MATH2222 WEEK 1 HOMEWORK

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(1) **Prove that $\sqrt{13}$ is irrational.**

This can be proven by contradiction.
Suppose that $\sqrt{13}$ is indeed rational. This implies that it can be written as a ratio of two integers, a and b (with $b \neq 0$), in some simplest form such that a and b share no common factors, i.e.,

$$\exists a, b \in \mathbb{Z}, b \neq 0, \text{ such that } \sqrt{13} = \frac{a}{b}$$

Begin by squaring both sides and isolating a.

$$13 = \frac{a^2}{b^2}$$
$$a^2 = 13b^2 \tag{1}$$

This shows that a^2 is divisible by 13. As 13 is prime, it can be further concluded that a is also divisible by 13. Therefore, a may be expressed as some integer k multiplied by 13.

$$\text{let } a = 13k, \text{ for some } k \in \mathbb{Z}$$

Substituting this into (1) and expanding,

$$(13k)^2 = 13b^2$$
$$13^2k^2 = 13b^2$$
$$b^2 = 13k^2$$

We now have a similar situation to (1); the same reasoning can be applied to show that b must be divisible by 13.
However, this means that both a and b are divisible by 13, contradicting the initial assumption that $\frac{a}{b}$ is in simplest form. Therefore, $\sqrt{13}$ is irrational. □

(2) **Let $\mathbf{f} : \mathbf{X} \longrightarrow \mathbf{Y}$ be a function and let $\mathbf{A}$ and $\mathbf{B}$ be subsets of $\mathbf{X}$.**
a) **Prove that $\mathbf{f(A \cup B) \subset f(A) \cup f(B)}$.**

We will prove more strongly that $\mathbf{f(A \cup B) = f(A) \cup f(B)}$; if this is proven it follows that the original statement is also true.
First, note that the set union between sets A and B can be written as,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$

but also (though generally unhelpfully) as,

$$A \cup B = \{x \mid x \in (A \cup B)\},$$

Date: February 2024.

as for any set A it holds that $A = \{x \mid x \in A\}$. This indicates that the constraint $(x \in A \text{ or } x \in B)$ is equivalent to the constraint $x \in (A \cup B)$.
Now to prove the original statement, consider the relevant images.

$$f(A) = \{f(x) \mid x \in A\}\,,$$
$$f(B) = \{f(x) \mid x \in B\}\,,$$
$$f(A \cup B) = \{f(x) \mid x \in (A \cup B)\}$$

Note that $f(x)$ implicitly represents the set of all $f(x) \in Y$, the codomain.
From this, by the definition of the set union, we have,

$$f(A) \cup f(B) = \{f(x) \mid x \in A \text{ or } x \in B\}\,,$$

or equivalently, as outlined previously,

$$f(A) \cup f(B) = \{f(x) \mid x \in (A \cup B)\}\,.$$

This, however, is precisely the definition of $f(A \cup B)$, therefore,

$$\mathbf{f}(\mathbf{A} \cup \mathbf{B}) = \mathbf{f}(\mathbf{A}) \cup \mathbf{f}(\mathbf{B}).$$

□

b) **Prove that $\mathbf{f}(\mathbf{A} \cap \mathbf{B}) \subset \mathbf{f}(\mathbf{A}) \cap \mathbf{f}(\mathbf{B})$.**

Begin by considering the image of the intersection of the two sets.

$$f(A \cap B) = \{f(x) \mid x \in (A \cap B)\}$$

Similarly to the previous proof, if x is in the set $A \cap B$, then it is, by definition, both in the sets A and B. Therefore,

$$f(A \cap B) = \{f(x) \mid x \in A \text{ and } x \in B\}\,.$$

Once again by definition of the set intersection, this is equivalent to,

$$f(A \cap B) = \{f(x) \mid x \in A\} \cap \{f(x) \mid x \in B\}\,,$$

which is precisely the definition of the set $f(A) \cap f(B)$, therefore the two are equivalent. □

3 **Prove the following.**

a) **If f and g are bounded, then $\mathbf{f} + \mathbf{g}$ is bounded.**

From the fact that both f and g are bounded, we know that there exists two real numbers N_f and N_g for which $|f(x)| < N_f$ and $|g(x)| < N_g$ hold for all $x \in \mathbb{R}$.
Combining these inequalities,

$$|f(x)| + |g(x)| < N_f + N_g.$$

By the triangle inequality,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| < N_f + Ng$$
$$|f(x) + g(x)| < N_f + Ng$$

We may now define a new bound $N = N_f + N_g$ and notice that,

$$|f(x) + g(x)| < N$$
$$|(f + g)(x)| < N$$

We have found a bound for $f + g$, therefore it is bounded. □

b) **If f and g are bounded, then fg is bounded**.

Once again from the fact that both f and g are bounded, we know that there exists two real numbers N_f and N_g for which $|f(x)| < N_f$ and $|g(x)| < N_g$ hold for all $x \in \mathbb{R}$.
As all values are positive in these inequalities, the two can be multiplied,

$$\begin{aligned} |f(x)||g(x)| &< N_f N_g \\ |f(x)g(x)| &< N_f N_g \\ |(fg)(x)| &< N_f N_g. \end{aligned}$$

Defining a new bound $N = N_f N_g$ shows that the function fg is in fact bounded.

$$|(fg)(x)| < N$$

□

c) **If $\mathbf{f+g}$ is bounded, then f and g are bounded**.

This can be disproved by counterexample.
Consider the functions $f(x) = x$ and $g(x) = -x$. As $x \longrightarrow \infty$, $f(x) \longrightarrow \infty$ and $g(x) \longrightarrow -\infty$, so f and g are unbounded.
The function $(f+g)(x) = f(x) + g(x) = x + (-x) = 0$, however is bounded by any positive and non-zero real number ϵ, i.e.,

$$|(f+g)(x)| = |0| = 0 < \epsilon \quad \forall x \in \mathbb{R}.$$

Therefore, there exists at least one pair of functions f and g such that they are individually unbounded, while their sum is bounded. □

d) **If fg is bounded, then f and g are bounded**.

This can be disproved by counterexample.
Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = x$. $f(x)$ has a vertical asymptote at $x = 0$, and so cannot be bounded, and $g(x)$ is unbounded as already shown in the previous proof.
Taking their product, however,

$$(fg)(x) = f(x)g(x) = \left(\frac{1}{x}\right) x = 1.$$

This can be bounded by any real number greater than 1, e.g.,

$$|(fg)(x)| = |1| = 1 < 2 \quad \forall x \in \mathbb{R}.$$

Therefore, there exists at least one pair of functions for which the product is bounded, while the individual functions are unbounded.

e) **If $\mathbf{f+g}$ is bounded and fg is bounded, then f and g are bounded**.

(Collaboration statement: after being stuck on this question, I spoke to Sam Roberts about the question, and eventually came to this solution)
This can be proven directly.
As $f+g$ is bounded and fg is bounded, the following inequalities may be written,

with $f+g$ bounded by a and fg bounded by b. Note that f and g here stand for $f(x)$ and $g(x)$ respectively.

$$|f+g| < a \tag{1}$$

$$|fg| < b \tag{2}$$

Note that if $fg \geq 0$, then $fg = |fg| < b$, and that if $fg \leq 0$, then $fg < |fg| < b \Rightarrow fg < b$. Therefore it is true in general that $fg < b$. This further means that $fg + k(x) = b \Rightarrow fg = b - k(x)$ for some bounded, strictly positive function $k : X \longrightarrow Y$.
Now take the square of the first inequality.

$$|f+g|^2 < a^2$$
$$f^2 + 2fg + g^2 < a^2$$

Substituting $fg = b - k(x)$,

$$f^2 + 2(b - k(x)) + g^2 < a^2$$
$$f^2 + g^2 < a^2 + 2k(x) - 2b$$

Adding to this two times equation (2),

$$f^2 + 2|fg| + g^2 < a^2 + 2k(x) - 2b + 2b$$
$$|f|^2 + 2|fg| + |g|^2 < a^2 + 2k(x)$$
$$(|f| + |g|)^2 < a^2 + 2k(x)$$

All terms are positive, so,

$$|f| + |g| < \sqrt{a^2 + 2k(x)}.$$

The function k is bounded by definition, so $|k(x)| < c \Rightarrow k(x) < c$ (as $k(x)$ is strictly positive) for some maximum value c, over all x. Therefore the right hand side of the above inequality may be replaced by $\sqrt{a^2 + 2c}$, giving a constant upper bound for $|f| + |g|$.

$$|f| + |g| < \sqrt{a^2 + 2c}.$$

Now, if f were unbounded, then $|f|$ would be unbounded in the positive direction. However, in $|f| + |g|$, $|g|$ may only add to $|f|$, it can be concluded that $|f| + |g|$ must also be unbounded in this case.
The same reasoning can be applied to g, and so if f or g are unbounded then $|f|+|g|$ must be unbounded also. The contra-positive of this is that $|f|+|g|$ being bounded must imply that both f and g are also bounded.
Therefore, as $|f| + |g|$ may be bounded by $\sqrt{a^2 + 2c}$, then both f and g must be bounded also. $\square$