MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 7

All exercises are worth 5 points. Due on Friday, December 8, at 20:00 via URM.

- **Exercise 1** is our first look at general Schrödinger operators $-\Delta + V$ which describe an electron in a potential landscape V. These operators are the basis of quantum mechanics. Understandings their spectral theory is a central motivation for building up the abstract machinery of Unit 2.
- **Exercise 2** provides us with a broad class of self-adjoint realizations of Schrödinger operators $-\Delta + V$, under essentially optimal assumptions on the potential V. This is the power of quadratic forms!
- **Exercise 3** generalizes our spectral considerations from the lecture about the position operator to any multiplication operator. Multiplication operators play the role of diagonal matrices in the infinite-dimensional theory, so we should expect their spectral theory to be “nice”.
- **Exercise 4** makes precise how we “diagonalize” the momentum operator by the Fourier transform.

Exercise 1. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a potential satisfying $V \in L^p(\mathbb{R}^3)$ for some $2 < p \leq \infty$. On the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$, consider the Schrödinger operator

$$H = -\Delta + V, \qquad D(H) = H^2(\mathbb{R}^3).$$

Show that $H : D(H) \to L^2(\mathbb{R}^3)$ is well-defined and symmetric.
Hint. Use the general Sobolev embedding theorem: $H^s(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ *whenever* $2 < q < \infty$ *satisfies* $\frac{1}{q} = \frac{1}{2} - \frac{s}{d}$. *Here, the Sobolev space* $H^s(\mathbb{R}^d)$ *is defined for any real number* $s \geq 0$ *through Fourier space, analogously to the case of integer* s.

Exercise 2. Let $d \geq 3$ and $V = v + w$ with $v \in L^{d/2}(\mathbb{R}^d)$ and $w \in L^\infty(\mathbb{R}^d)$. Consider the quadratic form

$$q(\psi) = \langle \nabla\psi, \nabla\psi \rangle + \langle \psi, V\psi \rangle$$

defined on $Q = H^1(\mathbb{R}^d)$.

(a) Prove that q is bounded from below by a constant.
Hint: Use the Sobolev embedding in the form $\|\psi\|_{2d/(d-2)} \leq C\|\nabla\psi\|$ *for some universal constant* $C > 0$.

(b) Construct a self-adjoint realization of the Schrödinger operator $-\Delta + V$.

Exercise 3. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a continuous function. On the Hilbert space $L^2(\mathbb{R}^d)$, define the multiplication operator $(\mathcal{M}_f, D(\mathcal{M}_f))$ on the usual domain

$$D(\mathcal{M}_f) = \{\psi \in L^2(\mathbb{R}^d) \,:\, f\psi \in L^2(\mathbb{R}^d)\}.$$

(a) Prove that $\sigma(\mathcal{M}_f) = \operatorname{ran} f$, where $\operatorname{ran} f$ is the range of f.
(b) Fix a number $a \in \mathbb{C}$. Give an example of a non-constant function $f : \mathbb{R} \to \mathbb{C}$ such that $\sigma_p(\mathcal{M}_f) = \{a\}$.

Remark. Part (a) generalizes to any measurable function f if the range is replaced by the essential range (i.e., the range up to zero-measure sets).

Exercise 4.

(a) Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary and $(T, D(T))$ an unbounded operator. Use Exercise 4 from exercise set 6 to prove that
$$\sigma(T) = \sigma(UTU^*), \qquad \sigma_{\#}(T) = \sigma_{\#}(UTU^*)$$
for any $\# \in \{p, c, r\}$.
(b) Consider the momentum operator
$$\left(-i\frac{d}{dx}, H^1(\mathbb{R})\right)$$
on the Hilbert space $L^2(\mathbb{R})$. Find a unitary transformation that takes this operator into a multiplication operator.
(c) Identify the spectrum, the point spectrum, and the continuous spectrum of the momentum operator.