MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 2

All exercises are worth 5 points. Due on Thursday, November 2, at 12:00 noon via URM.

- **Exercise 1** introduces an equivalent formalism for time evolution in quantum mechanics: the Heisenberg time evolution of operators. It allows to study time evolution of observables in a state-independent way, which can be helpful for interpretation, as in this example of the two-level Hamiltonian from the last exercise sheet.
- **Exercise 2** establishes a general version of the uncertainty principle in quantum mechanics for non-commuting observables.
- **Exercise 3** applies the general uncertainty principle from exercise 3 to the observables position and momentum on $\mathbb{R}^d$, in which case it becomes the famous Heisenberg uncertainty principle. As the last part shows, this has a special connection to Gaussian quantum states.
- **Exercise 4** studies solutions to the free Schrödinger equation when the initial data is a Gaussian bump (a prototyptical Schwartz function!). The result provides a useful explicit example for the behavior of the free Schrödinger evolution.

Exercise 1. Consider the two-level system of the previous exercise sheet. Let $\sigma_x, \sigma_y, \sigma_z$ the *Pauli matrices:*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Let $\hat{S}$ be the *spin operator:*

$$\hat{S} := (\hat{S}_x, \hat{S}_y, \hat{S}_z) , \qquad \hat{S}_i = \frac{1}{2}\sigma_i , \qquad i = x, y, z .$$

Given an observable O, its Heisenberg evolution is defined as:

$$O(t) := e^{iHt} O e^{-iHt} .$$

We can understand the time evolution of expectation values either by the Schrödinger evolution on states or by the Heisenberg evolution on observables:

$$\langle \psi, O(t)\psi \rangle = \langle \psi_t, O\psi_t \rangle .$$

(a) Compute the Heisenberg evolution of the spin operator:

$$\hat{S}(t) = e^{iHt}\hat{S}e^{-iHt} , \qquad H = -B\sigma_x , \qquad B \in \mathbb{R} .$$

(b) Suppose now that H has the following general form:

$$H = -\vec{B} \cdot \hat{S} , \qquad B \in \mathbb{R}^3 .$$

Let $\hat{S}(t)$ be the Heisenberg evolution of the spin operator with the Hamiltonian H. Prove that $\hat{S}(t)$ satisfies the following evolution equation:

$$\frac{d}{dt}\hat{S}(t) = \hat{S}(t) \times \vec{B} ,$$

with $\times$ the usual vector product. We say that the spin operator performs a "precession" at the frequency $|\vec{B}|$.

Hint. The Pauli matrices satisfy the following commutation relations (check them):

$$[\sigma_a, \sigma_b] = 2i \sum_{c \in \{x,y,z\}} \epsilon_{abc}\sigma_c.$$

Here $\epsilon_{xyz} = 1$ *and* $\epsilon_{abc} = sign(\pi\{a, b, c\})\epsilon_{xyz}$ *where* $sign(\pi\{a, b, c\})$ *is the sign of the permutation needed in order to turn the ordered triple* a, b, c *into* x, y, z. *The symbol* ϵ_{abc} *is called the Levi-Civita antisymmetric tensor.*

Exercise 2. This is an application of inequality (1) with $A = x$ and $B = -i\hbar\nabla$.

(a) Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\|\psi\|_2 = 1$. Prove that the conditions in exercise 3 are verified and so inequality (1) applies.
(For $\langle\cdot, \cdot\rangle$*, use the* L^2*-inner product.)*

(b) Calculate $[A, B]$.
(With this result, (1) *reduces to the Heisenberg uncertainty principle.)*

(c) Let $\alpha > 0$ and set $\psi_0(x) = e^{-\alpha x^2/2}$. Using exercise 1, compute the left side of (1) for $\psi_t(x)$. What do you find?
(Hint: Integration by parts is useful.)

Exercise 3. Let $\mathcal{H}$ be a Hilbert space and let $A, B : \mathcal{H} \to \mathcal{H}$ be quantum observables (self-adjoint operators). Let $\psi \in \mathcal{H}$ be a quantum state so that the following expressions are all well-defined: $A\psi$, $B\psi$, $A^2\psi$, $B^2\psi$, $AB\psi$, and $BA\psi$.

Recall the definition of the expectation value of a measurement of quantum observable C in the state ψ,

$$\langle C\rangle_\psi = \langle\psi, C\psi\rangle.$$

Similarly, the variance of the quantum measurement is

$$(\mathrm{Var}C)_\psi = \langle\psi, (C - \langle C\rangle_\psi)^2\psi\rangle$$

Prove that

$$(\mathrm{Var}A)_\psi(\mathrm{Var}B)_\psi \geq \frac{1}{4}|\langle[A, B]\rangle_\psi|^2 \tag{1}$$

where $[A, B] = AB - BA$ is the so-called commutator of the two operators.
(Hint: It is helpful to reduce to the case $\langle A\rangle_\psi = \langle B\rangle_\psi = 0$ *first.)*

Exercise 4. Consider the free Schrödinger equation on $\mathbb{R}^d$,

$$i\partial_t \psi_t = -\frac{1}{2}\Delta\psi_t \ ,$$

with initial data given by

$$\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{d/4} e^{-\frac{\alpha x^2}{2}} .$$

Find an explicit formula for the solution $\psi_t(x)$ that does not involve any integrals.