INTRODUCTION TO PDE **Assignment 1**

Due date: Thursday October 26th WS 2023-2024

Exercise 1. Write the Laplace operator on $\mathbb{R}^2$ and $\mathbb{R}^3$ in polar and spherical coordinates respectively.

Exercise 2. Let $u(x,y)$ be the height of a stationary membrane over the annulus

$$A := B_R(0) \setminus \bar{B}_1(0) = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < R\}$$

with fixed heights $u \equiv 0$ at $\partial B_1(0)$ and $u \equiv u_0 > 0$ at $\partial B_R(0)$.

(i) Solve explicitly the correspondent boundary value problem.

(ii) Show that the solution has less area than the correspondent conical area with same values at the boundary.

(iii) Solve the corresponding problem in arbitrary dimensions.

Hint: use the symmetry of the problem.

Exercise 3. Consider the one-dimensional Heat Equation

$$\frac{\partial}{\partial t} u(x,t) = \kappa \frac{\partial^2}{\partial x^2} u(x,t) \qquad 0 < x < a,\ 0 < t$$

for positive constants $\kappa, a > 0$ and with boundary data

$$\begin{aligned} u(0,t) = 0,\ u(a,t) &= 0 && t \geq 0\,, \\ u(x,0) &= f(x) && 0 < x < a\,, \end{aligned}$$

where $f(x)$ is a given fixed function. Find the general solution the the problem via separation of variables and obtain the explicit solution for the case where

$$f(x) = \sin\left(\frac{\pi}{a} x\right) - 3 \sin\left(\frac{2\pi}{a} x\right)\,.$$

Exercise 4. Let $\kappa > 0$ and consider the heat equation

$$\frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) \qquad \text{in } \mathbb{R}^n \times [0,\infty)\,.$$

(i) Show that for any $x_0 \in \mathbb{R}^n$ the *Heat Kernel*

$$\rho_{x_0}(x,t) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left(\frac{-|x-x_0|^2}{4\kappa t}\right), \qquad \kappa > 0,$$

solves the heat equation.

(ii) Show that the *total heat content*

$$\int_{\mathbb{R}} \rho_{x_0}(x,t)dx$$

is conserved in time.

(iii) Show that for arbitrary bounded and continuous function f,

$$u(x,t) = \int_{\mathbb{R}^n} f(y)\rho_y(x,t)\,dy$$

solves the heat equation with initial data $u(x,0) = f(x)$.