Exercises - PDEs

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Sheet 10

1 Exercise

Let us consider the problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - D_i a^{ij}(x,t) D_j u(x,t) - a^{ij}(x,t) D_i D_j u(x,t) - \\ \qquad\qquad\qquad\qquad - b^i(x,t) D_i u(x,t) = 0 \\ u(\cdot,0) = u_0(x) \\ u|_{\partial\Omega\times[0,T]} = 0 \end{cases} \tag{1}$$

assuming that the coefficients a^{ij}, b^i are smooth and bounded with a^{ij} uniformly elliptic. We want to show that

$$\sup_{\Omega\times(0,T]} |u(x,t)| \leq \sup_{\Omega} |u_0(x)|$$

Meaning that the maximum is attained on the temporal boundary.

Let us assume not, meaning that the maximum is attained at some point $(x_0,t_0) \in \Omega \times (0,T]$ (with T finite), and show that it implies a contradiction. Because $u(x_0,t_0)$ is a maximum than

$$\partial_t u(x_0,t_0) \geq 0$$

Moreover, for the same reason at (x_0,t_0) we have that $D_i u(x_0,t_0) = 0$ and $D_i D_j u(x_0,t_0) \leq 0$ (where $D_i D_j u$ is the hessian matrix). Then

$$D_i a^{ij}(x_0,t_0) D_j u(x_0,t_0) = 0$$
$$b^i(x_0,t_0) D_i u(x_0,t_0) = 0$$

In addition to that,

$$a^{ij} D_i D_j u(x_0,t_0) \leq 0$$

from uniform ellipticity. Then:

$$\begin{aligned} \partial_t u(x_0,t_0) - L_0 u(x_0,t_0) &:= \partial_t u(x_0,t_0) - D_i a^{ij}(x_0,t_0) D_j u(x_0,t_0) - \\ &- a^{ij}(x_0,t_0) D_i D_j u(x_0,t_0) - b^i(x_0,t_0) D_i u(x_0,t_0) \geq 0 \end{aligned}$$

Let us consider $u^\varepsilon(x,t) = u(x,t) - \varepsilon t$, with $\varepsilon > 0$ which is the solution of $\partial_t u^\varepsilon(x,t) - L_0 u^\varepsilon(x,t) = -\varepsilon$, with the same boundary condition as eq (1). Since at (x_0, t_0)

$$-\varepsilon = \partial_t u^\varepsilon - L_0 u^\varepsilon \geq 0$$

we have a contradiction. Meaning that $t_0 = 0$ and

$$\sup_{\Omega\times(0,T]} |u^\varepsilon(x,t)| = \sup_\Omega |u_0^\varepsilon(x)|$$

from which

$$\sup_{\Omega\times(0,T]} |u(x,t)| - \varepsilon T \leq \sup_{\Omega\times(0,T]} |u(x,t) - \varepsilon t| = \sup_\Omega |u_0(x)|$$

and

$$\sup_{\Omega\times(0,T]} |u(x,t)| \leq \sup_\Omega |u_0(x)| + \varepsilon T$$

since ε was arbitrary we can make it go to zero obtaining the statement.

2 Exercise

Let $u \in W^{1,2}(\Omega)$ be a weak solution to the equation:

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^i D_i u + du = D_i g^i - f$$

on $\Omega \in \mathbb{R}^n$ and with bounded coefficients. Suppose a^{ij} is uniformly elliptic and $g^i, f \in L^2$. We want to show that for every subdomain $\Omega' \subset\subset \Omega$ there exists a constant $C > 0$ depending only on the coefficients and the distance function such that:

$$\|u\|_{W^{1,2}(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|g^i\|_{L^2(\Omega)} \right)$$

Let us consider

$$\begin{aligned} \mathcal{L}(u,v) &:= \int_\Omega a^{ij} D_i u D_j v + b^i u D_i v - c^i D_i u v + duv dx = \\ &= \int_\Omega (D_i g^i - f) v dx =: \mathcal{F}(v) \end{aligned}$$

Let us define the test function $v := \eta^u$ with η the cut-off function in $\Omega' \subset\subset \Omega$. Then

$$\begin{aligned} \mathcal{L}(u,v) := \int_\Omega & a^{ij} \eta^2 D_i u D_j u + 2a^{ij} D_i u D_j \eta \eta u + 2b^i u^2 \eta D_i \eta + b^i \eta^2 u D_i u - \\ & - c^i u \eta^2 D_i u + du^2 \eta^2 dx = \int_\Omega D_i g^i u \eta^2 - f u \eta^2 dx = \\ & = - \int_\Omega g^i (D_i u \eta^2 + 2u\eta D_i \eta) - f u \eta^2 dx =: \mathcal{F}(v) \end{aligned}$$

integrating by parts. Using the ellipticity of a^{ij} we have

$$\begin{aligned}\lambda\int_\Omega \eta^2|Du|^2dx \leq \sup|a|\int_\Omega 2|Du||D\eta|\eta|u|dx+\\ +\sup|b|\left(\int_\Omega 2|u|^2\eta|D\eta|dx+\int_\Omega \eta^2|u||Du|dx\right)+\\ +\sup|c|\int_\Omega |u|\eta^2|Du|dx+\sup|d|\int_\Omega |u|^2\eta^2dx+\\ +\int_\Omega |g^i||Du|\eta^2+\int_\Omega 2|g^i||u|\eta|D\eta|+\int_\Omega |f||u|\eta^2dx\end{aligned}$$

We want to bound each term. Using Peter-Paul inequality we have that

$$\begin{aligned}\sup|a|\int_\Omega 2|Du||D\eta|\eta|u|dx &\leq \frac{\lambda}{10}\int_\Omega |Du|^2\eta^2dx+\frac{10}{\lambda}\|a\|_\infty^2\int_\Omega |Du|^2|\eta|^2dx\leq\\ &\leq \frac{\lambda}{10}\int_\Omega |Du|^2\eta^2dx+\frac{10}{\lambda}C(\sup|a|,d(\Omega',\partial\Omega))\|u\|_{L^2(\Omega)}\end{aligned}$$

since the square derivative of $|D\eta|^2 \leq \frac{c}{d(\Omega',\partial\Omega)}$. We can bound all the other terms using the same criteria. All but the last one which also require using Cauchy Schwarz

$$\int_\Omega |f||u|\eta^2dx \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$$

now using Peter-Paul

$$\int_\Omega |f||u|\eta^2dx \leq c\|f\|_{L^2(\Omega)}+\frac{1}{c}\|u\|_{L^2(\Omega)}$$

The final inequality becomes:

$$\begin{aligned}\lambda\int_\Omega |Du|^2\eta^2dx \leq \frac{\lambda}{10}\int_\Omega |Du|^2\eta^2dx+C(\lambda,\sup|a|,d(\Omega',\partial\Omega))\|u\|_{L^2(\Omega)}+\\ +C(\sup|b|,d(\Omega',\partial\Omega))\|u\|_{L^2(\Omega)}+\\ +\frac{\lambda}{10}\int_\Omega |Du|^2\eta^2dx+C(\lambda,\sup|b|,d(\Omega',\partial\Omega))\|u\|_{L^2(\Omega)}+\\ +\frac{\lambda}{10}\int_\Omega |Du|^2\eta^2dx+C(\lambda,\sup|c|,d(\Omega',\partial\Omega))\|u\|_{L^2(\Omega)}+\\ +C(\sup|d|)\|u\|_{L^2(\Omega)}+C\|f\|_{L^2(\Omega)}+\frac{1}{C}\|u\|_{L^2(\Omega)}+\\ +C(d(\Omega',\partial\Omega))\|u\|_{L^2(\Omega)}+C\|g^i\|_{L^2(\Omega)}+\\ +\frac{\lambda}{10}\int_\Omega |Du|^2\eta^2dx+C(\lambda)\|g^i\|_{L^2(\Omega)}\end{aligned}$$

which gives us

$$\int_\Omega |Du|^2\eta^2dx \leq C(\|u\|_{L^2(\Omega)}+\|g^i\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)})$$

where $C = C(\lambda, d(\Omega', \partial\Omega), \sup|a|, \sup|b|, \sup|c|, \sup|d|)$. Using the fact that $\Omega' \subset, \subset \Omega$ as well as Poincaré, which gives us the equivalence between the $\mathcal{H}$ and the $W^{1,2}(\Omega)$ norm, we obtain the claim.