Exercises - PDEs

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Sheet 8

1 Exercise

Let $\Omega \in \mathbb{R}^n$ be bounded and regular enough such that the embeddings $W^{1,p} \hookrightarrow L^q(\Omega)$ are compact for all $1 \leq p < n$ and $1 \leq q < \frac{np}{n-p}$ We want to show that there exists a constant C such that

$$\|u - \bar{u}\|_{\frac{np}{n-p}} \leq C\|Du\|_p$$

(all the norms are over the set Ω) holding for all $u \in W^{1,p}(\Omega)$ such that

$$\bar{u} := \frac{1}{|\Omega|}\int_\Omega u\, dx$$

Let us use a contradiction argument. Let us suppose that $\exists \{u_k\}_k \subset W^{1,p}(\Omega)$ such that

$$\|u - \bar{u}_k\|_p > k\|Dv_k\|_p$$

Let us take $\|u_k - \bar{u}_k\| = 1$ since otherwise we can always find a new sequence $v_k = u_k - \bar{u}_k/\|u_k - \bar{u}_k\|$ and if $\|u_k - \bar{u}_k\| = 0$ we already have an absurdity. Because of these considerations we have that $\bar{u}_k = 0$.
Moreover, we have that

$$\|Du_k\|_p < \frac{1}{k}$$

which means that u_k must be bounded in $W^{1,p}$. From the compactness of the embedding we can construct a subsequence $\{u_{k_j}\} \in W^{1,p}(\Omega)$ which converges to $u \in L^p(\Omega)$ in $L^p(\Omega)$, meaning that $\bar{u} = 0$ and $\|u\|_p = 1$. We therefore have that

$$k\|Du\|_p < 1$$

and since k is arbitrary we can find k big enough such that the inequality does not hold. Showing the contradiction. We proved that

$$\|u - \bar{u}_k\|_p \leq C\|Dv_k\|_p$$

since p is arbitrary, let us take $\frac{np}{n-p}$, Then since

$$\frac{np}{n-p} > p$$

we have that $L^{\frac{np}{n-p}} \subset L^p$ from a previous result meaning

$$\|u - \bar{u}_k\|_{\frac{np}{n-p}} \leq C\|Dv_k\|_p$$

2 Exercise

Let us consider a continuous embedding

$$X \hookrightarrow Y \hookrightarrow Z$$

between Banach spaces, we want to show that if $X \hookrightarrow Y$ is a compact embedding then $\forall \varepsilon > 0, \exists C_\varepsilon > 0 : \forall u \in X$

$$\|u\|_Y \leq \varepsilon\|u\|_X + C_\varepsilon\|u\|_Z$$

Let us consider the contradiction of our previous statement and show that it implies an absurdity. $\exists \varepsilon > 0 \,\&\, \exists \{u_k\} \in X$:

$$\|u_k\|_Y > \varepsilon\|u_k\|_X + k\|u_k\|_Z$$

By definition of continuous embedding we have that

$$\|u_k\|_Z \leq C_1\|u_k\|, \quad C_1 > 0$$

meaning that

$$\|u_k\|_Y - kC_1\|u_k\|_Y = (1 - kC_1)\|u_k\|_Y > \varepsilon\|u_k\|_X$$

Now we can assume that $\|u_k\|_X \neq 0$ since in this case we already have that the contradiction of the statement generates an absurdity since $\|u\|_V < C_2\|u\|_X$:

$$0 > 0$$

In this case it is always possible to find a sequence such that $\|u_k\| = 1$ since otherwise we can just take the sequence $v_k = u_k/\|u_k\|_Y$. It follows that

$$(1 - kC_1)\|u_k\| > \varepsilon$$

Because the embedding from X to Y is compact we know that $\|u_k\|_Y < 1$ meaning that $\exists u_{k_i} \to u$ in Y so that $\|u_{k_i} - u\| < \delta$, $\forall \delta$. Then

$$\|u_{k_i}\| \leq \|u\|_Y + \delta_{k_i}$$

Let us consider $\delta = \varepsilon/2$.

$$(1 - k_iC_1)(\frac{\delta}{2} + \|u\|_Y) > \varepsilon$$

$$(1 - k_iC_1)\|u\|_Y > \frac{\varepsilon}{2}$$

Then if $\|u\|_Y = 0$ we have the contradiction, while in the other case we can choose k big enough to make it not a contradiction.

3 Exercise

For $p > 1$, $\Omega \subset \mathbb{R}^n$ and $v \in W^{1,p}(\Omega)$ we want to find the Euler-Lagrange equation of the energy:

$$J(v) := \int_\Omega |v|^p \sqrt{1 + |Dv|^2} dx$$

From the theorem we have introduced about the Euler-Lagrange equation, in which

$$F(x, v(x), Dv(x)) = |v|^p \sqrt{1 + |Dv|^2},$$

since we know $F \in C^1(\Omega)$ because it is the product of differentiable function ($|v|^p$ is differentiable for $p > 1$, which it is by hypothesis). We then have that

$$\frac{\partial F}{\partial v} = pv|v|^{p-2} \sqrt{1 + |Dv|^2}$$

$$\begin{aligned} -D_i \frac{\partial F}{\partial D^i v} &= -D_i \left(|v|^p \frac{D_i v}{\sqrt{1 + |Dv|^2}} \right) \\ &= -\frac{|v|^p}{\sqrt{1 + |Dv|^2}} Av - \frac{|Dv|^2 p|v|^{p-2}}{\sqrt{1 + |Dv|^2}} \end{aligned}$$

where:

$$Av = a_{ij}(Dv) D_i D_j v$$

$$a_{ij}(Dv) = \frac{1}{\sqrt{1 + |Dv|^2}} \left(\delta_{ij} - \frac{D_i v D_j v}{1 + |Dv|^2} \right)$$

Putting them together

$$pv|v|^{p-2} \sqrt{1 + |Dv|^2} - \frac{|v|^p}{\sqrt{1 + |Dv|^2}} Av - \frac{|Dv|^2 p|v|^{p-2}}{\sqrt{1 + |Dv|^2}} = 0$$

Now since the only factor depending on $D_i D_j v$ is the one containing a_{ij} from the fact that $|v|^p \geq 0$ and $\sqrt{1 + |Dv|^2} \geq 0$ by definition, and $a_{ij}(Dv)\xi^i \xi^j > 0$ from class, then we proved that such a PDE is elliptical.

Now let us assume that $n = 1$ and $p = 1$. Then

$$pv|v|^{p-2} \sqrt{1 + |Dv|^2} - D_i \left(|v|^p \frac{D_i v}{\sqrt{1 + |Dv|^2}} \right) = 0$$

$$v|v|^{-1} \sqrt{1 + \left| \frac{\mathrm{d}v}{\mathrm{d}x} \right|^2} - \frac{\mathrm{d}}{\mathrm{d}x} \left(|v| \frac{\mathrm{d}v}{\mathrm{d}x} \frac{1}{\sqrt{1 + |\frac{\mathrm{d}v}{\mathrm{d}x}|^2}} \right) = 0$$

Let us assume that $v(x) = ce^{i\lambda x}$ solves the equation.

$$\begin{aligned}
&e^{i\lambda x}\sqrt{1+c^2\lambda^2{}^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(i\lambda c^2 e^{i\lambda x}\frac{1}{\sqrt{1+c^2\lambda^2}}\right) = 0\\
&e^{i\lambda x}\sqrt{1+c^2\lambda^2{}^2} = -\lambda^2 c^2 e^{i\lambda x}\frac{1}{\sqrt{1+c^2\lambda^2}}\\
&1+c^2\lambda^2{}^2 = -\lambda^2 c^2
\end{aligned}$$

which tells us that $\lambda = \pm\frac{i}{c\sqrt{2}}$ and $v(x) = ce^{\pm x/c\sqrt{2}}$. Choosing $c = \frac{1}{2}$ we have that the combination of the two solutions gives us

$$v = \frac{1}{2}(e^{x\sqrt{2}} + e^{-x\sqrt{2}})$$

which is almost the form of a catenary.

$$y = a\cosh\frac{x}{a} = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$$

4 Exercise

Let us consider $\Omega \subset \mathbb{R}^n$ open with $\varphi \in C^2(\partial\Omega)$ and $\psi \in C^2(\bar{\Omega})$ such that $\psi|_{\partial\Omega} < \varphi$ Now let us consider the energy

$$J(v) := \int_\Omega |Dv|^2 dx$$

on the function space $K = \{v \in C^{0,1}(\bar{\Omega}) : v \geq \psi,\, v = \varphi\}$ with $u \in W^{2,\infty}(\Omega)$ such that minimize the energy on K.

a) We want to show that $\Delta u \leq 0$ a.e. To do so, we can use the variational characterization of minimizer which states that given a $u \in K$ to be the unique solution such that:

$$J(u) = \min_{v\in K} J(v)$$

Then

$$\int_\Omega DuD(v-u)dx \geq 0$$

We can take a test function $\eta \in C_c^\infty$ such that $v := u+\eta$, notice that if $u, \eta \in K$ than also v will be in K since

$$|v(x) - v(y)| \leq |u(x) - u(y)| + |\eta(x) - \eta(y)| < (C + C')|x - y|$$

then we can write:

$$\int_\Omega DuD\eta\, dx \geq 0$$

Since $u \in W^{2,\infty}(\Omega)$ we can integrate by parts

$$-\int_\Omega \Delta u\, \eta\, dx \geq 0$$

For $\eta \geq 0$ we have that $\Delta u \leq 0$ almost everywhere

b) To prove that in a subset of Ω such that $\Omega' := \{x \in \Omega : u(x) > \psi(x)\}$. Now if we fix a test function $\eta \in C_c^\infty(\Omega')$ and fix $|s|$ such that this holds $v = u + s\eta \geq \psi$. Using the variational characterization of minimizer we have that the inequality

$$s \int_\Omega DuD\eta \, dx \geq 0$$

is true for sufficiently small values of s both positive and negative, meaning that

$$0 \leq s \int_\Omega DuD\eta \, dx \leq 0$$

which implies

$$\int_\Omega DuD\eta \, dx = 0$$

Using integration by parts we get the result $\Delta u = 0$ a.e. on Ω'.