MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 1

All exercises are worth 5 points. Due on ***Friday, October 27****, at 12:00 noon via URM.*

- **Exercise 1** practices the fundamentals of quantum mechanics — computing time evolution and expectation values — for a simple system with only two degrees of freedom (also called a "qubit").
- **Exercise 2** reviews relations between a few function spaces we've seen in class. For L^p-spaces, the main message is that the power p simultaneously controls both the degree of possible singularities and the decay rate at spatial infinity. We also see that Schwartz functions have excellent integrability properties.
- **Exercise 3** proves a lemma from lecture about convergence in Schwartz space.
- **Exercise 4** is the arguably most important explicit calculation of a Fourier transform ("The Fourier transform of a Gaussian is a Gaussian".) While this is a fundamental fact, its proof relies on applying a kind of "trick", so be especially sure to find help here if you are stuck.

Exercise 1. Consider the Hilbert space $\mathcal{H} = \mathbb{C}^2$. A basis is given by the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ which in this context are called the "spin-up" and "spin-down" states. Consider the Hamiltonian

$$H = -B\sigma_x \ , \qquad \text{where } \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \text{ and } B \in \mathbb{R}.$$

Consider a quantum state described at $t = 0$ by the wave function

$$\psi = \begin{pmatrix}1\\0\end{pmatrix} \ .$$

Let ψ_t be the solution of the Schrödinger equation:

$$i\partial_t \psi_t = H\psi_t \ , \qquad \psi_0 = \psi \ .$$

(a) Compute ψ_t for all times.
(b) Let

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} .$$

Suppose we decide to perform a quantum measurement of the observable σ_z at time t. Then the possible measurement outcomes are its eigenvalues ± 1. What are the probabilities of measuring these outcomes?

(c) What is the expectation value $\langle \sigma_z \rangle_{\psi_t}$?

(d) Further information lies in the variances of the observables σ_x, σ_z in the state ψ_t:

$$\begin{aligned}(\Delta\sigma_x)_{\psi_t} &= \langle \psi_t, (\sigma_x - \langle \psi_t, \sigma_x \psi_t\rangle)^2 \psi_t\rangle \ ,\\ (\Delta\sigma_z)_{\psi_t} &= \langle \psi_t, (\sigma_z - \langle \psi_t, \sigma_z \psi_t\rangle)^2 \psi_t\rangle \ .\end{aligned}$$

Find the values of t for which their product $(\Delta\sigma_x)_{\psi_t}(\Delta\sigma_z)_{\psi_t}$ is minimal. What is the value of the minimum?

Reminder. If we measure a quantum observable A in quantum state ψ, then the expectation value of the measurement is denoted by $\langle A \rangle_\psi$ and equal to

$$\langle A \rangle_\psi = \langle \psi, A\psi \rangle.$$

This formula follows from the measurement postulate discussed in class. We like it because it does not require any information about eigenvalues or eigenspaces of A! Consequently, it is often more convenient to calculate expectation values than the probabilities of individual measurement outcomes.

Exercise 2. Prove the following statements about function spaces.

(a) If $1 \le p < q \le \infty$, then neither the inclusion $L^p(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d)$ nor $L^q(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$ is true.
Hint. Think about integrability of power functions $|x|^\alpha$.

(b) Suppose that $f \in L^2(\mathbb{R}^d)$ is supported in a ball $\{x \in \mathbb{R}^d : |x| \le R\}$. Then f is in $L^1(\mathbb{R}^d)$.

(c) Suppose that $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, $f \in L^p(\mathbb{R}^d)$ for all $1 < p < \infty$.

(d) Suppose that $f \in \mathcal{S}(\mathbb{R}^d)$. Then $f \in L^p(\mathbb{R}^d)$ for every $1 \le p \le \infty$.

(e) Suppose that $f \in \mathcal{S}(\mathbb{R}^d)$ and $p : \mathbb{R}^d \to \mathbb{C}$ is a polynomial. Prove that $pf \in \mathcal{S}(\mathbb{R}^d)$.

Exercise 3. Define the Schwartz space metric as

$$\mathrm{d}_{\mathcal{S}}(f,g) = \sum_{n=0}^{\infty} 2^{-n} \sup_{\substack{\alpha,\beta \in \mathbb{N}_0^d:\\ |\alpha|+|\beta|=n}} \frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}.$$

Prove the following equivalence.

$$d_{\mathcal{S}}(f_j, f) \to 0 \quad \Longleftrightarrow \quad \|f_j - f\|_{\alpha,\beta} \to 0 \text{ for all } \alpha,\beta \in \mathbb{N}_0^d$$

Exercise 4. Consider the Gaussian function $g_\lambda \in L^1(\mathbb{R}^d)$ defined by

$$g_\lambda(x) = e^{-\lambda \frac{|x|^2}{2}} \ , \qquad \lambda > 0.$$

Prove that its Fourier transform satisfies

$$\hat{g}_\lambda(k) = \lambda^{-d/2} e^{-\frac{|x|^2}{2\lambda}}.$$

Hint. It is helpful to complete a square in the exponent. To compute the resulting integral, you have different options. You can use (1) differentiation under the integral sign or (2) a contour deformation in the complex plane that removes the imaginary part…or you can surprise us!