MATH 260: ORDINARY DIFFERENTIAL EQUATIONS

FINAL EXAM SUMMER 2024

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Please read, write your name, and sign the following:

I, Fady Youssef, agree that I will not discuss this exam with anyone who has not yet taken this exam. This would be a violation of the Gonzaga University academic integrity policy.

Signed Fady

DIRECTIONS: Complete the following problems. Follow instructions and show your clearly organized work, and/or provide a well-reasoned explanation. Proper mathematical notation is expected. You may use a calculator. **This test is worth 200 points.**

Formulas

$$\vec{Y}(t) = c_1 e^{\alpha t}\left(\cos(\beta t)\vec{V}_1 - \sin(\beta t)\vec{V}_2\right) + c_2 e^{\alpha t}\left(\sin(\beta t)\vec{V}_1 + \cos(\beta t)\vec{V}_2\right)$$

$$\vec{Y}(t) = c_1 e^{\lambda t}\,\vec{V}_1 + c_2 e^{\lambda t}(\vec{V}_1 + t\,\vec{V}_2)$$

PART 1: MULTIPLE CHOICE Choose the letter of the answer that best matches your work. Whenever possible, include your work. Each question is worth 6 points.

1. Find the Laplace transform of $e^{-(t-5)}H(t-5)$.

e^{-5s}

(A) $\frac{1}{s+1}$

(B) $\frac{e^{-5s}}{s-1}$

(C) $\frac{e^{-5s}}{s+1}$ ✓

(D) $\frac{e^{-5s}}{s-5}$

2. Find the inverse Laplace transform of $Y(s) = \frac{1}{s-2} + \frac{5}{s+6}$

$e^{2t} + 5e^{-6t}$

(A) $e^{2t} + e^{6t/5}$

(B) $e^{2t} + 5e^{-6t}$

(C) $5e^{2t}e^{6t/5}$

(D) This cannot be done.

$\frac{dx}{dt} = x(t^2 - 4)$

$\frac{1}{x} dx = t^2 - 4\, dt$

8. (6 pts.) Which of these differential equations are separable? Equations A, C

(A) $\frac{dx}{dt} = xt^2 - 4x$
$x(t^2 - 4)$

(C) $\frac{dx}{dt} = 3x^2t^3$ $\longrightarrow$ $\frac{dx}{dt} = 3x^2t^3$

$dx = 3x^2t^3\,dt$

$\frac{1}{3} \cdot x^{-2}\, dx = t^3 dt$

~~(B)~~ $\frac{dx}{dt} = \sin(2tx)$

~~(D)~~ $\frac{dx}{dt} = t^4 \ln(5x)$

9. (6 pts.) Determine the order of the equation and whether the equation is linear or not.

$t^3y''' - e^y y'' + \sin(y)\, y' = 0$ Order: 3 *Circle one.* Linear Not linear

10. (20 pts) Solve the initial value problem for $t > 0$:

$$y' - \frac{2}{t} y = t, \qquad y(1) = 3$$

$\mu(t) = e^{-\int \frac{2}{t} dt}$

$= e^{-2\ln(t)}$

$= t^{-2} = \frac{1}{t^2}$

$\frac{1}{t^2} y = \cancel{t} \cdot \frac{1}{t^{\cancel{2}}} dt$

$\frac{1}{t^2} y = \frac{1}{t}\, dt$

$t^2 \cdot \cancel{\frac{1}{t^2}} y = (\ln(t) + c) \cdot t^2$

$y = t^2 \ln(t) + ct^2$

$y(1) = \cancel{\ln(1)} + c = 3$

$c = 3$

$\boxed{y = t^2 \ln(t) + 3t^2}$

6. A mass weighting 4 lb stretches a spring 2 in, so that the mass is $m = \frac{1}{8}$ lb-sec²/ft. Suppose that the mass is also attached to a viscous damper with a damping constant of 2 lb-sec/ft. The spring constant is $k = 24$ lb/ft. After the spring-mass system is allowed to come to equilibrium, the mass is pushed down an additional 6 in in the positive direction and then released (set in motion with no initial velocity).
Which of the following initial value problems describe the position $y(t)$ of the mass at time t?

(a) $\frac{1}{8}y'' + 24y' + 2y = 0,\ y(0) = 6,\ y'(0) = 0$

(b) $\frac{1}{8}y'' + 2y' + 24y = 0,\ y(0) = 6,\ y'(0) = 0$

(c) $\frac{1}{8}y'' + 2y' + 24y = 0,\ y(0) = \frac{1}{2},\ y'(0) = 2$

(d) $\frac{1}{8}y'' + 2y' + 24y = 0,\ y(0) = \frac{1}{2},\ y'(0) = 0$

(e) None of these

PART 2: PARTIAL CREDIT. Show all your work neatly. I reserve the right to deduct points if your answer is not clearly supported by your work (even if your final answer is correct).

7. (10 pts.) For the following differential equation, is the given function a solution? You must show your work to receive credit.

$y'' - \frac{2}{x^2}y = 0$ Is $y(x) = x^2 - \frac{1}{x}$ a solution? Circle one: Yes No

$$y'' - \frac{2}{x^2}\left(x^2 - \frac{1}{x}\right) = 0$$

$$y'' - 2 + \frac{2}{x} = 0$$

$$y'' + \frac{2}{x} = 2$$

$$2 - \frac{1}{x} + \frac{2}{x} = 2$$

$$2 + \frac{1}{x} = 2$$

$$\frac{1}{x} = 0$$

$$y' = 2x - \ln(x)$$

$$y'' = 2 - \frac{1}{x}$$

$$2 - \frac{1}{x} - \frac{2}{x^2}\left(x^2 - \frac{1}{x}\right) = 0$$

$$\cancel{2} - \frac{1}{x} - \cancel{2} + \frac{2}{x^3} = 0$$

$$-\frac{1}{x} + \frac{2}{x^3} = 0$$

$$\frac{2}{x^3} \neq \frac{1}{x}$$

$(s+2)^2$

s^2+4s+4

3. Find the inverse Laplace transform of $Y(s) = \frac{3s+7}{s^2+4s+13}$ $= \frac{3s+7}{(s+2)^2+9} = \frac{3(s+2)+1}{(s+2)^2+3^2}$

   (A) $3e^{-2t}\cos(3t)$

   (B) $3e^{-2t}\cos(3t)+1$

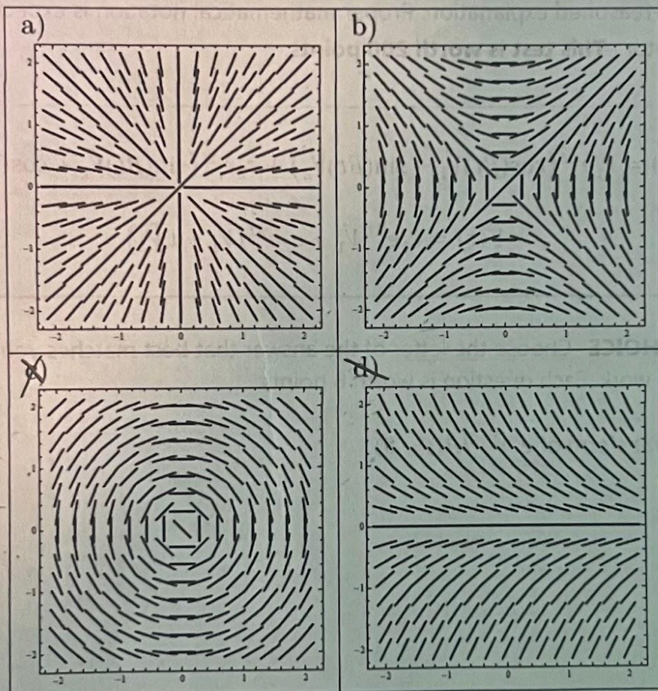
   (C) $3e^{-2t}\cos(3t)+e^{-2t}\sin(3t)$

   (D) $3e^{-2t}\cos(3t)+\frac{1}{3}e^{-2t}\sin(3t)$

$\Rightarrow 3\cdot\frac{s+2}{(s+2)^2+3^2}$ #26 $+ \frac{1}{(s+2)^2+3^2}$ #25

$3\cdot e^{-2t}\cos(3t) + \frac{1}{3}e^{-2t}\sin(3t)$

4. Which of the following is a direction field for the differential equation $\frac{dy}{dx} = \frac{x}{y}$? A



5. Which of the following forced 2nd order equations has solutions exhibiting resonance?

   (A) $y'' + 2y = 10\cos(2t)$

   (B) $y'' + 4y = 8\cos(2t)$

   (C) $y'' + 2y = 6\cos(4t)$

   (D) All of the above

   (E) None of the above

11. (28 pts.) Suppose that you have the autonomous differential equation

pos

$$\frac{dy}{dt} = y - y^2.$$

(a) What are the equilibrium solutions of the differential equation? $y=0$, $y=1$

$y(1-y)$

$(-1)(2) = -2$

$1(0) = 0$

$y=0$, $y=1$

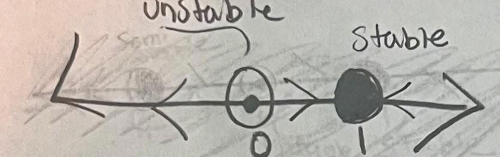
$3(-2) = -2$

(b) Sketch the phase line clearly labeling the equilibrium points and displaying the appropriate arrows. On your phase line, label each equilibrium solution as *stable, unstable,* or *semi-stable.*

below 0 : neg <

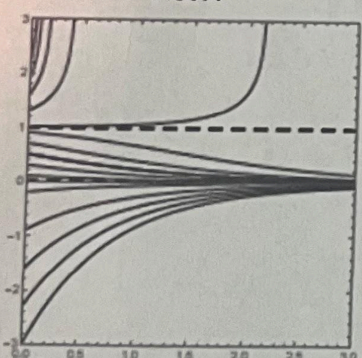
0 to 1 : pos >

above 1 : neg <

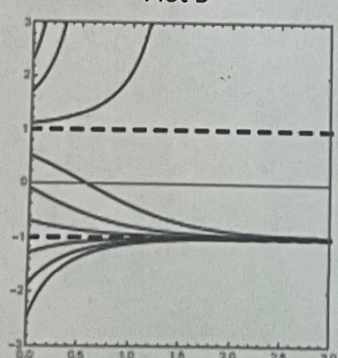


(c) Choose the appropriate plot for the solutions curves of the differential equation. C

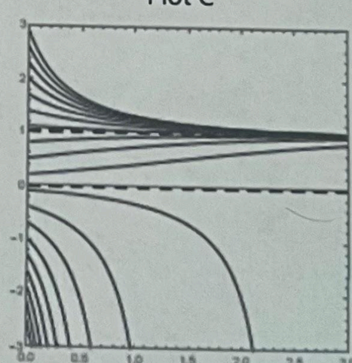
Plot A



Plot B



Plot C



(d) With initial condition $y(0) = -2$, the solution to $\frac{dy}{dt} = y - y^2$ will: (Circle one)

Diverge to $-\infty$ | Approach -1 | Approach 0

Approach 1 | Diverge to ∞ | None of these

15. (24 pts.) For the differential equation $y'' + 2y = 0$ with initial conditions $y(0) = 2,\ y'(0) = 1$

(a) Find the recurrence relationship that determines the coefficients a_n of a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$.

$y' = \sum_1^\infty n a_n x^{n-1}$

$y'' = \sum_2^\infty n(n-1) a_n x^{n-2}$

$\sum_2^\infty n(n-1) a_n x^{n-2} + 2\sum_0^\infty a_n x^n = 0$

$\sum_0^\infty (n+2)(n+1) a_{n+2} x^n + 2\sum_0^\infty a_n x^n$

$(n+2)(n+1) a_{n+2} + 2a_n = 0$

$(n+2)(n+1) a_{n+2} = -2a_n$

$a_{n+2} = \dfrac{-2a_n}{(n+2)(n+1)}$

Recurrence relation: $a_{n+2} = \dfrac{-2}{(n+2)(n+1)} a_n$ **for** $n \geq$ 0

(b) Use your recurrence relation to calculate the values of the first five coefficients.

$a_0 =$ 2 $\quad a_1 =$ 1 $\quad a_2 =$ −2

$a_3 = -\frac{1}{3}$ $\quad a_4 = \frac{1}{3}$

14. (20 pts) Consider the first-order system of equations: $\vec{Y}' = A\vec{Y}$ where $A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$

Solve this system with initial conditions $y_1(0) = 5, y_2(0) = 0$.

$Tr(A) = 1$

$det(A) = -20 + 14 = -6$

$\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2)$

$\lambda_1 = 3, \ \lambda_2 = -2$

$\lambda = 3: (A - 3I)\vec{v} = 0$

$\Rightarrow \begin{bmatrix} 2 & 7 \\ -2 & -7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{matrix} 2v_1 + 7v_2 = 0 \\ v_1 = 7, \ v_2 = -2 \end{matrix}\right\} \vec{v}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$\lambda = -2: (A + 2I)\vec{v}$

we're good here

$\Rightarrow \begin{bmatrix} 7 & 7 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{matrix} 7v_1 + 7v_2 = 0 \\ v_1 = 1, \ v_2 = -1 \end{matrix}\right\} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{v}_2$

$y_1(t) = c_1 e^{3t}\begin{pmatrix} 7 \\ -2 \end{pmatrix}$, $y_2(t) = c_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$y_1(0) = c_1\begin{pmatrix} 7 \\ -2 \end{pmatrix} = 5$

$7c_1 - 2c_1 = 5$

$5c_1 = 5$

$c_1 = 1$

$y_2(0) = c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

$c_2 - c_2 = 0$

$c_2 = 0?$

$y(t) = e^{3t}\begin{pmatrix} 7 \\ -2 \end{pmatrix}$

$= 7e^{3t} - 2e^{3t}$

$= 5e^{3t} //$

12. (6 pts.) For the following mass-spring equation, find the value of the damping coefficient μ that would make the system **critically damped.**

$$y'' + \mu y' + 2y = 0$$

$r^2 + \mu r + 2 = 0$

$\Delta = b^2 - 4ac = 0$

$\lambda^2 - 8 = 0$

$\lambda^2 = 8$

$\lambda = \sqrt{8}$

$\mu =$ $\sqrt{8}$

13. (24 pts.) Find the general solution of the differential equation $y'' - 2y' - 3y = -3e^{-t}$.

$r^2 - 2r - 3 = 0$

$(r-3)(r+1)$

$r_1 = 3$, $r_2 = -1$

$y_h = c_1 e^{3x} + c_2 e^{-x}$ //

$y_p = \cancel{Ae^{-t}}$

$y_p' = \cancel{-Ae^{-t}}$

$y_p'' = \cancel{Ae^{-t}}$

$y_p = \cancel{Ax^2e^{-t}}$

$y_p' = \cancel{-2Axe^{-t}}$

$y_p'' = \cancel{2Ae^{-t}}$

$y_p = Ate^{-t}$

$y_p' = Ae^{-t} - Ate^{-t}$

$y_p'' = -2Ae^{-t} + Ate^{-t}$

$Ae^{-t}(-2 + t - 2 + 2t - 3t)$

$-2Ae^{-t} + Ate^{-t} - 2Ae^{-t} + 2Ate^{-t} - 3Ate^{-t}$

$e^{-t}(-\cancel{2A} + At - \cancel{2A} + 2At - 3At)$

$e^{-t}(-4A + \cancel{3At - 3At}) = (-3)\, e^{-t}$

$-4A = -3$

$A = \frac{3}{4}$

$y_p = \frac{3}{4}te^{-t}$

$$\boxed{y(t) = c_1 e^{3x} + c_2 e^{-x} + \frac{3}{4}te^{-t}}$$ //

16. (20 pts.) Use Laplace transforms to solve the following initial value problem.

$$y'' + 9y = 2\delta(t-\pi), \qquad y(0) = 1,\ y'(0) = 4$$

$s^2 F(s) - s - 4 + 9F(s) = 2e^{-\pi s}$

$(s^2+9)F(s) = 2e^{-\pi s} + s + 4$

$F(s) = \dfrac{2e^{-\pi s}}{s^2+3^2} + \dfrac{\cancel{s}}{(s+3)^2} + \dfrac{4}{(s+3)^2}$

$\cos(3t)$ $\quad \frac{4}{3}\sin(3t)$

#10

$2 \cdot e^{-\pi s} \cdot \dfrac{1}{s^2+3^2}$

$(2)\ f(t-\pi)H(t-\pi)\ \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3^2}\right\}$

$2(H(t-\pi)) \cdot \frac{1}{3}\sin(3(t-\pi))$

$\Rightarrow \frac{2}{3}H(t-\pi)\sin(3(t-\pi)) + \cos(3t) + \frac{4}{3}\sin(3t)$ //