INTRODUCTION TO PDE **Assignment 2**

Due date: Thursday November 2nd WS 2023-2024

Exercise 1. **(Schwartz reflexion principle)** Let Ω^+ be a subdomain of the half-space $x^n > 0$ having as part of its boundary an open section T of the hyperplane $x^n = 0$. Suppose that u is harmonic in Ω^+, continuous in $\Omega^+ \cup T$, and that $u = 0$ on T. Show that the function U defined by

$$U(x^1, \dots, x^n) = \begin{cases} u(x^1, \dots, x^n) & x^n \geq 0 \\ -u(x^1, \dots, -x_n) & x^n < 0 \end{cases}$$

is harmonic in $\Omega^+ \cup T \cup \Omega^-$, where

$$\Omega^- = \{x \in \mathbb{R}^n \mid (x^1, \dots, -x^n) \in \Omega^+\}$$

is the reflexion of Ω^+ in $x^n = 0$.

Exercise 2. **(Exterior sphere condition)** Let $\Omega \subset \mathbb{R}^n$ open, bounded and with $\partial\Omega$ of class C^2. Show that there exists a uniform radius $R > 0$ such that for every point $x \in \partial\Omega$, there exists a ball $B_R(y)$ outside Ω with the property that $x \in \partial B_R(y)$.

Exercise 3. Consider the unbounded domain $\Omega \doteq \mathbb{R}^n \setminus B_1(0) = \{x \in \mathbb{R}^n \mid |x| > 1\}$ and a harmonic function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ with the property that $\lim_{|x| \to \infty} u(x) = 0$. Show that

$$\max_{\Omega} |u| = \max_{\partial\Omega} |u|\,.$$

Exercise 4. Determine the Green's function of a quadrant of Euclidean space, i.e.

$$\Omega \doteq \{x \in \mathbb{R}^n \mid x^i > 0\,,\ \forall i = 1, \dots, n\}$$