Hoofdstuk 7: bepaalde integratie

Sommen

$$Bovensom: U(N,f) = \sum_{i=1}^{n} M_i(f)\Delta x_i$$

$$Ondersom: L(N,f) = \sum_{i=1}^{n} m_i(f)\Delta x_i$$

$$Riemannsom: S(N,T,f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

Middelwaarde stellingen

$$Middelwaaardestelling: \int_a^b f(x)dx = (b-a)f(c)$$

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Oneigenlijke limieten

$$Oneigelijke\ integralen\ 1ste\ soort:$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to+\infty}\int_c^b f(x)dx$$

$$Oeigelijke\ integralen\ 2de\ soort:$$

$$\int_{a+}^{b-} f(x)dx = \int_{a+}^{c} f(x)dx + \int_{c}^{b-} f(x)dx = \lim_{\substack{p\to a\\ >}}\int_p^c f(x)dx + \lim_{\substack{q\to b\\ <}}\int_c^q f(x)dx$$

$$Oneigemijke\ integralen\ 3de\ soort:$$

$$\int_{a+}^{+\infty} f(x)dx = \int_{a+}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx = \lim_{\substack{p\to a\\ >}}\int_p^c f(x)dx + \lim_{q\to+\infty}\int_c^q f(x)dx$$

Numerieke integratie

$$Middelpuntsbenadering: \frac{b-a}{n}\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)$$

$$Fout: \frac{\|f''\|(b-a)^3}{24n^2}$$

$$Trapeziumbenadering: \frac{b-a}{2n}\left(f(x_0)+2\sum_{k=1}^{n} f(x_k)+f(x_n)\right)$$

$$Fout: \frac{\|f''\|(b-a)^3}{12n^2}$$

$$Simpsonbenadering: \frac{b-a}{3n}\big(f(x_0)+4f(x_1)+2f(x_2)+\dots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\big)$$

$$Fout: \frac{\|f^{iv}\|(b-a)^5}{180n^4}$$

Toepassingen

	Oppervlakte	**Omwentelingsvol.**	**Booglengte**
Cartesisch	$\int_a^b \lvert y\rvert dx$	$\pi\int_a^b y^2(x)dx$	$\int_a^b \sqrt{1+(y')^2}\,dx$
Parameter	$\int_{t1}^{t2} \lvert g(t)\rvert f'(t)dt$	$\pi\int_{t1}^{t2} \big(g(t)\big)^2 f'(t)ft$	$\int_{t1}^{t2} \sqrt{\big(f'(t)\big)^2+\big(g'(t)\big)^2}\,dt$
Pool	$\frac{1}{2}\int_\alpha^\beta r^2(\theta)d\theta$		$\int_a^b \sqrt{r^2+(r')^2}\,d\theta$

	Complanatie
Cartesisch	$2\pi\int_a^b \lvert y\rvert\sqrt{1+(y')^2}dx$
Parameter	$2\pi\int_{t1}^{t2} \lvert g(t)\rvert\sqrt{\big(f'(t)\big)^2+(g'(t))^2}dt$

Hoofdstuk 8: differentiaalvergelijkingen

$$P(x,y)dx + Q(x,y)dy = 0$$

Scheiden van veranderlijken

Naam	Voorwaarde	Vorm
Scheiding van veranderlijken	$y' = \frac{-P(x,y)}{Q(x,y)} = \frac{f(x)}{g(y)}$	
Homogene diff.vgl.	$F(\lambda x, \lambda y) = \lambda^n F(x,y)$	$u = \frac{y}{x} \Leftrightarrow y = ux$
Lineaire coef. 1ste graad + 1ste orde	$(a_1 x + b_1 y + c_1) + (a_2 x + b_2 y + c_2)y' = 0$ $Snijpunten\ van\ stelsel\ bepalen\ \Rightarrow \begin{cases} u = x - x_1 \\ v = y - y_1 \end{cases}$ $Als\ evenwijdig\ z = a_1 x + b_1 y + c_1$	

Exacte differentiaalvergelijkingen

$$P(x,y)dx + Q(x,y)dy = 0$$

$$Als\ diff.vgl.exact\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \begin{matrix} \int P(x,y)dx = \hat{P}(x,y) + c_y \\ \int Q(x,y)dx = \hat{Q}(x,y) + c_x \end{matrix}$$

$$met\ \varphi(x,y) = \hat{P}(x,y) = \hat{Q}(x,y)$$

Integrerende factoren

μ=μ(x)

$$\frac{\mu'(x)}{\mu(x)} = \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{Q} = \varphi(x)$$

μ=μ(y)= μ(x₂)

$$\frac{\mu'(x)}{\mu(x)} = \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{-P} = \varphi(x_2)$$

μ=μ(x+y)= μ(x₃)

$$\frac{\mu'(x)}{\mu(x)} = \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{Q - P} = \varphi(x_3)$$

μ=μ(xy)= μ(x₄)

$$\frac{\mu'(x)}{\mu(x)} = \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{xQ - yP} = \varphi(x_4)$$

$$\boxed{\mu(x_i) = e^{\int \varphi(x_i)dx_i}}$$

Naam	Vorm	
Scalaire lin. Diff.vgl.	$y' + P(x)y = Q(x)$	$\mu(x) = e^{\int P(x)dx}$
Bernouille	$y' + P(x)y = Q(x)y^m$	$\mu(x) = \frac{1-m}{y^m}$
Ricatti	$y' = f_0(x) + f_1(x)y + f_2(x)y^2$	$y = y_1 + \frac{1}{u}$

Lineaire differentiaalvergelijkingen van orde n >2

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$\overset{1}{\Rightarrow} \Phi(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 = 0 \overset{1}{\Rightarrow} \{u_i = x^{i-1}e^{tx}\}_{i=1}^{m} (m = multipiviteit)$$

$$e^{ibx} = \cos(bx) + i\sin(bx)$$

Euler

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0 \Rightarrow x = \pm e^z \quad voor\ x \gtrless 0$$

Het zoeken van 1 particuliere oplossing

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = e^{\alpha x}Q(x)$$

Onbepaalde coëfficiënten	$V(x) = \sum_{j=0}^{k+m} b_j x^j$ met $\begin{matrix} k = multipliciteit \\ m = graad \end{matrix}$ $y_p = V(x)e^{\alpha x}$
Variatie van parameters	$y_p = \sum_{i=0}^{n} u_i z_i$ met $u_i = opls.\ van\ KV\ van\ lineaire\ homogene\ deel$ $z_1 = \int \frac{1}{W(x)} \begin{vmatrix} 0 & u_2 \\ e^{\alpha x}Q(x) & u_2' \end{vmatrix}$ $z_2 = \int \frac{1}{W(x)} \begin{vmatrix} u_1 & 0 \\ u_1' & e^{\alpha x}Q(x) \end{vmatrix}$ met $W(x) = Wronkisiaan$
Reductie van orde	*Diff. Vgl van orde n omzetten naar orde n-1*

Hoofdstuk 9: Rijen en reeksen

Rij	Recursief	Niet-recursief
Rekenkundig	$x_n = x_{n-1} + v$	$x_n = x_p + (n-p)v$
Meetkundig	$x_n = q.x_{n-1}$	$x_n = q^{n-p}.x_p$

Someren van rijen

$$Rekenkundig: s_n = \frac{x_1 + x_n}{2}.n$$

$$Meetkundig: \; s_n = \frac{1-q^n}{1-q}.n$$

$$Eerste\ n\ natuurelijke\ getallen: \; S_1(n) = \frac{n(n+1)}{2}$$

$$Eerste\ kwadraten\ van\ \ n\ natuurelijke\ getallen: \; S_2(n) = \frac{n(n+1)(2n+1)}{6}$$

$$x_n = an^2 + bn + c \Rightarrow \; s_n = aS_2(n) + bS_1(n) + cn$$

$$S_3(n) = \frac{n^2(n+1)^2}{2} = (S_1(n))^2$$

Limieten van rijen

/

Reeksen

Criteria	Voorwaarde	
Negatief criterium	Algemeen $\sum x_n$	$Convergent: \lim_{n \to +\infty} x_n = 0$
Begrensde partieelsommen	Positieve reeks $\sum x_n$	*Verzameling part.sommen begrensd* → *convergent*
Verdichtingscriterium	Positieve reeks $\sum x_n$ + $(x_n)_n$ naar 0 dalend	$\sum x_n \sim \sum 2^n x_{2^n}$
Convergente majoranten en divergente minoranten	Positieve reeks $\sum x_n$	*Zie titel criterium*
Verhoudingscriterium	Twee positieve reeksen $\sum x_n \; en \; \sum y_n$	$Als \lim_{n \to \infty} \frac{x_n}{y_n} = 0$ *beide zelfde convergentie gedrag*
D'Alembert	Algemeen $\sum x_n$	$\lim_{n \to +\infty} \left\vert \frac{x_{n+1}}{x_n} \right\vert < 1 \rightarrow convergent$ $\lim_{n \to +\infty} \left\vert \frac{x_{n+1}}{x_n} \right\vert > 1 \rightarrow divergent$
Klein criterium van Cauchy	Positieve reeks $\sum x_n$	$\lim_{n \to +\infty} \sqrt[n]{x_n} < 1 \rightarrow convergent$ $\lim_{n \to +\infty} \sqrt[n]{x_n} > 1 \rightarrow divergent$
Integraalcriterium	$f: [0, \infty[\rightarrow [0, \infty[$ met f dalend naar 0	$\int_0^{+\infty} f(x)dx \;\; bestaat \rightarrow convergent$
Dirichelet	$\sum x_n$ waarvoor part.sommen begrensd + $(a_n)_n$ dalende rij zodat $\lim_{n \to +\infty} a_n = 0 \; \rightarrow \; \sum a_n x_n \, convergent$	
Abel	$\sum x_n \; convergent + (a_n)_n \; monotoon \; begrensde \; rij$ $\rightarrow \sum a_n x_n \, convergent$	
Leibnitz	$\sum x_n \;\; is \; alternerend \;\; met \; (\vert x_n \vert)_n \, een \; dalende \; rij$ waarvoor $\lim_{n \to +\infty} \vert x_n \vert = 0 \; \rightarrow \sum x_n \;\; convergent$	

Tylor- en Maclaurinreeksen

$$Tyloor\ polynoom: T_n(f,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$1. T_n(\lambda f + \mu g) = \lambda T_n(f) + \mu T_n(g)$$

$$2. \left(T_n(f)\right)' = T_{n-1}(f')$$

$$3. g(x) = \int_a^x f(t)dt \Rightarrow T_{n+1}(g)(x) = \int_a^x T_n(f)(t)dt$$

$$4. f(x) = P_n(x) + x^n g(x)$$

$$\boxed{Formule\ van\ Tylor: R_n(x) = f(x) - T_n(f)(x)}$$

$$Restterm\ van\ Tylor: R_n(x) = \frac{1}{n!}\int_a^x (x-t)^2 f^{(n+1)}(t)dt$$

$$Restterm\ van\ Lagrange: R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Puntsgewijze en uniforme convergentie

$$Puntsgewijze: f = \lim_{n\to\infty} f_n(x)$$

$$Uniforme: \lim_{n\to\infty} sup|f_n(x) - f(x)| = 0$$

Fourierreeksen

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$Kwadratische\ convergentie: \lim_{n\to\infty}\int_a^b |f_n(x) - f(x)|^2 dx$$

$$Perseval: \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{|a_0|}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$$

Hoofdstuk 10: Functies van meerdere veranderlijken

Metrieken

$$d_E(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_E = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

$$d_S(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_S = \sum_{i=1}^{n}|x_i-y_i|$$

$$d_M(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_M = \max_{i=1}|x_i-y_i|$$

Afleidbaarheid en differentieerbaarheid

$$Richtingsafgeliede: Df\big((a,b),\boldsymbol{h}\big) = \lim_{\lambda\to 0}\frac{f(a+\lambda h_1, b+\lambda h_2)-f(a,b)}{\lambda} \Rightarrow Afleidbaar$$

$$Raakvlak: z = \frac{\partial f}{\partial x}(\boldsymbol{a})(x-a)+\frac{\partial f}{\partial y}(\boldsymbol{b})(y-b)+f(\boldsymbol{a})$$

$$\lim_{\lambda\to 0}\frac{f(\boldsymbol{x})-f(\boldsymbol{a})-\frac{\partial f}{\partial x}(\boldsymbol{a})(x-a)-\frac{\partial f}{\partial y}(\boldsymbol{b})(y-b)}{\|\boldsymbol{x}-\boldsymbol{a}\|} = o \Rightarrow differentieerbaar$$

$$\boxed{Df(\boldsymbol{a},\boldsymbol{h}) = Df(\boldsymbol{a}).\boldsymbol{h}}$$

$$Kettingregel: D\boldsymbol{h}(\boldsymbol{a}) = D\boldsymbol{g}(\boldsymbol{b}).D\boldsymbol{f}(\boldsymbol{a})$$

$$Inverse: D\boldsymbol{f}^{-\mathbf{1}}(\boldsymbol{b}) = (D\boldsymbol{f}(\boldsymbol{a}))^{-\mathbf{1}} \quad met\ \boldsymbol{b} = f(\boldsymbol{a})$$

Krommen oppervlakken

$$de\ normaal\ van\ een\ opp.:\ \eta(u,v) = \frac{\partial\varphi}{\partial u}\mathrm{x}\frac{\partial\varphi}{\partial v}$$

De impliciete functie

$$Voor\ F(x,y)=0: y'(x) = \frac{dy}{dx}(x) = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)}$$

$$Voor\ F(x,y;z)=0: \left(\frac{\partial f}{\partial x}(x,y) \quad \frac{\partial f}{\partial y}(x,y)\right) = \begin{pmatrix} -\dfrac{\frac{\partial F}{\partial x}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)} & -\dfrac{\frac{\partial F}{\partial y}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)} \end{pmatrix}$$

$$Zij\ F(x,y,z) en\ G(x,y,z) met\ F,G\colon \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}\colon \begin{pmatrix} \dfrac{df}{dz}(z) \\ \dfrac{dg}{dz}(z) \end{pmatrix} = \begin{pmatrix} -\dfrac{\left|\dfrac{\partial(F,G)}{\partial(z,y)}(\boldsymbol{x_0})\right|}{\left|\dfrac{\partial(F,G)}{\partial(x,y)}(\boldsymbol{x_0})\right|} \\ -\dfrac{\left|\dfrac{\partial(F,G)}{\partial(z,y)}(\boldsymbol{x_0})\right|}{\left|\dfrac{\partial(F,G)}{\partial(x,y)}(\boldsymbol{x_0})\right|} \end{pmatrix}$$

Functie onderzoek

$$Hessiaanse\ matrix : H(x,y) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y) & \dfrac{\partial^2 f}{\partial y \partial x}(x,y) \\ \dfrac{\partial^2 f}{\partial x \partial y}(x,y) & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}$$

$$\Delta = H(\boldsymbol{a})$$

$$A = \frac{\partial^2 f}{\partial x^2}(\boldsymbol{a})$$

Multiplicatoren van Lagrange

$$F(x_1, x_2, \dots, x_{n-1}, x_n, \lambda) = f\ (x_1, x_2, \dots, x_{n-1}, x_n) + \lambda\phi(x_1, x_2, \dots, x_{n-1}, x_n)$$

$$met\ \phi(\boldsymbol{x}) = nevenvoorwaarde$$