INTRODUCTION TO PDE **Assignment 4**

Due date: Thursday, November 16th WS 2023-2024

Exercise 1. (Harnomic Polynomials)

(i) Find all the harmonic polynomials $p \in C^\infty(\mathbb{R}^3)$ with degree 2 or less.

(ii) Show that every harmonic function in $\mathbb{R}^n$ with

$$\sup_{B_R(0)} |u| \leq CR^{3-\delta}$$

for some $C = C(n)$ and $\delta > 0$ must be a polynomial of degree at most 2.

Exercise 2. (The Kelvin transform)[1] Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open set with $0 \notin \Omega$ and let $x^* = \frac{R^2}{|x|^2}x$ denote the inversion over the sphere $\partial B_R(0)$. For every $u \in C^0(\Omega)$ we define the Kelvin transform $u^* \in C^0(\Omega^*)$, $\Omega^* \doteq \{x^* : x \in \Omega\}$, by

$$u^*(x^*) \doteq \frac{u(x)}{|x^*|^{n-2}}\,.$$

(i) Show that the Kelvin transform of a harmonic function is again harmonic, and more generally that

$$\Delta u^*(x^*) = \frac{R^4}{|x^*|^{n+2}} \Delta u \left(\frac{R^2}{|x^*|^2} x^* \right).$$

Also, compute the Kelvin transform of harmonic polynomials up to degree 1.

(ii*) Let $0 \in \Omega \Subset \mathbb{R}^n$ and $u \in C^0(\Omega \setminus \{0\})$ harmonic such that $|u(x)| = o(|x|^{2-n})$ for $|x| \to 0$. Prove that u can be extended continuously through 0. *Hint: use the gradient estimate and the proof of the Mean Value Property.*

(iii) Let $\Omega \subset \mathbb{R}^3$ bounded and consider a harmonic function $u \in C^0(\mathbb{R}^3 \setminus \bar{\Omega})$ with $|u(x)| = o(1)$ as $|x| \to \infty$. Show that u has an asymptotic expansion of the form

$$u(x) = \frac{a}{|x|} + \frac{b_i}{|x|^3} x^i + o(|x|^{-2})\,. \tag{1}$$

[1] The exercise with a * sign is optional, but it adds points.

(iv) Let $\rho \in C_c^1(\mathbb{R}^3)$ be a nonnegative function representing the mass density of an isolated star. By Newton's law of gravity, the gravitational potential $u \in C^2(\mathbb{R}^3)$ satisfies

$$\begin{cases} \Delta u & = 4\pi\rho \\ u(x) & = o(1) \quad \text{as } |x| \to \infty \end{cases}.$$

Please, justify the the omission of the boundary terms in the (Dirichlet) Green's function representation formula for u and compute a and b^i from the expansion (1) of u in terms of the mass

$$m = \int_{\mathbb{R}^3} \rho(x)dx$$

and the center of mass

$$c^i = \frac{1}{m}\int_{\mathbb{R}^3} \rho(x)x^i dx\,.$$

Exercise 3. Let $\{u_1, u_2, \dots, u_N\}$ be a finite number of subharmonic functions. Show that

$$u(x) \doteq \max\{u_1(x), u_2(x), \dots, u_N(x)\}$$

is again subharmonic.

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ open with smooth boundary and let $f : \Omega \to \mathbb{R}$ and $\beta : \partial\Omega \to \mathbb{R}$ smooth. Consider the Neumann problem

$$\begin{cases} \Delta u & = f \quad \text{in } \Omega \\ \langle Du, \nu\rangle & = \beta \quad \text{on } \partial\Omega \end{cases} \tag{2}$$

where ν is the outward pointing unit normal to $\partial\Omega$.

(i) Show that the compatibility condition

$$\int_\Omega f(x)dx = \int_{\partial\Omega} \beta(x)ds$$

is a necessary condition for the solvability of (2). Try to give a physical explanation of such condition e.g. when u is a temperature distribution and f is a heat source.

(ii) Use the divergence theorem to show that the solution to (2), if it exists, it must be unique up to a constant (in each connected component).

(iii) Find the Green's representation formula for solutions to the Neumann problem in $\Omega \doteq \{(x, y) \in \mathbb{R}^2 : y > 0\}$, assuming that all functions are smooth and bounded as $|x| \to \infty$.