Nieuwe verdeling

$X \sim (1, 2, .. 6)$ $\qquad Y \sim (1, ..., 6)$

new $\to$ $Z = X + Y$ Som v/d ogen

convolutie product

onafhankelijk

$\mathbb{P}(X=k) \to \mathbb{P}(X+Y=k) = \sum_i \mathbb{P}(X=i \text{ en } Y=k-i)$

kng dobbelsteen gelijke kans. Binomiaal

$= \sum_{i=0}^{k} \mathbb{P}(X=i)\ \mathbb{P}(Y=k-i)$

$\mathbb{P}(X=k) = \binom{n}{k} p^k q^{n-k}$

$\binom{n}{k} p^k q^{n-k}$

$= \sum_{i=0}^{k} \binom{n}{i} p^i q^{n-i} \cdot \binom{m}{k-i} p^{k-i} q^{m-(k-i)}$

$= \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} p^k q^{n-i+m-(k-i)}$

$= \sum_{i=0}^{k} \binom{n+m}{k} p^k q^{m+n-k}$

Verwachtingswaarde

$\mathbb{E}[X] = \sum_k k\,\mathbb{P}(X=k)$

lemma 1

$i\binom{n}{i} = n\binom{n-1}{i-1}$

$\text{JJ} = i\binom{n}{i} = i\,\frac{n!}{i!(n-k)!}$

$= \frac{n!}{(i-1)!(n-k)!}$

$= \frac{n \cdot (n-1)!}{(i-1)!(n-1)!}$

$= n\binom{n-1}{i-1}$

lemma 2

$i^2\binom{n}{i} = n(n-1)\binom{n-2}{i-2} + n\binom{n-1}{i-1}$

lemma 1

$\text{JJ}^2 = i \cdot i\binom{n}{i}$

$= i\,n\binom{n-1}{i-1}$

$= n \cdot i\binom{n-1}{i-1}$ $\qquad$ CA.B

$(j+1-1)$ $\qquad$ $CB \cdot (A-1) + BAC$

$= n(i-1)\binom{n-1}{i-1} + n\binom{n-1}{i-1}$

lemma 1

$= n(n-1)\binom{n-2}{n-2} + n\binom{n-1}{i-1}$

Uniform E[x]

$$\mathbb{E}[x] = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$= \sum_{i=1}^{n} k\, \mathbb{P}(X = k) = \sum_{i=1}^{n} k \cdot \frac{1}{n} \qquad \frac{1}{6}(1+2+3+4+5+6)$$

Som 2 verW. toevalsveranderlijken

$$\mathbb{E}[x+y] \overset{?}{=} \mathbb{E}[x] + \mathbb{E}[y] \qquad \mathbb{E}[x] = \sum_k k\, \mathbb{P}(x = k)$$

convolutie product

$$\mathbb{E}[x+y] = \sum_k k\, \mathbb{P}(x+y = k)$$

$$= \sum_k k \sum_l \mathbb{P}(x = l \wedge y = k-l)$$

$$= \sum_k \boxed{k} \sum_l \mathbb{P}(x = l)\, \mathbb{P}(y = k-l)$$

$$= \sum_{x,z} z\mathbb{P}(X = x \text{ en } Y = z-x)$$

$$\overset{y=z-x}{=} \sum_{x,y} (x+y)\mathbb{P}((X=x)\cap(Y=y))$$

$y = k - l$

$$= \sum_k \sum_l (x+y)\, \mathbb{P}(X = x \wedge y = y)$$

$$= \sum_k \sum_l x\, \mathbb{P}(x = x \wedge y = y) + \sum_k \sum_l y\, \mathbb{P}(x = x \wedge y = y)$$

$$= \sum_k x \sum_l \mathbb{P}(x = x \wedge y = y) + \sum_k y \sum \mathbb{P}(x = x \wedge y = y)$$

$$= \sum_k x \sum \mathbb{P}(x = x) + \sum_k y \sum_l \mathbb{P}(y = y)$$

Product E[x]

$$\mathbb{E}[aX] \overset{?}{=} a\,\mathbb{E}[x]$$

$$= \sum ax\,\mathbb{P}(x=x)$$

$$= a\sum_{x} x\,\mathbb{P}(X=x) = a\,\mathbb{E}[x]$$

E[x] Binomiale verdeling

n

$x \sim B(n,p)$ $\qquad \mathbb{P}(x=k) = \binom{n}{k} p^k q^{n-k}$

lemma 1

$$j\binom{n}{j} = n\binom{n-1}{j-1}$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

lemma 2

$$j^2\binom{n}{j} = n(n-1)\binom{n-2}{j-2} + n\binom{n-1}{j-1}$$

$$\mathbb{E}[x] = \sum_{k=0}^{n} k\,\mathbb{P}(x=k)$$

$$= \sum_{k=0}^{n} k\binom{n}{k} p^k q^{n-k}$$

$k = l+1$

$$= \sum_{k=0}^{n} \boxed{n}\binom{n-1}{k-1} p^k q^{n-k}$$

$$= n\sum_{k=1}^{n-1} \binom{n-1}{k-1} p^{l+\boxed{1}} q^{n-(l+1)}$$

$$= np\sum_{k=1}^{n-1} \binom{n-1}{k-1} p^{l} q^{n-l-1}$$

$k = l+1$

$$= np\sum_{k=1}^{n-1} \binom{n-1}{k-1} p^{k-1} q^{n-1-k}$$

$$= n\cdot p\,(p+q)^{n-1} = n\cdot p$$

$$=$$

Binomium newton

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Stappen

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

$\sum_{k=1}^{n}$

$k = l+1$ $\quad \hat{p}$

Binomium van newton

Var(x) binomiale verd.

$$VAR(x) = n.p.q$$

$$VAR(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$= \sum_{k=0}^{n} k^2 \, \mathbb{P}(x=k) - (np)^2$$

$$= \sum_{k=1}^{n=0} k^2 \binom{n}{k} q^k p^{n-k} - (np)^2$$

$$= \sum_{k=1}^{n=0} k \, n \binom{n-1}{k-1} q^k p^{n-k} - (np)^2$$

$$= n \sum_{k=1}^{n=0} k \binom{n-1}{k-1} q^k p^{n-k} - (n.p)^2$$

$$= n \sum_{k=1}^{n} (k-1+1) \binom{n-1}{k-1} p^k q^{n-k} - (n.p)^2$$

zoals bij $\mathbb{E}[x]$

$$= n \sum_{k=1}^{n} (k-1) \binom{n-1}{k-1} p^k q^{n-k} + n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{n-k} - (n.p)^2$$

$k = l+1$

$$= n \sum_{k=1}^{n} (n-1) \binom{n-2}{k-2} p^k q^{n-k} + n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{n-k} - (n.p)^2 \qquad -(n.p)^2$$

$$= n(n-1) \sum_{k=1}^{n} \binom{n-2}{k-2} p^k q^{n-k} + n \sum_{k=1}^{n} \binom{n-1}{k-1} p^{l+\boxed{1}} q^{n-1-l} - (n.p)^2$$

$k = l+2$

$$= n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} p^{l+\boxed{2}} q^{n-2-k} + np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-1-k} \qquad k-1 = l \qquad -(n.p)^2$$

$$= n(n-1)p^2 \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} p^{n-2-k} + np \, (p+q)^{n-1} - (n.p)^2$$

$$= n(n-1)p^2 \underbrace{(p+q)}_{1}{}^{n-2} + np \underbrace{(p+q)}_{1}{}^{n-1} - (n.p)^2$$

$$= n^2p^2 - np^2 + np - np^2$$

$$= n.p.q$$

Stap 1

k^2

$k \, n(\cdots)$

$((k-1)+1)$

$k = l+2$ $\qquad$ $\mathbb{E}[x]$ $\quad k = l+1$

E[x] Neg. Binomiaal

$\mathbb{E}[x] = n/p$

$$\mathbb{E}[x] = \sum_{k=0}^{\infty} k \, \mathbb{P}(x=k)$$

$$\binom{k-1}{n-1} p^n q^{k-n}$$

$$= \sum_{k=n}^{\infty} k \binom{k-1}{n-1} p^n q^{k-n}$$

$$= \sum_{k=n}^{\infty} n \binom{k}{n} p^n q^{k-n}$$

$$= n \cdot p^n \sum_{k=n}^{\infty} \binom{k}{n} q^{k-n}$$

$$j\binom{n}{j} = n\binom{n-1}{j-1}$$

$$n\binom{k}{n} = k\binom{k-1}{n-1}$$

$$= n \cdot p^n \sum_{k=n}^{\infty} (-1)^{k-n} \binom{-(n+1)}{k-n} q^{k-n}$$

$\ell = k-n$

$$= n \cdot p^n \sum_{k=n}^{\infty} (-1)^{\ell} \binom{-(n+1)}{\ell} (q)^{\ell} \cdot 1^{-(n+1)-\ell}$$

$$\binom{k}{n} = (-1)^{k-n} \binom{-(n+1)}{k-n}$$

$$= n \cdot p^n (1-q)^{-(n+1)} = n p^n p^{-(n+1)} = \frac{n}{p}$$

$$\binom{k}{n} = (-1)^{k-n} \binom{-(n+1)}{k-n}$$

Var(x) Neg. Binomiaal

$$\mathrm{Var}(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$= \sum_{k=n}^{\infty} k^2 \binom{k-1}{n-1} p^n q^{n-k} \quad + \left(\frac{n}{p}\right)^2$$

$$= \sum_{k=n}^{\infty} k \cdot n \binom{n}{k} p^n q^{n-k}$$

$$k\binom{k-1}{n-1} = n\binom{k}{n}$$

$$= n p^n \sum_{k=n}^{\infty} (k+1-1) \binom{k}{n} q^{k-n} - (n/p)^2 \qquad (k+1)\binom{k}{n} = (n+1)\binom{k+1}{n+1}$$

$$= n p^n \sum_{k=n}^{\infty} (k+1) \binom{k}{n} q^{k-n} - n p^n \sum_{k=n}^{\infty} \binom{k}{n} q^{k-n} - (n/p)^2$$

wat je doet bij $\mathbb{E}[x]$

$$= n p^n \sum_{k=n}^{\infty} (n+1) \binom{k+1}{n+1} q^{k-n} - n p^n \sum_{k=n}^{\infty} \binom{k}{n} q^{k-n} - (n/p)^2$$

$k=n+j$

$$= n p^n \sum_{j=0}^{\infty} (n+1) \binom{(n+j)+1}{n+1} q^{k-n} - n p^n \sum_{k=n}^{\infty} (-1)^{k-n} \binom{-(n+1)}{k-n} q^{k-n} - (n/p)^2$$

$$= n p^2 \sum (n+1) \binom{-(n+2)}{j} -q^j - n p^n \sum_{k=n}^{\infty} (-1)^\ell \binom{-(n+1)}{\ell} q^\ell \cdot 1^{-(n+1)-\ell} - (n/p)^2$$

$$= n(n+1)p^2 \sum_{j=0}^{\infty} \binom{-(n+2)}{j} (-q)^j \cdot 1^{-(n+2)-j}$$

$$= n(n+1)p^2 (\underbrace{1-q}_{p})^{-(n+2)} - n p^n (\underbrace{1-q}_{p})^{-(n+1)} - (n/p)^2$$

$$= \frac{nq}{p^2}$$

$$\binom{k+1}{n+1} = (-1)^{k+1-n+1} \binom{+(n+2)}{k-n}$$

$$k^2 \binom{k-1}{n-1}$$

$$k \, n \binom{k}{n}$$

$$(k+1) \qquad -1$$

$$-(n+2)$$

$\mathbb{E}[x]$

$-(n+1)$

E[x] poisson verdeling

∞

$$\mathbb{P}(x=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$\mathbb{E}[x] = \lambda$$

$$\mathbb{E}[x] = \sum_{k=n}^{\infty} k\,(x=k)$$

$$= \sum_{k=n}^{\infty} \cancel{k}\, e^{-\lambda}\frac{\lambda^k}{\cancel{k!}}\ k-1!$$

$$= \sum_{k=n}^{\infty} e^{-\lambda}\frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=n}^{\infty} e^{-\lambda}\,\lambda\,\frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda}\lambda\sum_{k=n}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}$$

$$\ell = k-1$$

$$= e^{-\lambda}\lambda\sum_{\ell=0}^{\infty}\frac{\lambda^{\ell}}{\ell!}\ \Big]\ e^{x}$$

$$= \cancel{e^{-\lambda}}\ \lambda\ \cancel{e^{\lambda}} = \lambda$$

Var(x) poisson verdeling

$$\mathrm{Var}(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$= \sum_k k^2\,\mathbb{P}(x=k) - \lambda^2$$

$$= \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!} - \lambda^2$$

$$= \sum_{k=1}^{\infty} k\, e^{-\lambda}\frac{\lambda^k}{(k-1)!} - \lambda^2$$

$$= e^{-\lambda}\sum_{k=1}^{\infty}(k-1+1)\frac{\lambda^k}{(k-1)!} - \lambda^2$$

$$= e^{-\lambda}\sum_{k=1}^{\infty}(k-1)\frac{\lambda^k}{(k-2)!} + e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}\ \Big]\ e^{\lambda} - \lambda^2$$

$$= e^{-\lambda}\sum_{k=1}^{\infty}\lambda^2\frac{\lambda^{k-2}}{(k-2)!} + e^{-\lambda}\cdot\lambda\, e^{\lambda} - \lambda^2$$

$$= \cancel{e^{-\lambda}}\ \cancel{\lambda^2}\ \cancel{e^{\lambda}} + \cancel{e^{-\lambda}}\cdot\lambda\ \cancel{e^{\lambda}}\ \cancel{-\lambda^2} = \lambda$$

E[x] uniforme verdeling

$$\sum k \mathbb{P}(x=k)$$

$$\frac{1}{n}\sum_{j=1}^{n} k$$

Var(x) uniforme verdeling

$$Var(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$= \sum_{k=1}^{n} k^2 \frac{1}{n} - \left(\frac{1}{n}\sum_{k=1}^{n} k\right)^2$$

$$= \frac{1}{n}\sum_{k=1}^{n} k^2 + \left(\frac{1}{n}\sum k\right)^2$$

$$= \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)(2n+1)}{6} + \frac{1}{\cancel{n^2}} \cdot \frac{\cancel{n^2}(n+1)^2}{4}$$

$$= \frac{(n+1)(2n+1)}{6} + \frac{(n+1)^2}{4}$$

$$= (n+1)\left(\frac{(2n+1)}{6} + \frac{(n+1)}{4}\right)$$

$$= (n+1)\left(\frac{n-1}{12}\right) = \frac{n^2-1}{12}$$

$$\cancel{\frac{1}{n}} \frac{\cancel{n}(n+1)(2n+1)}{6} + \cancel{\frac{1}{n^2}} \frac{n^2(n+1)^2}{4} = \frac{n^2-n}{12}$$

Stelling Chebychef
grote machten
benadering
Rest van Var en $\mathbb{E}[x]$

Stelling van chebychev

$\mathbb{P}(X - \mathbb{E}[X] > \alpha \sigma_x) < \frac{1}{\alpha^2}$ nul % buiten interval

Bewijs: $y = [x - \mathbb{E}[x]]^2$ $\mathbb{E}[y] = \mathbb{E}[x - \mathbb{E}[x]^2] = VAR(x) = \sigma_x^2$

$A = "|x - \mathbb{E}[x] > \alpha \sigma_x|"$ $\mathbb{P}(A) = 0$ ok ✓

$\Rightarrow \mathbb{P}(A) > 0$

TB: $\mathbb{P}(A) < \frac{1}{\alpha^2}$ $\Rightarrow \sigma_x > 0$

$$\mathbb{E}[y] = \sum_{a \in \Omega} y(a)\,\mathbb{P}(a) = \sum_{a \in A} y(a)\,\mathbb{P}(a) + \underbrace{\sum_{a \notin A} \underbrace{y(a)}_{\geq 0 \text{ kwadraat}}\,\underbrace{\mathbb{P}(a)}_{\geq 0 \text{ (kans)}}}_{\geq 0}$$

a = atomaire gebeurtenissen

$$y(a) > \alpha^2 \sigma_x^2 \overset{a \in A}{\Longrightarrow} y = [x - \mathbb{E}[x]]^2 > \alpha^2 \sigma_x^2$$

$$\Rightarrow \sigma_x^2 > \alpha^2 \sigma_x^2 \underbrace{\sum_{a \in A} \mathbb{P}(a)}_{\mathbb{P}(A)}$$

$/ \sigma_x^2$

$1 > \alpha^2\,\mathbb{P}(A)$

$\Rightarrow \mathbb{P}(A) < \frac{1}{\alpha^2}$

De wet van de grote getallen

$\forall \varepsilon, \delta > 0$, je kan n zo groot mogelijk kiezen zodat

$$P(|p-\bar{p}| < \varepsilon) > 1-\delta$$

$$P(|p-\bar{p}| < \varepsilon) > 1-\delta$$

chebychef

$$(|x - E[x]| \le \alpha\sigma) > 1 - \frac{1}{\alpha^2}$$

Bewijs

$$x_1, x_2, \dots \sim B(1, p)$$

$$y = \bar{p} = \sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n} x_i$$

neem ε, δ zeer klein

$$|y - E[x]| = |p - \bar{p}|$$

kies α zodat

$$\delta \le \frac{1}{\alpha^2}$$

$$\alpha^2 \ge \frac{1}{\delta}$$

$$\alpha \ge \sqrt{\frac{1}{\delta}}$$

kies n zodat $n \ge \frac{\alpha^2}{4\varepsilon^2}$

$$\varepsilon^2 \ge \frac{\alpha^2}{4n}$$

$$\varepsilon^2 \ge \alpha^2 \cdot \sigma_y^2$$

$$\Rightarrow \alpha\sigma_y = \varepsilon$$

bernoulli:

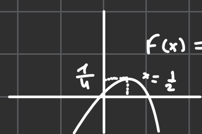
$$E[y] = E\left[\frac{1}{n}\sum_{i=0}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[\overset{p}{\widetilde{x}}] = \frac{1}{n}(n \cdot p) = p$$

$$\sqrt{Var(y)} = \sqrt{Var\left(\boxed{\frac{1}{n}}\sum_{i=1}^{n} x_i\right)} = \sqrt{\frac{1}{n^2} var\left(\sum_{i=1}^{n} x_i\right)}$$

$$= \sqrt{\frac{n \cdot p \cdot q}{n^2}} = \sqrt{\frac{pq}{n}}$$

$$= \sqrt{\frac{p(1-p)}{n}}$$

$f(x) = x(1-x)$



$$\le \sqrt{\frac{1}{4n}}$$

$$= \frac{1}{2\sqrt{n}}$$

stafwyking

$$\boxed{\sigma_x \le \frac{1}{2\sqrt{n}}}$$

VAR

$$\boxed{\sigma_x^2 \le \frac{1}{4n}}$$

$$P(|p-\bar{p}| \le \varepsilon) \ge P(|p-\bar{p}| \le \alpha\sigma_y) > 1 - \frac{1}{\alpha^2} \ge 1-\delta$$

dus

$$\Rightarrow P(|p-\bar{p}| \le \varepsilon) \ge 1-\delta$$

binomiaal => Poisson

TB:

$$\lim_{\substack{n \to \infty \\ p \to 0 \\ \lambda = n.p \text{ cst}}} \binom{n}{k} p^k (1-p)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

k vast

$\lambda = n.p$ vast

$\frac{\lambda}{n} = p$

Bewijs

$$= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$= \underset{\text{vast}}{\boxed{\frac{\lambda^k}{k!}}} \underbrace{\frac{n!}{(n-k)!\,n^k}}_{A} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{B} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{C}$$

$$A = \frac{n(n-1)\ldots(n-k+1)\cancel{(n-k)}\ldots\cancel{2.1}}{n.n.n\ldots\cancel{(n-k)}\ldots\cancel{2.1}}$$

$$= 1\left(1 - \underbrace{\tfrac{1}{n}}_{=0}\right)\left(1 - \underbrace{\tfrac{2}{n}}_{=0}\right) \xrightarrow{n \to \infty} 1$$

$$\lim\left(1 + \frac{1}{n}\right) = e$$

$$\lim_{n \to \infty}\left(1 + \frac{x}{n}\right)^n = e^x$$

$$B = \left(1 - \frac{\lambda}{n}\right)^n = \left(1 + \frac{(-\lambda)}{n}\right)^n \xrightarrow{n \to \infty} e^{-\lambda}$$

$$C = \left(1 - \frac{\lambda}{n}\right)^{-k} \xrightarrow{n \to \infty} 1$$

hieruit volgt

$$\lim_{\substack{n \to \infty \\ p \to 0 \\ \lambda = n.p \text{ cst}}} \binom{n}{k} p^k (1-p)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

De wet van de grote getallen

$\forall \delta, \varepsilon > 0$

$\mathbb{P}(|p - \bar{p}| < \varepsilon) > 1 - \delta$

Bewijs

$x_1, x_2, \dots \; B(1, p)$

bernoulli:

$$\mathbb{E}[y] = \mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n} x_i\right] = \frac{1}{n}\sum_{i=0}^{n} \overset{p}{\mathbb{E}[x]} = \frac{np}{n} = p$$

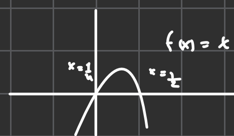
$$y = \color{red}{\bar{p}} = \sum_{k=0}^{n} x_k = \frac{1}{n}\sum_{k=0}^{n} x_i$$

$$\sigma = \sqrt{var(y)} = \sqrt{var\left(\frac{1}{n}\sum_{i=0}^{n} x_i\right)} = \sqrt{\frac{1}{n^2}\, var\left(\sum_{i=0}^{n} x_i\right)}$$

$$= \sqrt{\frac{np \cdot q}{n^2}} = \sqrt{\frac{p(1-p)}{n}}$$

$$\leq \sqrt{\frac{1}{4n}}$$

$$= \frac{1}{2\sqrt{n}}$$



$f(x) = x(1-x)$

kies nu ε, δ zodat

$|y - \mathbb{E}[x]| = |p - \bar{p}|$

kies δ zodat

$\delta \searrow \; \leq 1 - \frac{1}{\alpha^2}$

$\alpha^2 \geq \frac{1}{\delta}$

$\alpha \geq \sqrt{\frac{1}{\delta}}$

$$\boxed{\sigma_y = \frac{1}{2\sqrt{n}}}$$

$$\boxed{\sigma_y^2 = \frac{1}{4n}}$$

kies n zodat $n \geq \frac{\alpha^2}{4\varepsilon^2}$

$\varepsilon^2 \geq \frac{\alpha^2}{4n}$

$\varepsilon^2 \geq \alpha^2 \cdot \sigma_y^2$

$\mathbb{P}(|p - \bar{p}| < \varepsilon) \geq \mathbb{P}(|p - \bar{p}| \leq \alpha\sigma_y)$

$> \frac{1}{\alpha^2} \geq 1 - \delta$

Wat is var(x)

$$\text{VAR}(x) = \sum_x (x - \mathbb{E}[x])^2 \cdot \mathbb{P}(X = x)$$

$$= \underbrace{\sum_x x^2 \cdot \mathbb{P}(X=x)}_{\mathbb{E}[x^2]} - 2\,\underbrace{\mathbb{E}[x] \sum_x x\, P(x=x)}_{\mathbb{E}[x]^2} + \mathbb{E}[x]^2 \underbrace{\sum_x \mathbb{P}(x=x)}_{1}$$

$$= \mathbb{E}[x^2] - 2\mathbb{E}[x]^2 + \mathbb{E}[x]^2$$

$$= \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

Var(x+y)

$$\text{VAr}(x+y) \overset{?}{=} \text{VAr}(x) + \text{VAr}(y)$$

$$\text{VAr}(x+y) = \mathbb{E}[(x+y)^2] - \mathbb{E}[x+y]^2$$

$$= \mathbb{E}[x^2 + 2xy + y^2] - (\mathbb{E}[x] + \mathbb{E}[y])^2$$

$$= \underbrace{\mathbb{E}[x^2] - \mathbb{E}[x^2]}_{\text{VAr}\,x} + \underbrace{\mathbb{E}[y^2] - \mathbb{E}[y]^2}_{\text{VAr}\,y} + \mathbb{E}[\cancel{2xy}] - \cancel{2\mathbb{E}[x]}\,\mathbb{E}[y]$$

$$= \text{VAr}(x) + \text{VAr}(y)$$

E[xy]

$$\mathbb{E}[xy] = \sum_y \sum_x xy \ \mathbb{P}(X = x \text{ en } Y = y)$$

$$= \sum_y y \, \mathbb{P}(y = y) + \sum_x x \, \mathbb{P}(x = x)$$

$$= \mathbb{E}[y] + \mathbb{E}[x]$$

Binomium van Newton

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Basisgeval

$LL = 1 = RL$

IH $\quad k = n$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$k = n+1$

$$LL = (x+y)^{n+1} = (x+y)^n (x+y) \overset{IH}{=} \left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right)(x+y)$$

$$= \left(\sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k}\right) + \left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}\right)$$

$l = k+1$
$k = l-1$

$$= \sum_{l=1}^{n} \binom{n}{l-1} x^l y^{n+1-l} + \sum_{k=0} \binom{n}{k} x^k y^{n+1-k}$$

$n = k-1$

$$= \binom{n}{n} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k}$$

$$\binom{n}{0} x^0 y^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k}$$

$$= \binom{n+1}{n+1} x^{n+1} y^0 + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] x^k y^{n+1-k}$$

$$+ \binom{n+1}{0} x^0 y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{(n+1)-k}$$

Partitie

(1) $\forall a \in A : [a] \neq \emptyset$

(2) $\bigcup_{a \in A} [a] = A$

(3)

ⓐ $\forall a,b \in A : [a] = [b] \iff a \sim b$

$\Rightarrow b \in [b] = [a]$

$\Rightarrow a \sim b$

$\Leftarrow$ $[a] \subset [b]$

neem een willekeurig $x \in [a] \Rightarrow a \sim x$

geg $a \sim b \overset{(S)}{\Rightarrow} b \sim a$ $\Big\}$ (T) $b \sim x \Rightarrow x \in [b]$

$[b] \subset [a]$

neem x willekeurig $x \in [b]$

geg $b \sim a \Rightarrow a \sim b$ $\Big\}$ (T) $a \sim x \Rightarrow x \in [a]$

ⓑ $\forall a,b \in A : [a] \neq [b] \iff [a] \cap [b] = \emptyset$

Stel dat $[a] \neq [b]$ en $[a] \cap [b] \neq \emptyset$

$\Rightarrow \exists_x \in [a] \cap [b]$

$\Rightarrow x \in [a] \wedge x \in [b]$

$\Rightarrow a \sim x \wedge b \sim x$

$\Rightarrow (S)\ b \sim x$

$\Rightarrow a \sim b$

$\Rightarrow [a] = [b]$ contradictie

$\lightning$

Def inj, bij, surj

Injectief

$\forall_{a,a'} \in A : f(a) = f(a') \Rightarrow a = a'$

Surjectief

$\forall b \in B, \exists a \in A : f(a) = b$