MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 5

All exercises are worth 5 points. Due on Friday, November 24, at 12:00 noon via URM.

- **Exercise 1** shows that the prototypical SCOPUG $P_f(t)$ is *not* continuous in operator norm sense — so strong continuity is the right notion.
- **Exercise 2** contains two simple, but essential facts about the Laplacian: it is a symmetric, positive semidefinite operator. Part (c) extends the idea to general Fourier multipliers.
- **Exercise 3** cleanly characterizes Hilbert space adjoints and self-adjointness for *bounded* linear operators. It is an important caveat for later that the situation is much more subtle for unbounded operators — an *unbounded* symmetric operator is not necessarily self-adjoint.
- **Exercise 4** explores two further paradigmatic examples of SCOPUGs: finite-dimensional e^{-itH} and the translation group on $L^2(\mathbb{R})$
- The **bonus exercise** is for everyone who is curious to see a sequence of operators that converges weakly but not strongly — which shows that weak convergence is really weaker than strong convergence.

Exercise 1.

(a) Let $g \in L^\infty(\mathbb{R}^d)$. Prove that $T_g : L^1(\mathbb{R}^d) \to \mathbb{C}$ with

$$\|T_g\| = \|g\|_\infty$$

(b) Prove that $\{P_f(t) | t \in \mathbb{R}\}$ is *not* continuous in the operator norm-sense

Exercise 2. Consider the Hilbert space $L^2(\mathbb{R}^2)$.

(a) Let $H_0 = -\Delta$ with domain $D(H_0) = H^2(\mathbb{R}^2)$. Prove that H_0 is a symmetric operator.

(b) Prove that H_0 is positive semidefinite, i.e.,

$$\langle \psi, H_0 \psi \rangle \geq 0, \qquad \forall \psi \in D(H_0).$$

(c) More generally, let $g \in C^\infty_{poly}(\mathbb{R}^d)$ and consider the associated Fourier multiplier $A_g = g(-i\nabla_x)$ on the domain $D(A_g) = \mathcal{S}(\mathbb{R}^d)$. Under what conditions on g is A_g symmetric? Under what conditions is A_g positive semidefinite?

Exercise 3.

Date: Update Nov. 20: Fixed typo in exercise 1, added definition of adjoint in exercise 3.

Let $A, B \in \mathcal{L}(\mathcal{H})$ be bounded linear operators on a Hilbert space $\mathcal{H}$. Let A be the Hilbert space adjoint defined by

$$A^* = J^{-1}A'J$$

where $A' : \mathcal{H}' \to \mathcal{H}'$ is the dual operator and $J : \mathcal{H} \to \mathcal{H}'$ is the anti-unitary isometry

$$J\varphi = \langle \varphi, \cdot \rangle$$

(a) Prove that $(AB)^* = B^*A^*$.
(b) Prove that $\|A^*\| = \|A\|$.
(c) Prove that the Hilbert space adjoint $A^* \in \mathcal{L}(\mathcal{H})$ is uniquely characterized by the relation

$$\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle, \qquad \forall \psi, \varphi \in \mathcal{H}.$$

(d) Prove that $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint if and only if A is symmetric.

Remark. Recall that we abbreviate $\mathcal{L}(\mathcal{H}) \equiv \mathcal{L}(\mathcal{H}, \mathcal{H})$.

Important caveat: The implication "symmetric $\Rightarrow$ self-adjoint" is only true for bounded operators!

Exercise 4.

(a) Consider the Hilbert space $\mathcal{H} = \mathbb{C}^n$ and let H be an $n \times n$ Hermitian matrix. Prove that $U(t) = e^{-itH}$ defined as on homework set 4 is a strongly continuous one-parameter unitary group (SCOPUG) with generator $(H, \mathbb{C}^n)$.
(b) Use exercise 3 to show that $H = H^*$, so the generator of this SCOPUG is self-adjoint.
(c) Prove that $U(t) = T_t$ is a SCOPUG with the generator $(-i\frac{\mathrm{d}}{\mathrm{d}x}, H^1(\mathbb{R}))$.
For this, you can assume that the translations are strongly continuous. This is a fact from measure theory.

Bonus exercise (5 points). Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. For measurable, bounded subsets A and B of $\mathbb{R}^d$, let χ_A and χ_B denote their the indicator functions

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Let $Q_{A,B} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denote the linear operator

$$(Q_{A,B}\psi)(x) = \chi_B(x)\langle \chi_A, \psi \rangle.$$

(a) Prove that every such $Q_{A,B}$ is a bounded operator and compute its norm.
(b) Fix a non-zero vector $z \in \mathbb{R}^d$ and write $B_n = B + nz$ for the set B shifted by nz, for every $n \in \mathbb{N}$. Prove that $Q_n \xrightarrow{w} 0$.
(c) Prove that $Q_n \overset{s}{\not\to} 0$.