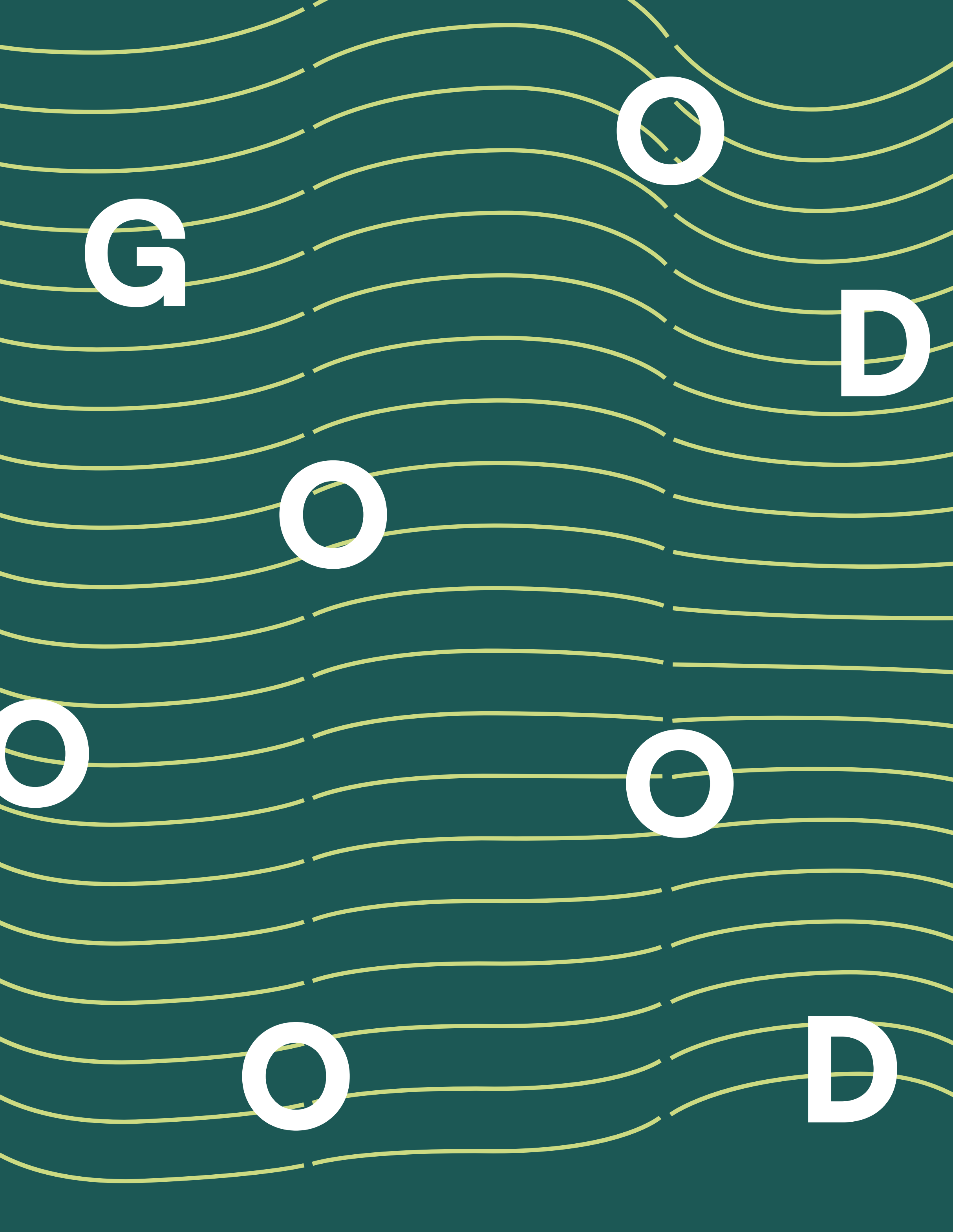
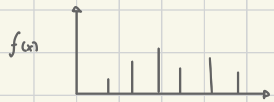

O
G
D
O
O
O
D
O

Discrete verdelingen



"beperkt aantal reele getallen met strikt positieve kant"

$f(x) = \mathbb{P}(X = x)$

$\hookrightarrow \quad > 0$

$\mathbb{P}(a \leq X \leq b) = \sum_{k=a}^{b} \mathbb{P}(X = k)$

Totale Som = 1

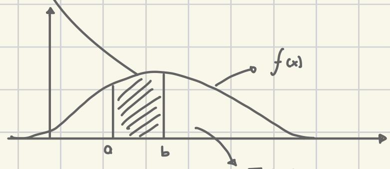
Continue verdelingen

Op een (=eventueel oneindig) interval zijn alle waarden mogelijk

$\mathbb{P}(X = k) = 0$

$\mathbb{P}(a < X < b) = ?$

$\mathbb{P}(-\infty \leq X \leq +\infty) = 1$



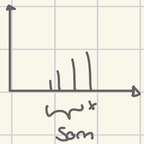
Totale opp. onder $f(x)$ is 1

$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,dx$

$\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Discrete verdelingen

$F_x(x) = \mathbb{P}(x \leq f(x))$



Verwachtingswaarde

$$\mathbb{E}[x] = \sum_k k \, \mathbb{P}(x = k)$$

Variantie

$$Var(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

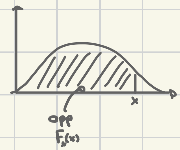
$$= \sum_k k^2 \, \mathbb{P}(x = k) - \mathbb{E}[x]^2$$

$\sigma_{(x)} = \sqrt{Var(x)}$

Continue verdelingen

$$F(x) = \mathbb{P}(\underbrace{x \leq x}_{\text{interval}})$$

$$= \mathbb{P}(-\infty < X < x)$$



$$\mathbb{E}[x] = \int_{-\infty}^{+\infty} x \, f_x(x) \, dx$$

$$Var(x) = \int_{-\infty}^{+\infty} x^2 \, f_x(x) \, dx - \mathbb{E}[x]^2$$

$\sigma_{(x)} = \sqrt{Var(x)}$

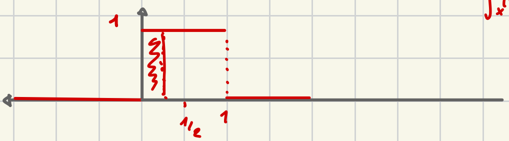
Continue verdeling

Uniforme (continue)

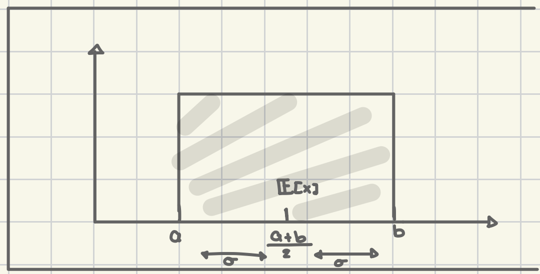
bv Randgetal in phyton
↳ tussen 0 of 1

$X \sim U([0,1])$

$$f_X(x) = 1_{[0,1]}(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$

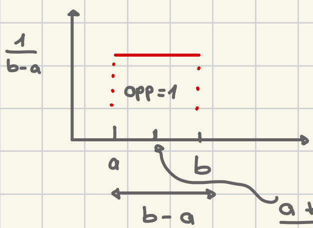


bv $P(0 < x < \frac{1}{3})$



Algemeen

$X \sim U([a,b])$



$\frac{a+b}{2} = \mathbb{E}[x]$

$$f_X(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}[x]_a^b = \frac{1}{b-a}(b-a) = 1$$

$$\mathbb{E}[x] = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$\mathrm{Var}(x) = \cancel{\mathbb{E}[x^2] - \mathbb{E}[x]^2} = \int x^2 f_X(x)\,dx - \mathbb{E}[x]^2 = \int_a^b \frac{x^2}{b-a}\,dx - \left(\frac{b+a}{2}\right)^2 = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{1}{3(b-a)}(b^3 - a^3) - \frac{(a+b)^2}{4} = \frac{(b^2 + ab + a^2)}{3} - \frac{(a+b)^2}{4} = \frac{a^2 - 2ab + b^2}{12} = \frac{(a-b)^2}{12}$$

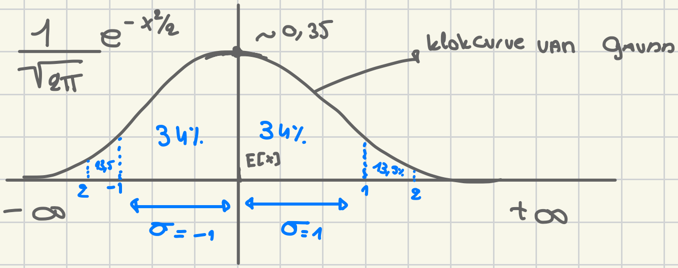
$$\sigma = \sqrt{\mathrm{Var}(x)} = \sqrt{\frac{(a-b)^2}{12}} = \frac{a-b}{2\sqrt{3}} = \frac{\sqrt{3}(b-a)}{6}$$

Continue verdeling

Standaard normale verdeling

- Symmetrisch t.o.v y-As (gemiddelde 0)
- opp onder $f_x(x) = 1$

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = 1 \qquad \left(\frac{1}{\sqrt{2\pi}} \approx 0{,}39 \right)$$

gemiddelde:

$$\mathbb{E}[x] = \int_{-\infty}^{+\infty} x \; f_x(x) \, dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi} x \, e^{-x^2/2} dx$$

$= \ldots$

$= 0$

Substitutie

$t = -x^2$

$dt = -2x \, dx$

Variatie

$$\int_{-\infty}^{+\infty} x^2 f_x(x) \, dx - \underbrace{\mathbb{E}[x]^2}_{=0}$$

$$= \int_{-\infty}^{+\infty} x^2 \frac{1}{2\sqrt{2\pi}} e^{-x^2/2} dx$$

$= 1$

Standaard Afwijking:

$$\sigma = \sqrt{1} = 1$$

Continue verdeling

Normale verdeling

bv. lengt van Vlaamse mannen

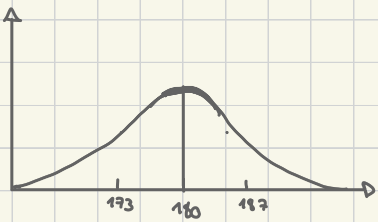
gemiddeld = 180

Stand. Afwyking = 7 cm

μ

gemiddeld

$X \sim N(180, 7)$ σ



E[x] opschuiving

$\frac{1}{7\sqrt{2\pi}} e^{-\left(\frac{x-180}{7}\right)^2/2}$

St Af. opschuiving

Algemeen

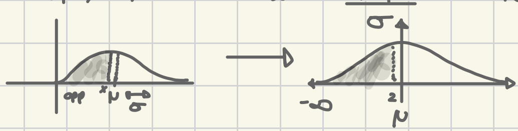
$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$X \sim N(\mu, \sigma)$

- opp = 1
- $E[x] = \mu$
- $Var(x) = \sigma^2$
- $\sigma_x(x) = \sigma$

Berekeningen

$x \sim N(\mu, \sigma) \longrightarrow z = \frac{x-\mu}{\sigma} \sim N \sim (0,1)$



vb.

$X \sim N(180, 7)$

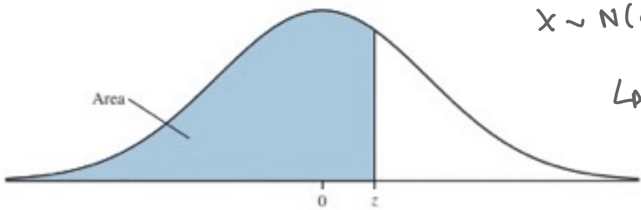
$P(187 \leq x \leq 197{,}5)$ omzetten nr z-scores $z = \frac{x-N}{\sigma}$

$= P\left(\frac{187-180}{7} \leq \frac{x-180}{7} \leq \frac{197{,}5-180}{7}\right)$

$= P(1 \leq Z \leq 2{,}5) = \Phi(2{,}5) - \Phi(1) = 0{,}9938 - 0{,}8413 = 0{,}1525$

ongeveer 15%

Cumulative Normal Distribution (continued)



$X \sim N(0,1)$

$\hookrightarrow z = 0{,}73$

$\mathbb{P}(z \leq 0{,}73) = 0{,}7367$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7 or more	.9999									

Continue verdeling

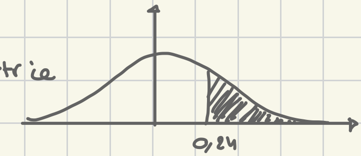
Normale verdeling

$Z > 0 \rightarrow$ lees uit het de tabel af met

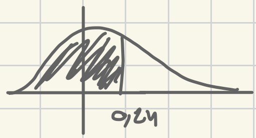
$\Phi(z) = F_Z(z)$ is $\qquad \Phi(0,73) = 0,7673$

$Z < 0 \rightarrow \quad Z = -0,24$

Symmetrie $\longrightarrow$



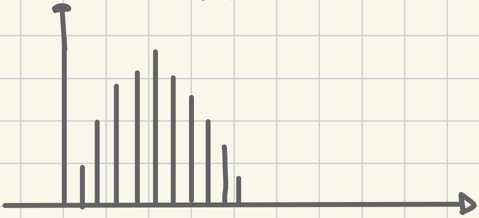
$= 1 -$



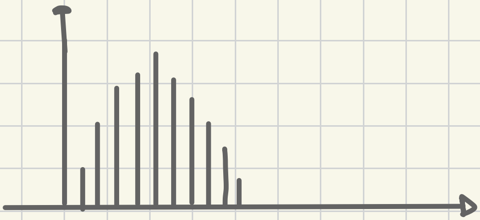
$$\boxed{\begin{aligned} &P(a < Z < b) \\ &= \Phi(b) - \Phi(a) \end{aligned}}$$

$= 1 - \Phi(0,24) = 1 - 5948 = 0,4052$

$X \sim B(n, p)$



$X \sim P(\lambda)$



40 keer keto herhalen + kans op succes 10%

$\mathbb{E}[x]$ $\quad\quad$ $np = 4$

$\mathbb{I}[x]$

PASUNA 114

Continue verdeling

Eigenschap

→ poisson verdeling is een goede benadering van de binomiale verdeling
→ Als p klein is

(VAAK : $n \geqslant 30$ $\quad n.p \leq 5$)

$\mathbb{P}(x = k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$

$\mathbb{P}(x = k) = \binom{n}{k} p^k q^{n-k}$

Als de Formules op elkaar lijken goede benadering

Eig Voor vaste k en voor $\lambda = np$ vast

$$\lim_{\substack{n \to \infty \\ p \to 0 \\ \lambda = np \text{ const}}} \binom{n}{k} p^k (1-p)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

Bewijs

$\lambda = n \cdot p$

$p = \dfrac{\lambda}{n}$

$k = \text{vast}$

$\lambda = \text{vast}$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$= \overset{\text{vast}}{\boxed{\frac{\lambda^k}{k!}}} \underbrace{\frac{n!}{(n-k)!\,n^k}}_{A} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{B} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{C}$$

$$A = \frac{n(n-1) \ldots (n-k+1) \cancel{(n-k)} \ldots \cancel{2} \cdot \cancel{1}}{n.n.n \ldots n. \cancel{(n-k)} \ldots \cancel{2} \cdot \cancel{1}}$$

$$= 1 \left(1 - \underset{n\infty}{\frac{1}{n}}\right) \left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{k-1}{n}\right) \xrightarrow{n \to \infty} 1$$

$$B = \left(1 - \frac{\lambda}{n}\right)^n = \left(1 + \frac{(-\lambda)}{n}\right)^n \xrightarrow{n \to \infty} e^{-\lambda}$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \Big| \quad \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

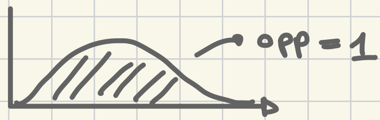
$$C = \left(1 - \frac{\lambda}{n}\right)^{-k} \xrightarrow{n \to \infty} 1$$

→ dit vervolledigt het bewijs

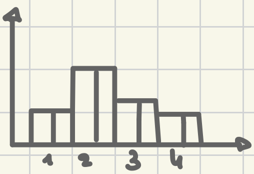
Discrete verdeling benaderen door een contenue

Discrete

Contenue



Omvorming

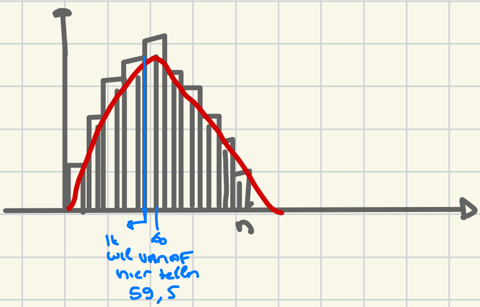


Staafjes met dikte 1

→ opp = 1

Binomiale verdeling

→ benaderen adhv $x \sim N(\mu, \sigma)$



$n.p = \mathbb{E}[x] = \mathbb{E}[x'] = \mu$

$n.p.q = VAR(x) = VAR(x') = \sigma^2$

$$\begin{cases} \sigma = \sqrt{n.pq} \\ \mu = n.p \end{cases}$$

Vb

$X \sim B(300, \frac{1}{4})$

$\mu = \frac{300}{4} = 75$

$\sigma = \sqrt{\frac{900}{4}} = \frac{30}{2} = 15$ ≠ 7.5

$\mathbb{P}(60 < x \leq 75) \approx \mathbb{P}(60 \leq x' \leq 75)$

0,5136 ≠ 0,4772

~~X~~ geen goede manier

Vervolg

$X \sim N(75, \frac{15}{2})$

$\hookrightarrow \quad Z = \frac{x - 75}{15/2} \sim N(0,1)$

$\mathbb{P}\left(\frac{-15}{2}/2 \leq \frac{x' - 75}{15/2} \leq \frac{0/15}{2}\right) = \mathbb{P}(-2 \leq Z \leq 0)$

$= \Phi(0) - \Phi(-2) = \Phi(0) - (1 - \Phi(2)) = 0{,}5 - (\underbrace{1 - 0{,}9772}_{0{,}0228})$

$= 0{,}4772$

Discrete verdeling benaderen door een contenue

Betere manier

↳ "Continuïteitscorrectie"

want we willen een gehele getal

$$\mathbb{P}(60 \leq x \leq 75) = \mathbb{P}(59{,}5 \leq x \leq 75{,}5)$$

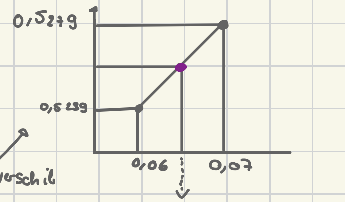
$$\approx \mathbb{P}(59{,}5 \leq x' \leq 75{,}5)$$

$$= \mathbb{P}\left(\frac{-15{,}5}{15/2} \leq Z \leq \frac{0{,}5}{15/2}\right)$$

$$= \mathbb{P}(-2{,}066 \leq Z \leq 0{,}066)$$

$$= \Phi(0{,}066) - (1 - \Phi(2{,}066))$$

$$= \Phi(0{,}06) = 0{,}5239$$
$$\Phi(0{,}07) = 0{,}5279$$



verschil

"Ik wil 6 tiende verder"

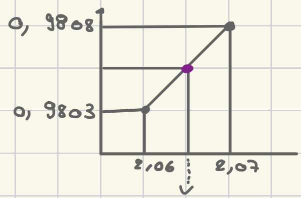
$$\Phi(0{,}066) = 0{,}5239 + \underbrace{\frac{6}{10} \cdot 0{,}0040}_{0{,}0024}$$

$$= 0{,}53263$$

$$\frac{\overset{AB}{0{,}0066} - \overset{x_1}{0{,}06}}{\underset{x_2}{0{,}07} - \underset{x_1}{0{,}06}} = \frac{6}{10}$$

$$\Phi(2{,}066)$$

$$= \Phi(2{,}06) = 0{,}9803$$
$$= \Phi(2{,}07) = 0{,}9808$$



$$\Phi(2{,}066) = 0{,}9803 + \frac{3}{5} \cdot 0{,}0005$$

$$= 0{,}9806$$

2,066
||
6/10
||
3/5

$$\mathbb{P}(59{,}5 \quad x' \quad 75{,}5)$$
$$=$$
$$\Phi(0{,}66) - 1 + \Phi(2{,}066)$$
$$=$$
$$0{,}5263 - 1 + 0{,}9806$$
$$=$$
$$0{,}5069$$

Discrete verdeling benaderen door een contenue

Algemeen

$X \sim B(n, p)$

$\mathbb{P}(a \leq X \leq b)$

$X' \sim N(np, \sqrt{npq})$

$\mathbb{P}(a - \frac{1}{2} \leq X \leq b + \frac{1}{2})$

uitzondering

Als $\mathbb{P}(a < x < b) \approx \mathbb{P}(a + \frac{1}{2} \leq X' \leq b - \frac{1}{2})$

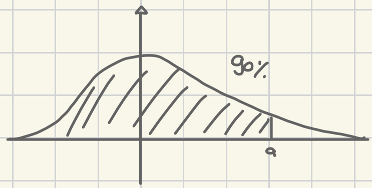
2) z-code

3) $\Phi(b) - \Phi(a)$

Discrete verdeling benaderen door een contenue

Tabel onder som gebruiken

$Z \sim N(0,1)$



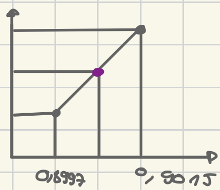
$\mathbb{P}(Z \leq a) = 90\%$

$\longrightarrow$ zoek 0,9000 in tabel

$\Phi(1{,}28) = 0{,}8997$

$\Phi(1{,}29) = 0{,}9015$

$\frac{0{,}0003}{0{,}0018}$



zoek

$$\frac{0{,}9000 - \overset{x_1}{0{,}8997}}{\underset{x_2}{0{,}9015} - \underset{x_1}{0{,}8997}} = \frac{0{,}0003}{0{,}0018} = \frac{3}{18} \approx \frac{2}{10} \Rightarrow a \simeq 1{,}282$$

Centrale limietstelling

$X_1, X_2, X_3, \ldots$ onafhankelijke toevalsveranderlijken

met $\mathbb{E}[X_i] = \mu$ en $\sigma(X_i) = \sigma$

$$\lim_{n \to +\infty} \mathbb{P}\left(\frac{\sum_{i=0}^{n} x_i - n\mu}{\sigma\sqrt{n}} \leq a \right) = \Phi(a)$$

$\longrightarrow$ in ons geval

$X_i \sim B(1,p)$ Bernoulli

$$\sum_{i=1}^{n} x_i \sim B(n,p)$$