INTRODUCTION TO PDE **Assignment 3**

Due date: Thursday, November 9th WS 2023-2024

Exercise 1. (Liouville's theorem) Show that every bounded harmonic function $u : \mathbb{R}^n \to \mathbb{R}$ is a constant function.

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ open and $u \in C^1(\Omega)$. Show that for all compact subset $K \subset \Omega$ there exists some $C < \infty$ such that

$$|u(x) - u(y)| \leq C|x - y|$$

for every $x, y \in K$.

Exercise 3. Let $u \in C^2_c(\mathbb{R}^n)$ (i.e. $u \in C^2(\mathbb{R}^n)$ with compact support) be non-negative. Prove that there exists a constant $C < \infty$ such that

$$|Du|^2(x) \leq Cu(x)$$

for every $x \in \mathbb{R}^n$.

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ open and $u \in C^0(\Omega)$. Show that

$$u(x) \leq \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, dy \qquad \forall B_r(x) \subset \Omega$$

if and only if for every ball $B_r(x) \subset\subset \Omega$ and every function h harmonic in $B_r(x)$ with $u \leq h$ in $\partial B_r(x)$, it holds that $u \leq h$ in $B_r(x)$.
Hint: use the Poisson integral formula.