Exercises - Mathematical Quantum Theory

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Sheet 10

1 Exercise

(a) Given a PVM $\{P(\Omega)\}_{\Omega\in\mathcal{B}(\mathbb{R})}$ let $\Phi : S(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be the associated measurable functional calculus. Which by definition have the form:

$$\Phi(f) = \int_{\mathbb{R}} f(\lambda)dp(\lambda) := \sum_i c_i P(\Omega_i)$$

where in this case f is a simple function. We want to prove that Φ is a C^*-algebra homeomorphism, i.e.:

- Φ is linear: from the linearity of integrals we have that it is linear.
- $\Phi(1) = \mathbb{1}_{\mathcal{H}}$:

$$\Phi(\mathbf{1}) = \int_{\mathbb{R}} 1dp(\lambda) = \int_{\mathbb{R}} dp(\lambda) = P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$$

- For all $f, g \in S(\mathbb{R})$, we have $\Phi(fg) = \Phi(f)\Phi(g)$: from the definition of a simple function we can express $f = \sum_j \alpha_j \chi_{\Omega_j}$ and $g = \sum_k \beta_k \chi_{\Gamma_k}$ which allows us to consider

$$\Phi(fg) = \Phi\left(\sum_j \alpha_j \chi_{\Omega_j} \sum_k \beta_k \chi_{\Gamma_k}\right) = \Phi\left(\sum_{j,k} \alpha_j \beta_k \chi_{\Omega_j} \chi_{\Gamma_k}\right) =$$
$$= \Phi\left(\sum_{j,k} \alpha_j \beta_k \chi_{\Omega_j\cap\Gamma_k}\right)$$

using now the definition of functional calculus for simple functions:

$$\Phi(fg) = \sum_{j,k} \alpha_j\beta_k P(\Omega_j \cap \Gamma_k) = \sum_{j,k} \alpha_j\beta_k P(\Omega_j)P(\Gamma_k) = \Phi(f)\Phi(g)$$

- For all $f \in S(\mathbb{R}), \Phi(\bar{f}) = \Phi(f)^*$: again using the definition a simple function we have

$$f = \sum_j \alpha_j \chi_{\Omega_j}, \quad \bar{f} = \overline{\sum_j \alpha_j \chi_{\Omega_j}} = \sum_j \overline{\alpha_j} \chi_{\Omega_j}$$

from the properties of complex conjugate and the fact that the characteristic function real valued. Then, by the definition of functional calculus for simple functions:

$$\Phi(\bar{f}) = \Phi\left(\sum_j \overline{\alpha_j}\chi_{\Omega_j}\right) = \sum_j \overline{\alpha_j}P(\Omega_j) = \sum_j \overline{\alpha_j}P(\Omega_j)^* = \Phi(f)^*$$

(b) Now let $f, g : \mathbb{R} \to \mathbb{C}$ be measurable, but not necessarily bounded. We want to show that the $\Phi(f)\Phi(g) = \Phi(g)\Phi(f)$ on $\mathcal{D}_f \cap \mathcal{D}_g$:

Let us consider two sequences $f_k \subset \mathcal{D}_f, g_k \subset \mathcal{D}_g$ converging respectively to $f \in \mathcal{D}_f, g \in \mathcal{D}_g$ and such that $f_k = f\chi_{\Omega_k}, g_k = g\chi_{\Gamma_k}$. By the definition of $(\Phi(\cdot), \mathcal{D})$ we set $\Phi(f) = \lim_{k\to\infty}\Phi(f_k)$, then:

$$\begin{aligned}\Phi(f)\Phi(g) &= \lim_{k\to\infty}\Phi(f_k)\Phi(g_k) = \lim_{k\to\infty}\Phi(f\chi_{\Omega_k})\Phi(g\chi_{\Gamma_k}) = \\ &= \lim_{k\to\infty}\Phi(f\chi_{\Omega_k}\, g\,\chi_{\Gamma_k}) = \lim_{k\to\infty}\Phi(g\chi_{\Gamma_k}) = \Phi(f\chi_{\Omega_k}) = \\ &= \lim_{k\to\infty}\Phi(g_k)\Phi(f_k) = \Phi(f)\Phi(g)\end{aligned}$$

since from their definition, f_k, g_k are bounded.

2 Exercise

Let us take $\mathcal{H} = L^2([0,1])$ and consider the Volterra operator

$$Vf(x) = \int_0^1 f(y)dy.$$

(a) we want to show that such an operator is bounded on $\mathcal{H}$.

The operator norm

$$\begin{aligned}\|V\|^2 &= \sup_{\|f\|_{L^2}=1}\|Vf\|^2_{L^2} = \\ &= \sup_{\|f\|_{L^2}=1}\int_0^1\left|\int_0^x f(y)dy\right|^2 dx \le \sup_{\|f\|_{L^2}=1}\int_0^1\left(\int_0^x |f(y)|dy\right)^2 dx\end{aligned}$$

by Cauchy-Schwartz:

$$\int_0^1\left(\int_0^x |f(y)|dy\right)^2 dx \le \int_0^1 \|f(x)\|_{L^2} x\, dx < \infty$$

(b)

3 Exercise

4 Exercise