CHAPTER 13

MOMENTUM ANALYSIS OF FLOW SYSTEMS

When dealing with engineering problems, it is desirable to obtain fast and accurate solutions at minimal cost. Most engineering problems, including those associated with fluid flow, can be analyzed using one of three basic approaches: differential, experimental, and control volume. In *differential approaches*, the problem is formulated accurately using differential quantities, but the solution of the resulting differential equations is difficult, usually requiring the use of numerical methods with extensive computer codes. *Experimental approaches* complemented with dimensional analysis are highly accurate, but they are typically time-consuming, and expensive. The *finite control volume approach* described in this chapter is remarkably fast and simple, and usually gives answers that are sufficiently accurate for most engineering purposes. Therefore, despite the approximations involved, the basic finite control volume analysis performed with a paper and pencil has always been an indispensable tool for engineers.

In this chapter, we present the finite control volume momentum analysis of fluid flow problems. First we give an overview of the conservation relations for mass, linear momentum, angular momentum, and energy, and derive the Reynolds transport theorem. Then we develop the linear and angular momentum equations for control volumes, and use them to determine the forces and moments associated with fluid flow.

CONTENTS

13–1 ▪ NEWTON'S LAWS AND CONSERVATION OF MOMENTUM

Newton's laws are relations between motions of bodies and the forces acting on them. Newton's first law states that "*a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero*." Therefore, a body tends to preserve its state or inertia. Newton's second law states that "*the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass*." Newton's third law states "*when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first*." Therefore, the direction of an exposed reaction force depends on the body taken as the system.

For a rigid body of mass m, Newton's second law is expressed as

Newton's second law: $$\vec{F} = m\vec{a} = m\frac{d\vec{\mathcal{V}}}{dt} = \frac{d(m\vec{\mathcal{V}})}{dt} \qquad \textbf{(13–1)}$$

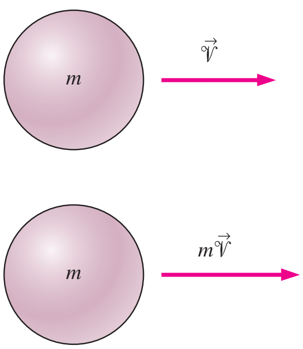


FIGURE 13–1
Linear momentum is the product of mass and velocity, and its direction is the direction of velocity.

where $\vec{F}$ is the net force acting on the body and $\vec{a}$ is the acceleration of the body under the influence of $\vec{F}$.

The product of the mass and the velocity of a body is called the *linear momentum* or just the *momentum* of the body. The momentum of a rigid body of mass m moving with a velocity $\vec{\mathcal{V}}$ is $m\vec{\mathcal{V}}$ (Fig. 13–1). Then Newton's second law expressed in Eq. 13–1 can also be stated as "*the rate of change of the momentum of a body is equal to the net force acting on the body*" (Fig. 13–2). This statement is more in line with Newton's original statement of the second law, and it is more appropriate for use in fluid mechanics when studying the forces generated as a result of velocity changes of fluid streams. Therefore, in fluid mechanics, Newton's second law is usually referred to as the *linear momentum equation*.

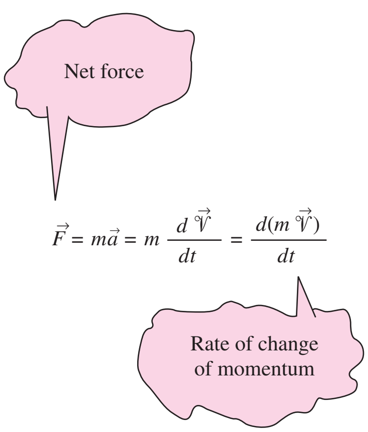


FIGURE 13–2
Newton's second law is also expressed as *the rate of change of the momentum of a body is equal to the net force acting on it.*

The momentum of a system remains constant when the net force acting on it is zero, and thus the momentum of such systems is conserved. This is known as the *conservation of momentum principle*. This principle has proven to be a very useful tool when analyzing collisions such as those between balls; between balls and rackets, bats, or clubs; and between atoms or subatomic particles; and explosions such as those that occur in rockets, missiles, and guns. The momentum of a loaded rifle, for example, must be zero after shooting since it is zero before shooting, and thus the rifle must have a momentum equal to that of the bullet in the opposite direction so that the vector sum of the two is zero.

Note that force, acceleration, velocity, and momentum are vector quantities, and as such they have direction as well as magnitude. Also, momentum is a constant multiple of velocity, and thus the direction of momentum is the direction of velocity. Any vector equation can be written in scalar form for a specified direction using magnitudes, e.g., $F_x = ma_x = d(m\mathcal{V}_x)/dt$ in the x direction.

The counterpart of Newton's second law for rotating rigid bodies is expressed as $\vec{M} = I\vec{\alpha}$, where $\vec{M}$ is the net torque applied on the body, I is the moment of inertia of the body about the axis of rotation, and $\vec{\alpha}$ is the angular acceleration. It can also be expressed in terms of the rate of change of angular momentum $d\vec{H}/dt$ as

Angular momentum equation:
$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt} \qquad \textbf{(13–2)}$$

where $\vec{\omega}$ is the angular velocity. For a rigid body rotating about a fixed x-axis, the angular momentum equation can be written in scalar form as

Angular momentum about x-axis:
$$M_x = I_x\frac{d\omega_x}{dt} = \frac{dH_x}{dt} \qquad \textbf{(13–3)}$$

The angular momentum equation can be stated as "*the rate of change of the angular momentum of a body is equal to the net torque acting on it*" (Fig. 13–3).

The total angular momentum of a rotating body remains constant when the net torque acting on it is zero, and thus the angular momentum of such systems is conserved. This is known as the *conservation of angular momentum principle*, and is expressed as $I\omega$ = *constant*. Many interesting phenomena such as an ice skater spinning faster when she brings her arms close to her body and a diver rotating faster when he curls after the jump can be explained easily with the help of the conservation of angular momentum principle (in both cases, the moment of inertia I is decreased and thus the angular velocity ω is increased as the outer parts of the body are brought closer to the axis of rotation).

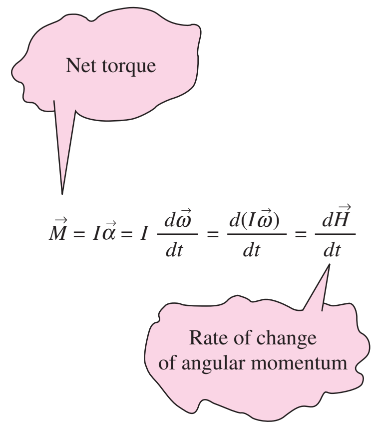


FIGURE 13–3
The rate of change of the angular momentum of a body is equal to the net torque acting on it.

13–2 ▪ THE REYNOLDS TRANSPORT THEOREM

In thermodynamics and solid mechanics we often work with a **system** (also called a **closed system**), defined as a *quantity of matter of fixed identity*. In fluid dynamics, it is more common to work with a **control volume** (also called an **open system**), defined as a *region in space chosen for study.* The size and shape of a system may change during a process, but no mass crosses its boundaries. A control volume, on the other hand, allows mass to flow in or out across its boundaries, which are called the **control surface**. A control volume may also move and deform during a process, but many real-world applications involve fixed, nondeformable control volumes.

Figure 13–4 illustrates both a system and a control volume for the case of deodorant being sprayed from a spray can. When analyzing the spraying process, a natural choice for our analysis is either the moving, deforming fluid (a system) or the volume bounded by the inner surfaces of the can (a control volume). These two choices are identical before the deodorant is sprayed. When some contents of the can are discharged, the system approach considers the discharged mass as part of the system, and tracks it (a difficult job indeed), and thus the mass of the system remains constant. Conceptually this is equivalent to attaching a flat balloon to the nozzle of the can, and letting the spray

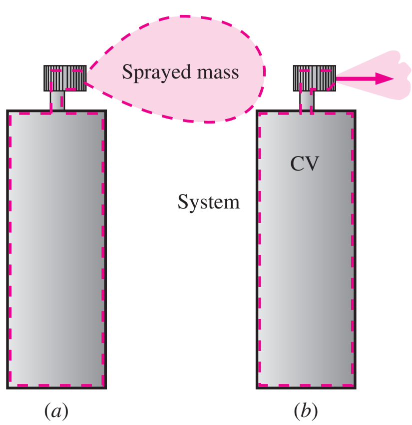


FIGURE 13–4
Two methods of analyzing the spraying of deodorant from a spray can: (*a*) We follow the fluid as it moves and deforms. This is the *system approach*—no mass crosses the boundary and the total mass of the system remains fixed. (*b*) We consider a fixed interior volume of the can. This is the *control volume approach*—mass crosses the boundary.

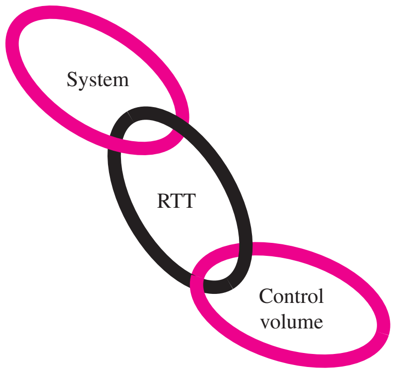


FIGURE 13–5
The *Reynolds transport theorem* provides a link between the system approach and the control volume approach.

inflate the balloon. The inner surface of the balloon now becomes part of the boundary of the system. The control volume approach, however, is not concerned at all with the deodorant that has escaped the can (other than its properties at the exit), and thus the mass of the control volume decreases during this process while its volume remains constant. Therefore, the system approach treats the spraying process as an expansion of the system's volume, whereas the control volume approach considers it as a fluid discharge through the control surface of the fixed control volume.

Most principles of fluid mechanics are adopted from solid mechanics, where the physical laws dealing with the time rates of change of extensive properties are expressed for systems. In fluid mechanics, it is usually more convenient to work with control volumes, and thus there is a need to relate the changes in a control volume to the changes in a system. The relationship between the time rates of change of an extensive property for a system and for a control volume is expressed by the **Reynolds transport theorem** (abbreviated **RTT**), which provides the link between the system and control volume approaches (Fig. 13–5).

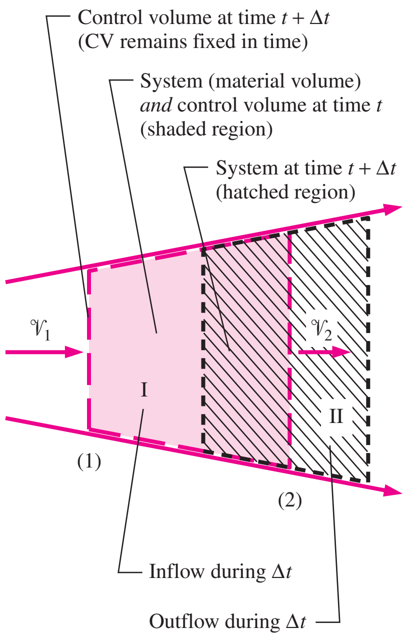
The general form of the Reynolds transport theorem can be derived by considering a general system with an arbitrary shape and arbitrary interactions, but the derivation is rather involved. To help you grasp the fundamental meaning of the theorem, we derive it first in a straightforward manner using a simple geometry, and then generalize the results.

Consider fluid flow from left to right through a diverging (expanding) portion of a flow field as sketched in Fig. 13–6. The upper and lower bounds of the fluid under consideration are *streamlines* of the flow, and we assume uniform flow through any cross section between these two streamlines. We choose the control volume to be fixed between sections (1) and (2) of the flow field. Both (1) and (2) are normal to the direction of flow. At some initial time t, the system coincides with the control volume, and thus the system and control volume are identical (the shaded region in Fig. 13–6). During time interval Δt, the system moves in the flow direction at uniform speeds $\mathcal{V}_1$ at section (1) and $\mathcal{V}_2$ at section (2). The system at this later time is indicated by the hatched region. The region uncovered by the system during this motion is designated as section I (which is still part of the CV), and the new region covered by the system is designated as section II (not part of the CV). Therefore, at time $t + \Delta t$, the system consists of the same fluid, but it occupies the region CV—I + II. The control volume is fixed in space, and thus it remains as the shaded region marked CV at all times.

Let B represent any **extensive property** (such as mass, energy, or momentum), and let $b = B/m$ represent the corresponding **intensive property**. Noting that extensive properties are additive, the extensive property B of the system at times t and $t + \Delta t$ can be expressed as

$$B_{\text{sys},\,t} = B_{\text{CV},\,t} \quad \text{(the system and CV concide at time t)}$$
$$B_{\text{sys},\,t+\Delta t} = B_{\text{CV},\,\text{t}+\Delta\text{t}} - B_{\text{I},\,t+\Delta t} + B_{\text{II},\,t+\Delta t} \qquad \textbf{(13–4)}$$

Subtracting the first equation here from the second one and dividing by Δt gives



At time t: Sys = CV
At time $t + \Delta t$: Sys = CV − I + II

FIGURE 13–6
A moving *system* (hatched region) and a fixed *control volume* (shaded region) in a diverging portion of a flow field at times t and $t + \Delta t$. The upper and lower bounds are streamlines of the flow.

$$\frac{B_{\text{sys},\, t+\Delta t} - B_{\text{sys},\, t}}{\Delta t} = \frac{B_{\text{CV},\, t+\Delta t} - B_{\text{CV},\, t}}{\Delta t} - \frac{B_{\text{I},\, t+\Delta t}}{\Delta t} + \frac{B_{\text{II},\, t+\Delta t}}{\Delta t} \tag{13–5}$$

Taking the limit as $\Delta t \to 0$, while using the definition of derivative yields

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} - \dot{B}_{\text{in}} + \dot{B}_{\text{out}} \tag{13–6}$$

or

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} - b_1 \rho_1 \mathcal{V}_1 A_1 + b_2 \rho_2 \mathcal{V}_2 A_2 \tag{13–7}$$

since

$$B_{\text{I},\, t+\Delta t} = b_1 m_{\text{I},\, t+\Delta t} = b_1 \rho_1 \mathcal{V}_{\text{I},\, t+\Delta t} = b_1 \rho_1 \mathcal{V}_1 \Delta t A_1$$
$$B_{\text{II},\, t+\Delta t} = b_2 m_{\text{II},\, t+\Delta t} = b_2 \rho_2 \mathcal{V}_{\text{II},\, t+\Delta t} = b_2 \rho_2 \mathcal{V}_2 \Delta t A_2$$

and

$$\dot{B}_{in} = \dot{B}_I = \lim_{\Delta t \to 0} \frac{B_{I,\, t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{b_1 \rho_1 \mathcal{V}_1 \Delta t A_1}{\Delta t} = b_1 \rho_1 \mathcal{V}_1 A_1$$

$$\dot{B}_{out} = \dot{B}_{II} = \lim_{\Delta t \to 0} \frac{B_{II,\, t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{b_2 \rho_2 \mathcal{V}_2 \Delta t A_2}{\Delta t} = b_2 \rho_2 \mathcal{V}_2 A_2$$

Equation 13–6 states that *the time rate of change of the property B of the system is equal to the time rate of change of B of the control volume plus the net flux of B by mass across the control surface.* This is the desired relation since it relates the change of a property of a system to the change of that property for a control volume. Note that Eq. 13–6 applies at any instant of time, where it is assumed that the system and the control volume occupy the same space at that particular instant of time.

The influx $\dot{B}_{in}$ and outflux $\dot{B}_{out}$ of the property B in this case are easy to determine since there is only one inlet and one exit, and the velocities are normal to the surfaces at sections (1) and (2). In general, however, we may have several inlet and exit ports, and the velocity may not be normal to the control surface at the point of entry. Also, the velocity may not be uniform. To generalize the process, we consider a differential surface area dA on the control surface, and denote its **unit outer normal** by $\vec{n}$. The flow rate of the property b through dA is $\rho b \vec{\mathcal{V}} \cdot \vec{n}\, dA$ since the dot product $\vec{\mathcal{V}} \cdot \vec{n}$ gives the normal component of the velocity. Then the net rate of flow through the entire control surface is determined by integration to be (Fig. 13–7)

$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{\mathcal{V}} \cdot \vec{n}\, dA \quad \text{(inflow if negative)} \tag{13–8}$$

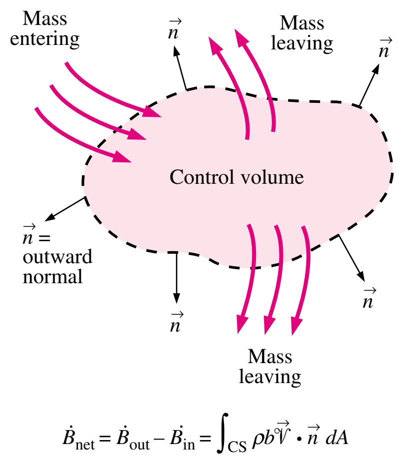


FIGURE 13–7
The integral of $b\rho\vec{\mathcal{V}} \cdot \vec{n}\, dA$ over the control surface gives the net amount of the property B flowing out of the control volume (into the control volume if it is negative) per unit time.

An important aspect of this relation is that it automatically subtracts the inflow from the outflow, as explained next. The dot product of the velocity vector at a point on the control surface and the outer normal at that point is $\vec{\mathcal{V}} \cdot \vec{n} = |\vec{\mathcal{V}}|\,|\vec{n}| \cos\theta = |\vec{\mathcal{V}}| \cos\theta$, where θ is the angle between the velocity vector and the outer normal, as shown in Fig. 13–8. For $\theta < 90°$, we have $\cos\theta > 0$ and thus $\vec{\mathcal{V}} \cdot \vec{n} > 0$ for outflow of mass from the control volume, and for $\theta > 90°$, we have $\cos\theta < 0$ and thus $\vec{\mathcal{V}} \cdot \vec{n} < 0$ for inflow of mass into the control volume. Therefore, the differential quantity $\rho b \vec{\mathcal{V}} \cdot \vec{n}\, dA$ is positive for mass flowing out of the control volume, and negative for mass flowing into the control volume, and its integral over the entire control surface gives the rate of net outflow of the property B by mass.

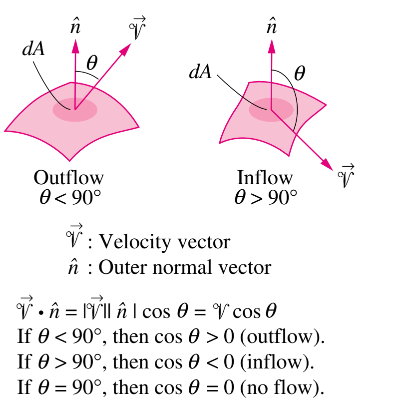


FIGURE 13–8
Inflow and outflow of mass across the differential area of a control surface.

The properties within the control volume may vary with position, in general. In such a case, the total amount of property B within the control volume must be determined by integration:

$$B_{CV} = \int_{CV} \rho b\, dV \tag{13–9}$$

The term $\dfrac{dB_{CV}}{dt}$ in Eq. 13–6 is thus equal to $\dfrac{d}{dt}\displaystyle\int_{CV} \rho b\, dV$, and represents the time rate of change of the property B content of the control volume. A positive value for dB_{CV}/dt indicates an increase in the B content, and a negative value indicates a decrease. Substituting Eqs. 13–8 and 13–9 into Eq. 13–6 yields the Reynolds transport theorem, also known as the *system-to-control-volume transformation* for a fixed control volume:

RTT, fixed CV:
$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho \, b\vec{\mathcal{V}} \cdot \vec{n} \, dA \tag{13–10}$$

When the control volume is not moving or deforming with time, the time derivative on the right-hand side can be moved inside the integral since the domain of integration does not change with time. (In other words, it is irrelevant whether we differentiate or integrate first.) But the time derivative in that case must be expressed as a *partial* derivative ($\partial/\partial t$) since density and the quantity b may depend on the position within the control volume. Thus, an alternate form of the Reynolds transport theorem for a fixed control volume is

Alternate RTT, fixed CV:
$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{\text{CS}} \rho b\vec{\mathcal{V}} \cdot \vec{n} \, dA \tag{13–11}$$

Equation 13–10 was derived for a *fixed* control volume. However, many practical systems such as turbine and propeller blades involve nonfixed control volumes. Fortunately, Eq. 13–10 is also valid for *moving* and/or *deforming* control volumes provided that the absolute fluid velocity $\vec{\mathcal{V}}$ in the last term be replaced by the **relative velocity** $\vec{\mathcal{V}}_r$,

Relative velocity:
$$\vec{\mathcal{V}}_r = \vec{\mathcal{V}} - \vec{\mathcal{V}}_{\text{CS}} \tag{13–12}$$

where $\vec{\mathcal{V}}_{\text{CS}}$ is the local velocity of the control surface (Fig. 13–9). The most general form of the Reynolds transport theorem is thus

RTT, nonfixed CV:
$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho b\vec{\mathcal{V}}_r \cdot \vec{n} \, dA \tag{13–13}$$

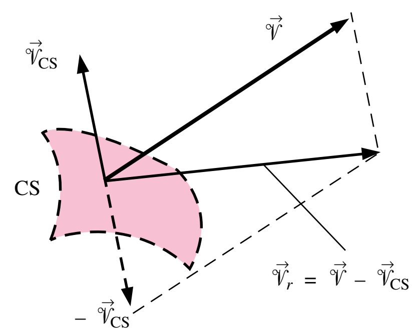


FIGURE 13–9
Relative velocity crossing a control surface is found by vector addition of the absolute velocity of the fluid and the negative of the local velocity of the control surface.

Note that for a control volume that moves and/or deforms with time, the time derivative must be applied *after* integration, as in Eq. 13–13. As a simple example of a moving control volume, consider a toy car moving at a constant absolute velocity $\vec{\mathcal{V}}_c = 10$ km/h to the right. A high-speed jet of water (absolute velocity $= \vec{\mathcal{V}}_{\text{jet}} = 25$ km/h to the right) strikes the back of the car and propels it (Fig. 13–10). If we draw a control volume around the car, the relative velocity is $\vec{\mathcal{V}}_r = 25 - 10 = 15$ km/h to the right. This represents the velocity at which an observer moving with the control volume (moving with the car) would observe the fluid crossing the control surface. In other words, $\vec{\mathcal{V}}_r$ is the fluid velocity expressed relative to a coordinate system moving *with* the control volume.

Finally, by application of Leibnitz theorem, it can be shown that the Reynolds transport theorem for a general moving and/or deforming control volume (Eq. 13–13) is equivalent to the form given by Eq. 13–11, which is repeated here:

Alternate RTT, nonfixed CV:
$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{\text{CS}} \rho b\vec{\mathcal{V}} \cdot \vec{n} \, dA \tag{13–14}$$

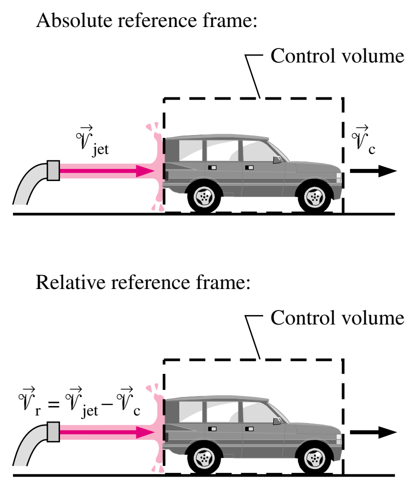


FIGURE 13–10
Reynolds transport theorem applied to a control volume moving at constant velocity.

We emphasize that the velocity vector $\vec{\mathcal{V}}$ in Eq. 13–14 must be taken as the *absolute* velocity (as viewed from a fixed reference frame) in order to apply to a nonfixed control volume.

Special Case 1: Steady Flow

During steady flow, the amount of the property B within the control volume remains constant, and thus the time derivative in Eq. 13–10 becomes zero. Then the Reynolds transport theorem reduces to

RTT, steady flow:
$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CS}} \rho b \vec{\mathcal{V}} \cdot \vec{n}\, dA \tag{13–15}$$

Note that unlike the control volume, the property B content of the system may still change with time during a steady process. But in this case the change must be equal to the net property transported by mass across the control surface (an advective rather than an unsteady effect).

Special Case 2: Uniform Inlets and Outlets

In most practical applications of the RTT, fluid crosses the boundary of the control volume at a finite number of inlets and outlets. Furthermore, there are some applications where there are nearly *uniform properties* over the cross sections where fluid enters or leaves the control volume (Fig. 13–11). In such cases, the control surface integral of Eq. 13–14 can be replaced by an algebraic summation. Since ρ, $\mathcal{V}$, and b are uniform across the inlet or outlet, the Reynolds transport theorem in this case reduces to

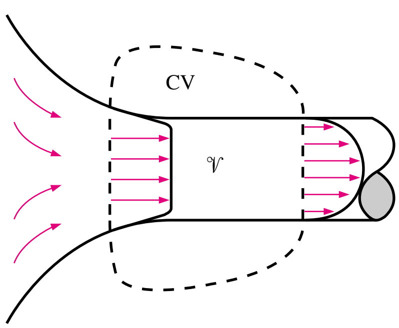


FIGURE 13–11
The *uniform-flow approximation* is valid across a limited type of inlet and/or outlet. Examples include the jet exiting from a well-designed nozzle, uniform free-stream flow in wind tunnels, and well-rounded inlets to pipes and ducts. In the pipe entrance region sketched here, the uniform-flow approximation is reasonable at the inlet of the control volume, but would be a poor approximation at the control volume outlet since the flow there is *not* uniform.

RTT, uniform inlets and outlets:
$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \sum_{\text{out}} \underbrace{\rho b \mathcal{V} A}_{\text{for each outlet}} - \sum_{\text{in}} \underbrace{\rho b \mathcal{V} A}_{\text{for each inlet}} \tag{13–16}$$

or

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \sum_{\text{out}} \dot{m} b - \sum_{\text{in}} \dot{m} b \tag{13–17}$$

since at any inlet or outlet, $\dot{m} = \rho \mathcal{V} A$.

Note that the assumption of uniform inlets and outlets simplifies the analysis greatly, but may not always be valid. In most real cases, the inflow or outflow is *not* uniform. In such cases we are tempted to simply let ρ, $\mathcal{V}$, and b be the mean density, mean velocity, and mean property across the inlet or outlet respectively, and apply Eq. 13–16. However, this leads to errors in the analysis since the control surface integral of Eq. 13–14 becomes *nonlinear* when property b contains a velocity term (e.g., in the linear momentum equation $b = \vec{\mathcal{V}}$). Fortunately we can eliminate the errors by including correction factors in Eq. 13–16, as discussed later in this chapter.

Equations 13–15 through 13–17 apply to fixed or moving control volumes, but as discussed previously the relative velocity must be used for the case of a nonfixed control volume. In Eq. 13–17 for example, the mass flow rate is relative to the (moving) control surface.

The Reynolds transport theorem can be applied to any scalar or vector property. In this chapter, we will apply the Reynolds transport theorem to conservation of mass, linear momentum, and angular momentum by choosing parameter B to be mass, linear momentum, or angular momentum, respectively. In this fashion we can easily convert from the fundamental system conservation laws (Lagrangian view point) to forms that are valid and useful in a control volume analysis (Eulerian view point).

An Application: Conservation of Mass

The general conservation of mass relation for a control volume can also be derived using the Reynolds transport theorem by taking the property B to be the mass, m. Then we have $b = 1$ since dividing the mass by mass to get the property per unit mass gives unity. Also, the mass of a system is constant, and thus its time derivative is zero. That is, $dm_{\text{sys}}/dt = 0$. Then the Reynolds transport equation in this case reduces to (Fig. 13–12)

General conservation of mass:

$$\frac{d}{dt}\int_{\text{CV}} \rho \, dV + \int_{\text{CS}} \rho(\vec{\mathcal{V}} \cdot \vec{n}) \, dA = 0 \tag{13–18}$$

It states that *the time rate of change of mass within the control volume plus the net mass flow rate through the control surface is equal to zero*.

This demonstrates that the Reynolds transport theorem is a very powerful tool, and we can use it with confidence.

Splitting the surface integral in Eq. 13–18 into two parts—one for the outgoing flow streams (positive) and one for the incoming streams (negative)—the general conservation of mass relation can also be expressed as

$$\frac{d}{dt}\int_{\text{CV}} \rho \, dV + \sum_{\text{out}} \int_A \rho \mathcal{V}_n \, dA - \sum_{\text{in}} \int_A \rho \mathcal{V}_n \, dA = 0 \tag{13–19}$$

Using the definition of mass flow rate, it can also be expressed as

$$\frac{d}{dt}\int_{\text{CV}} \rho \, dV = \sum_{\text{out}} \dot{m} - \sum_{\text{in}} \dot{m} \qquad \text{or} \qquad \frac{dm_{\text{CV}}}{dt} = \sum_{\text{out}} \dot{m} - \sum_{\text{in}} \dot{m} \tag{13–20}$$

where the summation signs are used to emphasize that *all* the inlets and outlets are to be considered.

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho b(\vec{\mathcal{V}} \cdot \vec{n}) \, dA$$

$B = m \qquad b = 1 \qquad b = 1$

$$\frac{dm_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho \, dV + \int_{\text{CS}} \rho(\vec{\mathcal{V}} \cdot \vec{n}) \, dA$$

FIGURE 13–12
The conservation of mass equation is obtained by replacing B in the Reynolds transport theorem by mass m, and b by 1 (m per unit mass $= m/m = 1$).

13–3 ▪ CHOOSING A CONTROL VOLUME

We now briefly discuss how to *wisely* select a control volume. A control volume can be selected as any arbitrary region in space through which fluid flows, and its bounding control surface can be fixed, moving, and even deforming during flow. The application of a basic conservation law is simply a systematic procedure for bookkeeping or accounting of the quantity under consideration, and thus it is extremely important that the boundaries of the control volume are well defined during an analysis. Also, the flow rate of any quantity into or out of a control volume depends on the flow velocity *relative*

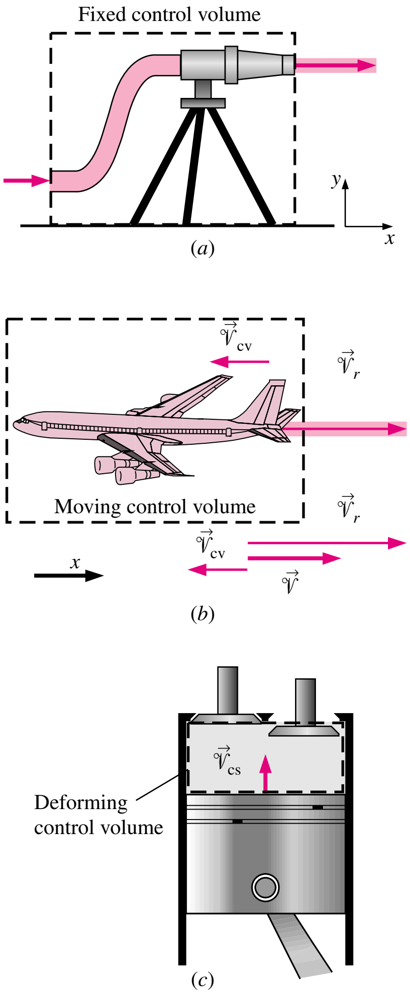


FIGURE 13–13
Examples of (*a*) fixed, (*b*) moving, and (*c*) deforming control volumes.

to the control surface, and thus it is essential to know if the control volume remains at rest during flow or if it moves.

Many flow systems involve stationary hardware firmly fixed to a stationary surface, and such systems are best analyzed using *fixed* control volumes. When determining the reaction force acting on a tripod holding the nozzle of a hose, for example, a natural choice for the control volume is one that passes perpendicularly through the nozzle exit flow and through the bottom of the tripod legs (Fig. 13–13*a*). This is a fixed control volume, and the water velocity relative to a fixed point on the ground is the same as the water velocity relative to the nozzle exit plane.

When analyzing flow systems that are moving or deforming, it is usually more convenient to allow the control volume to *move* or *deform*. When determining the thrust developed by the jet engine of an airplane cruising at constant velocity, for example, a wise choice of control volume is one that encloses the airplane and cuts through the nozzle exit plane (Fig. 13–13*b*). The control volume in this case moves with velocity $\vec{V}_{\text{CV}}$, which is identical to the cruising velocity of the airplane relative to a fixed point on earth. When determining the quantity of exhaust gases leaving the nozzle, the proper velocity to use is the velocity of the exhaust gases relative to the nozzle exit plane, that is the *relative velocity* $\vec{V}_r$. Since the entire control volume moves at velocity $\vec{V}_{\text{CV}}$, Eq. 13–12 becomes $\vec{V}_r = \vec{V} - \vec{V}_{\text{CV}}$, where $\vec{V}$ is the *absolute velocity* of the exhaust gases, i.e., the velocity relative to a fixed point on earth. Note that $\vec{V}_r$ is the fluid velocity expressed relative to a coordinate system moving *with* the control volume. Also, this is a vector equation, and velocities in opposite direction have opposite signs. For example, if the airplane is cruising at 500 km/h to the left, and the velocity of the exhaust gases is 800 km/h to the right relative to the ground, the velocity of the exhaust gases relative to the nozzle exit is

$$\vec{V}_r = \vec{V} - \vec{V}_{\text{CV}} = 800\vec{i} - (-500\vec{i}) = 1300\vec{i}\ \text{km/h}$$

That is, the exhaust gases leave the nozzle at 1300 km/h to the right relative to the nozzle exit (in the direction opposite to that of the airplane); this is the velocity that should be used when evaluating the outflow of exhaust gases through the control surface (Fig. 13–13*b*). Note that the exhaust gases would appear motionless to an observer on the ground if the relative velocity were equal in magnitude to the airplane velocity.

When analyzing the purging of exhaust gases from a reciprocating internal combustion engine, a wise choice for the control volume is one that comprises the space between the top of the piston and the cylinder head (Fig. 13–13*c*). This is a *deforming* control volume, since part of the control surface moves relative to other parts. The relative velocity for an inlet or outlet on the deforming part of a control surface is then given by Eq. 13–12, $\vec{V}_r = \vec{V} - \vec{V}_{\text{CS}}$, where $\vec{V}$ is the absolute fluid velocity and $\vec{V}_{\text{CS}}$ is the control surface velocity, both relative to a fixed point outside the control volume. Note that $\vec{V}_{\text{CS}} = \vec{V}_{\text{CV}}$ for moving but nondeforming control volumes, and $\vec{V}_{\text{CS}} = \vec{V}_{\text{CV}} = 0$ for fixed ones.

13–4 FORCES ACTING ON A CONTROL VOLUME

The forces acting on a control volume consist of **body forces** that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as the pressure and viscous forces and reaction forces at points of contact).

In control volume analysis, the sum of all forces acting on the control volume at a particular instant of time is represented by $\sum \vec{F}$, and is expressed as

Total force acting on control volume:
$$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} \qquad \text{(13–21)}$$

Body forces act on each volumetric portion of the control volume. The body force acting on a differential element of fluid of volume dV within the control volume is shown in Fig. 13–14, and we must perform a volume integral to account for the net body force on the entire control volume. *Surface forces* act on each portion of the control surface. A differential surface element of area dA and unit outward normal $\vec{n}$ on the control surface is shown in Fig. 13–14, along with the surface force acting on it. We must perform an area integral to obtain the net surface force acting on the entire control surface. As sketched, the surface force may act in a direction independent of that of the outward normal vector.

The most common body force is that of **gravity,** which exerts a downward force on every differential element of the control volume. While other body forces, such as electric and magnetic forces may be important in some analyses, we consider only gravitational forces here.

A careful selection of the control volume enables us to write the total force acting on the control volume, $\sum \vec{F}$, as the sum of more readily available quantities like weight, pressure, and reaction forces. We recommend the following for control volume analysis:

Total force:
$$\underbrace{\sum \vec{F}}_{\text{total force}} = \underbrace{\sum \vec{F}_{\text{gravity}}}_{\text{body force}} + \underbrace{\sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}}}_{\text{surface forces}} \qquad \text{(13–22)}$$

The first term on the right-hand side of Eq. 13–22 is the body force *weight*, since gravity is the only body force we are considering. The other three terms combine to form the net surface force; they are pressure forces, viscous forces, and "other" forces acting on the control surface. $\sum \vec{F}_{\text{other}}$ is composed of reaction forces required to turn the flow, forces at bolts, cables, struts, or walls through which the control surface cuts, etc.

All of these surface forces arise as the control volume is isolated from its surroundings for analysis, and the effect of any detached object is accounted for by a force at that location. This is similar to drawing a free-body-diagram in your statics and dynamics classes. We should choose the control volume such that the forces that we are not interested in remain internal, and thus they do not complicate the analysis. A well-chosen control volume exposes only

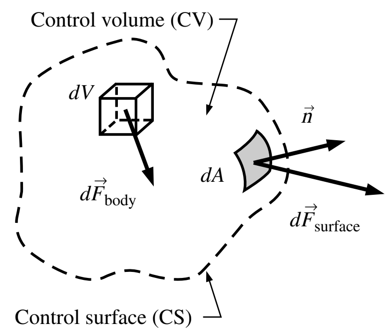


FIGURE 13–14
The total force acting on a control volume is composed of body forces and surface forces; body force is shown on a differential volume element, and surface force is shown on a differential surface element.

the forces that are to be determined (such as reaction forces) and a minimum number of other forces.

Only external forces are considered in the analysis. The internal forces (such as the pressure force between a fluid and the inner surfaces of the flow section) are not considered in a control volume analysis unless they are exposed by passing the control surface through that area.

A common simplification in the application of Newton's laws of motion is to subtract the *atmospheric pressure* and work with gage pressures. This is because the atmospheric pressure acts in all directions, and its effect cancels out in every direction (Fig. 13–15). This means we can also ignore the pressure forces at outlet sections where the fluid is discharged to the atmosphere since the discharge pressures in such cases will very nearly be atmospheric pressure at subsonic velocities.

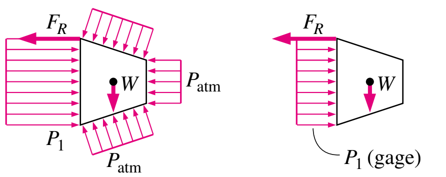


FIGURE 13–15
The atmospheric pressure acts in all directions, and thus it can be ignored when performing force balances since its effect cancels out in every direction.

As an example of how to wisely choose a control volume, consider control volume analysis of water flowing steadily through a faucet with a partially closed gate valve spigot (Fig. 13–16). It is desired to calculate the net force on the flange to ensure that the flange bolts are strong enough. There are many possible choices for the control volume. Some authors restrict their control volumes to the fluid itself, as indicated by CV A (the colored control volume). With this control volume, there are pressure forces that vary along the control surface, there are viscous forces along the pipe wall and at locations inside the valve, and there is a body force, namely, the weight of the water in the control volume. Fortunately, to calculate the net force on the flange, we do *not* need to integrate the pressure and viscous stresses all along the control surface. Instead, we can lump the unknown pressure and viscous forces together into one reaction force, representing the net force of the walls on the water. This force, plus the weight of the faucet and the water, is equal to the net force on the flange. (We must be very careful with our signs, of course.)

When choosing a control volume, you are not limited to the fluid alone. Often it is more convenient to slice the control surface *through* solid objects such as walls, struts, or bolts as illustrated by CV B (the gray control volume) in Fig. 13–16. A control volume may even surround an entire object, like the one shown here. Control volume B is a wise choice because we are not concerned with any details of the flow or even the geometry inside the control volume. For the case of CV B, we assign a net reaction force acting at the portions of the control surface that slice through the flange. Then, the only other things we need to know are the gage pressure of the water at the flange (the inlet to the control volume) and the weights of the water and the faucet assembly. The pressure everywhere else along the control surface is atmospheric (zero gage pressure), and cancels out. This problem is revisited in Section 13–5.

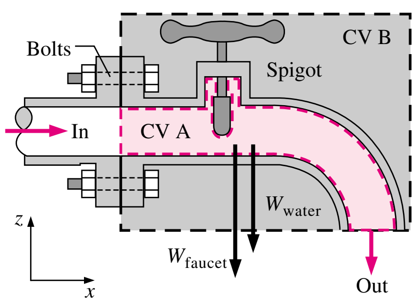


FIGURE 13–16
Cross section through a faucet assembly, illustrating the importance of choosing a control volume wisely; CV B is much easier to work with than CV A.

13–5 ▪ THE LINEAR MOMENTUM EQUATION

Newton's second law for a system of mass m subjected to a net force $\vec{F}$ is expressed as

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V}) \tag{13–23}$$

where $m\vec{\mathcal{V}}$ is the **linear momentum** of the system. Noting that both the density and velocity may change from point to point within the system, Newton's second law can be expressed more generally as

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{sys}} \vec{\mathcal{V}}\rho\, dV \tag{13–24}$$

where $\delta m = \rho\, dV$ is the mass of a differential volume element dV, and $\vec{\mathcal{V}}\rho\, dV$ is its momentum. Therefore, Newton's second law can be stated as *the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system*. This statement is valid for a coordinate system that is at rest or moves with a constant velocity, called an *inertial coordinate system* or *inertial reference frame*. Accelerating systems such as aircraft during takeoff are best analyzed using noninertial (or accelerating) coordinate systems fixed to the aircraft. Note that Eq. 13–24 is a vector relation, and thus the quantities $\vec{F}$ and $\vec{\mathcal{V}}$ have direction as well as magnitude.

The preceding relation is for a given mass of a solid or fluid, and is of limited use in fluid mechanics since most flow systems are analyzed using control volumes. The *Reynolds transport theorem* developed in Section 13–2 provides the necessary tools to shift from the system formulation to the control volume formulation. Setting $b = \vec{\mathcal{V}}$ and thus $B = m\vec{\mathcal{V}}$, the Reynolds transport theorem can be expressed for linear momentum as (Fig. 13–17)

$$\frac{d(m\vec{\mathcal{V}})_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{\mathcal{V}}\, dV + \int_{\text{CS}} \rho\vec{\mathcal{V}}\,(\vec{\mathcal{V}}_r \cdot \vec{n}\,)\, dA \tag{13–25}$$

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \int_{\text{CS}} \rho b(\vec{\mathcal{V}}_r \cdot \vec{n}\,)\, dA$$

$$B = m\vec{\mathcal{V}} \qquad b = \vec{\mathcal{V}} \qquad b = \vec{\mathcal{V}}$$

$$\frac{d(m\vec{\mathcal{V}})_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{\mathcal{V}}\, dV + \int_{\text{CS}} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}}_r \cdot \vec{n}\,)\, dA$$

FIGURE 13–17

The linear momentum equation is obtained by replacing B in the Reynolds transport theorem by the total momentum $m\vec{\mathcal{V}}$, and b by the momentum per unit mass $\vec{\mathcal{V}}$.

But the left-hand side of this equation is, from Eq. 13–23, equal to $\Sigma\vec{F}$. Substituting, the general form of the linear momentum equation that applies to fixed, moving, or deforming control volumes is obtained to be

General:
$$\sum \vec{F} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{\mathcal{V}}\, dV + \int_{\text{CS}} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}}_r \cdot \vec{n}\,)\, dA \tag{13–26}$$

which can be stated as

$$\left(\begin{array}{c}\text{The sum of all}\\ \text{external forces}\\ \text{acting on a CV}\end{array}\right) = \left(\begin{array}{c}\text{The time rate of change of}\\ \text{the linear momentum of the}\\ \text{contents of the CV}\end{array}\right) + \left(\begin{array}{c}\text{The net flow rate of}\\ \text{linear momentum out of the}\\ \text{control surface by mass flow}\end{array}\right)$$

Here $\vec{\mathcal{V}}_r = \vec{\mathcal{V}} - \vec{\mathcal{V}}_{\text{CS}}$ is the fluid velocity relative to the control surface (for use in mass flow rate calculations at all locations where the fluid crosses the control surface), and $\vec{\mathcal{V}}$ is the fluid velocity as viewed from a fixed reference frame. The product $\rho(\vec{\mathcal{V}}_r \cdot \vec{n})\, dA$ represents the mass flow rate through area element dA into or out of the control volume.

For a fixed control volume (no motion or deformation of control volume), $\vec{\mathcal{V}}_r = \vec{\mathcal{V}}$ and the linear momentum equation becomes

Fixed CV:
$$\sum \vec{F} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{\mathcal{V}}\, dV + \int_{\text{CS}} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}} \cdot \vec{n})\, dA \tag{13–27}$$

Note that the momentum equation is a *vector equation*, and thus each term should be treated as a vector. Also, the components of this equation can be resolved along orthogonal coordinates (such as x, y, and z in the rectangular coordinate system) for convenience. The force $\vec{F}$ in most cases consists of weights, hydrostatic pressure forces, and reaction forces (Fig. 13–18). The momentum equation is commonly used to calculate the forces (usually on support systems or connectors) induced by the flow.

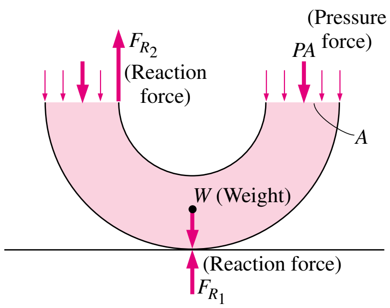


FIGURE 13–18
In most cases, the force $\vec{F}$ consists of weights, hydrostatic pressure forces, and reaction forces.

Special Cases

During *steady flow*, the amount of momentum within the control volume remains constant, and thus the time rate of change of linear momentum of the contents of the control volume (the derivative in Eq. 13–26) is zero. It gives

Steady flow:
$$\sum \vec{F} = \int_{\text{CS}} \rho \vec{V} (\vec{V}_r \cdot \vec{n})\, dA \qquad \textbf{(13–28)}$$

Most momentum problems considered in this text are steady.

While Eq. 13–27 is exact for fixed control volumes, it is not always convenient when solving practical engineering problems because of the integrals. Instead, as we did for conservation of mass, we would like to rewrite Eq. 13–27 in terms of mean velocities and mass flow rates through inlets and outlets. In other words, our desire is to rewrite the equation in *algebraic* rather than *integral* form. In many practical applications, fluid crosses the boundaries of the control volume at one or more inlets and one or more outlets, and carries with it some momentum into or out of the control volume. For simplicity, we always draw our control surface such that it slices normal to the inflow or outflow velocity at each such inlet or outlet (Fig. 13–19).

The mass flow rate $\dot{m}$ into or out of the control volume across an inlet or outlet is

Mass flow rate across an inlet or outlet:
$$\dot{m} = \int_{A_c} \rho(\vec{V} \cdot \vec{n})\, dA_c = \rho V_m A_c \qquad \textbf{(13–29)}$$

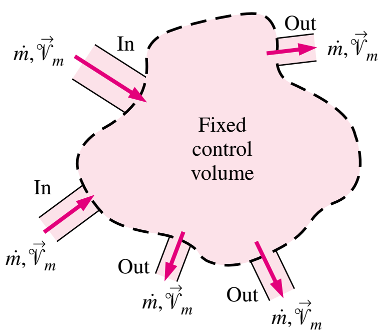


FIGURE 13–19
In a typical engineering problem, the control volume may contain many inlets and outlets; at each inlet or outlet we define the mass flow rate $\dot{m}$ and the mean velocity $\vec{V}_m$.

Comparing Eq. 13–29 with Eq. 13–27, we notice an extra velocity in the control surface integral of Eq. 13–27. If $\vec{V}$ were uniform ($\vec{V} = \vec{V}_m$) across the inlet or outlet, we could simply take it outside the integral. Then we could write the rate of inflow or outflow of momentum through the inlet or outlet in simple algebraic form,

Momentum flow rate across a uniform inlet or outlet:
$$\int_{A_c} \rho \vec{V} (\vec{V} \cdot \vec{n})\, dA_c = \rho V_m A_c \vec{V}_m = \dot{m}\vec{V}_m \qquad \textbf{(13–30)}$$

The uniform flow approximation is reasonable at some inlets and outlets, e.g., the well-rounded entrance to a pipe, the flow at the entrance to a wind tunnel test section, and a slice through a water jet moving at nearly uniform speed through air (Fig. 13–20). At each such inlet or outlet, Eq. 13–30 can be applied directly.

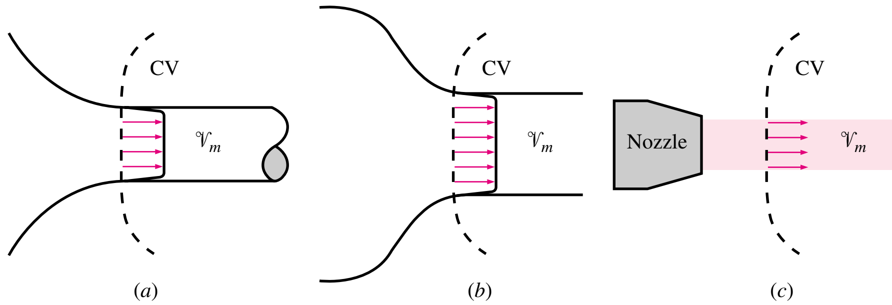


FIGURE 13–20
Examples of inlets or outlets in which the uniform flow approximation is reasonable: (*a*) the well-rounded entrance to a pipe, (*b*) the entrance to a wind tunnel test section, and (*c*) a slice through a free water jet in air.

Momentum-Flux Correction Factor, β

Unfortunately, the velocity across many inlets and outlets of practical engineering interest is *not* uniform. Nevertheless, it turns out that we can still convert the control surface integral of Eq. 13–27 into algebraic form, but a dimensionless correction factor β, called the **momentum-flux correction factor,** is required. The algebraic form of Eq. 13–27 for a fixed control volume is then written as

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{\mathcal{V}}\, dV + \sum_{\text{out}} \beta\dot{m}\vec{\mathcal{V}}_m - \sum_{\text{in}} \beta\dot{m}\vec{\mathcal{V}}_m \qquad \textbf{(13–31)}$$

where a unique value of momentum-flux correction factor is applied to each inlet and outlet in the control surface. Note that $\beta = 1$ *for the case of uniform flow* over an inlet or outlet, as in Fig. 13–20. For the general case, we define β such that the integral form of the momentum flux into or out of the control surface at an inlet or outlet of cross-sectional area A_c can be expressed in terms of mass flow rate $\dot{m}$ through the inlet or outlet and mean velocity $\vec{\mathcal{V}}_m$ through the inlet or outlet,

Momentum flux across an inlet or outlet:

$$\int_{A_c} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}} \cdot \vec{n})\, dA_c = \beta\dot{m}\vec{\mathcal{V}}_m \qquad \textbf{(13–32)}$$

For the case in which density is uniform over the inlet or outlet, we solve Eq. 13–32 for β,

$$\beta = \frac{\rho\displaystyle\int_{A_c} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}} \cdot \vec{n})\, dA_c}{\dot{m}\vec{\mathcal{V}}_m} = \frac{\rho\displaystyle\int_{A_c} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}} \cdot \vec{n})\, dA_c}{\rho\mathcal{V}_m A_c\vec{\mathcal{V}}_m} \qquad \textbf{(13–33)}$$

where we have substituted $\rho\mathcal{V}_m A_c$ for $\dot{m}$ in the denominator. The densities cancel and since $\mathcal{V}_m$ is constant, it can be brought inside the integral. Furthermore, if the control surface slices normal to the inlet or outlet area, we have $(\vec{\mathcal{V}} \cdot \vec{n})\, dA_c = \mathcal{V} dA_c$. Thus, Eq. 13–33 simplifies to

Momentum flux correction factor:

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{\mathcal{V}}{\mathcal{V}_m}\right)^2 dA_c \qquad \textbf{(13–34)}$$

Note that we have also assumed that $\vec{\mathcal{V}}$ is in the same direction as $\vec{\mathcal{V}}_m$ over the inlet or outlet. It turns out that for any velocity profile you can imagine, β is always greater than or equal to unity.

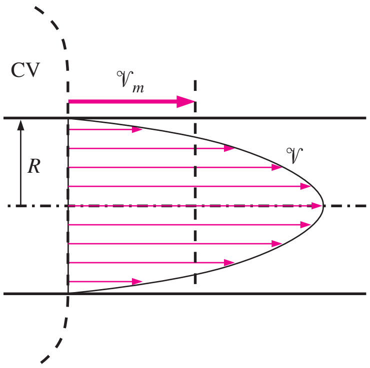


FIGURE 13–21
Velocity profile over a cross section of pipe in which the flow is fully developed and laminar.

EXAMPLE 13–1 Momentum-Flux Correction Factor for Laminar Pipe Flow

Consider laminar flow through a very long straight section of round pipe. It is shown in Chap. 12 that the velocity profile through a cross-sectional area of the pipe is parabolic (Fig. 13–21), with axial velocity component given by

$$\mathcal{V} = 2\mathcal{V}_m\left(1 - \frac{r^2}{R^2}\right) \tag{1}$$

where R is the radius of the inner wall of the pipe and $\mathcal{V}_m$ is the mean velocity. Calculate the momentum-flux correction factor through a cross section of the pipe for the case in which the pipe flow represents an outlet of the control volume, as sketched in Fig. 13–21.

SOLUTION For a given velocity distribution we are to calculate the momentum-flux correction factor.

Assumptions **1** The flow is incompressible and steady. **2** The control volume slices through the pipe normal to the pipe axis, as sketched in Fig. 13–21.

Analysis We substitute the given velocity profile for $\mathcal{V}$ in Eq. 13–34 and integrate, noting that $dA_c = 2\pi r\, dr$,

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{\mathcal{V}}{\mathcal{V}_m}\right)^2 dA_c = \frac{4}{\pi R^2}\int_0^R\left(1 - \frac{r^2}{R^2}\right)^2 2\pi r\, dr \tag{2}$$

Defining a new integration variable $y = 1 - r^2/R^2$ and thus $dy = -2r\, dr/R^2$ (also, $y = 1$ at $r = 0$, and $y = 0$ at $r = R$) and performing the integration, the momentum-flux correction factor for fully-developed laminar flow becomes

Laminar flow:
$$\beta = -4\int_1^0 y^2 dy = -4\left[\frac{y^3}{3}\right]_1^0 = \frac{4}{3} \tag{3}$$

Discussion We have calculated β for an outlet, but the same result would have been obtained if we had considered the cross section of the pipe as an *inlet* to the control volume.

From Example 13–1 we see that β is not very close to unity for fully developed laminar pipe flow, and ignoring β could potentially lead to significant error. If we were to perform the same kind of integration as in Example 13–1 but for fully developed *turbulent* rather than laminar pipe flow, we would find that β ranges from about 1.01 to 1.04. Since these values are so close to unity, many practicing engineers completely disregard the momentum-flux correction factor. While the neglect of β in turbulent flow calculations may have an insignificant effect on the final results, it is wise to keep it in our equations. Doing so not only improves the accuracy of our calculations, but reminds us to include the momentum-flux correction factor when solving laminar flow control volume problems.

For turbulent flow β may have an insignificant effect at inlets and outlets, but for laminar flow β may be important and should not be neglected. It is wise to include β in all momentum control volume problems.

Steady Flow

If the flow is also *steady*, the time derivative term in Eq. 13–31 vanishes and we are left with

Steady linear momentum equation:

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{\mathcal{V}} - \sum_{\text{in}} \beta \dot{m} \vec{\mathcal{V}} \tag{13–35}$$

where we dropped the subscript m from mean velocity for convenience. Equation 13–35 states that *the net force acting on the control volume during steady flow is equal to the difference between the rates of outgoing and incoming momentum flows*. This statement is illustrated in Fig. 13–22. It can also be expressed for any direction, since Eq. 13–35 is a vector equation.

Steady Flow with One Inlet and One Outlet

Many practical problems involve just one inlet and one outlet (Fig. 13–23). The mass flow rate for such **single-stream systems** remains constant, and Eq. 13–35 reduces to

One inlet and one outlet:

$$\sum \vec{F} = \dot{m}(\beta_2 \vec{\mathcal{V}}_2 - \beta_1 \vec{\mathcal{V}}_1) \tag{13–36}$$

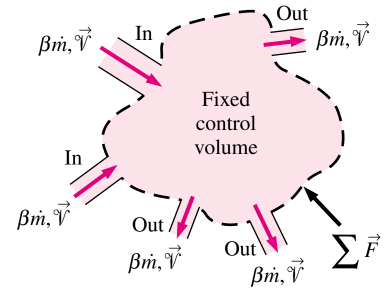
where we have adopted the usual convention that subscript 1 implies the inlet and subscript 2 the outlet, and $\vec{\mathcal{V}}_1$ and $\vec{\mathcal{V}}_2$ denote the *mean* velocities across the inlet and outlet, respectively.

We emphasize again that all the preceding relations are *vector* equations, and thus all the additions and subtractions are *vector* additions and subtractions. Recall that subtracting a vector is equivalent to adding it after reversing its direction (Fig. 13–24). Also, when writing the momentum equation along a specified coordinate (such as the *x*-axis), we use the projections of the vectors on that axis. For example, Eq. 13–36 can be written along the *x*-coordinate as

Along x-coordinate:

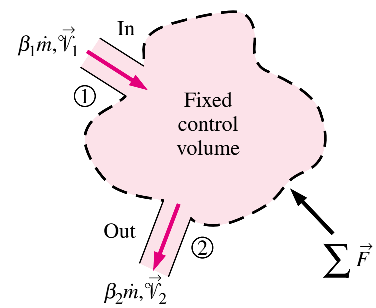
$$\sum \vec{F}_x = \dot{m}(\beta_2 \vec{\mathcal{V}}_{2,x} - \beta_1 \vec{\mathcal{V}}_{1,x}) \tag{13–37}$$

where $\Sigma \vec{F}_x$ is the vector sum of the x-components of the forces, and $\vec{\mathcal{V}}_{2,x}$ and $\vec{\mathcal{V}}_{1,x}$ are the x-components of the outlet and inlet velocities of the fluid stream, respectively. The force or velocity components in the positive x-direction are positive quantities, and those in the negative x-direction are negative quantities. Also, it is good practice to take the direction of unknown forces in the positive directions (unless the problem is very straightforward). A negative value obtained for an unknown force indicates that the assumed direction is wrong, and should be reversed.



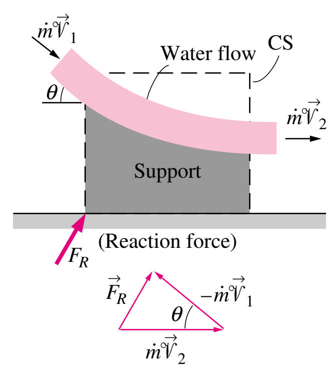
$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{\mathcal{V}} - \sum_{\text{in}} \beta \dot{m} \vec{\mathcal{V}}$$

FIGURE 13–22
The net force acting on the control volume during steady flow is equal to the difference between the outgoing and the incoming momentum fluxes by mass.



$$\sum \vec{F} = \dot{m}(\beta_2 \vec{\mathcal{V}}_2 - \beta_1 \vec{\mathcal{V}}_1)$$

FIGURE 13–23
A control volume with only one inlet and one outlet.



Note: $\vec{\mathcal{V}}_2 \neq \vec{\mathcal{V}}_1$ even if $|\vec{\mathcal{V}}_2| = |\vec{\mathcal{V}}_1|$

FIGURE 13–24
The determination of the reaction force on the support caused by a change of direction of water by vector addition.

No External Forces

An interesting situation arises when there are no external forces such as weight, pressure, and reaction forces acting on the body in the direction of motion—a common situation for space vehicles and satellites. For a control volume with uniform inlets and exits, Eq. 13–29 reduces in this case to

No external forces:
$$0 = \frac{d(m\vec{V})_{CV}}{dt} + \sum_{out} \beta \dot{m}\vec{V} - \sum_{in} \beta \dot{m}\vec{V} \qquad \textbf{(13–38)}$$

This is an expression of the conservation of momentum principle, which can be stated as *in the absence of external forces, the rate of change of the momentum of a control volume is equal to the difference between the rates of incoming and outgoing momentum flow rates.*

When the mass m of the control volume remains constant, the first term of the equation above simply becomes mass times acceleration since

m_{CV} = constant:
$$\frac{d(m\vec{V})_{CV}}{dt} = m_{CV}\frac{d\vec{V}_{CV}}{dt} = (m\vec{a})_{CV} \qquad \textbf{(13–39)}$$

Therefore, the control volume in this case can be treated as a solid body, with a net force $\vec{F} = m\vec{a} = \sum_{in} \beta \dot{m}\vec{V} - \sum_{out} \beta \dot{m}\vec{V}$ (due to a change of momentum) acting on it. This approach can be used to determine the linear acceleration of space vehicles when a rocket is fired.

EXAMPLE 13–2 The Force to Hold a Deflector Elbow in Place

A reducing elbow is used to deflect water flow at a rate of 14 kg/s in a horizontal pipe upward 30° while accelerating it (Fig. 13–25). The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 113 cm^2 at the inlet and 7 cm^2 at the exit. The elevation difference between the centers of the exit and the inlet is 30 cm. The weight of the elbow and the water in it is considered to be negligible. Determine (*a*) the gage pressure at the center of the inlet of the elbow and (*b*) the anchoring force needed to hold the elbow in place.

SOLUTION A reducing elbow deflects water upward and discharges it to the atmosphere. The pressure at the inlet of the elbow and the force needed to hold the elbow in place are to be determined.

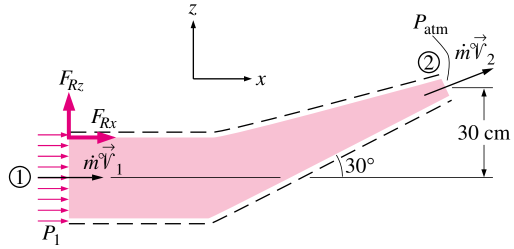


FIGURE 13–25
Schematic for Example 13–2.

Assumptions **1** The flow is steady, and the frictional effects are negligible. **2** The weight of the elbow and the water in it is negligible. **3** The water is discharged to the atmosphere, and thus the gage pressure at the exit is zero. **4** The effect of the momentum-flux correction factor is negligible, and thus $\beta \approx 1$.

Properties We take the density of water to be 1000 kg/m³.

Analysis (*a*) We take the elbow as the control volume and designate the inlet by 1 and the outlet by 2. We also take the *x*- and *z*-coordinates as shown. The continuity equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m} = 14$ kg/s. Noting that $\dot{m} = \rho A \mathcal{V}$, the inlet and outlet velocities of water are

$$\mathcal{V}_1 = \frac{\dot{m}}{\rho A_1} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(0.0113 \text{ m}^2)} = 1.24 \text{ m/s}$$

$$\mathcal{V}_2 = \frac{\dot{m}}{\rho A_2} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(7 \times 10^{-4} \text{ m}^2)} = 20.0 \text{ m/s}$$

We use the Bernoulli equation as a first approximation to calculate the pressure. In a later chapter we will learn how to include frictional losses along the walls. Taking the center of the inlet cross section as the reference level ($z_1 = 0$) and noting that $P_2 = P_{atm}$, the Bernoulli equation for a streamline going through the center of the elbow is expressed as

$$\frac{P_1}{\rho g} + \frac{\mathcal{V}_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{\mathcal{V}_2^2}{2g} + z_2$$

$$P_1 - P_2 = \rho g \left(\frac{\mathcal{V}_2^2 - \mathcal{V}_1^2}{2g} + z_2 - z_1 \right)$$

$$P_1 - P_{atm} = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)\left(\frac{(20 \text{ m/s})^2 - (1.24 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 0.3 - 0 \right)\left(\frac{1 \text{ kN}}{1000 \text{ kg}\cdot\text{m/s}^2} \right)$$

$$P_1 = 202.2 \text{ kN/m}^2 = \mathbf{202.2 \text{ kPa}} \quad \text{(gage)}$$

(*b*) The momentum equation for steady one-dimensional flow is

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{\mathcal{V}} - \sum_{\text{in}} \beta \dot{m} \vec{\mathcal{V}}$$

We let the *x*- and *z*-components of the anchoring force of the elbow be F_{Rx} and F_{Rz}, and assume them to be in the positive direction. We also use gage pressure since the atmospheric pressure acts on the entire control surface. Then the momentum equations along the *x*- and *z*-axes become

$$F_{Rx} + P_1 A_1 = \dot{m}\mathcal{V}_2 \cos\theta - \dot{m}\mathcal{V}_1$$
$$F_{Rz} = \dot{m}\mathcal{V}_2 \sin\theta$$

Solving for F_{Rx} and F_{Rz}, and substituting the given values,

$$\begin{aligned} F_{Rx} &= \dot{m}(\mathcal{V}_2 \cos\theta - \mathcal{V}_1) - P_1 A_1 \\ &= (14 \text{ kg/s})[(20 \cos 30° - 1.24) \text{ m/s}]\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2} \right) - (202{,}200 \text{ N/m}^2)(0.0113 \text{ m}^2) \\ &= 225 - 2285 = \mathbf{-2060 \text{ N}} \end{aligned}$$

$$F_{Rz} = \dot{m}\mathcal{V}_2 \sin\theta = (14 \text{ kg/s})(20 \sin 30° \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2} \right) = \mathbf{140 \text{ N}}$$

The negative result for F_{Rx} indicates that the assumed direction is wrong, and it should be reversed. Therefore, F_{Rx} acts in the negative x-direction.

Discussion There is a nonzero pressure distribution along the inside walls of the elbow, but since the control volume is outside the elbow, these pressures do not appear in our analysis. The actual value of $P_{1,\text{gage}}$ will be higher than that calculated here because of frictional and other irreversible losses in the elbow.

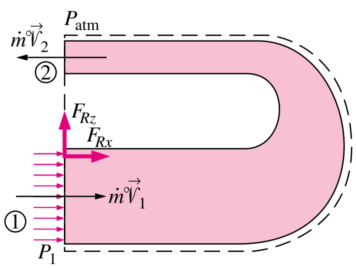


FIGURE 13–26
Schematic for Example 13–3.

EXAMPLE 13–3 The Force to Hold a Reversing Elbow in Place

The deflector elbow in the previous example is replaced by a reversing elbow such that the fluid makes a 180° U-turn before it is discharged, as shown in Fig. 13–26. The elevation difference between the centers of the inlet and the exit sections is still 0.3 m. Determine the anchoring force needed to hold the elbow in place.

SOLUTION The inlet and the exit velocities and the pressure at the inlet of the elbow remain the same, but the vertical component of the anchoring force at the connection of the elbow to the pipe is zero in this case ($F_{Rz} = 0$) since there is no other force or momentum flux in the vertical direction. The horizontal component of the anchoring force is determined from the momentum equation written in the x-direction. Noting that the exit velocity is negative since it is in the negative x-direction, we have

$$F_{Rx} + P_1A_1 = \dot{m}(-\mathcal{V}_2) - \dot{m}\mathcal{V}_1$$

Solving for F_{Rx} and substituting the known values,

$$\begin{aligned} F_{Rx} &= -\dot{m}(\mathcal{V}_2 + \mathcal{V}_1) - P_1A_1 \\ &= -(14\ \text{kg/s})[(20 + 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2) \\ &= -297 - 2285 = -\mathbf{2582\ N} \end{aligned}$$

Therefore, the horizontal force on the flange is 2582 N acting in the negative x-direction (trying to separate the elbow from the pipe). This force is equivalent to the weight of about 260 kg mass, and thus the connectors (such as bolts) used must be strong enough to withstand this force.

Discussion The reaction force in the x-direction is larger than that of Example 13–2 since the walls turn the water over a much greater angle. If the reversing elbow is replaced by a straight nozzle (like one used by firefighters) such that water is discharged in the positive x-direction, the momentum equation in the x-direction becomes

$$F_{Rx} + P_1A_1 = \dot{m}\mathcal{V}_2 - \dot{m}\mathcal{V}_1 \quad \rightarrow \quad F_{Rx} = \dot{m}(\mathcal{V}_2 - \mathcal{V}_1) - P_1A_1$$

since both $\mathcal{V}_1$ and $\mathcal{V}_2$ are in the positive x-direction. This shows the importance of using the correct sign (positive if in the positive direction and negative if in the opposite direction) for velocities and forces.

EXAMPLE 13–4 Water Jet Striking a Stationary Plate

Water accelerated by a nozzle strikes a stationary vertical plate at a rate of 10 kg/s with a normal velocity of 20 m/s (Fig. 13–27). After the strike, the water stream splatters off in all directions in the plane of the plate. Determine the force needed to prevent the plate from moving horizontally due to the water stream.

SOLUTION A water jet strikes a vertical stationary plate normally. The force needed to hold the plate in place is to be determined.

Assumptions **1** The flow is steady and one-dimensional. **2** The water splatters in directions normal to the approach direction of the water jet. **3** The water jet is exposed to the atmosphere, and thus the pressure of the water jet and the splattered water is atmospheric pressure, which is disregarded since it acts on the entire system. **4** The vertical forces and momentum fluxes are not considered since they have no effect on the horizontal reaction force. **5** The effect of the momentum-flux correction factor is negligible, and thus $\beta \approx 1$.

Analysis We draw the control volume for this problem such that it contains the entire plate and cuts through the water jet and the support bar normally. The momentum equation for steady one-dimensional flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{\mathcal{V}} - \sum_{\text{in}} \beta \dot{m} \vec{\mathcal{V}}$$

Writing it for this problem along the *x*-direction (without forgetting the negative sign for forces and velocities in the negative *x*-direction) and noting $\mathcal{V}_{1,x} = \mathcal{V}_1$ and $\mathcal{V}_{2,x} = 0$ gives

$$-F_R = -\dot{m}\mathcal{V}_1$$

Substituting the given values,

$$F_R = \dot{m}\mathcal{V}_1 = (10 \text{ kg/s})(+20 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{200\ N}$$

Therefore, the support must apply a 200-N horizontal force (equivalent to the weight of about a 20-kg mass) in the negative *x*-direction (the opposite direction of the water jet) to hold the plate in place.

Discussion The plate absorbs the full brunt of the momentum of the water jet since the *x*-direction momentum at the outlet of the control volume is zero. If the control volume were drawn instead along the interface between the water and the plate, there would be additional (unknown) pressure forces in the analysis. By cutting the control volume through the support, we avoid having to deal with this additional complexity. This is an example of a "wise" choice of control volume.

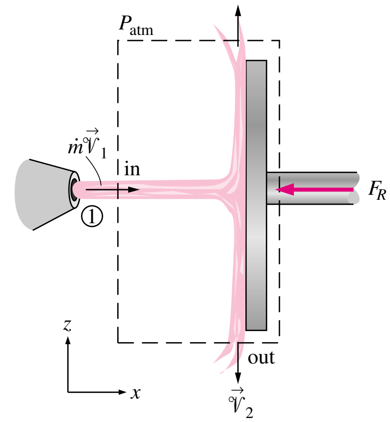


FIGURE 13–27
Schematic for Example 13–4.

EXAMPLE 13–5 Power Generation and Wind Loading of a Wind Turbine

A wind generator with a 30-foot-diameter blade span has a cut-in wind speed (minimum speed for power generation) of 7 mph, at which velocity the turbine generates 0.4 kW of electric power (Fig. 13–28). Determine (*a*) the efficiency of the wind turbine-generator set and (*b*) the horizontal force exerted by the wind on the supporting mast of the wind turbine. What is the effect of doubling the wind velocity to 14 mph on power generation and the force exerted? Assume the efficiency remains the same, and take the density of air to be 0.076 lbm/ft^3.

SOLUTION The power generation and loading of a wind turbine are to be analyzed. The efficiency and the force exerted on the mast are to be determined, and the effects of doubling the wind velocity are to be investigated.

Assumptions **1** The wind flow is steady, one-dimensional, and incompressible. **2** The efficiency of the turbine-generator is independent of wind speed. **3** The frictional effects are negligible, and thus none of the incoming kinetic energy is converted to thermal energy. **4** The average velocity of air through the wind turbine is the same as the wind velocity (actually, it is considerably less—see the discussion that follows the example). **5** The wind flow is uniform and thus the momentum-flux correction factor is $\beta = 1$.

Properties The density of air is given to be 0.076 lbm/ft^3.

Analysis Kinetic energy is a mechanical form of energy, and thus it can be converted to work entirely. Therefore, the power potential of the wind is its kinetic energy, which is $\mathcal{V}^2/2$ per unit mass and $\dot{m}\mathcal{V}^2/2$ for a given mass flow rate:

$$\mathcal{V}_1 = (7 \text{ mph})\left(\frac{1.4667 \text{ ft/s}}{1 \text{ mph}}\right) = 10.27 \text{ ft/s}$$

$$\dot{m} = \rho_1 \mathcal{V}_1 A_1 = \rho_1 \mathcal{V}_1 \frac{\pi D^2}{4} = (0.076 \text{ lbm/ft}^3)(10.27 \text{ ft/s}) \frac{\pi (30 \text{ ft})^2}{4} = 551.7 \text{ lbm/s}$$

$$\dot{W}_{\max} = \dot{m}\text{ke}_1 = \dot{m}\frac{\mathcal{V}_1^2}{2}$$

$$= (551.7 \text{ lbm/s}) \frac{(10.27 \text{ ft/s})^2}{2} \left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right)\left(\frac{1 \text{ kW}}{737.56 \text{ lbf} \cdot \text{ft/s}}\right) = 1.225 \text{ kW}$$

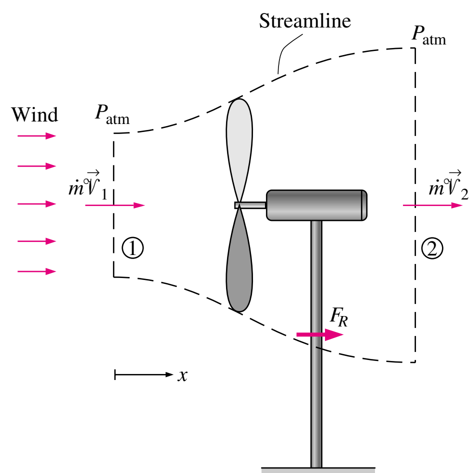


FIGURE 13–28
Schematic for Example 13–5.

Therefore, the available power to the wind turbine is 1.225 kW at the wind velocity of 7 mph. Then the turbine-generator efficiency becomes

$$\eta_{\text{wind turbine}} = \frac{\dot{W}_{\text{act}}}{\dot{W}_{\text{max}}} = \frac{0.4 \text{ kW}}{1.225 \text{ kW}} = \mathbf{0.327} \qquad (\text{or } \mathbf{32.7\%})$$

(*b*) The frictional effects are assumed to be negligible, and thus the portion of incoming kinetic energy not converted to electric power leaves the wind turbine as outgoing kinetic energy. Noting that the mass flow rate remains constant, the exit velocity is determined to be

$$\dot{m}\text{ke}_2 = \dot{m}\text{ke}_1(1 - \eta_{\text{wind turbine}}) \qquad \rightarrow \qquad \dot{m}\frac{\mathcal{V}_2^2}{2} = \dot{m}\frac{\mathcal{V}_1^2}{2}(1 - \eta_{\text{wind turbine}})$$

or

$$\mathcal{V}_2 = \mathcal{V}_1\sqrt{1 - \eta_{\text{wind turbine}}} = (10.27 \text{ ft/s})\sqrt{1 - 0.327} = 8.43 \text{ ft/s}$$

We draw a control volume around the wind turbine such that the wind is normal to the control surface at the inlet and the exit and the entire control surface is at the atmospheric pressure. The momentum equation for steady one-dimensional flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta\dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \beta\dot{m}\vec{\mathcal{V}}$$

Writing it along the *x*-direction (without forgetting the negative sign for forces and velocities in the negative *x*-direction) and noting that $\mathcal{V}_{1,x} = \mathcal{V}_1$ and $\mathcal{V}_{2,x} = \mathcal{V}_2$ give

$$F_R = \dot{m}\mathcal{V}_2 - \dot{m}\mathcal{V}_1 = \dot{m}(\mathcal{V}_2 - \mathcal{V}_1)$$

Substituting the known values gives

$$F_R = \dot{m}(\mathcal{V}_2 - \mathcal{V}_1) = (551.7 \text{ lbm/s})(8.43 - 10.27 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right)$$
$$= \mathbf{-31.5 \text{ lbf}}$$

The negative sign indicates that the reaction force acts in the negative *x*-direction, as expected.

The power generated is proportional to $\mathcal{V}^3$ since the mass flow rate is proportional to $\mathcal{V}$ and the kinetic energy to $\mathcal{V}^2$. Therefore, doubling the wind velocity to 14 mph will increase the power generation by a factor of $2^3 = 8$ to $0.4 \times 8 = 3.2$ kW. The force exerted by the wind on the support mast is proportional to $\mathcal{V}^2$. Therefore, doubling the wind velocity to 14 mph will increase the wind force by a factor of $2^2 = 4$ to $31.5 \times 4 = 126$ lbf.

Discussion To gain more insight into the operation of devices with propellers such as helicopters, wind turbines, hydraulic turbines, and turbofan engines, we reconsider the wind turbine and draw the streamlines, as shown in Fig. 13–29. (In the case of power-consuming devices such as a fan and a

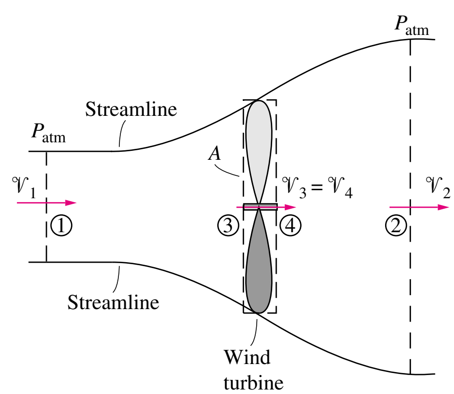


FIGURE 13–29
The large and small control volumes for the analysis of a propeller bounded by the streamlines.

helicopter, the streamlines converge rather than diverge since the exit velocity will be higher and thus the exit area will be lower.) The upper and lower streamlines can be considered to form an "imaginary duct" for the flow of air through the propeller. Sections 1 and 2 are sufficiently far from the propeller so that $P_1 = P_2 = P_{atm}$. The momentum equation for this large control volume between sections 1 and 2 was obtained to be

$$F_R = \dot{m}(\mathcal{V}_2 - \mathcal{V}_1) \tag{1}$$

The smaller control volume between sections 3 and 4 encloses the propeller, and $A_3 = A_4 = A$ and $\mathcal{V}_3 = \mathcal{V}_4$ since it is so slim. The propeller is a device that causes a pressure change, and thus the pressures P_3 and P_4 are different. The momentum equation applied to the smaller control volume gives

$$F_R + P_3 A - P_4 A = 0 \quad \rightarrow \quad F_R = (P_4 - P_3)A \tag{2}$$

The Bernoulli equation is not applicable between sections 1 and 2 since the path crosses a propeller, but it is applicable separately between sections 1 and 3 and sections 4 and 2:

$$\frac{P_1}{\rho g} + \frac{\mathcal{V}_1^2}{2g} + z_1 = \frac{P_3}{\rho g} + \frac{\mathcal{V}_3^2}{2g} + z_3 \quad \text{and} \quad \frac{P_4}{\rho g} + \frac{\mathcal{V}_4^2}{2g} + z_4 = \frac{P_2}{\rho g} + \frac{\mathcal{V}_2^2}{2g} + z_2$$

Adding these two equations and noting that $z_1 = z_2 = z_3 = z_4$, $\mathcal{V}_3 = \mathcal{V}_4$, and $P_1 = P_2 = P_{atm}$ gives

$$\frac{\mathcal{V}_2^2 - \mathcal{V}_1^2}{2} = \frac{P_4 - P_3}{\rho} \tag{3}$$

Substituting $\dot{m} = \rho A \mathcal{V}_3$ into (1) and then combining it with (2) and (3) gives

$$\mathcal{V}_3 = \frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \tag{4}$$

Thus we conclude that *the mean velocity of a fluid through a propeller is the arithmetic average of the upstream and downstream velocities.* Of course, the validity of this result is limited by the applicability of the Bernoulli equation.

Now back to the wind turbine. The velocity through the propeller can be expressed as $\mathcal{V}_3 = \mathcal{V}_1(1 - a)$, where $a < 1$ since $\mathcal{V}_3 < \mathcal{V}_1$. Combining this expression with (4) gives $\mathcal{V}_2 = \mathcal{V}_1(1 - 2a)$. Also, the mass flow rate through the propeller becomes $\dot{m} = \rho A \mathcal{V}_3 = \rho A \mathcal{V}_1(1 - a)$. When the frictional effects and losses are neglected, the power generated by a wind turbine is simply the difference between the incoming and the outgoing kinetic energies:

$$\begin{aligned}\dot{W} = \dot{m}(\text{ke}_1 - \text{ke}_2) &= \frac{\dot{m}(\mathcal{V}_1^2 - \mathcal{V}_2^2)}{2} = \frac{\rho A \mathcal{V}_1(1 - a)[\mathcal{V}_1^2 - \mathcal{V}_1^2(1 - 2a)^2]}{2} \\ &= 2\rho A \mathcal{V}_1^3 a(1 - a)^2\end{aligned}$$

Dividing this by the available power of the wind $\dot{W}_{max} = \dot{m}\mathcal{V}_1^2/2$ gives the efficiency of the wind turbine in terms of a,

$$\eta_{\text{wind turbine}} = \frac{\dot{W}}{\dot{W}_{max}} = \frac{2\rho A \mathcal{V}_1^3 a(1 - a)^2}{(\rho A \mathcal{V}_1)\,\mathcal{V}_1^2/2}$$

The value of a that maximizes the efficiency is determined by setting the derivative of $\eta_{\text{wind turbine}}$ with respect to a equal to zero and solving for a. It gives $a = \frac{1}{3}$. Substituting this value into the efficiency relation above gives $\eta_{\text{wind turbine}} = \frac{16}{27} = 0.593$, which is the upper limit for the efficiency of wind turbines and other propellers. This is known as the **Betz limit.** The efficiency of actual wind turbines is about half of this ideal value.

EXAMPLE 13–6 Repositioning of a Satellite

An orbiting satellite system has a mass of $m_{\text{sat}} = 5000$ kg and is traveling at a constant velocity of $\mathcal{V}_0$. To alter its orbit, an attached rocket discharges $m_f = 100$ kg of solid fuel at a velocity $\mathcal{V}_f = 3000$ m/s relative to $\mathcal{V}_0$ in a direction opposite to $\mathcal{V}_0$ (Fig. 13–30). The fuel discharge rate is constant for two seconds. Determine (*a*) the acceleration of the system during this two-second period, (*b*) the change of velocity of the satellite system during this time period, and (*c*) the thrust exerted on the system.

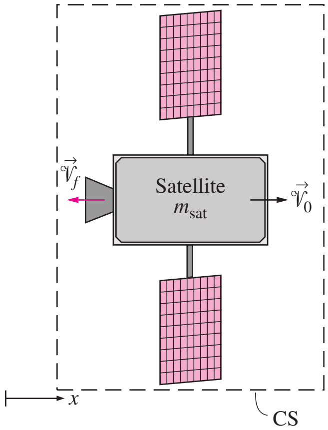


FIGURE 13–30
Schematic for Example 13–6.

SOLUTION The rocket of a satellite is fired in the opposite direction to motion. The acceleration, the velocity change, and the thrust are to be determined.

Assumptions **1** The flow of combustion gases is steady and one-dimensional during the firing period. **2** There are no external forces acting on the satellite, and the effect of the pressure force at the nozzle exit is negligible. **3** The mass of discharged fuel is negligible relative to the mass of the satellite, and thus the satellite may be treated as a solid body with a constant mass. **4** The effect of the momentum-flux correction factor is negligible, and thus $\beta \approx 1$.

Analysis (*a*) A body moving at constant velocity can be considered to be stationary for convenience. Then the velocities of fluid streams become simply their velocities relative to the moving body. We take the direction of motion of the satellite as the positive direction along the x-axis. There are no external forces acting on the satellite and its mass is nearly constant. Therefore, the satellite can be treated as a solid body with constant mass, and the momentum equation in this case is simply Eq. 13–38,

$$0 = \frac{d(m\vec{\mathcal{V}})_{\text{CV}}}{dt} + \sum_{\text{out}} \beta \dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \beta \dot{m}\vec{\mathcal{V}} \quad \rightarrow \quad m_{\text{sat}} \frac{d\vec{\mathcal{V}}_{\text{sat}}}{dt} = -\dot{m}_f \vec{\mathcal{V}}_f$$

Noting that the motion is on a straight line and the discharged gases move in the negative x-direction, we can write the momentum equation using magnitudes as

$$m_{\text{sat}} \frac{d\mathcal{V}_{\text{sat}}}{dt} = \dot{m}_t \mathcal{V}_t \quad \rightarrow \quad \frac{d\mathcal{V}_{\text{sat}}}{dt} = \frac{\dot{m}_t}{m_{\text{sat}}} \mathcal{V}_t = \frac{m_t/\Delta t}{m_{\text{sat}}} \mathcal{V}_t$$

Substituting, the acceleration of the satellite during the first two seconds is determined to be

$$a_{\text{sat}} = \frac{d\mathcal{V}_{\text{sat}}}{dt} = \frac{m_t/\Delta t}{m_{\text{sat}}} \mathcal{V}_t = \frac{(100 \text{ kg})/(2 \text{ s})}{5000 \text{ kg}} (3000 \text{ m/s}) = \mathbf{30 \text{ m/s}^2}$$

(*b*) Knowing acceleration, which is constant, the velocity change of the satellite during the first two seconds is determined from the definition of acceleration $a_{\text{sat}} = d\mathcal{V}_{\text{sat}}/dt$ to be

$$d\mathcal{V}_{\text{sat}} = a_{\text{sat}}\, dt \quad \rightarrow \quad \Delta\mathcal{V}_{\text{sat}} = a_{\text{sat}}\, \Delta t = (30 \text{ m/s}^2)(2 \text{ s}) = \mathbf{60 \text{ m/s}}$$

(*c*) The thrust exerted on the system is simply the momentum flux of the combustion gases in the reverse direction:

$$\text{Thrust} = F_R = -\dot{m}_f \mathcal{V}_f = -(100/2 \text{ kg/s})(-3000 \text{ m/s})\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{150 \text{ kN}}$$

Discussion Note that if this satellite were attached somewhere, it would exert a force of 150 kN (equivalent to the weight of 15 tons of mass) to its support. This can be verified by taking the satellite as the system, and applying the momentum equation.

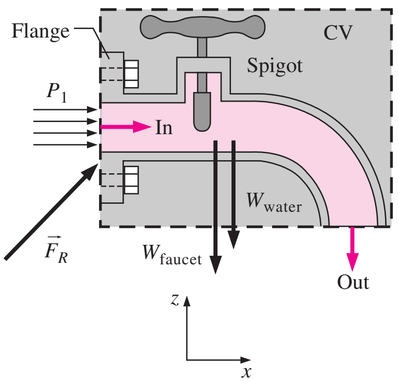


FIGURE 13–31
Control volume for Example 13–7 with all forces shown; gage pressure is used for convenience.

EXAMPLE 13–7 Net Force on a Flange

Water flows at a rate of 18.5 gallons per minute through a flanged faucet with a partially closed gate valve spigot (Fig. 13–31). The inner diameter of the pipe at the location of the flange is 0.780 in (= 0.0650 ft), and the pressure at that location is measured to be 13.0 psig. The total weight of the faucet assembly plus the water within it is 12.8 lbf. Calculate the net force on the flange.

SOLUTION Water flow through a flanged faucet is considered. The net force acting on the flange is to be calculated.

Assumptions **1** The flow is steady and incompressible. **2** The flow at the inlet and at the outlet is uniform. **3** The pipe diameter at the outlet of the faucet is the same as that at the flange. **4** The effect of the momentum-flux correction factor is negligible, and thus $\beta \approx 1$.

Properties The density of water at room temperature is 62.3 lbm/ft^3.

Analysis We choose the faucet and its immediate surroundings as the control volume, as shown in Fig. 13–31 along with all the forces acting on it. These forces include the weight of the water and the weight of the faucet assembly, the gage pressure force at the inlet to the control volume, and the net force of the flange on the control volume, which we call $\vec{F}_R$. We use gage pressure for convenience since the gage pressure on the rest of the control surface is zero (atmospheric pressure). Note that the pressure through the outlet of the control volume is also atmospheric since we are assuming incompressible flow; hence the gage pressure is also zero through the outlet.

We now apply the control volume conservation laws. Conservation of mass is trivial here since there is only one inlet and one outlet; namely, the mass flow rate into the control volume is equal to the mass flow rate out of the control volume. Also, the outflow and inflow mean velocities are identical since the inner diameter is constant and the water is incompressible, and are determined to be

$$\mathcal{V}_2 = \mathcal{V}_1 = \mathcal{V} = \frac{\dot{V}}{A_c} = \frac{\dot{V}}{\pi D^2/4} = \frac{18.5 \text{ gal/min}}{\pi(0.065 \text{ ft})^2/4}\left(\frac{0.1337 \text{ ft}^3}{1 \text{ gal}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 12.42 \text{ ft/s}$$

Also,

$$\dot{m} = \rho\dot{V} = (62.3 \text{ lbm/ft}^3)(18.5 \text{ gal/min})\left(\frac{0.1337 \text{ ft}^3}{1 \text{ gal}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 2.568 \text{ lbm/s}$$

Next we apply the momentum equation. The momentum equation for steady flow with uniform properties at the inlets and the exits is

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \beta \dot{m}\vec{\mathcal{V}}$$

We let the x- and z-components of the force acting on the flange be F_{Rx} and F_{Rz}, and assume them to be in the positive directions. We also use gage pressures since the atmospheric pressure acts on the entire control surface, and thus it can be ignored. The magnitude of the velocity in the x-direction is $+\mathcal{V}_1$ at the inlet, but zero at the outlet. The magnitude of the velocity in the y-direction is zero at the inlet, but $-\mathcal{V}_2$ at the outlet. Also, the weight of the faucet assembly and the water within it acts in the $-y$-direction as a body force. No pressure or viscous forces act on the control volume in the y-direction.

Then the momentum equations along the x- and y-directions become

$$F_{Rx} + P_1A_1 = 0 - \dot{m}(+\mathcal{V}_1)$$
$$F_{Rz} - W_{\text{faucet}} - W_{\text{water}} = \dot{m}(-\mathcal{V}_2) - 0$$

Solving for F_{Rx} and F_{Rz}, and substituting the given values,

$$\begin{aligned}
F_{Rx} &= -\dot{m}\mathcal{V}_1 - P_1A_1 \\
&= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right) - (13 \text{ lbf/in}^2)\,\frac{\pi(0.780 \text{ in})^2}{4} \\
&= -7.20 \text{ lbf} \\
F_{Rz} &= -\dot{m}\mathcal{V}_2 + W_{\text{faucet+water}} \\
&= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right) + 12.8 \text{ lbf} = 11.8 \text{ lbf}
\end{aligned}$$

Then the net force of the flange on the control volume can be expressed in vector form as

$$\vec{F}_R = F_{Rx}\vec{i} + F_{Rz}\vec{k} = -7.20\vec{i} + 11.8\vec{k} \text{ lbf}$$

From Newton's third law, the force of the faucet assembly exerts on the flange is the negative of $\vec{F}_R$,

$$\vec{F}_{\text{faucet on flange}} = -\vec{F}_R = 7.20\vec{i} - 11.8\vec{k} \text{ lbf}$$

Discussion The faucet assembly pulls to the right and down; this agrees with our intuition. Namely, the water exerts a high pressure at the inlet, but the outlet pressure is atmospheric. In addition, the momentum of the water at the inlet in the x-direction is lost in the turn, causing an additional force to the right on the pipe walls. The faucet assembly weighs much more than the momentum effect of the water, so we expect the force to be downward. Note that labeling forces such as "faucet on flange" clarifies the direction of the force.

13–6 ▪ THE ANGULAR MOMENTUM EQUATION

The linear momentum equation discussed earlier was useful in determining the relationship between the linear momentum of flow streams and the resultant forces. Many engineering problems involve the moment of the linear momentum of flow streams, and the rotational effects caused by them. Such problems are best analyzed by the angular momentum equation, also called the moment of momentum equation. An important class of fluid devices, called turbomachines, which include centrifugal pumps, turbines, and fans, is analyzed by the angular momentum equation.

The *moment of a force* $\vec{F}$ about a point O is the vector (or cross) product (Fig. 13–32)

Moment of a force: $$\vec{M} = \vec{r} \times \vec{F} \quad \textbf{(13–40)}$$

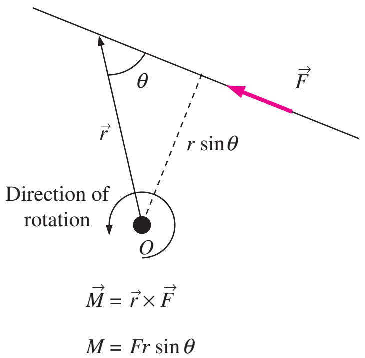


FIGURE 13–32
The moment of a force $\vec{F}$ about a point O is the vector product of the position vector $\vec{r}$ and $\vec{F}$.

where $\vec{r}$ is the position vector from point O to any point on the line of action of $\vec{F}$. The vector product of two vectors is a vector whose line of action is normal to the plane that contains the crossed vectors ($\vec{r}$ and $\vec{F}$ in this case) and whose magnitude is

Magnitude of the moment of a force: $$M = Fr \sin \theta \quad \textbf{(13–41)}$$

where θ is the angle between the lines of action of the vectors $\vec{r}$ and $\vec{F}$. Therefore, the magnitude of the moment about point O is equal to the magnitude of the force multiplied by the normal distance of the line of action of the force from the point O. The sense of the moment vector $\vec{M}$ is determined by the right-hand rule: when the fingers of the right hand are curled in the direction that the force tends to cause rotation, the thumb points the direction of the moment vector (Fig. 13–33). Note that a force whose line of action passes through point O produces zero moment about point O.

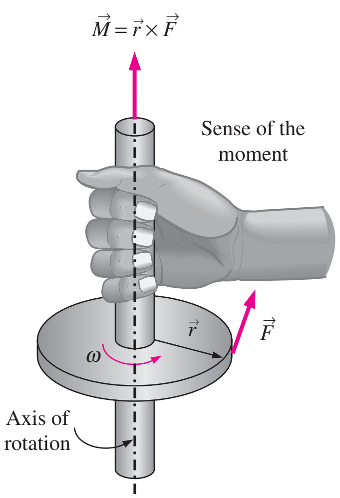


FIGURE 13–33
The determination of the direction of the moment by the right-hand rule.

Replacing the vector $\vec{F}$ in Eq. 13–40 by the momentum vector $m\vec{\mathcal{V}}$ gives the *moment of momentum*, also called the *angular momentum,* about a point O as

Moment of momentum: $$\vec{H} = \vec{r} \times m\vec{\mathcal{V}} \quad \textbf{(13–42)}$$

Therefore, $\vec{r} \times \vec{\mathcal{V}}$ represents the angular momentum per unit mass, and the angular momentum of a differential mass $dm = \rho\, dV$ is $d\vec{H} = (\vec{r} \times \vec{\mathcal{V}})\rho\, dV$. Then the angular momentum of a system is determined by integration to be

Moment of momentum (system): $$\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{\mathcal{V}})\rho\, dV \quad \textbf{(13–43)}$$

The rate of change of the moment of momentum is

Rate of change of moment of momentum: $$\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt}\int_{sys} (\vec{r} \times \vec{\mathcal{V}})\rho\, dV \quad \textbf{(13–44)}$$

The angular momentum equation for a system was expressed in Eq. 13–2 as

$$\sum \vec{M} = \frac{d\vec{H}_{sys}}{dt} \quad \textbf{(13–45)}$$

where $\sum \vec{M} = \sum \vec{r} \times \vec{F}$ is the net torque applied on the system, which is the vector sum of the moments of all forces acting on the system, and $d\vec{H}_{sys}/dt$ is the rate of change of the angular momentum of the system. It is stated as *the* *rate of change of angular momentum of a system is equal to the net torque acting on the system.* This equation is valid for a fixed quantity of mass. This statement is valid for an inertial reference frame, i.e., a reference frame that is fixed or moves with a constant velocity in a straight path.

The general control volume formulation of the angular momentum equation is obtained by setting $b = \vec{r} \times \vec{V}$ and thus $B = \vec{H}$ in the general Reynolds transport theorem. It gives (Fig. 13–34)

$$\frac{dH_{sys}}{dt} = \frac{d}{dt}\int_{CV} \vec{r} \times \vec{V}\rho dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})dA \qquad \textbf{(13–46)}$$

The left-hand side of this equation is, from Eq. 13–45, equal to $\Sigma\vec{M}$. Substituting, the angular momentum equation for a general control volume (fixed or moving, fixed shape or distorting) is obtained to be

General:
$$\sum\vec{M} = \frac{d}{dt}\int_{CV} \vec{r} \times \vec{V}\rho\, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA \qquad \textbf{(13–47)}$$

which can be stated as

$$\begin{pmatrix}\text{The sum of all}\\ \text{external moments}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change of}\\ \text{the angular momentum of the}\\ \text{contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{angular momentum out of the}\\ \text{control surface by mass flow}\end{pmatrix}$$

Again, $\vec{V}_r = \vec{V} - \vec{V}_{CS}$ is the fluid velocity relative to the control surface (for use in mass flow rate calculations at all locations where the fluid crosses the control surface), and $\vec{V}$ is the fluid velocity as viewed from a fixed reference frame. The product $\rho(\vec{V}_r \cdot \vec{n})\, dA$ represents the mass flow rate through dA into or out of the control volume, depending on the sign.

For a fixed control volume (no motion or deformation of control volume), $\vec{V}_r = \vec{V}$ and the angular momentum equation becomes

Fixed CV:
$$\sum\vec{M} = \frac{d}{dt}\int_{CV} \vec{r} \times \vec{V}\rho\, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V} \cdot \vec{n})\, dA \qquad \textbf{(13–48)}$$

Also, note that the forces acting on the control volume consist of *body forces* that act throughout the entire body of the control volume such as gravity, and *surface forces* that act on the control surface such as the pressure and reaction forces at points of contact. The net torque consists of the moments of these forces as well as the torques applied on the control volume.

Special Cases

During *steady flow*, the amount of angular momentum within the control volume remains constant, and thus the time rate of change of angular momentum of the contents of the control volume is zero. Then,

Steady flow:
$$\sum\vec{M} = \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA \qquad \textbf{(13–49)}$$

In many practical applications, the fluid crosses the boundaries of the control volume at a certain number of inlets and outlets, and it is convenient to replace

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b(\vec{V}_r \cdot \vec{n})\, dA$$

$$B = \vec{H} \qquad b = \vec{r} \times \vec{V} \qquad b = \vec{r} \times \vec{V}$$

$$\frac{dH_{sys}}{dt} = \frac{d}{dt}\int_{CV} \vec{r} \times \vec{V}\rho\, dV + \int_{CS} \vec{r} \times \vec{V}\rho(\vec{V}_r \cdot \vec{n})\, dA$$

FIGURE 13–34
The angular momentum equation is obtained by replacing B in the Reynolds transport theorem by the total angular momentum $\vec{H}$, and b by the angular momentum per unit mass $\vec{r} \times \vec{V}$.

the area integral by an algebraic expression written in terms of the average properties over the cross-sectional areas where the fluid enters or leaves the control volume. In such cases, the angular momentum flow rate can be expressed as the difference between the angular momentums of outgoing and incoming streams. The angular momentum equation for this *uniform flow* case reduces to

Uniform flow: $$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{\mathcal{V}})\rho\, dV + \sum_{out} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{in} \vec{r} \times \dot{m}\vec{\mathcal{V}} \qquad \textbf{(13–50)}$$

If the flow is *steady* as well as *uniform*, the relation above further reduces to (Fig. 13–35)

Steady and uniform flow: $$\sum \vec{M} = \sum_{out} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{in} \vec{r} \times \dot{m}\vec{\mathcal{V}} \qquad \textbf{(13–51)}$$

$$\sum \vec{M} = \sum_{out} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{in} \vec{r} \times \dot{m}\vec{\mathcal{V}}$$

FIGURE 13–35
The net torque acting on a control volume during steady flow is equal to the difference between the outgoing and incoming angular momentum flows.

It states that *the net torque acting on the control volume during steady flow is equal to the difference between the outgoing and incoming angular momentum flow rates.* This statement can also be expressed for any specified direction.

In many problems, all the significant forces and momentum flows are in the same plane, and thus all giving rise to moments in the same plane and about the same axis. For such cases, Eq. 13–51 can be expressed in scalar form as

$$\sum M = \sum_{out} r\dot{m}\mathcal{V} - \sum_{in} r\dot{m}\mathcal{V} \qquad \textbf{(13–52)}$$

where r represents the normal distance between the point about which moments are taken and the line of action of the force or velocity, provided that the sign convention for the moments is observed. That is, all moments in the counterclockwise direction are positive, and all moment in the clockwise direction are negative.

No External Moments

When there are no external moments applied, the angular momentum equation 13–50 reduces to

No external moments: $$0 = \frac{d\vec{H}_{CV}}{dt} + \sum_{out} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{in} \vec{r} \times \dot{m}\vec{\mathcal{V}} \qquad \textbf{(13–53)}$$

This is an expression of the conservation of angular momentum principle which can be stated as *in the absence of external moments, the rate of change of the angular momentum of a control volume is equal to the difference between the incoming and outgoing angular momentum fluxes.*

When the moment of inertia I of the control volume remains constant, the first term of the last equation simply becomes moment of inertia times angular acceleration, $I\vec{\alpha}$. Therefore, the control volume in this case can be treated as a solid body, with a net torque of $\vec{M} = I\vec{\alpha} = \sum_{in} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{out} \vec{r} \times \dot{m}\vec{\mathcal{V}}$ (due to

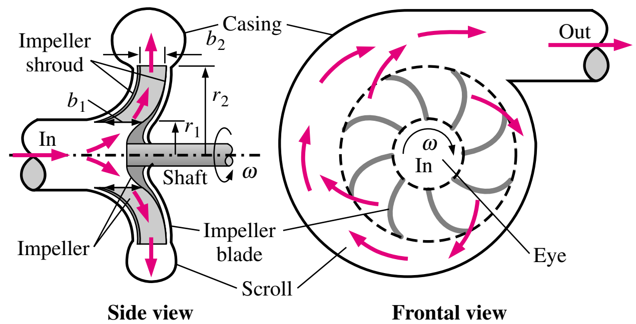


FIGURE 13–36
Side and frontal views of a typical centrifugal pump.

a change of angular momentum) acting on it. This approach can be used to determine the angular acceleration of space vehicles and aircraft when a rocket is fired in a direction different than the direction of motion.

Radial-Flow Devices

Many rotary flow devices such as the centrifugal pumps and fans involve flow in the radial direction normal to the axis of rotation, and are called radial-flow devices. In a centrifugal pump, for example, the fluid enters the device in the axial direction through the eye of the impeller, turns outward as it flows through the passages between the blades of the impeller, collects in the scroll, and is discharged in the tangential direction, as shown in Fig. 13–36. The axial-flow devices are easily analyzed using the linear momentum equation. But the radial-flow devices involve large changes in angular momentum of the fluid, and are best analyzed with the help of the angular momentum equation.

To analyze the centrifugal pump, we choose the annular region that encloses the impeller section as the control volume, as shown in Fig. 13–37. Note that the mean flow velocity, in general, will have normal and tangential components at both the inlet and the exit of the impeller section. Also, when the shaft rotates at an angular velocity of ω, the impeller blades will have a tangential velocity of ωr_1 at the inlet and ωr_2 at the outlet. For steady incompressible flow, the conservation of mass equation can be written as

$$\dot{V}_1 = \dot{V}_2 = \dot{V} \quad \rightarrow \quad (2\pi r_1 b_1)\mathcal{V}_{1,\,n} = (2\pi r_2 b_2)\mathcal{V}_{2,\,n} \tag{13–54}$$

where b_1 and b_2 are the flow widths at the inlet where $r = r_1$ and the outlet where $r = r_2$, respectively. (Note that the actual circumferential cross-sectional area is somewhat less than $2\pi rb$ since the blade thickness is not zero.) Then the average normal components $\mathcal{V}_{1,\,n}$ and $\mathcal{V}_{2,\,n}$ of absolute velocity can be expressed in terms of the volumetric flow rate $\dot{V}$ as

$$\mathcal{V}_{1,\,n} = \frac{\dot{V}}{2\pi r_1 b_1} \quad \text{and} \quad \mathcal{V}_{2,\,n} = \frac{\dot{V}}{2\pi r_2 b_2} \tag{13–55}$$

The normal velocity components $\mathcal{V}_{1,\,n}$ and $\mathcal{V}_{2,\,n}$ as well pressure acting on the inner and outer circumferential areas pass through the shaft center, and thus they do not contribute to torque about the origin. Then only the tangential

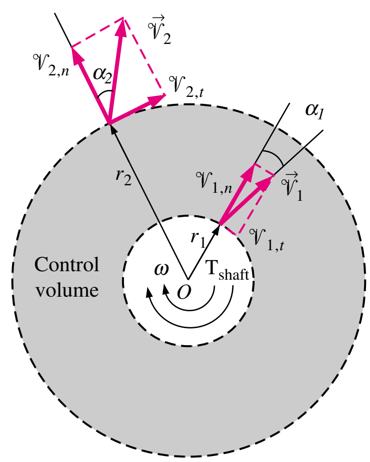


FIGURE 13–37
An annual control volume that encloses the impeller section of a centrifugal pump.

velocity components contribute to torque, and the application of the angular momentum equation $\sum M = \sum_{\text{out}} r\dot{m}\mathcal{V} - \sum_{\text{in}} r\dot{m}\mathcal{V}$ to the control volume gives

$$\mathrm{T}_{\text{shaft}} = \dot{m}(r_2\mathcal{V}_{2,\,t} - r_1\mathcal{V}_{1,\,t}) \tag{13–56}$$

which is known as **Euler's turbine formula**. When the angles α_1 and α_2 between the direction of absolute flow velocities and the radial direction are known, it becomes

$$\mathrm{T}_{\text{shaft}} = \dot{m}(r_2\mathcal{V}_2 \sin \alpha_2 - r_1\mathcal{V}_1 \sin \alpha_1) \tag{13–57}$$

In the idealized case of the tangential fluid velocity being equal to the blade angular velocity both at the inlet and the exit, we have $\mathcal{V}_{1,\,t} = \omega r_1$ and $\mathcal{V}_{2,\,t} = \omega r_2$, and the torque becomes

$$\mathrm{T}_{\text{shaft, ideal}} = \dot{m}\omega(r_2^2 - r_1^2) \tag{13–58}$$

where $\omega = 2\pi\dot{n}$ is the angular velocity of the blades. When the torque is known, the shaft power can be determined from $\dot{W}_{\text{shaft}} = \omega\mathrm{T}_{\text{shaft}} = 2\pi\dot{n}\mathrm{T}_{\text{shaft}}$.

EXAMPLE 13–8 The Moment Acting at the Base of a Water Pipe

Underground water is pumped to a sufficient height through a 10-cm-diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section, as shown in Fig. 13–38. Water discharges to atmospheric air at a velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the moment acting at the base of the pipe (point *A*), and the required length of the horizontal section that will make the moment at point *A* zero.

SOLUTION Water is pumped through a piping section. The moment acting at the base and the required length of the horizontal section to make this moment zero is to be determined.

Assumptions **1** The flow is steady and uniform. **2** The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero.

Properties We take the density of water to be 1000 kg/m^3.

Analysis We take the entire L-shaped pipe as the control volume, and designate the inlet by 1 and the outlet by 2. We also take the *x*- and *y*-coordinates as shown. The control volume and the reference frame are fixed.

The conservation of mass equation for this one-inlet one-outlet steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}$, and $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$ since A_c = constant. The mass flow rate and the weight of the horizontal section of the pipe are

$$\dot{m} = \rho A_c\mathcal{V} = (1000\ \text{kg/m}^3)[\pi(0.10\ \text{m})^2/4](3\ \text{m/s}) = 23.56\ \text{kg/s}$$

$$W = mg = (12\ \text{kg/m})(1\ \text{m})(9.81\ \text{m/s}^2)\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = 118\ \text{N}$$

To determine the moment acting on the pipe at point *A*, we need to take the moment of all forces and momentum flows about that point. This is a steady and uniform flow problem, and all forces and momentum flows are in the same

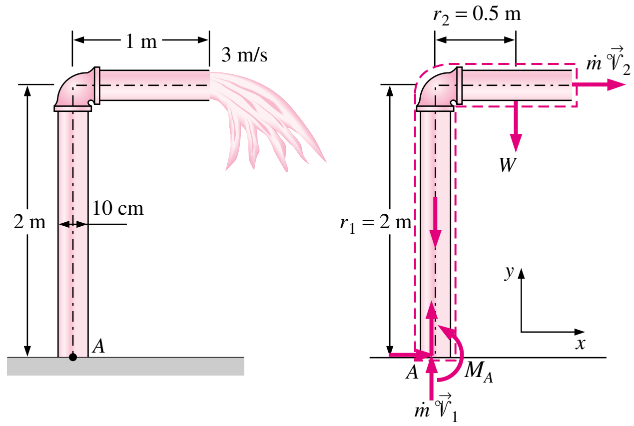


FIGURE 13–38
Schematic for Example 13–8 and the free-body diagram.

plane. Therefore, the angular momentum equation in this case can be expressed as

$$\sum M = \sum_{\text{out}} r\dot{m}\mathcal{V} - \sum_{\text{in}} r\dot{m}\mathcal{V}$$

where r is the moment arm, all moments in the counterclockwise direction are positive, and all in the clockwise direction are negative.

The free-body diagram of the L-shaped pipe is given in Fig. 13–38. Noting that the moments of all forces and momentum flows passing through point A are zero, the only force that will yield a moment about point A is the weight W of the horizontal pipe section, and the only momentum flow that will yield a moment is the exit stream (both are negative since both moments are in the clockwise direction). Then the angular momentum equation about point A becomes

$$M_A - r_1 W = -r_2\dot{m}\mathcal{V}_2$$

Solving for M_A and substituting give

$$\begin{aligned} M_A &= r_1 W - r_2\dot{m}\mathcal{V}_2 \\ &= (0.5\text{ m})(118\text{ N}) - (2\text{ m})(23.56\text{ kg/s})(3\text{ m/s})\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right) = 58.9 - 141.4 \\ &= -\mathbf{82.5\ N\cdot m} \end{aligned}$$

The negative sign indicates that the assumed direction for M_A is wrong, and should be reversed. Therefore, a moment of 82.5 N · m acts at the stem of the pipe in the clockwise direction. That is, the concrete base must apply a 82.5 N · m moment on the pipe stem in the clockwise direction to counteract the excess moment caused by the exit stream.

The weight of the horizontal pipe is $W = 118$ N per m length. Therefore, the weight for a length of L m is LW with a moment arm of $r_1 = L/2$. Setting $M_A = 0$ and substituting, the length L of the horizontal pipe that will cause the moment at the pipe stem to vanish is determined to be

$$0 = r_1 W - r_2\dot{m}\mathcal{V}_2 \quad \rightarrow \quad 0 = (L/2)LW - r_2\dot{m}\mathcal{V}_2$$

or

$$L = \sqrt{\frac{2r_2\dot{m}\mathcal{V}_2}{W}} = \sqrt{\frac{2 \times 141.4\ \text{N}\cdot\text{m}}{118\ \text{N/m}}} = \mathbf{2.40\ m}$$

Discussion Note that the pipe weight and the momentum of the exit stream cause opposing moments at point *A*. This example shows the importance of accounting for the moments of momentums of flow streams when performing a dynamic analysis and evaluating the stresses in pipe materials at critical cross sections.

EXAMPLE 13–9 Power Generation from a Sprinkler System

A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head, as shown in Fig. 13–39. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s, and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.

SOLUTION A four-armed sprinkler is used to generate electric power. For a specified flow rate and rotational speed, the power produced is to be determined.

Assumptions **1** The flow is uniform and cyclically steady (i.e., steady from a frame of reference rotating with the sprinkler head). **2** The water is discharged to the atmosphere, and thus the gage pressure at the nozzle exit is zero. **3** Generator losses and air drag of rotating components are neglected.

Properties We take the density of water to be 1000 kg/m^3 = 1 kg/L.

Analysis We take the disk that encloses the sprinkler arms as the control volume, which is a stationary control volume.

The conservation of mass equation for this steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}$. Noting that the four nozzles are identical, we have $\dot{m}_{\text{nozzle}} = \dot{m}/4$ or $\dot{V}_{\text{nozzle}} =$

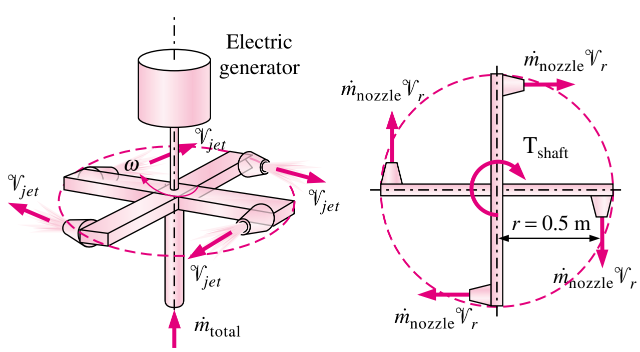


FIGURE 13–39
Schematic for Example 13–9 and the free-body diagram.

$\dot{V}/4$ since the density of water is constant. The average jet exit velocity relative to the nozzle is

$$\mathcal{V}_{\text{jet}} = \frac{\dot{V}_{\text{nozzle}}}{A_{\text{jet}}} = \frac{5\text{ L/s}}{[\pi(0.01\text{ m})^2/4]}\left(\frac{1\text{ m}^3}{1000\text{ L}}\right) = 63.66\text{ m/s}$$

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300\text{ rev/min})\left(\frac{1\text{ min}}{60\text{ s}}\right) = 31.42\text{ rad/s}$$

$$\mathcal{V}_{\text{nozzle}} = r\omega = (0.6\text{ m})(31.42\text{ rad/s}) = 18.85\text{ m/s}$$

That is, the water in the nozzle is also moving at a velocity of 18.85 m/s in the opposite direction when it is discharged. Then the velocity of water jet relative to the control volume (or relative to a fixed location on earth) becomes

$$\mathcal{V}_r = \mathcal{V}_{\text{jet}} - \mathcal{V}_{\text{nozzle}} = 63.66 - 18.85 = 44.81\text{ m/s}$$

Noting that this is a steady and uniform flow problem, and all forces and momentum flows are in the same plane, the angular momentum equation can be expressed as $\sum M = \sum_{\text{out}} r\dot{m}\mathcal{V} - \sum_{\text{in}} r\dot{m}\mathcal{V}$ where r is the moment arm, all moments in the counterclockwise direction are positive, and all in the clockwise direction are negative.

The free-body diagram of the disk that contains the sprinkler arms is given in Fig. 13–39. Note that the moments of all forces and momentum flows passing through the axis of rotation are zero. The momentum flows via the water jets leaving the nozzles yield a moment in the clockwise direction and the effect of the generator on the control volume is a moment also in the clockwise direction (thus both are negative). Then the angular momentum equation about the axis of rotation becomes

$$-\mathrm{T}_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}\mathcal{V}_r \qquad \text{or} \qquad \mathrm{T}_{\text{shaft}} = r\dot{m}_{\text{total}}\mathcal{V}_r$$

Substituting, the torque transmitted through the shaft is determined to be

$$\mathrm{T}_{\text{shaft}} = r\dot{m}_{\text{total}}\mathcal{V}_r = (0.6\text{ m})(20\text{ kg/s})(44.81\text{ m/s})\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right) = 537.7\text{ N}\cdot\text{m}$$

since $\dot{m}_{\text{total}} = \rho\dot{V}_{\text{total}} = (1\text{ kg/L})(20\text{ L/s}) = 20\text{ kg/s}$.

Then the power generated becomes

$$\dot{W} = 2\pi\dot{n}\mathrm{T}_{\text{shaft}} = \omega\mathrm{T}_{\text{shaft}} = (31.32\text{ rad/s})(537.7\text{ N}\cdot\text{m})\left(\frac{1\text{ kW}}{1000\text{ N}\cdot\text{m/s}}\right) = 16.9\text{ kW}$$

Therefore, this sprinkler-type turbine has the potential to produce 16.9 kW of power.

Discussion To put the result obtained in perspective, we consider two limiting cases. In the first limiting case, the sprinkler is stuck and thus the angular velocity is zero. The torque developed will be maximum in this case since $\mathcal{V}_{\text{nozzle}} = 0$ and thus $\mathcal{V}_r = \mathcal{V}_{\text{jet}} = 63.66$ m/s, giving $\mathrm{T}_{\text{shaft, max}} = 764\text{ N}\cdot\text{m}$. But the power generated will be zero since the shaft does not rotate.

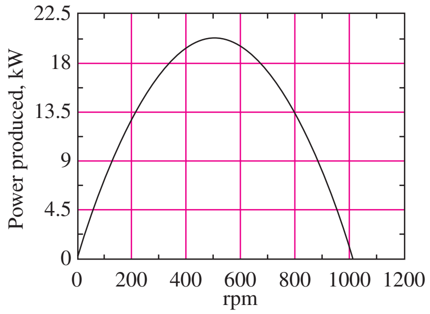


FIGURE 13–40
The variation of power produced with angular speed.

In the second limiting case, the shaft is disconnected from the generator (and thus both the torque and power generation are zero) and the shaft accelerates until it reaches an equilibrium velocity. Setting $T_{shaft} = 0$ in the angular momentum equation gives $\mathcal{V}_r = 0$ and thus $\mathcal{V}_{jet} = \mathcal{V}_{nozzle} = 63.66$ m/s. The corresponding angular speed of the sprinkler is

$$\dot{n} = \frac{\omega}{2\pi} = \frac{\mathcal{V}_{nozzle}}{2\pi r} = \frac{63.66 \text{ m/s}}{2\pi(0.6 \text{ m})}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 1013 \text{ rpm}$$

At this rpm, the velocity of the jet will be zero relative to an observer on earth (or relative to the fixed disk-shaped control volume selected).

The variation of power produced with angular speed is plotted in Fig. 13–40. Note that the power produced increases with increasing rpm, reaches a maximum (at about 500 rpm in this case), and then decreases.

SUMMARY

This chapter deals mainly with the conservation of momentum for finite control volumes. The forces acting on the control volume consist of *body forces* that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and *surface forces* that act on the control surface (such as the pressure forces and reaction forces at points of contact). The sum of all forces acting on the control volume at a particular instant of time is represented by $\sum \vec{F}$, and is expressed as

$$\underbrace{\sum \vec{F}}_{\text{total force}} = \underbrace{\sum \vec{F}_{\text{gravity}}}_{\text{body forces}} + \underbrace{\sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}}}_{\text{surface forces}}$$

Newton's second law can be stated as the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system. Setting $b = \vec{\mathcal{V}}$ and thus $B = m\vec{\mathcal{V}}$ in the Reynolds transport theorem and utilizing Newton's second law gives the *linear momentum equation* for a control volume as

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{\mathcal{V}}\, dV + \int_{CS} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}}_r \cdot \vec{n})\, dA$$

It reduces to the following special cases:

Steady flow: $$\sum \vec{F} = \int_{CS} \rho\vec{\mathcal{V}}(\vec{\mathcal{V}}_r \cdot \vec{n})\, dA$$

Unsteady flow (*algebraic form*):

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{\mathcal{V}}\, dV + \sum_{\text{out}} \beta\dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \beta\dot{m}\vec{\mathcal{V}}$$

Steady flow (*algebraic form*): $$\sum \vec{F} = \sum_{\text{out}} \beta\dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \beta\dot{m}\vec{\mathcal{V}}$$

No external forces: $$0 = \frac{d(m\vec{\mathcal{V}})_{CV}}{dt} + \sum_{\text{out}} \beta\dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \beta\dot{m}\vec{\mathcal{V}}$$

where β is the momentum correction factor whose value is nearly 1 for most flows encountered in practice. A control volume whose mass m remains constant can be treated as a solid body, with a net force of $\vec{F} = m\vec{a} = \sum_{\text{in}} \dot{m}\vec{\mathcal{V}} - \sum_{\text{out}} \dot{m}\vec{\mathcal{V}}$ acting on it.

Newton's second law can also be stated as the rate of change of angular momentum of a system is equal to the net torque acting on the system. Setting $b = \vec{r} \times \vec{\mathcal{V}}$ and thus $B = \vec{H}$ in the general Reynolds transport theorem gives the *angular momentum equation* as

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{\mathcal{V}})\rho\, dV + \int_{CS} (\vec{r} \times \vec{\mathcal{V}})\rho(\vec{\mathcal{V}}_r \cdot \vec{n})\, dA$$

It reduces to the following special cases:

Steady flow: $$\sum \vec{M} = \int_{CS} (\vec{r} \times \vec{\mathcal{V}})\rho(\vec{\mathcal{V}}_r \cdot \vec{n})\, dA$$

Uniform flow:

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{\mathcal{V}})\rho\, dV + \sum_{\text{out}} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{\mathcal{V}}$$

Steady and uniform flow:

$$\sum \vec{M} = \sum_{\text{out}} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{\mathcal{V}}$$

Scalar form for one direction:

$$\sum M = \sum_{\text{out}} r\dot{m}\mathcal{V} - \sum_{\text{in}} r\dot{m}\mathcal{V}$$

No external moments: $$0 = \frac{d\vec{H}_{CV}}{dt} + \sum_{\text{out}} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{\mathcal{V}}$$

A control volume whose moment of inertial I remains constant can be treated as a solid body, with a net torque of $\vec{M} = I\vec{\alpha} = \sum_{\text{in}} \vec{r} \times \dot{m}\vec{\mathcal{V}} - \sum_{\text{out}} \vec{r} \times \dot{m}\vec{\mathcal{V}}$ acting on it. This relation can be used to determine the angular acceleration of spacecraft when a rocket is fired.

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2. R. W. Fox and A. T. McDonald. *Introduction to Fluid Mechanics.* 5th ed. New York: Wiley, 1999.
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PROBLEMS*

Newton's Laws and Conservation of Momentum

13–1C Name four physical quantities that are conserved, and two quantities that are not conserved during a process.

13–2C Express Newton's first, second, and third laws.

13–3C Is momentum a vector? If so, in what direction does it point?

13–4C Express the conservation of momentum principle. What can you say about the momentum of a body if the net force acting on it is zero?

13–5C Express Newton's second law of motion for rotating bodies. What can you say about the angular velocity and angular momentum of a rotating nonrigid body of constant mass if the net torque acting on it is zero?

13–6C Consider two rigid bodies having the same mass and angular speed. Do you think these two bodies must have the same angular momentum? Explain.

Linear Momentum Equation

13–7C Explain the importance of the Reynolds transport theorem in fluid mechanics, and describe how the linear momentum equation is obtained from it.

13–8C Describe body forces and surface forces, and explain how the net force acting on control volume is determined. Is fluid weight a body force or surface force? How about pressure?

*Problems designated by a "C" are concept questions, and students are encouraged to answer them all. Problems designated by an "E" are in English units, and the SI users can ignore them. Problems with a CD-EES icon are solved using EES, and complete solutions together with parametric studies are included on the enclosed CD. Problems with a computer-EES icon are comprehensive in nature, and are intended to be solved with a computer, preferably using the EES software that accompanies this text.

13–9C How do surface forces arise in the momentum analysis of a control volume? How can we minimize the number of surface forces exposed during analysis?

13–10C What is the importance of the momentum-flux correction factor in the momentum analysis of slow systems? For which type of flow is it significant and must be considered in analysis: laminar flow, turbulent flow, or jet flow?

13–11C Write the momentum equation for steady one-dimensional flow for the case of no external forces and explain the physical significance of its terms.

13–12C In the application of the momentum equation, explain why we can usually disregard the atmospheric pressure and work with gage pressures only.

13–13C Two firemen are fighting a fire with identical water hoses and nozzles, except that one is holding the hose straight so that the water leaves the nozzle in the same direction it comes, while the other holds it backward so that the water makes a U-turn before being discharged. Which fireman will experience a greater reaction force?

13–14C A rocket in space (no friction or resistance to motion) can expel gases relative to itself at some high velocity $\mathcal{V}$. Is $\mathcal{V}$ the upper limit to the rocket's ultimate velocity?

13–15C Describe in terms of momentum and airflow why a helicopter hovers.

FIGURE P13–15C

13–16C Does it take more, equal, or less power for a helicopter to hover at the top of a high mountain than it does at sea level? Explain.

13–17C In a given location, would a helicopter require more energy in summer or winter to achieve a specified performance? Explain.

13–18C A horizontal water jet from a nozzle of constant exit cross section impinges normally on a stationary vertical flat plate. A certain force F is required to hold the plate against the water stream. If the water velocity is doubled, will the necessary holding force also be doubled? Explain.

13–19C A constant velocity horizontal water jet from a stationary nozzle impinges normally on a vertical flat plate that is held in a frictionless track. As the water jet hits the plate, it begins to move due to the water force. Will the acceleration of the plate remain constant or change? Explain.

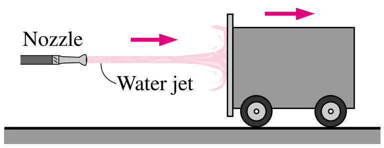

FIGURE P13–19C

13–20C A horizontal water jet of constant velocity $\mathcal{V}$ from a stationary nozzle impinges normally on a vertical flat plate that is held in a frictionless track. As the water jet hits the plate, it begins to move due to the water force. What is the highest velocity the plate can attain? Explain.

13–21 Show that the force exerted by a liquid jet on a stationary nozzle as it leaves with a velocity $\mathcal{V}$ is proportional to $\mathcal{V}^2$ or, alternatively, to $\dot{m}^2$.

13–22 A horizontal water jet of constant velocity $\mathcal{V}$ impinges normally on a vertical flat plate and splashes off the sides in the vertical plane. The plate is moving toward the oncoming water jet with velocity ½ $\mathcal{V}$. If a force F is required to maintain the plate stationary, how much force is required to move the plate toward the water jet?

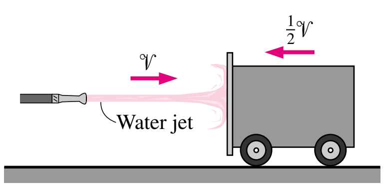

FIGURE P13–22

13–23 A 90° elbow is used to direct water flow at a rate of 25 kg/s in a horizontal pipe upward. The diameter of the entire elbow is 10 cm. The elbow discharges water into the atmosphere, and thus the pressure at the exit is the local atmospheric pressure. The elevation difference between the centers of the exit and the inlet of the elbow is 35 cm. The weight of the elbow and the water in it is considered to be negligible. Determine (*a*) the gage pressure at the center of the inlet of the elbow and (*b*) the anchoring force needed to hold the elbow in place.

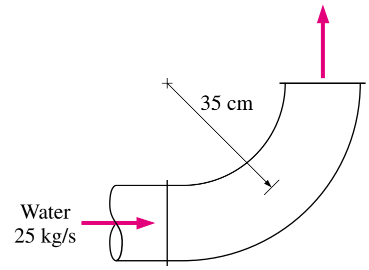

FIGURE P13–23

13–24 Repeat Prob. 13–23 for the case of another (identical) elbow being attached to the existing elbow so that the fluid makes a U-turn. *Answers:* (*a*) 6.87 kPa, (*b*) 213 N

13–25E A horizontal water jet impinges against a vertical flat plate at 30 ft/s, and splashes off the sides in the vertical plane. If a horizontal force of 350 lbf is required to hold the plate against the water stream, determine the volume flow rate of the water.

13–26 A reducing elbow is used to deflect water flow at a rate of 30 kg/s in a horizontal pipe upward by an angle $\theta = 45°$ from the flow direction while accelerating it. The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 150 cm² at the inlet and 25 cm² at the exit. The elevation difference between the centers of the exit and the inlet is 40 cm. The mass of the elbow and the water in it is 50 kg. Determine the anchoring force needed to hold the elbow in place.

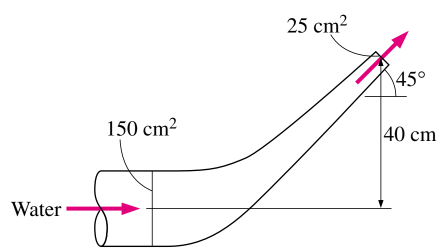

FIGURE P13–26

13–27 Repeat Prob. 13–26 for the case of $\theta = 110°$.

13–28 Water accelerated by a nozzle to 15 m/s strikes the vertical back surface of a cart moving horizontally at a constant velocity of 5 m/s in the flow direction. The mass flow rate of water is 25 kg/s. After the strike, the water stream splatters off in all directions in the plane of the back surface. (*a*) Determine the force that needs to be applied on the brakes of the cart to

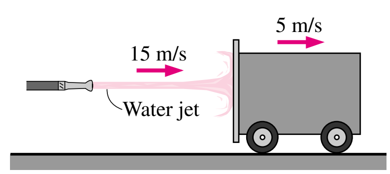

FIGURE P13–28

prevent it from accelerating. (*b*) If this force were used to generate power instead of wasting it on the brakes, determine the maximum amount of power that can be generated.
Answers: (*a*) 250 N, (*b*) 1.25 kW

13–29 Reconsider Prob. 13–28. If the mass of the cart is 300 kg and the brakes fail, determine the acceleration of the cart when the water first strikes it. Assume the mass of water that wets the back surface is negligible.

13–30E A 100-ft^3/s water jet is moving in the positive *x*-direction at 20 ft/s. The stream hits a stationary splitter, such that half of the flow is diverted upward at 45° and the other half is directed downward, and both streams have a final speed of 20 ft/s. Disregarding gravitational effects, determine the *x*- and *y*-components of the force required to hold the splitter in place against the water force.

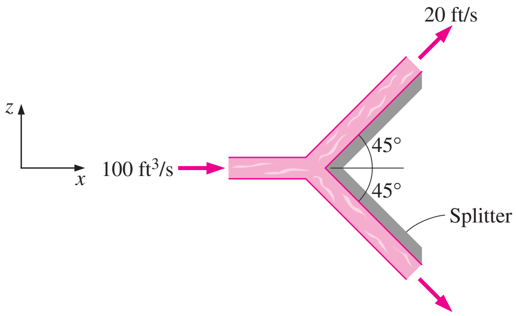

FIGURE P13–30E

13–31E Reconsider Prob. 13–30E. Using EES (or other) software, investigate the effect of splitter angle on the force exerted on the splitter in the incoming flow direction. Let the half splitter angle vary from 0 to 180° in increments of 10°. Tabulate and plot your results, and draw some conclusions.

13–32 A horizontal 5-cm-diameter water jet with a velocity of 18 m/s impinges normally upon a vertical plate of mass 1000 kg. The plate is held in a frictionless track and is initially stationary. When the jet strikes the plate, the plate begins to move in the direction of the jet. The water always splatters in the plane of the retreating plate. Determine (*a*) the acceleration of the plate when the jet first strikes it (time = 0), (*b*) the time it will take for the plate to reach a velocity of 9 m/s, and (*c*) the plate velocity 20 s after the jet first strikes the plate. Assume the velocity of the jet relative to the plate remains constant.

13–33 Water flowing in a horizontal 30-cm-diameter pipe at 5 m/s and 300 kPa gage enters a 90° bend reducing section, which connects to a 15-cm-diameter vertical pipe. The inlet of the bend is 50 cm above the exit. Neglecting any frictional and gravitational effects, determine the net resultant force exerted on the reducer by the water.

13–34 Commercially available large wind turbines have blade span diameters as large as 100 m and generate over 3 MW of electric power at peak design conditions. Consider a wind turbine with a 90-m blade span subjected to 25 km/h steady winds. If the combined turbine-generator efficiency of the wind turbine is 32 percent, determine (*a*) the power generated by the turbine and (*b*) the horizontal force exerted by the wind on the supporting mast of the turbine. Take the density of air to be 1.25 kg/m^3, and disregard frictional effects.

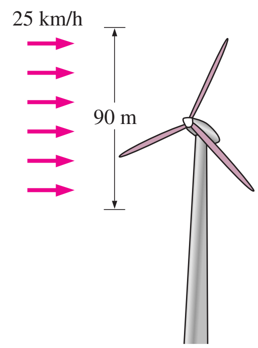

FIGURE P13–34

13–35E A 3-in-diameter horizontal water jet having a velocity of 140 ft/s strikes a curved plate, which deflects the water back in its original direction. How much force is required to hold the plate against the water stream?

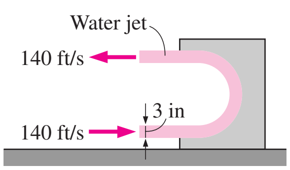

FIGURE P13–35E

13–36E A 3-in-diameter horizontal jet of water, with velocity 140 ft/s, strikes a bent plate, which deflects the water by 135° from its original direction. How much force is required to hold the plate against the water stream and what is its direction? Disregard frictional and gravitational effects.

13–37 Firemen are holding a nozzle at the end of a hose while trying to extinguish a fire. If the nozzle exit diameter is 6 cm and the water flow rate is 5 m^3/min, determine (*a*) the average water exit velocity and (*b*) the horizontal resistance force required of the firemen to hold the nozzle.
Answers: (*a*) 29.5 m/s, (*b*) 2457 N

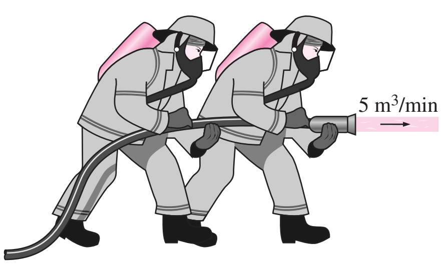

FIGURE P13–37

13–38 A 5-cm-diameter horizontal jet of water with a velocity of 30 m/s strikes a flat plate that is moving in the same direction as the jet at a velocity of 10 m/s. The water splatters in all directions in the plane of the plate. How much force does the water stream exert on the plate?

13–39 Reconsider Prob. 13–38. Using EES (or other) software, investigate the effect of the plate velocity on the force exerted on the plate. Let the plate velocity vary from 0 to 30 m/s, in increments of 3 m/s. Tabulate and plot your results.

13–40E A fan with 24-in-diameter blades moves 2000 cfm (cubic feet per minute) of air at 70°F at sea level. Determine (*a*) the force required to hold the fan and (*b*) the minimum power input required for the fan. Choose the control volume sufficiently large to contain the fan, and the gage pressure and the air velocity on the inlet side to be zero. Assume air approaches the fan through a large area with negligible velocity, and air exits the fan with a uniform velocity at atmospheric pressure through an imaginary cylinder whose diameter is the fan blade diameter.
Answers: (*a*) 0.82 lbf, (*b*) 5.91 W

13–41 An unloaded helicopter of mass 10,000 kg hovers at sea level while it is being loaded. In the unloaded hover mode, the blades rotate at 400 rpm. The horizontal blades above the helicopter cause a 15-m-diameter air mass to move downward at an average velocity proportional to the overhead blade rotational velocity (rpm). A load of 15,000 kg is loaded onto the helicopter, and the helicopter slowly rises. Determine (*a*) the volumetric airflow rate downdraft that the helicopter generates during unloaded hover and the required power input and (*b*) the rpm of the helicopter blades to hover with the 15,000-kg load and the required power input. Take the density of atmospheric air to be 1.18 kg/m³. Assume air approaches the blades from the top through a large area with negligible velocity, and air is forced by the blades to move down with a uniform velocity through an imaginary cylinder whose base is the blade span area.

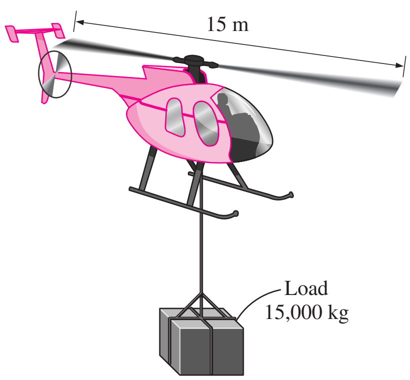

FIGURE P13–41

13–42 Reconsider the helicopter in Prob. 13–41, except that it is hovering on top of a 3000-m-high mountain where the air density is 0.79 kg/m³. Noting that the unloaded helicopter blades must rotate at 400 rpm to hover at sea level, determine the blade rotational velocity to hover at the higher altitude. Also determine the percent increase in the required power input to hover at 3000-m altitude relative to that at sea level.
Answers: 489 rpm, 22%

13–43 A sluice gate, which controls flow rate in a channel by simply raising or lowering a vertical plate, is commonly used in irrigation systems. A force is exerted on the gate due to the difference between the water heights y_1 and y_2 and the flow velocities $\mathcal{V}_1$ and $\mathcal{V}_2$ upstream and downstream from the gate, respectively. Disregarding the wall shear forces at the channel surfaces, develop relations for $\mathcal{V}_1$, $\mathcal{V}_2$, and the force acting on a sluice gate of width w during steady and uniform flow.
Answer: $F_R = \dot{m}(\mathcal{V}_1 - \mathcal{V}_2) + \frac{w}{2}\rho g(y_1^2 - y_2^2)$

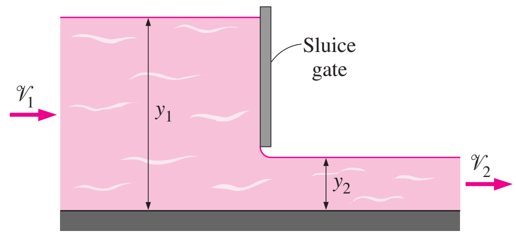

FIGURE P13–43

13–44 Water enters a centrifugal pump axially at atmospheric pressure at a rate of 0.12 m³/s and at a velocity of 7 m/s, and leaves in the normal direction along the pump casing, as shown in the figure. Determine the force acting on the shaft (which is also the force acting on the bearing of the shaft) in the axial direction.

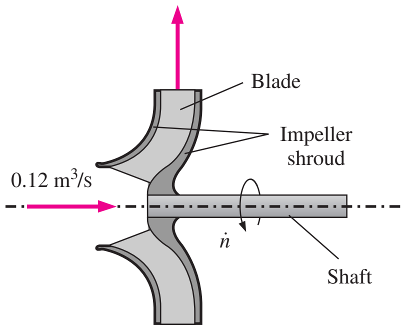

FIGURE P13–44

Angular Momentum Equation

13–45C How is the angular momentum equation obtained from Reynolds transport equations?

13–46C Express the unsteady angular momentum equation in vector form for a control volume that has a constant moment of inertia I, no external moments applied, and one outgoing uniform flow stream of velocity $\vec{\mathcal{V}}$, and mass flow rate $\dot{m}$.

13–47C Express the angular momentum equation in scalar form about a specified axis of rotation for a fixed control volume for steady and uniform flow.

13–48 Water is flowing through a 12-cm-diameter pipe that consists of a 3-m-long vertical and 2-m-long horizontal section with a 90° elbow at the exit to force the water to be discharged downward, as shown in the figure, in the vertical direction. Water discharges to atmospheric air at a velocity of 4 m/s, and the mass of the pipe section when filled with water is 15 kg per meter length. Determine the moment acting at the intersection of the vertical and horizontal sections of the pipe (point A). What would your answer be if the flow were discharged upward instead of downward?

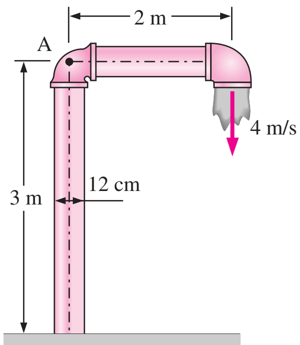

FIGURE P13–48

13–49E A large lawn sprinkler with two identical arms is used to generate electric power by attaching a generator to its rotating head. Water enters the sprinkler from the base along the axis of rotation at a rate of 8 gal/s, and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 250 rpm in a horizontal plane. The diameter of each jet is 0.5 in, and the normal distance between the axis of rotation and the center of each nozzle is 2 ft. Determine the electric power produced.

13–50E Reconsider the lawn sprinkler in Prob. 13–49E. If the rotating head is somehow stuck, determine the moment acting on the head.

13–51 A lawn sprinkler with three identical arms is used to water a garden by rotating in a horizontal plane by the impulse caused by water flow. Water enters the sprinkler along the axis of rotation at a rate of 40 L/s, and leaves the 1.2-cm-diameter nozzles in the tangential direction. The bearing applies a retarding torque of $T_0 = 50\ \text{N} \cdot \text{m}$ due to friction at the anticipated operating speeds. For a normal distance of 40 cm between the axis of rotation and the center of the nozzles, determine the angular velocity of the sprinkler shaft.

13–52 Pelton wheel turbines are commonly used in hydroelectric power plants to generate electric power. In these turbines, a high-speed jet at a velocity of $\mathcal{V}_j$ impinges on buckets, forcing the wheel to rotate. The buckets reverse the direction of the jet, and the jet leaves the bucket making an angle β with the direction of the jet, as shown in the figure. Show that the power produced by a Pelton wheel of radius r rotating steadily at an angular velocity of ω is $\dot{W}_{\text{shaft}} = \rho\omega r\dot{\mathcal{V}}(\mathcal{V}_j - \omega r)(1 - \cos\beta)$, where ρ is the density and $\dot{V}$ is the volumetric flow rate of the fluid. Obtain the numerical value for $\rho = 1000\ \text{kg/m}^3$, $r = 2$ m, $\dot{\mathcal{V}} = 10\ \text{m}^3\text{/s}$, $\dot{n} = 150$ rpm, $\beta = 160°$, and $\mathcal{V}_j = 50$ m/s.

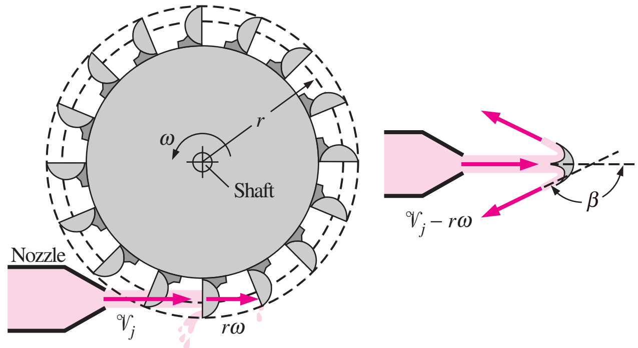

FIGURE P13–52

13–53 Reconsider Prob. 13–52. The turbine will have the maximum efficiency when $\beta = 180°$, but this is not practical. Investigate the effect of β on the power generation by allowing it to vary from 0° to 180°. Do you think we are wasting a large fraction of power by using buckets with a β of 160°?

13–54 The impeller of a centrifugal blower has a radius of 15 cm and a blade width of 6.1 cm at the inlet, and a radius of 30 cm and a blade width of 3.4 cm at the outlet. The blower delivers atmospheric air at 20°C and 95 kPa. Disregarding any losses and assuming the tangential components of air velocity at the inlet and the outlet to be equal to the impeller velocity at respective locations, determine the volumetric flow rate of air when the rotational speed of the shaft is 800 rpm, and the power consumption of the blower is 120 W. Also determine the normal components of velocity at the inlet and outlet of the impeller.

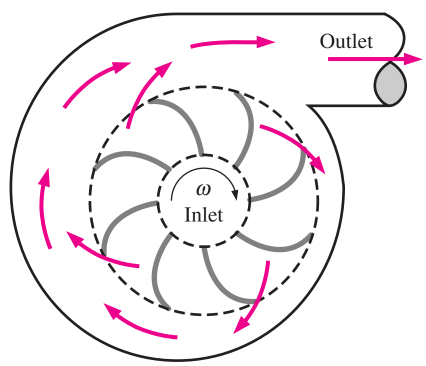

FIGURE P13–54

13–55 Consider a centrifugal blower that has a radius of 20 cm and a blade width of 8.2 cm at the impeller inlet, and a radius of 45 cm and a blade width of 5.6 cm at the outlet. The blower delivers air at a rate of 0.70 m³/s at a rotational speed of 700 rpm. Assuming the air to enter the impeller in radial direction and to exit at an angle of 50° from the radial direction, determine the minimum power consumption of the blower. Take the density of air to be 1.25 kg/m³.

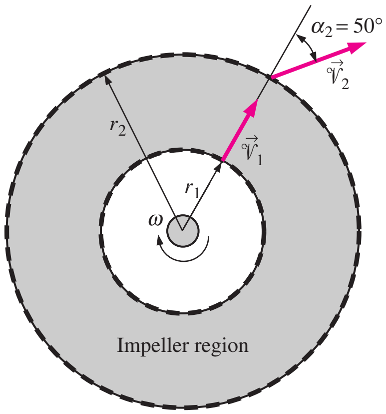

FIGURE P13–55

13–56 Reconsider Prob. 13–55. For the specified flow rate, investigate the effect of discharge angle α_2 on the minimum power input requirements. Assume the air to enter the impeller in radial direction ($\alpha_1 = 0°$), and vary α_2 from 0° to 85° in increments of 5°. Plot the variation of power input versus α_2, and discuss your results.

13–57E Water enters the impeller of a centrifugal pump radially at a rate of 80 cfm when the shaft is rotating at 500 rpm. The tangential component of absolute velocity of water at the exit of the 2-ft outer diameter impeller is 180 ft/s. Determine the torque applied to the impeller.

13–58 The impeller of a centrifugal pump has inner and outer diameters of 13 cm and 30 cm, respectively, and a flow rate of 0.15 m³/s at a rotational speed of 1200 rpm. The blade width of the impeller is 8 cm at the inlet and 3.5 cm at the outlet. If water enters the impeller in the radial direction and exits at an angle of 60° from the radial direction, determine the minimum power requirement for the pump.

Review Problems

13–59 Water is flowing into and discharging from a pipe U-section as shown in the figure. At flange (1), the total absolute pressure is 200 kPa, and 30 kg/s flows into the pipe. At flange (2), the total pressure is 150 kPa. At location (3), 8 kg/s of water discharges to the atmosphere, which is at 100 kPa. Determine the total x- and z-forces at the two flanges connecting the pipe. Discuss the significance of gravity force for this problem.

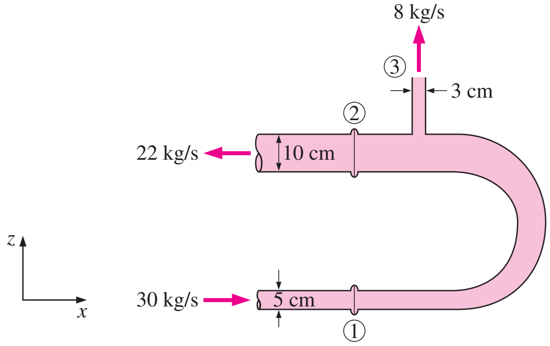

FIGURE P13–59

13–60 A tripod holding a nozzle, which directs a 5-cm-diameter stream of water from a hose, is shown in the figure. The nozzle mass is 10 kg when filled with water. The tripod is rated to provide 1800 N of holding force. A fireman was standing 60 cm behind the nozzle and was hit by the nozzle when the tripod suddenly failed and released the nozzle. You have been hired as an accident reconstructionist and, after testing the tripod, have determined that as water flow rate increased, it did collapse at 1800 N. In your final report you must state the water velocity and the flow rate consistent with the failure and the nozzle velocity when it hit the fireman.
Answers: 30.2 m/s, 0.0593 m³/s, 14.7 m/s

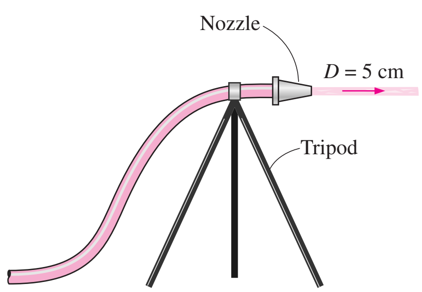

FIGURE P13–60

13–61 Consider an airplane with a jet engine attached to the tail section that expels combustion gases at a rate of 18 kg/s with a velocity of $\mathcal{V} = 250$ m/s relative to the plane. During landing, a thrust reverser (which serves as a brake for the aircraft and facilitates landing on a short runway) is lowered in the path of the exhaust jet, which deflects the exhaust from rearward to 160°. Determine (*a*) the thrust (forward force) that the engine produces prior to the insertion of the thrust reverser and (*b*) the braking force produced after the thrust reverser is deployed.

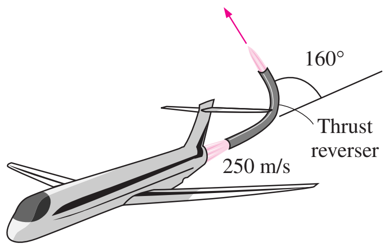

FIGURE P13–61

13–62 Reconsider Prob. 13–61. Using EES (or other) software, investigate the effect of thrust reverser angle on the braking force exerted on the airplane. Let the reverser angle vary from 0° (no reversing) to 180° (full reversing) in increments of 10°. Tabulate and plot your results, and draw conclusions.

13–63E A spacecraft cruising in space at a constant velocity of 1500 ft/s has a mass of 18,000 lbm. To slow down the spacecraft, a solid fuel rocket is fired, and the combustion gases leave the rocket at a constant rate of 150 lbm/s at a velocity of 5000 ft/s in the same direction as the spacecraft for a period of 5 s. Assuming the mass of the spacecraft remains constant, determine (*a*) the deceleration of the spacecraft during this 5-s period, (*b*) the change of velocity of the spacecraft during this time period, and (*c*) the thrust exerted on the spacecraft.

13–64 A 5-cm-diameter horizontal water jet having a velocity of 30 m/s strikes a vertical stationary flat plate. The water splatters in all directions in the plane of the plate. How much force is required to hold the plate against the water stream?

13–65 A 5-cm-diameter horizontal jet of water, with velocity 30 m/s, strikes the tip of a horizontal cone, which deflects the water by 45° from its original direction. How much force is required to hold the cone against the water stream?

13–66 A 60-kg ice skater is standing on ice with ice skates (no friction). She is holding a flexible hose (essentially weightless) that directs a 2-cm-diameter stream of water horizontally parallel to her skates. The water velocity at hose outlet is 10 m/s. If she is initially standing still, determine (*a*) the velocity of the skater and the distance she travels in 5 s and (*b*) how long it will take to move 5 m and the velocity at that moment. *Answers:* (*a*) 2.62 m/s, 6.54 m, (*b*) 4.4 s, 2.3 m/s

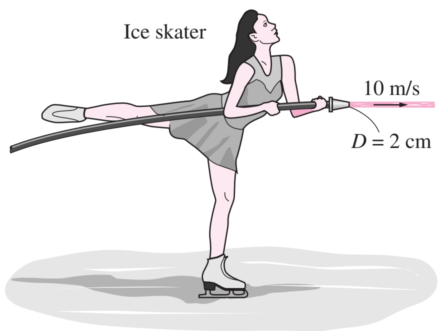

FIGURE P13–66

13–67 The apocryphal Indiana Jones needs to ascend a 10-m-high building. There is a large hose filled with pressurized water hanging down from the building top. He builds a square platform and mounts four 5-cm-diameter nozzles pointing down at each corner. By connecting hose branches, a water jet with 15 m/s velocity can be produced from each nozzle. Jones, the platform, and the nozzles have a combined mass of 150 kg. Determine (*a*) the minimum water jet velocity needed to raise the system, (*b*) how long it will take for the system to rise 10 m when the water jet velocity is 15 m/s and the velocity of the system at that moment, and (*c*) how much higher the momentum will raise Jones if he shuts off the water at the moment the platform reaches 10 m above the ground. How much time does he have to jump from the platform to the roof?
Answers: (*a*) 13.7 m/s, (*b*) 3.2 s, (*c*) 2.1 m, 1.3 s

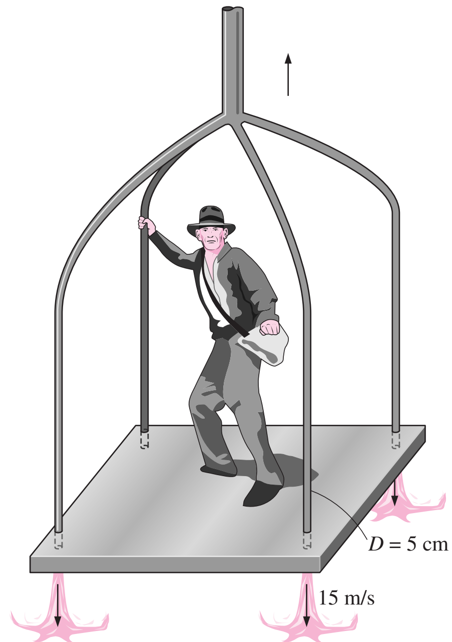

FIGURE P13–67

13–68E An engineering student considers using a fan as a levitation demonstration. He plans to face the box-enclosed fan so the air blast is directed face down through a 3-ft-diameter blade span area. The system weights 5 lbf, and he will secure the system from rotating. By increasing the power to the fan, he plans to increase the blade rpm and air exit velocity until the exhaust provides sufficient upward force to cause the box fan to hover in the air. Determine (*a*) the air exit velocity to produce 5 lbf, (*b*) the volumetric flow rate needed, and (*c*) the minimum mechanical power that must be supplied to the airstream. Take the air density to be 0.078 lbm/ft^3.

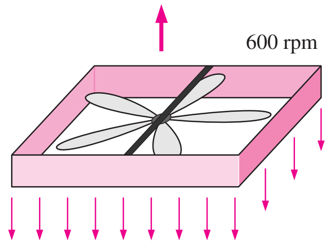

FIGURE P13–68E

13–69 A soldier jumps from a plane and opens his parachute when his velocity reaches the terminal velocity $\mathcal{V}_T$. The parachute slows him down to his landing velocity of $\mathcal{V}_F$. After the parachute is deployed, the air resistance is proportional to the velocity squared (i.e., $F = k\mathcal{V}^2$). The soldier, his parachute, and his gear have a total mass of m. Show that $k = \dfrac{mg}{\mathcal{V}_F^2}$

and develop a relation for the soldier's velocity after he opens the parachute at time $t = 0$.

Answer: $\mathcal{V} = \mathcal{V}_F \dfrac{\mathcal{V}_T + \mathcal{V}_F + (\mathcal{V}_T - \mathcal{V}_F)e^{-2gt/\mathcal{V}_F}}{\mathcal{V}_T + \mathcal{V}_F - (\mathcal{V}_T - \mathcal{V}_F)e^{-2gt/\mathcal{V}_F}}$

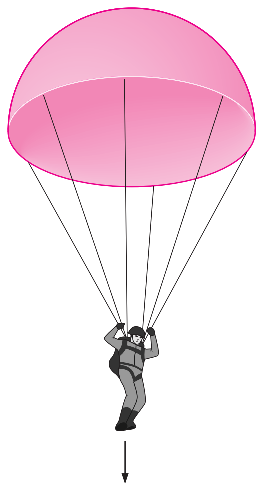

FIGURE P13–69

13–70 A horizontal water jet with a flow rate of $\dot{V}$ and cross-sectional area of A will drive a covered cart of mass m_c along a level and frictionless path. The jet enters a hole at the rear of the cart, and all water that enters the cart is retained, increasing the system mass. The relative velocity between the jet of constant velocity $\mathcal{V}_J$ and the cart of variable velocity $\mathcal{V}$ is $\mathcal{V}_J - \mathcal{V}$. If the cart is initially empty and stationary when the jet action is initiated, develop a relation (integral form is acceptable) for cart velocity versus time.

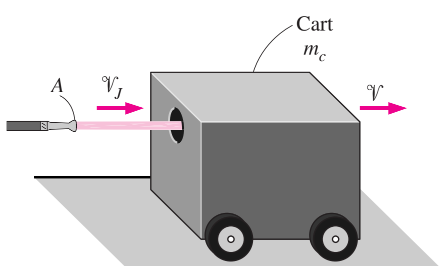

FIGURE P13–70

13–71 Frictionless vertical guide rails maintain a plate of mass m_p in a horizontal position, such that it can slide freely in the vertical direction. A nozzle can direct a water stream of area A against the plate underside. The water jet splatters in the plate plane, applying an upward force against the plate. The water flow rate $\dot{m}$ (kg/s) can be controlled. Assume that times are short, so the velocity of the rising jet can be considered constant with height. (*a*) Determine the minimum mass flow rate $\dot{m}_{\min}$ necessary to just levitate the plate and obtain a relation for the steady-state velocity of the upward moving plate for $\dot{m} > \dot{m}_{\min}$. (*b*) At time $t = 0$, the plate is at rest, and the water jet with $\dot{m} > \dot{m}_{\min}$ is suddenly turned on. Apply a force balance to the plate and obtain the integral that relates velocity to time (do not solve).

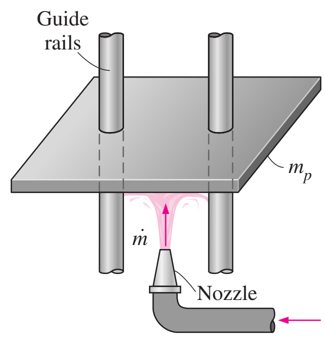

FIGURE P13–71

13–72 Water enters a mixed flow pump axially at a rate of 0.2 m^3/s and at a velocity of 5 m/s, and is discharged to the atmosphere at an angle of 60° from the horizontal, as shown in the figure. If the discharge flow area is half the inlet area, determine the force acting on the shaft in the axial direction.

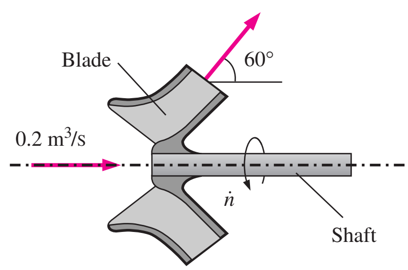

FIGURE P13–72

13–73 Water accelerated by a nozzle enters the impeller of a turbine through its outer edge of diameter D with a velocity of $\mathcal{V}$ making an angle α with the radial direction at a mass flow rate of $\dot{m}$. Water leaves the impeller in the radial direction. If the angular speed of the turbine shaft is $\dot{n}$, show that the maximum power that can be generated by this radial turbine is $\dot{W}_{\text{shaft}} = \pi \dot{n} \dot{m} D \mathcal{V} \sin \alpha$.

13–74 Water enters a two-armed lawn sprinkler along the vertical axis at a rate of 60 L/s, and leaves the sprinkler nozzles as 2-cm diameter jets at an angle of θ from the tangential direction, as shown in the figure. The length of each sprinkler arm is 0.45 m. Disregarding any frictional effects, determine the rate of rotation $\dot{n}$ of the sprinkler in rev/min for (*a*) $\theta = 0°$, (*a*) $\theta = 30°$, and (*a*) $\theta = 60°$.

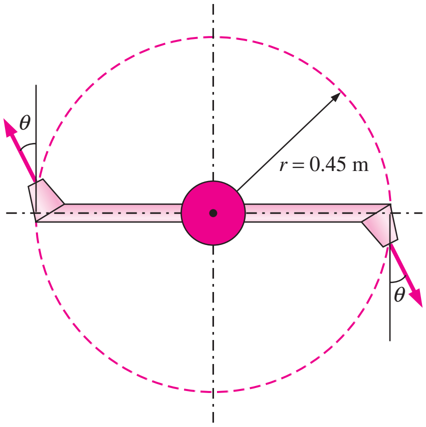

FIGURE P13–74

13–75 Reconsider Prob. 13–74. For the specified flow rate, investigate the effect of discharge angle θ on the rate of rotation $\dot{n}$ by varying θ from 0° to 90° in increments of 10°. Plot the rate of rotation versus θ, and discuss your results.

13–76 A stationary water tank of diameter D is mounted on wheels and is placed on a frictionless level surface. A smooth hole of diameter D_o near the bottom of the tank allows water to jet horizontally and rearward, and the water jet force propels the system forward. The water in the tank is much heavier than the tank-and-wheel assembly, so only the mass of water remaining in the tank needs to be considered in this problem. Considering the decrease in the mass of water with time, develop relations for (*a*) the acceleration, (*b*) the velocity, and (*c*) the distance traveled by the system as a function of time.

Design and Essay Problem

13–77 Visit a fire station and obtain information about flow rates through hoses and discharge diameters. Using this information, calculate the impulse force the firemen are subjected to.