The Evolution of Mathematics

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The story of math started with numbers. The concept of numbers has been fundamental to human civilization since ancient times. The earliest known evidence of humans having the idea of counting is **Ishango Bone** which dates back to 20,000 years old. It was discovered in 1950, in the Democratic Republic of Congo, and is named after the region it was founded. The bone is 10 cm long and has a series of notches or etchings used for counting. Almost 10,000 years ago in ancient Mesopotamia, modern-day Iran, Kuwait, Syria, and Turkey, merchants started using small, 3D clay objects as counters to represent certain quantities, units, or goods. However, the clay objects would get lost or damaged. Then about 6,000 years ago, Mesopotamians came up with the first unambiguous notations for numbers. Ideas of position-based numeral systems with the base 60 started to rise in 3,200 BC in the Sumerian empire, the southern part of modern-day Iraq. It is thought the number 60 is related to the origin of the number 12, which is the number of joints on 4 fingers of a hand, the thumb being free to count. Five repeated hand counts, using five fingers of the other hand, deliver the number 60 which was used as the base for counting large numbers. Babylonians, people of southern Mesopotamia sharing the northern border with Sumer, adopted the Sumerian counting system and gave rise to Babylonian numerals in 2000 BC. Babylonians also adopted this system because dividing the whole of 60 into parts was easy as 60 has a lot of factors: 2, 3, 4, 5, 6, 10, 12, 15, 20, 30. From this derives the modern-day usage of 60 seconds in a minute, 60 minutes in an hour, and 360 (60×6) degrees in a circle, as well as the use of seconds and minutes of arc to denote fractions of a degree.

Meanwhile, Egyptians had started developing an intuition of geometry by associating numbers with lengths. They measured long distances using ropes with knots at regular intervals and counted knots to measure distance. As the distance need not be an integral multiple of knots, Egyptians started exploring the idea of fractions. By 1800 BC, the Egyptian civilization and the Indus Valley civilization had developed a numeral system with base 10, especially because working with base 10 made dealing with fractions easier. The Egyptians also developed a rudimentary understanding of arithmetic operations, including multiplication and division, which are illustrated in the Rhind Papyrus.

The **Rhind Papyrus** from ancient Egypt is a 2-meter-long papyrus that contains 84 problems about multiplication, division, fractions, and basic geometry. It is one of the oldest mathematical texts discovered, providing insight into the mathematical knowledge and practices of ancient Egyptians. It was written around 1650 BC by a scribe named *Ahmose*. One of the most notable sections is a 2/n table, which lists the value of 2 divided by odd numbers up to 101, demonstrating the Egyptians' understanding of unit fractions. However, people didn't know how to write fractions in a decimal positional system until the method was invented by the Greek polymath, **Archimedes**, in 250 BC. The papyrus also includes problems related to areas and volumes, suggesting an early understanding of geometric concepts. The papyrus is named after Scottish antiquarian Alexander Henry Rhind, who purchased it in Luxor, Egypt, in 1858. Today, most of its remains are located at the British Museum in London, where it has been studied extensively by historians and mathematicians to gain insights into the mathematical knowledge and practices of ancient Egyptian civilization.

Around the same time of 1600 BC, the Indus Valley civilization had fallen, likely due to environmental changes and potential conflicts. By 1100 BC, it started to flourish again due to slow migration towards the region, particularly in the area of present-day Pakistan and northwestern India. In this period till 500 BC, scholars compiled the past learnings and the scattered pieces of information into four main Vedas *(Rig Veda, Yajur Veda, Sama Veda, and Atharva Veda)* and six auxiliary Vedangas. The Vedangas were supplementary texts that provided rules and explanations for various aspects of the Vedas, including rituals, grammar, phonetics, and astronomy. The second Vedanga in the series named *Kalpa*, presumably written in 900 BC, includes the rituals with major life events such as birth, wedding, and death in the family. One segment of Kalpa named Shulba Sutras (or *Aphorisms of the Cord*) describes the construction of fire-altars and the precise measurements required for various ritualistic purposes. The Shulba Sutras are considered the earliest known texts on geometry in India and demonstrate a sophisticated understanding of geometric principles, including the Pythagorean theorem, which predates its attribution to Pythagoras. Fractions such as 1/2, 1/4, and 1/8 are used in determining the dimensions of the altars, indicating an advanced knowledge of fractions and their applications. The Shulba Sutras also provide methods for constructing various geometric shapes, such as squares, rectangles, and circles, using rope and peg techniques. So, we can safely assume that by 800 BC, Egyptians, and the neighboring civilizations, including the Indus Valley civilization, had a decent understanding of geometric concepts like shapes, areas, parallel lines, perpendicular lines, angles, and the use of fractions in practical applications.

Around 800 BC, the Greek civilization started to emerge, particularly in the regions of mainland Greece and the Aegean islands. From 800 BC to 600 BC, Greeks were more focused on the literature and mythology. Homer, considered one of the greatest poets of ancient Greece, in this period produced the foundations of modern Greek literature through his epic poems, Iliad and Odyssey, which remain influential works to this day. The rise of literature and philosophical thought brought forth three notable philosophers in the fifth, fourth, and third centuries BC, namely – Socrates, Plato, and Aristotle, who laid the groundwork for Western philosophy and influenced numerous fields, including mathematics. However, it is generally thought that the neighboring Babylonian and Egyptian civilizations, with their advanced knowledge of astronomy, geometry, and arithmetic, significantly influenced the younger Greek tradition in mathematics.

Greek mathematics began around 600 BC with **Thales** of Miletus, widely regarded as the first true mathematician. Thales, who learned mathematics in Babylon and Egypt, formulated what we today know as Thales' Theorem, a fundamental principle in geometry related to similar triangles. Then, around 550 BC, **Pythagoras** of Samos, who had studied in Egypt and Babylonia, came up with the famous Pythagorean Theorem, a fundamental relationship between the sides of a right-angled triangle. He and his followers, known as the Pythagoreans, made significant contributions to the study of numbers, including the discovery of irrational numbers and the theory of proportions. Lastly, **Euclid**, who lived in Alexandria around 300 BC, compiled the foundational work of Greek mathematics in his book *Elements*, significantly influenced by the teachings of Pythagoras and the Pythagorean school of thought. Elements established what we now know as the **Euclidean Geometry**, a comprehensive system of geometry based on a set of axioms and logical reasoning. Euclid was also interested in the properties of numbers such as divisibility, prime numbers, etc. This gave rise to the field of what we know today as **Number Theory**. However, Number Theory, as developed by Euclid and other Greek mathematicians, mainly focused on integers and rational numbers, which were the classes of numbers known by that point in time.

The place-value system for writing numbers, after the invention of the mathematical concept of zero by the Indian mathematician **Aryabhata** in 499 AD, brought arithmetic operations to the masses, revolutionizing the way numbers were represented and calculations were performed. The world was working on understanding the shapes better and applying these ideas to lay the foundational work in astronomy and navigation. **Trigonometry**, the study of relationships between the sides and angles of triangles, became increasingly popular and important, as it enabled more accurate calculations and measurements for applications such as surveying, astronomy, and architecture. Aryabhata also worked towards finding an accurate value of pi, the mathematical constant representing the ratio of a circle's circumference to its diameter. In the second part of his seminal work *Aryabhatiyam, Ganitpada 10*, he writes: "Add four to 100, multiply by eight, and then add 62,000. By this rule, the circumference of a circle with a diameter of 20,000 can be approached." This ingenious algorithm implies that for a circle whose diameter is 20,000 units, the circumference will be 62,832 units, giving an approximation for pi as 62,832/20,000 = 3.1416, which was remarkably accurate for that time, considering the limited computational resources available. Aryabhata's method for calculating pi, along with his other contributions to mathematics and astronomy, such as the concept of sine function and the study of solar and lunar eclipses, had a profound impact on the development of these fields in India and influenced subsequent scholars and scientists.

Scholars had ideas about equations and ways of finding unknown values, also known as solutions to those equations, as early as the ancient Babylonian and Egyptian civilizations. But nobody had given this technique a formal treatment or compiled all the techniques in an orderly manner until the 9th century AD. When we begin working with the unknowns, formulae, and the process of balancing the terms on the left and the right-hand sides of the equality, arithmetic transitions into the realm of **Algebra**, a more generalized and abstract form of mathematics. In 820 AD, the Persian mathematician **Muhammad ibn Musa al-Khwarizmi**, in his seminal work *Al-Jabr wa'l-Muqābala* (also known as *Al-Jabr*, meaning *the reunion of broken parts*, and *The Compendious Book on Calculation by Completion and Balancing*) provided an exhaustive account of solving for the positive roots of polynomial equations up to the second degree. Al-Khwarizmi's work systematically covered the solution of linear and quadratic equations, introducing the concept of algebraic operations and laying the foundations of the algebraic method for solving equations. This unlocked a whole new set of ideas and paved the way for the development of algebra as a distinct branch of mathematics.

However, scholars got stuck while finding the non-positive roots because that often got them to a position where they were supposed to find the square root of a negative number, a concept that was not yet understood and was considered impossible at the time. As a result, negative and complex roots were initially ignored or left as unsolved cases. It wasn't until 1543 that the Italian mathematician **Niccolo Tartaglia** made a significant breakthrough by showing that even to find positive solutions to certain cubic equations, it is required sometimes to manipulate the square roots of negative numbers, which he treated as formal algebraic quantities. His work on solving cubic equations, now known as the *Cardano-Tartaglia formula*, paved the way for further exploration of complex numbers and their role in algebra. Tartaglia's work was further extended and refined by **Galileo Galilei** in the late 16th century.

The next big breakthrough happened in 1637 AD, when the French mathematician and philosopher **René Descartes** published his seminal work *La Géométrie*, which laid the foundations for what is now known as **Cartesian or Analytic Geometry**. Descartes ingeniously used the concept of algebraic equations to describe and analyze geometric shapes and curves, effectively bridging the gap between algebra and Euclidean geometry. By introducing a coordinate system, with perpendicular x and y axes intersecting at the origin, Descartes was able to represent geometric objects using algebraic equations, unlocking the world of Cartesian geometry for the world. This revolutionary idea allowed for the algebraic manipulation and analysis of geometric concepts, paving the way for a deeper understanding and exploration of geometric properties through algebraic methods. While solving for equations involving square roots of negative numbers, Descartes recognized the need to treat these quantities as legitimate mathematical objects, coining the term "imaginary numbers" to refer to the square roots of negative numbers, a terminology that still holds to this day. Although initially perceived as a mathematical curiosity, imaginary numbers later proved to be invaluable in the study of complex analysis and other branches of mathematics.

What's particularly interesting about Cartesian geometry is that it gave rise to the idea of a *system of linear equations*, which can be used to represent and analyze lines, planes, and other linear objects in space. Solving and understanding linear equations, a fundamental concept in Cartesian geometry, paved the way for the development of **Linear Algebra**, a powerful branch of mathematics dealing with the study of vector spaces, matrices, and linear transformations. Linear algebra, with its concepts of vectors, matrices, and their operations, emerged as a crucial tool for describing and manipulating geometric objects, as well as finding applications in various fields, including physics, engineering, computer science, and economics. The ideas and techniques of linear algebra, rooted in Descartes' groundbreaking work, continue to play a vital role in modern mathematics and its numerous applications.

In the late 17th century, English mathematician **Isaac Newton** and German mathematician **Gottfried Wilhelm Leibniz** independently developed the basic ideas of **Calculus**, a revolutionary branch of mathematics that deals with the study of rates of change and the accumulation of quantities over time. They published their groundbreaking works around the same time, with Newton's seminal work *Philosophiæ Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy)* appearing in 1687, and Leibniz's *Nova Methodus pro Maximis et Minimis (New Method for Maxima and Minima)* in 1684. At that point, there was a great controversy over which mathematician among the two deserved credit for the invention of calculus. Newton claimed that Leibniz had stolen ideas from his unpublished notes on calculus, which Newton had shared with a few members of the Royal Society during his earlier visits. This controversy, known as the *Calculus Controversy* or the *Newton-Leibniz Controversy*, divided English-speaking mathematicians from continental European mathematicians for many years, to the detriment of the development of English mathematics. A careful examination of the papers of Leibniz and Newton by modern historians and mathematicians shows that they arrived at their results independently, with Leibniz starting first with the concept of integration and the calculation of areas under curves (known as the *integral calculus*), while Newton began with the concept of differentiation and the study of rates of change (known as the *differential calculus*). Both their approaches were based on the idea of infinitesimal quantities. It is Leibniz, however, who gave the new discipline its name *Calculus*, derived from the Latin word *calculus*, meaning small stone or pebble, referring to the use of pebbles for counting and calculation in ancient times. Leibniz's notation, using the integral sign $\int$ for integration and the differential operator $\frac{d}{dx}$ for differentiation, became widely adopted and is still in use today.

The theory of *Calculus* had many holes in it. It lacked the rigorous development from axioms, theorems, and definitions for the next 150 years. This is because we didn't have a sound understanding of what are functions. The definition of function itself was evolving till the 19th century. For example, the concept of a function was initially tied to the idea of a single algebraic expression, but later expanded to include more general mappings between sets. Nonetheless, throughout the 19th century, significant contributions were made by the French mathematician **Augustin-Louis Cauchy** and German mathematicians **Karl Weierstrass** and **Bernhard Riemann** to fix the gaps in the theory of Calculus. Cauchy's work on the precise definition of limits and continuity, Weierstrass's rigorous formulation of the epsilon-delta definition of limits, and Riemann's integral, which generalized the notion of integration, all played a crucial role in putting calculus on a firm foundation. Riemann in his latter half of life explored the theory of **Complex Analysis**, which saw heavy applications in Physics due to the works of French mathematician **Joseph Fourier** before him. Fourier's work on the analysis of periodic functions using trigonometric series laid the groundwork for the study of complex analysis and its applications in solving differential equations arising in physics.

One important discovery that happened in 1761, was Swiss mathematician **Johann Heinrich Lambert** proving that π is irrational. The significance of this result was that we were still struggling with understanding the nature of irrational numbers, which are numbers that cannot be expressed as the ratio of two integers. Lambert's proof showed that even a seemingly simple constant like π had a deep, non-repeating decimal representation, highlighting the complexity of irrational numbers. As the understanding of numbers and functions was evolving, the understanding of Algebra was evolving as well. French mathematician **Évariste Galois**, German mathematician **Johann Carl Friedrich Gauss**, Norwegian mathematician **Niels Henrik Abel**, and French mathematician **Henri Poincaré** contributed to this new algebra, which we know today as **Abstract Algebra**. Galois theory, for example, provided a powerful framework for understanding the solvability of polynomial equations by radicals, while group theory, pioneered by Gauss and further developed by others, became a fundamental tool in abstract algebra and its applications. Similarly, the understanding of geometry expanded from the *Euclidean* and *Cartesian* worlds, to a generic world of **Topology**. It started in 1736 by Swiss Mathematician **Leonhard Euler**, but further contributions were made by *Cauchy, Riemann, and Poincaré* in their period. Topology deals with the study of properties of geometric objects that are preserved under continuous deformations.

In the late 20th century, Russian mathematician **Georg Ferdinand Ludwig Phillip Cantor** introduced **Set Theory** to give functions a modern rigorous definition. His works challenged the existing understanding of mathematicians and caused a lot of controversy. However, his work has today become the fundamental theory of mathematics. Due to his works, for the first time, we got an understanding of real numbers, irrational numbers, and the concept of infinity. Because the *Calculus Theory*, *Abstract Algebra*, *Topology* - all relied heavily upon the basic notion of function, which changed after Cantor, almost all ideas had to be revisited and revised wherever necessary. While doing so, the works of French mathematician **Henri Lebesgue** rose to prominence, which gave rise to a whole new field of **Measure Theory.**

When Cantor left his theory and when mathematicians were revising the old works, they stumbled upon problems that remained unsolved for the next 100 years to come. In 1931, Austrian mathematician **Kurt Gödel** came up with his **Incompleteness Theorem** which changed the viewpoint of mathematicians. In his theorem, Gödel proved that in every axiomatic framework of mathematics, you can come up with logical-mathematical statements that cannot be proved. Certain of Cantor's statements belong to this class only and hence cannot be proved ever. This groundbreaking result shook the foundations of mathematics, showing that there are inherent limitations to formal systems, no matter how powerful they may seem. Gödel's Incompleteness Theorem has profound implications for various branches of mathematics and computer science. For example, it implies that there are limits to what can be computed by algorithms, as demonstrated by **Alan Turing**'s work on the *Halting Problem.* Gödel's theorem also sparked further research into the nature of mathematical truth and the boundaries of human knowledge.

Finally, the last big dent in the universe of mathematics was made by the Russian mathematician **Andrey Kolmogorov**, who laid the foundations of the **Modern Axiomatic Theory of Probability**. He proved how the **Classical Probability**'s definition is flawed and updated the framework of probability. The *Classical Probability* which was started in 1654 by French mathematicians **Blaise Pascal** and **Pierre de Fermat** and was heavily contributed by French mathematicians **Jacob Bernoulli** and *Carl Guass*, got a new interpretation after *Kolmogorov's work.* Kolmogorov's theory finds a special place in the field of **Stochastic Calculus** that has revolutionized the financial markets.

So, if I have to summarise the *map of mathematics*, I would split the attempt to understand this map into two parts: *the classical theory* and *the modern theory.* Ideally, you should finish the classical theory first and then move on to modern theory, but there are no hard and fast boundaries.

Classical Theory	Modern Theory
1. Euclidean Geometry	1. Set Theory & Discrete Mathematics
2. Polynomial Algebra & Cartesian Geometry	2. Abstract Algebra
3. Linear Algebra	3. Real Analysis & Measure Theory
4. Number Theory	4. Complex Analysis
5. Calculus & Analytic Geometry	5. Modern Axiomatic Probability & Stochastic Calculus
6. Classical Probability & Statistics	6. Topology

While studying through the history of mathematics, I was simply awestruck by the fact of how far we have come, starting from humble origins in ancient civilizations to the modern-day complexities. The understanding of domains like Modern Probability and Stochastic Calculus allow us to optimize, and sometimes even predict, future-decisions - making us more powerful than ever. But at the same time, Gödel's Incompleteness Theorem humbles us and reminds us of the fact that we are still bound by the universal supreme.