Tips and Tricks for Partial Fraction Decomposition

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Partial Fractions the Normal Way

Let $f(x)$ be a rational function in the form

$$f(x) = \frac{P(x)}{Q(x)}$$

The partial fraction decomposition theorem states that if the degree of $P(x)$ is less than the degree of $Q(x)$ then we may decompose the rational function into a sum of simpler rational functions.

Case One: Simple Unique Factors. If $Q(x)$ has k simple unique factors, i.e $Q(x) = (x - x_0)(x - x_1)(x - x_2)\dots(x - x_k)$ and $x_0 \neq x_1 \neq \dots \neq x_k$, then we may write:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_0}{x - x_0} + \frac{A_1}{x - x_1} + \dots + \frac{A_k}{x - x_k}$$

Example: Find the partial fraction decomposition for the function

$$f(x) = \frac{1}{x^2 - 3x + 2}$$

Solution: We factorise the denominator as $(x - 1)(x - 2)$ then we may write

$$f(x) = \frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

Next we multiply the entire equality by $(x - 1)(x - 2)$ to obtain

$$1 = A(x-2) + B(x-1) = (A+B)x - 2A - B$$

To proceed we may use coeffecient matching. Notice that the left hand side is a zeroth degree polynomial. Therefore we need the first order polynomial coeffecient on the right hand side to go to zero. Thus we arrive at the following system of equations:

$$\begin{cases} A + B = 0 \\ -(2A + B) = 1 \end{cases}$$

It is easy to see that this system has solution $A = -1, B = 1$. Therefore we may write

$$f(x) = \frac{1}{x^2 - 3x + 2} = \frac{1}{x-2} - \frac{1}{x-1}$$

Case Two: Simple Factors With Multiplicity. If $Q(x)$ has k simple factors, some of which have a multiplicity greater than one, we may still expand the rational expression. If $x_0, x_1, x_2, \dots, x_{k-1}$ are simple unique roots and x_k is a root of order/multiplicity n then we may write:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_0}{x - x_0} + \dots + \frac{A_{k-1}}{x - x_{k-1}} + \frac{A_k}{x - x_k} + \frac{A_{k+1}}{(x - x_k)^2} + \dots + \frac{A_{k+n}}{(x - x_k)^n}$$

Example: Find the partial fraction decomposition for the function

$$f(x) = \frac{1}{x(x-1)^3}$$

Solution: Since $x = 1$ is a factor of order 3, we may write

$$f(x) = \frac{1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

To proceed we multiply the entire expression by $x(x-1)^3$ to obtain

$$1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

To proceed we must – tediously – expand the right hand side:

$$1 = A(x^3 - 3x^2 + 3x - 1) + B(x^3 - 2x^2 + x) + C(x^2 - x) + Dx$$

$$1 = (A+B)x^3 - (3A + 2B - C)x^2 + (3A + B - C + D)x - A$$

Thus via coeffecient matching we obtain the following system of equations:

$$\begin{cases} A + B = 0 \\ 3A + 2B - C = 0 \\ 3A + B - C + D = 0 \\ A = -1 \end{cases}$$

We are lucky to have that $A = -1$, which immediately gives $B = 1$. It is thus trivial to solve the other two equations which yield $C = -1$ and $D = 1$. Therefore we have the following decomposition:

$$f(x) = \frac{1}{x(x-1)^3} = -\frac{1}{x} + \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$$

Case Three: No Real Factors. If $Q(x)$ has k simple factors and it also has a factor $J(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ which has no real number factors, then we may write:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_0}{x - x_0} + \frac{A_1}{x - x_1} + \cdots + \frac{A_k}{x - x_k} + \frac{b_0 + b_0 x + \cdots + b_{n-1}x^{n-1}}{J(x)}$$

Example: Find the partial fraction decomposition of the function

$$f(x) = \frac{1}{(x-1)(x^2+x+1)}$$

Solution: The denominator of the function f has simple factor $x - 1$ and an irreducible factor of degree two. Therefore we may write

$$f(x) = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+x+1}$$

We once again multiply the entire expression by $(x-1)(x^2+x+1)$ to obtain

$$1 = A(x^2 + x + 1) + (Bx + C)(x - 1) = A(x^2 + x + 1) + Bx^2 - Bx + Cx - C$$

We therefore may write

$$1 = (A + B)x^2 + (A - B + C)x + A - C$$

which yields the following system of equations:

$$\begin{cases} A + B = 0 \\ A - B + C = 0 \\ A - C = 1 \end{cases}$$

To proceed we add the first two equations to obtain $2A + C = 0$ and we then add this equation to the third equation to obtain $3A = 1$, i.e $A = 1/3$. The rest of the coeffecients are easily obtained as $B = -1/3$ and $C = -2/3$. Therefore we have the following expansion:

$$f(x) = \frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{x+2}{x^2+x+1}$$

This method of coeffecient matching seems like a fine method at first, however it can get quite involved. Hence we proceed to the first simplifying observation.

Simplifying the Process: Test Numbers

The first key realization we make when trying to simplify the process of finding partial fraction decomposition is the following. Whatever the coeffecients $A_0, A_1, \ldots, A_k$ are for a partial fraction decomposition, they must be the same regardless of x – *the coefficients themselves are never functions of* x. This is best illustrated via example.

Example: Find the partial fraction decomposition of the function

$$f(x) = \frac{x^2 + 4x + 4}{x(x-1)(x-2)(x+1)}$$

Solution: This would be a bit of a monster to solve with the coefficient matching method. We proceed with the usual method to write

$$f(x) = \frac{(x+2)^2}{x(x-1)(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x+1}$$

Next we multiply the entire expression by $x(x-1)(x-2)(x+1)$ to obtain

$$(x+2)^2 = A(x-1)(x-2)(x+1) + Bx(x-2)(x+1) + Cx(x-1)(x+1) + Dx(x-1)(x-2)$$

To proceed we recall that the coefficients are not functions of x, i.e they do not vary for particular x. Therefore we may evaluate the entire expression above at points which greatly simplify it. Notice that the point $x = 0$ will allow us to instantly solve for A:

$$x = 0 : \ 2^2 = A(-1)(-2)(1) \Longrightarrow A = 2$$

Now notice that the point $x = 1$ will immediately yield B:

$$x = 1 : \ 3^2 = B(1)(-1)(2) \Longrightarrow B = -\frac{9}{2}$$

Next notice that the point $x = 2$ will immediately yield C :

$$x = 2 : \ 4^2 = C(2)(1)(3) \Longrightarrow C = \frac{16}{6} = \frac{8}{3}$$

Finally we notice that the point $x = -1$ will immediately yield D :

$$x = -1 : \ 1 = D(-1)(-2)(-3) \Longrightarrow D = \frac{1}{6}$$

Thus the final result:

$$f(x) = \frac{x^2 + 4x + 4}{x(x-1)(x-2)(x+1)} = \frac{2}{x} - \frac{9}{2}\left(\frac{1}{x-1}\right) + \frac{8}{3}\left(\frac{1}{x-2}\right) + \frac{1}{6}\left(\frac{1}{x+1}\right)$$

Example: Find the partial fraction decomposition for the function

$$f(x) = \frac{x^2}{(x-1)(x^2+1)}$$

Solution: We write the usual partial fraction decomposition:

$$f(x) = \frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}$$

Then we multiply the entire expression by $(x-1)(x^2+1)$ to obtain

$$x^2 = A(x^2+1) + (Bx+C)(x-1)$$

This expression isn't quite as nice as the previous example. However we can still obtain A immediately by taking the point $x = 1$:

$$x = 1: \ 1 = A(2) \Longrightarrow A = \frac{1}{2}$$

To obtain C we take the point $x = 0$:

$$x = 0: \ 0 = A + C(-1) = \frac{1}{2} - C \Longrightarrow C = \frac{1}{2}$$

To obtain B we use any point that we want. Take $x = 2$:

$$x = 2: \ 4 = 5A + (2B + C) = \frac{5}{2} + \frac{1}{2} + 2B \Longrightarrow B = \frac{1}{2}$$

Therefore the final expression is

$$f(x) = \frac{x^2}{(x-1)(x^2+1)} = \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{x+1}{x^2+1}\right)$$

Two Equations from one Test Point: Complex Numbers

We may make the test point method more powerful by realizing that we may take complex test points. This is because the method of partial fraction decomposition works for complex functions $f(z)$ where $z = x + jy$ and j is the number such that $j^2 = -1$. Formally, we would find the partial fraction decomposition for the complex function $f(z)$ then restrict $f(z \to x)$ but that isn't necessary. Consider the previous example at the step:

$$x^2 = A(x^2+1) + (Bx+C)(x-1)$$

We may use $x = j$ to get the B, C terms immediately:

$$j^2 = -1 = (jB + C)(j-1) = -B - jB + jC - C = -(C+B) + j(C-B)$$

Complex numbers are essentially vectors due to their two dimensional nature. Therefore we may equate components like vectors, i.e we set real equal to real and imaginary equal to imaginary:

$$\begin{cases} C + B = 1 \\ C - B = 0 \end{cases}$$

Therefore with one test point we obtained two variables:

$$C = \frac{1}{2}, \ B = \frac{1}{2}$$

Then we immediately obtain A by using $x = 1$ and we obtained the same expression as above.

Simplifying the Process: Limits

The next fact we may make use of when simplifying the process of partial fractions is the use of limits. A function and its partial fraction decomposition must have the same limiting behaviour since they are the same function. This method is best illustrated via example as well.

Example: Find again the partial fraction expansion of the function

$$f(x) = \frac{x^2}{(x-1)(x^2+1)}$$

Solution: We proceed as usual

$$f(x) = \frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

However instead of multiplying by the entire denominator of the left hand side, we instead only multiply by $x-1$:

$$\frac{x^2}{x^2+1} = A + (x-1)\left(\frac{Bx+C}{x^2+1}\right)$$

These two functions must have the same asymptotic behaviour:

$$\lim_{x\to\infty} \frac{x^2}{x^2+1} = 1 = A + \lim_{x\to\infty} \frac{Bx^2 + Cx - Bx - C}{x^2+1} = A + B$$

Therefore we immediately note that $A + B = 1$ and A is easily solved by the test point $x = 1$ to be $1/2$ and all that remains is solving for C to obtain the results above. This method can allow one to easily find equations by merely inspecting the function given.

Using Derivative to Compute a Partial Fraction Decomposition

Using derivatives we may essentially derive a formula that computes the coefficients of a partial fraction decomposition, no equations or test points required. Let $f(x) = P(x)/Q(x)$ have $k > 1$ repeated roots at $x = x_0$, then we may write

$$f(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{J(x)(x - x_0)^k}$$

where $J(x)$ is a polynomial where $J(x_0) \neq 0$. For simplicity we take $J(x) = 1$. We may write this expression as

$$f(x) = \frac{P(x)}{J(x)(x - x_0)^k} = \sum_{n=0}^{k-1} \frac{A_{k-n}}{(x - x_0)^{n+1}}$$

Let us multiply this expression by $(x - x_0)^k$:

$$f(x)(x - x_0)^k = \sum_{n=0}^{k-1} \frac{A_{k-n}}{(x - x_0)^{n-k+1}} = A_k(x - x_0)^{k-1} + A_{k-1}(x - x_0)^{k-2} + \cdots + A_2(x - x_0) + A_1$$

Notice that we can find A_1 immediately by evaluating $f(x)(x - x_0)^k$ at the point x_0. Then to find the next coeffecient all we would have to do is differentiate $f(x)(x - x_0)^k$ and evaluate at x_0. To find any A_k we take the $k - 1$th derivative.

$$\frac{d^{k-1}}{dx^{k-1}}\Big(f(x)(x - x_0)^k\Big) = \frac{d^{k-1}}{dx^{k-1}} \sum_{n=0}^{k-1} \frac{A_{k-n}}{(x - x_0)^{n+1}} = (k - 1)!A_k$$

Therefore we may write

$$A_k = \frac{1}{(k - 1)!} \lim_{x \to x_0} \frac{d^{k-1}}{dx^{k-1}}\Big(f(x)(x - x_0)^k\Big)$$