Discrete Random Variable X	Continuous Random Variable X
X takes the values $x_1, x_2, \dots, x_k$	X takes the values on an interval or the real numbers
Probability Mass Function (PMF) $p(x) = P(X = x)$	**Probability Density Function (PDF)**: a non-negative function $f_X(x)$ for which $P(a < X < b) =$ area under $f_X(x)$ over (a, b)
The sum of all values of $p(x)$ is 1: $\sum_{i=1}^{k} p(x_i) = 1$	$\int_{-\infty}^{\infty} f_X(x)dx = 1$
$\mu_X = E(X) = \sum_{i=1}^{k} x_i \cdot p(x_i)$	$\mu_X = E(X) = \int_{-\infty}^{\infty} x \overset{\text{PDF}}{f_X(x)} dx$
$\sigma_X^2 = Var(X) = E[(X-\mu)^2] = \sum_{i=1}^{k} (x_i - \mu)^2 \cdot p(x_i)$	$\sigma_X^2 = Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx$
$E[h(X)] = \sum_{i=1}^{k} h(x_i) \cdot p(x_i)$	$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx$

Any random variable has a cumulative Distribution Function (CDF): $F(x) = P(X \le x)$ for any real number x.

Shortcut formula $\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$

Standard deviation (SD) of X: $\sigma_X = \sqrt{\sigma_X^2}$

Special case: Linear Transformations $h(X) = aX + b$

$E[aX + b] = a\,E(X) + b$ or $\mu_{aX+b} = a\,\mu_X + b$

$Var[aX + b] = a^2\,Var(X)$ or $\sigma_{aX+b}^2 = a^2\,\sigma_X^2$

Standard deviation: $\sigma_{aX+b} = |a|\sigma_X$

$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x f(x)\,dx$

1. An appliance dealer sells 3 different models of upright freezers having 16, 18, and 20 cubic feet of storage, respectively.
Let X = the rated capacity of a freezer of this brand sold at a certain store. Suppose that X has the pmf:

x	16	18	20
$p(x)$	0.2	0.5	0.3

a) Compute $E(X)$ and $E(X^2)$.

$E(x) = 16 \cdot (.2) + 18(.5) + 20(.3)$
$\Rightarrow 18.2$

$E(x^2) = 16^2(.2) + 18^2(.5) + 20^2(.3)$
$\Rightarrow 333.2$

b) Calculate $\sigma^2 = Var(X)$ in two ways:

b1) Using the definition of variance: $\text{Var(X)} = E[(X-\mu)^2] = \sum_{i=1}^{k}(x_i - \mu)^2 \cdot p(x_i)$

$\mu = 18.2$

$$\left.\begin{aligned} (16-18.2)^2 &= (-2.2)^2 = 4.84 \\ (18-18.2)^2 &= (-.2)^2 = .04 \\ (20-18.2)^2 &= (1.8)^2 = 3.24 \end{aligned}\right\}$$

$Var(x) = 4.84(.2) + .04(.5) + 3.24(.3) = .968 + .02 + .972$
$= 1.96$ //

b2) Using the shortcut formula $\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$

$\Rightarrow 333.2 - 18.2^2$

$E(X) = 18.2$

$\sigma^2 = 333.2 - 331.24 = 1.96$

$E(x^2) = 333.2$

c) If the price of a freezer having capacity X is $Y = 70X - 650$, what is the expected price $E(Y)$ paid by the next customer to buy a freezer?

$Y = 70x - 650$ $E(x) = 18.2$

$E(Y) = E(70X - 650)$

$= 70E(x) - 650$

$= 70(18.2) - 650 \Rightarrow 1274 - 650 \rightarrow E(Y) = \$624$

d) What is the standard deviation of the price $70X - 650$ paid by the next customer?

Standard deviation

$= \sigma_X = \sqrt{Var(x)} = \sqrt{1.96}$

$= 1.4$

$\sigma_Y = |a|\sigma_X$

$= 70 \cdot 1.4 = 98$

e) Suppose that although the rated capacity of a freezer is X, the actual capacity is $h(X) = X - 0.008X^2$. What is the expected actual capacity $E(h(X))$ of a freezer purchased by the next customer?

$h(x) = x - .008x^2$

$E(h(x)) = E(x - .008x^2)$

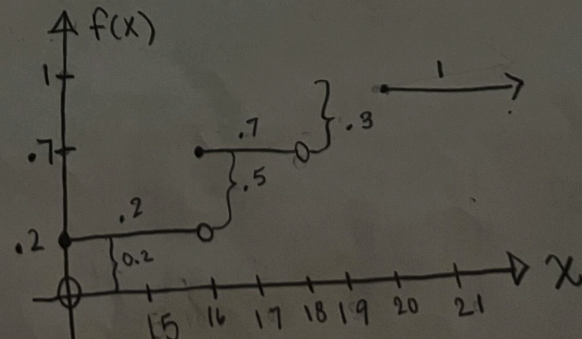
$= E(x) - .008E(x^2)$

$= 18.2 - .008(333.2)$

$= 18.2 - 2.6656 = 15.5344 = E(h(x))$

f) Find the Cumulative Distribution Function (CDF) of a random variable is **$F(x) = P(X \le x)$ for any real number x.**
Make a table first, then write $F(x)$ as a piecewise function and graph it.

$$F(x) = \begin{cases} 0, & x < 16 \\ 0.2, & 16 \le x < 18 \\ .2 + .5 = .7, & 18 \le x < 20 \\ .2 + .5 + .3 = 1, & x \ge 20 \end{cases}$$



CDF = Cummulative Distribution Function

2. Toss a fair coin three times in succession. The sample space is
$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$
Let X = Number of heads rolled.

X	0	1	2	3
P(x)	1/8	3/8	3/8	1/8
f(x)	1/8	4/8	7/8	1

a) What set of outcomes corresponds to $X = 0$?

1/8 TTT

b) What set of outcomes corresponds to $X = 1$?

3/8 HTT, THT, TTH

c) What set of outcomes corresponds to $X = 2$?

3/8 HHT, HTH, THH

d) What set of outcomes corresponds to $X = 3$?

1/8 HHH

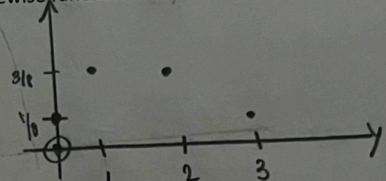
e) What is the probability $P(X \geq 1)$?

7/8

$P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$

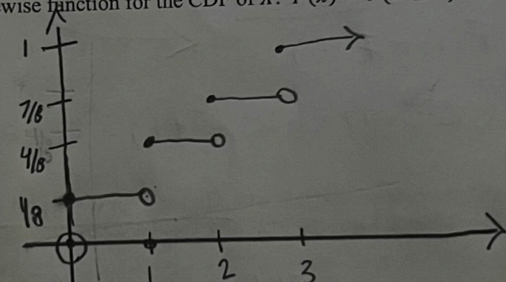
f) Make a table and write a piecewise function for the PMF $p(x)$ of X.

$$P(x) = \begin{cases} 1/8, & x = 0 \\ 3/8, & x = 1 \\ 3/8, & x = 2 \\ 1/8, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$



f) Make a table and write a piecewise function for the CDF of X: $F(x) = P(X \leq x)$. Also graph $F(x)$.

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/8, & 0 \leq x < 1 \\ 4/8, & 1 \leq x < 2 \\ 7/8, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$



f) Calculate $\mu = E(X)$ and $E(X^2)$

$E(X) = \sum x\,p(x) = 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot \frac{1}{8}$
$= 0 + 3/8 + 6/8 + \frac{3}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$ //

$E(X^2) = \sum x^2 p(x) = 0^2 \cdot 1/8 + 1^2 \cdot 3/8 + 2^2 \cdot 3/8 + 3^2 \cdot 1/8$
$= 0 + 3/8 + 12/8 + 9/8 = 24/8 = 3$ //

g) Calculate $\sigma^2 = Var(X)$ and standard deviation σ of X.

$\sqrt{\sigma^2} = Var(x) = E(X^2) - (E(X))^2 = 3 - (1.5)^2 = 3 - 2.25 = \sqrt{.75}$

$\sigma = \sqrt{.75} \approx .866$ //

Done

(PDF)

3. Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} ke^{-3x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) What is the value of k? Hint: $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\lim_{M\to\infty} \int_0^{\infty} Ke^{-3x}dx = 1 \Rightarrow K\cdot\left[\frac{e^{-3x}}{-3}\right]_0^M \Rightarrow K\left(\frac{e^{M}}{-3} - \frac{e^{0}}{-3}\right) = 1$$

$$K\left(\frac{\cancel{\infty}}{\cancel{-3}} + \frac{1}{3}\right) = 1$$

$$\frac{K}{3} = 1 \Rightarrow K = 3$$

b) Find $P(0.5 \le X \le 1)$ using the PDF.

$$P(.5 \le x \le 1) = \int_{.5}^{1} 3e^{-3x}dx = 3\cdot\frac{e^{-3x}}{-3}\Bigg|_{.5}^{1}$$

$$= 3\ \cancel{3}\cdot\left(\frac{e^{-3}}{\cancel{3}} + \frac{e^{-1.5}}{\cancel{3}}\right) = -e^{-3} + e^{-1.5} \approx .1733$$

c) Find the CDF $F(x)$.

$$F(x) = \int_0^x 3e^{-3t}dt$$

$$3\cdot\left(\frac{e^{-3x}}{-3} + \frac{1}{3}\right) \Rightarrow -e^{-3x} + 1 = F(x)$$

d) Find $P(0.5 \le X \le 1)$ using the CDF.

$$P(.5 \le x \le 1) = F(1) - F(.5)$$

$$= (-e^{-3} + 1) - (-e^{-1.5} + 1) = -e^{-3} + e^{-1.5} \approx .1733$$

4. If $X \sim U(0,1)$, the uniform random variable with $f(x) = 1$ for $0 \le x \le 1$ and 0 otherwise, find the expected value and the variance. Hint: What is the PDF of X?

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^1 x\,dx = \frac{x^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

$$E(x^2) = \int_0^1 x^2dx = \frac{x^3}{3}\Bigg|_0^1 = \frac{1}{3}$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$