical solution to the problem at hand. Typically the easiest way to formulate an equivalent IE for a PDE is to have knowledge of the fundamental behaviour of solutions to the PDE – Green's functions.

We proceed by briefly reviewing Green's functions and Dirac delta distributions. Let $\hat{L}$ be a linear differential operator acting on u and let f be a sourcing term on domain Ω which has boundary $\partial\Omega$. For now we assume that $\hat{L}u = f$ is constrained under the following boundary value problem (BVP):

$$\hat{L}u = f(\mathbf{r}), \quad \mathbf{r} \in \Omega \tag{I.1}$$

$$u\Big|_{\partial\Omega} = 0 \tag{I.2}$$

Then the **Green's function**, $G(\mathbf{r} - \mathbf{r}')$, for $\hat{L}$ is the solution to the following BVP:

$$\hat{L}G = \delta(\mathbf{r} - \mathbf{r}') \tag{I.3}$$

$$G\Big|_{\partial\Omega} = 0 \tag{I.4}$$

A decent high level analogy can be drawn here if one recalls the unit impulse response based approach of analysing LTIC systems in an introductory signals class. By inputting into the PDE (the system) a delta distribution, information about the "fundamental" response of the system can be ascertained and thereby the resonse to any sourcing function. However, the singular excitation can be anywhere in the system, hence the $\delta(\mathbf{r} - \mathbf{r}')$. [1] Recall that the Dirac delta distribution satisfies the following properties:

$$\int_{x-\epsilon}^{x+\epsilon} \delta(x - x')\, dx' = 1, \ \epsilon > 0 \tag{I.5}$$

$$\lim_{x \to \epsilon} \delta(x - \epsilon) = \infty \tag{I.6}$$

and the distribution has the following useful convolution property:

$$\int_{-\infty}^{\infty} f(x')\delta(x - x')\, dx' = f(x) \tag{I.7}$$

[1] Note $\delta(\mathbf{r})$ is short hand notation in coordinate systems larger than 1D. For example in 3D Cartesian coordinates: $\delta(\mathbf{r}) = \delta(x, y, z) = \delta(x)\delta(y)\delta(z)$

Using this property, we can construct the sourcing term in BVP (I.1) as follows:

$$f(\mathbf{r}) = \int_\Omega f(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,d\Omega' \tag{I.8}$$

However via (I.3) we have

$$f(\mathbf{r}) = \int_\Omega f(\mathbf{r}')\hat{L}\left\{G\right\}\,d\Omega' = \hat{L}\left\{\int_\Omega f(\mathbf{r}')G(\mathbf{r}-\mathbf{r}')\,d\Omega'\right\} \tag{I.9}$$

Here the operator $\hat{L}$ can be interchanged with the integral as it operates on $\mathbf{r}$ coordinates, and the integral is with respect to the primed coordinates. Thus plugging (I.9) into (I.1) we obtain:

$$\hat{L}u = \hat{L}\left\{\int_\Omega f(\mathbf{r}')G(\mathbf{r}-\mathbf{r}')\,d\Omega'\right\} \tag{I.10}$$

Or rather let's state this another way:

$$\hat{L}\left\{u\right\} = \hat{L}\left\{u \text{ in disguise}\right\} \tag{I.11}$$

It is clear then that the two quantities under the operator must be the same thing (more precisely up to a constant). This isn't the most precise derivation, but at at least a surface level of mathematical intuition it should be clear that for the BVP (I.1) we have the following:

$$\boxed{u(\mathbf{r}) = \int_\Omega f(\mathbf{r}')G(\mathbf{r}-\mathbf{r}')\,d\Omega'} \tag{I.12}$$

Now to phrase (I.12) more coherently we can say **we have transformed a PDE in u into an IE in f**. To briefly hint at one application of the BEM approach: If u is known on the boundary, then enough information is present to find f. The reader may wonder when is this useful? After all the typical case in PDE problems is the desire to see a systems evolution in response to sources, and fluxes and fixtures on the boundary. Here's a simple inverse problem that arises in electromagnetics: What is the charge distribution on a conductor held at some fixed potential V? Consider that in electromagnetics potential, V, is related to charge distribution, ρ, via the following PDE:

$$\nabla^2 V = -\rho(\mathbf{r})/\epsilon \tag{I.13}$$

Consider that we know the Green's function for such a problem, G, then we can write (assuming homogeneous Dirichlet conditions):

$$u(\mathbf{r}) = -\int_\Omega \frac{\rho(\mathbf{r}')}{\epsilon_0}G(\mathbf{r}|\mathbf{r}')\,d\Omega' \text{ }^{2} \tag{I.14}$$

If V is known on the boundary of the problem, *then it is the Green's function and weighting functions which transform the information on the boundary into information about the rest of the domain.* Thus to find the charge distribution *everywhere* it suffices to only know the potential on the boundary (which is common). This process will be shown in the next section.

Aside: This process of transforming boundary information into function information on a domain by investigating singularities (Green's functions) should ring alarm bells from an introductory complex analysis course: Cauchy's integral formula allows one to determine function values in a domain from the boundary by investigating singularities. We'll leave this thought behind for now, but I encourage any reader of these notes to keep this connection in mind because it is no coincidence and at the very least to comfort those uneasy with the soundness of the approach.

While an introductory course in PDE's may cover (I.12), these courses may neglect to mention that (I.12) is the simplest version of an I.E that arises from a Green's function approach. To understand why consider that in the initial BVP posed, we assumed there was homogeneous Dirichlet boundary conditions, what happens if some other linear operator $\hat{B}$ operates on the boundary in an inhomogeneous fashion?

For this set of notes the easiest case will be generalized, i.e the 3D case of Poisson's equation. Green's second identity allows us to relate information in the domain to information about fluxes and fixtures on the surface. Let u, v be two scalar fields defined over the volume Ω bounded by the surface $S = \partial\Omega$, then Green's second identity states:

$$\int_\Omega (u\nabla^2 v - v\nabla^2 u)\,d\Omega = \oint_S (u\nabla v\cdot\mathbf{n} - v\nabla u\cdot\mathbf{n})\,dS \tag{I.15}$$

[2]From this point onward we denote $G(\mathbf{r}-\mathbf{r}')$ as $G(\mathbf{r}|\mathbf{r}')$ which is typically standard.

here $\mathbf{n}$ denotes the normal vector to the surface $S = \partial\Omega$. Note that sometimes the following notation is used to denote the gradient-normal vector dot product:

$$\nabla v \cdot \mathbf{n} := \frac{\partial v}{\partial n} \tag{I.16}$$

$$\nabla u \cdot \mathbf{n} := \frac{\partial u}{\partial n} \tag{I.17}$$

This is the notation I am familiar with in the study of PDEs. So we re-write I.15 as

$$\int_\Omega (u\nabla^2 v - v\nabla^2 u)\, d\Omega = \oint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS \tag{I.18}$$

In a heuristic sense, we can see that (I.18) has this property of relating boundary information to volume information. With this identity in mind we now consider the following PDE:

$$\nabla^2 u = -f(\mathbf{r}) \tag{I.19}$$

and we let

$$v = G(\mathbf{r}|\mathbf{r}') \Longrightarrow \nabla^2 v = \nabla^2 G = -\delta(\mathbf{r} - \mathbf{r}') \tag{I.20}$$

In the typical fashion of a Green's function approach we pass the coordinates to primed coordinates:

$$\int_\Omega -u(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') + G(\mathbf{r}|\mathbf{r}')f(\mathbf{r}')\, d\Omega' = \oint_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS' \tag{I.21}$$

On the left hand side we note

$$\int_\Omega -u(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') + G(\mathbf{r}|\mathbf{r}')f(\mathbf{r}')\, d\Omega' = \int_\Omega u(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')\, d\Omega' + \int_\Omega f(\mathbf{r}')G(\mathbf{r}|\mathbf{r}')\, d\Omega' = -u(\mathbf{r}) + \int_\Omega f(\mathbf{r}')G(\mathbf{r}|\mathbf{r}')\, d\Omega' \tag{I.22}$$

Thus in equation (I.21) we can solve for $u(\mathbf{r})$:

$$\boxed{u(\mathbf{r}) = \int_\Omega f(\mathbf{r}')G(\mathbf{r}|\mathbf{r}')\, d\Omega' - \oint_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS'} \tag{I.23}$$

Note: u and G under the surface integral refer to their values *on the boundary*. Compare this equation with the homogeneous case (I.12). Note that (I.23) reduces to (I.12) should u and G be zero on the boundary, which is exactly the homogeneous case we started with. Further note that (I.12) applies to any linear operator $\hat{L}$ with homogeneous boundary conditions and no time dependence, however (I.23) is specifically the I.E for (I.19) with potential for non-homogeneous boundary conditions.

Example 1: Constructing an IE for a Physical Problem. Consider a charged conducting plate situated in free space with dimensions $w \times t$ and zero thickness located in the plane $z = 0$. Suppose the plate is held at a constant potential Φ. Find the integral equation for the charge density on the plate.

For this problem two methods of obtaining the integral equation will be shown. The first method illustrates that IEs can be derived from physical arguments of superposition, and the second utilizes the Green's function approach outlined above.

Solution One, Superposition Argument: Consider the following diagram:

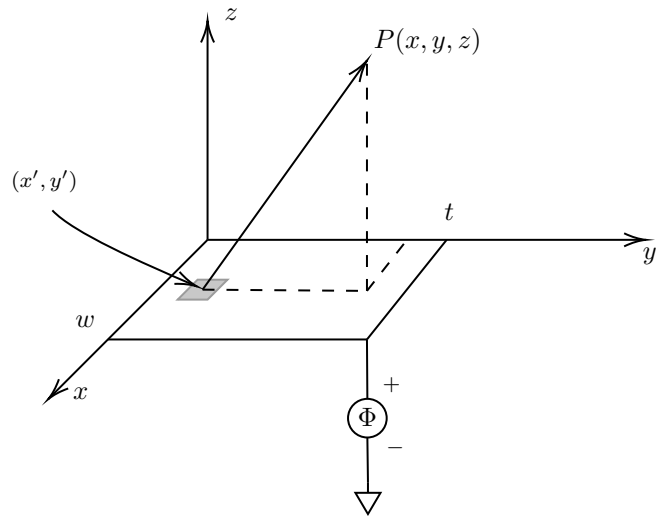

Fig. 1. Plate at potential Φ. We consider a differential area around the "source point" (x', y') (and its local charge density $\rho(x', y')$) and we consider its contribution to the potential at the observation point P.

The potential, u, at point P from this small region of charge can be approximated as:

$$du = \frac{\rho(x', y')}{4\pi\epsilon_0\sqrt{(x-x')^2 + (y-y')^2 + z^2}}\, dx' dy'$$

The total potential at point P is therefore the super position of all possible charge elements (regions around (x', y')), i.e

$$u(x, y, z) = \int_0^t \int_0^w \frac{\rho(x', y')}{4\pi\epsilon_0\sqrt{(x-x')^2 + (y-y')^2 + z^2}}\, dx'\, dy'$$

We know that the plate is held at potential, Φ, therefore:

$$\boxed{u(x, y, 0) = \Phi = \int_0^t \int_0^w \frac{\rho(x', y')}{4\pi\epsilon_0\sqrt{(x-x')^2 + (y-y')^2}}\, dx'\, dy', \quad 0 \le x \le w,\ 0 \le y \le t}$$

This is the integral equation which describes the problem, as it will yield the charge distribution on the plate (more on this later).

Solution Two, Green's Function Approach: A standard PDE course including Green's functions, an electromagnetics course or simply a table would reveal that the Green's function for Poisson's equation in 3D free-space is:

$$G(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

Since the source is always at the plane, we have $z' = 0$ and:

$$G(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi\sqrt{(x-x')^2 + (y-y')^2 + z^2}}$$

Consider (I.23) since the boundary is at infinity, and since we require $|G| \mapsto 0$, $|u| \mapsto 0$ as $\mathbf{r} \mapsto \infty$ we know that the surface integral component of (I.23) does not apply. Hence:

$$u(\mathbf{r}) = \int_\Omega \frac{\rho(\mathbf{r}')}{\epsilon_0} G(\mathbf{r}|\mathbf{r}')\, d\Omega'$$

Thus we have the following expression for potential:

$$u(x, y, z) = \int_0^t \int_0^w \frac{\rho(x', y')}{4\pi\epsilon_0\sqrt{(x-x')^2 + (y-y')^2 + z^2}}\, dx'\, dy'$$

Finally applying the condition $u(x, y, 0) = \Phi$ we obtain the integral equation for the problem:

$$\boxed{\Phi = \int_0^t \int_0^w \frac{\rho(x', y')}{4\pi\epsilon_0\sqrt{(x-x')^2 + (y-y')^2}}\, dx'\, dy', \quad 0 \le x \le w,\ 0 \le y \le t}$$

II. Introduction to the Boundary Element Method

This section introduces the boundary element method using weak solutions to IEs that arises in problems. This method basically identical to Galerkin's method[3] in Finite Element Analysis, or Fourier analysis. Let $\hat{L}^{(i)}$ denote an integral operator, and consider an integral equation of the form:

$$\hat{L}^{(i)}\rho = u \tag{II.1}$$

Typically this equation can be arrived at with Green's function information, or via physical arguments if the Green's function is not readily available. Next with choice of global basis function $\phi_j(\mathbf{r})$ we expand the desired quantity $\rho(\mathbf{r})$ under these weighting functions:

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} \rho_j \phi_j(\mathbf{r}) \tag{II.2}$$

where the computational domain is composed of N boundary elements. This is then substituted into the IE to obtain:

$$\hat{L}^{(i)} \left\{ \sum_{j=1}^{N} \rho_j \phi_j(\mathbf{r}) \right\} = u \tag{II.3}$$

Next we multiply (II.3) by some weighting function w_i (sometimes this is chosen to be ϕ_i^*) and take the inner product over the computational domain:

$$\sum_{j=1}^{N} \rho_j \langle \hat{L}^{(i)} \phi_j, w_i \rangle = \langle u, w_i \rangle, \quad i = 1, 2, 3, \ldots N \tag{II.4}$$

This equation can then be converted to a matrix equation $\mathbf{A}\vec{\rho} = \mathbf{b}$ where

$$A_{ij} = \langle \hat{L}^{(i)} \phi_j, w_i \rangle \tag{II.5}$$

$$b_i = \langle u, w_i \rangle \tag{II.6}$$

From this the quantity ρ can be solved over the entire computational domain. This is all easier said than done.

Example 2: Computing the Charge Density in Example One. Consider the scenario outlined in Example 1. After dividing the plate into N sub-surfaces ΔS_j, define the following global basis functions:

$$\phi_j(x, y) = \begin{cases} 1, & (x, y) \in \Delta S_j \\ 0, & (x, y) \notin \Delta S_j \end{cases}$$

Using a method called point matching (setting the weighting function $w_i = \delta(\mathbf{r} - \mathbf{r}_i)$), determine a matrix equation which when solved determines the charge distribution on the plate.

Solution: We start by expanding the charge distribution $\rho(x, y)$ under the pulse basis function (common basis functions in BEM) as in (II.2):

$$\rho(x, y) = \sum_{j=1}^{N} \rho_j \phi(x, y)$$

Recall that in example one the integral equation for this scenario is as follows:

$$\Phi = \int_0^t \int_0^w \frac{\rho(x', y')}{4\pi\epsilon_0 \sqrt{(x - x')^2 + (y - y')^2}} \, dx' \, dy', \quad 0 \le x \le w, \ 0 \le y \le t$$

Substitution of the charge density expansion gives

$$\Phi = \sum_{j=1}^{N} \rho_j \left(\int_0^t \int_0^w \frac{\phi(x', y')}{4\pi\epsilon_0 \sqrt{(x - x')^2 + (y - y')^2}} \, dx' \, dy' \right)$$

Now note that $\phi_j = 1$ if $(x, y) \in \Delta S_j$ and zero otherwise thus we have the following:

$$\Phi = \sum_{j=1}^{N} \rho_j \iint_{S_j} \frac{1}{4\pi\epsilon_0 \sqrt{(x - x')^2 + (y - y')^2}} \, dx' \, dy'$$

[3] See appendix A for an outline of Galerkin's method

To simplify the problem, we define:

$$\beta_j(x,y) = \iint\limits_{S_j} \frac{1}{4\pi\epsilon_0\sqrt{(x-x')^2+(y-y')^2}}\,dx'\,dy'$$

Which simplifies the integral equation to the following:

$$\Phi = \sum_{j=1}^{N} \rho_j \beta_j(x,y)$$

Now the point matching method is used:

$$\langle \Phi, \delta(x-x_i)\delta(y-y_i)\rangle = \sum_{j=1}^{N} \rho_j \langle \beta_j(x,y), \delta(x-x_i)\delta(y-y_i)\rangle$$

Consider the right hand inner product:

$$\langle \beta_j(x,y), \delta(x-x_i)\delta(y-y_i)\rangle = \int_0^t \int_0^w \beta_j(x,y)\delta(x-x_i)\delta(y-y_i)\,dx\,dy = \beta_j(x_i,y_i)$$

and the left hand inner product is trivial. Thus we have the following equation:

$$\Phi = \sum_{j=1}^{N} \rho_j \beta_j(x_i,y_i)$$

We denote A_{ij} to be the following quantity:

$$A_{ij} = \beta_j(x_i,y_i) = \iint\limits_{\Delta S_j} \frac{dx'dy'}{4\pi\epsilon_0\sqrt{(x_i-x')^2+(y_i-y')^2}}$$

Then the equation for node i is simply:

$$\Phi = \sum_{j=1}^{N} A_{ij}\rho_j$$

To actually evaluate A_{ij}, we can make some easy simplifications to the problem. Firstly, assume that all the charge on the element ΔS_j is located at the source point (x', y') (this is more accurate the more we refine the boundary mesh). This allows us to use a mid-point approximation to evaluate A_{ij}:

$$\boxed{A_{ij} \approx \frac{\Delta S_j}{4\pi\epsilon_0\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}} = \frac{\Delta S_j}{4\pi\epsilon_0 R_{ij}},\ i \neq j}$$

Note that here R_{ij} is the distance between the centers of elements ΔS_i and ΔS_j. The case where $i = j$ is more complicated as the denominator of the integrand tends to zero. The approach here would either be computing the integral manually and taking a limit, or using symbolic integration in CAS. I chose the latter. To set up the integral, we assume that each ΔS_j is of the same size: $\Delta x \times \Delta y$, then

$$A_{ii} = \int_0^{\Delta y}\int_0^{\Delta x} \frac{dx'dy'}{4\pi\epsilon_0\sqrt{(x_i-x')^2+(y_i-y')^2}}$$

Let $\eta = x' - x_i$ and $\zeta = y' - y_i$, then the following expression arises:

$$A_{ii} = \int_{-y_i}^{\Delta y - y_i}\int_{-x_i}^{\Delta x - x_i} \frac{d\eta\,d\zeta}{4\pi\epsilon_0\sqrt{\eta^2+\zeta^2}}$$

Since the coordinates x_i, y_i are at the center of each boundary square, we have:

$$\boxed{A_{ii} = \int_{-\Delta y/2}^{\Delta y/2}\int_{-\Delta x/2}^{\Delta x/2} \frac{d\eta\,d\zeta}{4\pi\epsilon_0\sqrt{\eta^2+\zeta^2}} = \frac{1}{2\pi\epsilon_0}\left[\Delta x \ln\left(\frac{\Delta+\Delta y}{\Delta x}\right) + \Delta y \ln\left(\frac{\Delta+\Delta x}{\Delta y}\right)\right]}$$

where

$$\Delta = \sqrt{\Delta x^2 + \Delta y^2}$$

This integral was evaluated with CAS assistance. With expressions for A_{ij}, we can now write a Python simulation for a particular Φ, w and t. We let $w = t = 1\,\text{mm}$ and $\Phi = 10\,\text{V}$, and in Python we proceed to obtain the charge density.

Example 3: The Simple Dipole Antenna.

APPENDIX

Appendix A – Overview of Galerkin's Method

Galerkin's method is a method of formulating a "weak solution" to a given IBP at hand.