# Exercises - MQT

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Sheet 9

## 1 Exercise

Let $P : \mathcal{B}(R) \to \mathcal{L}(\mathcal{H})$ be a projection-valued measure. Let $\Omega_1, \Omega_2, \Omega$ be Borel sets.

(a) By definition of complement $\mathbb{R} = \Omega^c \sqcup \Omega$\$ then

$$P(\Omega^c \sqcup \Omega) = P(\Omega^c) + P(\Omega) = P(\mathbb{R}) = 1$$

$$P(\Omega^c) = 1 - P(\Omega)$$

Now since $\emptyset$ is the complement of $\mathbb{R}$ then, by the previous result

$$P(\emptyset) = 1 - P(\mathbb{R}) = 0$$

(b) From the definition of disjoint union we have that $(\Omega_1 \cap \Omega_2) \sqcup (\Omega_1 \cup \Omega_2) = \Omega_1 \sqcup \Omega_2$, then:

$$P(\Omega_1 \cap \Omega_2) + P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2)$$

$$P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2) - P(\Omega_1 \cap \Omega_2)$$

(c) From the previous case we have that $P(\Omega_1 \cap \Omega_2) + P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2)$, squaring the right-hand side of the equation we find:

$$P^2(\Omega_1 \sqcup \Omega_2) = P(\Omega_1) + P(\Omega_2) + 2P(\Omega_1)(\Omega_2)$$

from which we observe that $P(\Omega_1) \cdot (\Omega_2) = 0$ if the two sets are disjoint.[1] Then let us consider $\Omega_1 = (\Omega_1 \setminus \Omega_2) \sqcup (\Omega_1 \cap \Omega_2)$ and similarly $\Omega_2 = (\Omega_2 \setminus \Omega_1) \sqcup (\Omega_1 \cap \Omega_2)$ Then:

$$P(\Omega_1) \cdot P(\Omega_2) = (P(\Omega_1 \setminus \Omega_2) + P(\Omega_1 \cap \Omega_2)) \cdot (P(\Omega_2 \setminus \Omega_1) + P(\Omega_1 \cap \Omega_2))$$

Expanding the product we find that all the factors are multiplications between disjoint elements except for $P^2(\Omega_1 \cap \Omega_2)$, meaning that this is the only factor of

[1] Let us call $P_1 = P(\Omega_1), P_2 = P(\Omega_2)$. Then using the properties of orthogonal projection $P_1 P_2 = (P_1 P_2)^* = P_2^* P_1^* = P_2 P_1$. This allows us to write $(P_1 + P_2)^2 = P_1^2 + P_2^2 + 2P_1 P_2$.

the multiplication which does not vanish, and by the properties of the orthogonal projection
$$P(\Omega_1) \cdot P(\Omega_2) = P^2(\Omega_1 \cap \Omega_2) = P(\Omega_1 \cap \Omega_2)$$
(d) Now let us assume that $\Omega_1 \subset \Omega_2$ then:
$$(P(\Omega_1) - P(\Omega_2))^2 > 0$$
$$P^2(\Omega_1) + P^2(\Omega_2) - 2P(\Omega_1)P(\Omega_2) = P(\Omega_1) + P(\Omega_2) - 2P(\Omega_1)P(\Omega_2) > 0$$
Using the previous result we have that $P(\Omega_1)P(\Omega_2) = P(\Omega_1 \cap \Omega_2)$ which in our case is just $P_{\Omega_1}$, then
$$P(\Omega_1) + P(\Omega_2) - 2P(\Omega_1 \cap \Omega_2) = P(\Omega_1) + P(\Omega_2) - 2P(\Omega_2) > 0$$
which gives us $P(\Omega_1) < P(\Omega_2)$ Proving the statement.

## 2 Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. We want to show that:
$$P(\Omega) = \chi_{f^{-1}(\Omega)}$$
is a projection-valued measure.

To show this we need to show that it satisfies the definition of projection-valued measure.

(1)
$$P^2(\Omega) = \chi^2_{f^{-1}(\Omega)} = \chi_{f^{-1}(\Omega)} = P(\Omega)$$
from the definition of characteristic function. Moreover, since it is a real-valued function we have that:
$$P^*(\Omega) = \chi^*_{f^{-1}(\Omega)} = \chi_{f^{-1}(\Omega)} = P(\Omega)$$

(2) Now from the existence of the inverse, $(f^{-1})$ we know that $f$ must be bijective which implies that $f^{-1}(\mathbb{R}) = \mathbb{R}$ and therefore
$$P(\mathbb{R}) = \chi_{f^{-1}(\mathbb{R})} = 1$$

(3) We now want to show that $P(\Omega_1 \sqcup \Omega_2) = P(\Omega_1) + P(\Omega_2)$, which is equivalent to show that $f^{-1}(\Omega_1 \sqcup \Omega_2) = f^{-1}(\Omega_1) \sqcup f^{-1}(\Omega_2)$.
Let $A \subset B \subset X$ and $g : X \to X$ be a function. Taking $x \in f(A)$, and $x = f(a)$ with $a \in A$. However, by hypothesis, $A \subset B$ meaning that $a \in B$ and $f(a) \in f(B)$, therefore $x = f(a) \in f(B)$ showing that $f(A) \subset f(B)$. Now since
$$\Omega_1 \subset \Omega_1 \sqcup \Omega_2, \quad \Omega_2 \subset \Omega_1 \sqcup \Omega_2$$
we have
$$f^{-1}(\Omega_1) \subset f^{-1}(\Omega_1 \sqcup \Omega_2)$$

$$f^{-1}(\Omega_2) \subset f^{-1}(\Omega_1 \sqcup \Omega_2)$$

Meaning that

$$f^{-1}(\Omega_1) \sqcup f^{-1}(\Omega_2) \subset f^{-1}(\Omega_1 \sqcup \Omega_2)$$

Similarly if we take $x \in \Omega_1 \sqcup \Omega_2$ then it will be either in $\Omega_1$ or $\Omega_2$ which means that $f^{-1}(x) \in f^{-1}(\Omega_1 \sqcup \Omega_2)$ will be in $f^{-1}(\Omega_1)$ or $f^{-1}(\Omega_2)$ meaning

$$f^{-1}(\Omega_1) \sqcup f^{-1}(\Omega_2) \supset f^{-1}(\Omega_1 \sqcup \Omega_2)$$

and therefore $f^{-1}(\Omega_1) \sqcup f^{-1}(\Omega_2) = f^{-1}(\Omega_1 \sqcup \Omega_2)$. From the properties of the characteristic function we have

$$\begin{aligned} P(\Omega_1 \sqcup \Omega_2) &= \chi_{f^{-1}(\Omega_1 \sqcup \Omega_2)} = \chi_{f^{-1}(\Omega_1) \sqcup f^{-1}(\Omega_2)} = \\ &= \chi_{f^{-1}(\Omega_1)} + \chi_{f^{-1}(\Omega_2)} = P(\Omega_1) + P(\Omega_2) \end{aligned}$$

Which proves the statement.

# 3 Exercise

Consider the self-adjoint operator $T = -\Delta$ with $D(T) = H^2(\mathbb{R}^d)$ we define its PVM as:

$$P(\Omega) = \mathcal{F}^{-1} \chi_\Omega(k) \mathcal{F}$$

which satisfies the definition of PVM directly from the properties of the characteristic function. Moreover, given $\psi \in L^2(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}} |k|^2 \|\chi_\Omega(k)\psi\| dk$$

is bounded since $\chi_\Omega(k)\psi \in L^2(\Omega)$ and $|k|^2$ is just a constant. It is also possible to express

$$\langle \hat{\varphi}, -\Delta \hat{\psi} \rangle = \int_{\mathbb{R}} k \langle \hat{\varphi}, \chi_\Omega(k) \hat{\psi} \rangle dk$$

# 4 Exercise

Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator.
(a) Liouville's theorem from complex analysis says that any bounded analytic function $f : \mathbb{C} to \mathbb{C}$ is constant. Using this fact we want to show that $\sigma(T) \neq \emptyset$.

We can look for a function $g$ such that

$$\mathcal{F}^{-1} \mathcal{M}_g \mathcal{F} \psi = T\psi$$

which is equivalent to say

$$\mathcal{M}_g \mathcal{F} \psi = \mathcal{F} T \psi.$$

Taking the $\mathcal{H}$ norm of both sides we have that:

$$\|\mathcal{M}_g \mathcal{F} \psi\| = \|\mathcal{F} T \psi\| = \|T\psi\| < C\|\psi\|$$

meaning that $\mathcal{M}_g\mathcal{F}$ is bounded which is true only if $g$ is bounded. Using Liouville's theorem we obtain that $g$ must be a constant function, i.e. $g(x) = c$. From a previous exercise sheet we have that $\sigma(T) = \sigma(UTU^*)$ and that $\sigma(\mathcal{M}_g) = \overline{\text{ran }(g)}$. Now

$$\sigma(T) = \sigma(\mathcal{F}T\mathcal{F}^{-1}) = \sigma(\mathcal{M}_c) = \{c\}$$

proving that $\sigma(T) \neq \emptyset$.

(b) Let $r(T)$ be the spectral radius of $T$, defined as

$$r(T) = \sup_{\lambda\in\sigma(T)} |\lambda|$$

We want to show that $\|T\| \geq r(T)$.

Since it is bounded we know that

$$\|T\| = \sup_{\psi\in\mathcal{H}} \frac{\|T\psi\|_{\mathcal{H}}}{\|\psi\|_{\mathcal{H}}} \geq \frac{\|T\psi\|_{\mathcal{H}}}{\|\psi\|_{\mathcal{H}}} = \frac{\|\lambda\psi\|_{\mathcal{H}}}{\|\psi\|_{\mathcal{H}}} = |\lambda|\frac{\|\psi\|_{\mathcal{H}}}{\|\psi\|_{\mathcal{H}}}$$

since it has to be true for all $\lambda \in \sigma(T)$:

$$\|T\| \geq \sup_{\lambda\in\sigma(T)} |\lambda|\frac{\|\psi\|_{\mathcal{H}}}{\|\psi\|_{\mathcal{H}}} = \sup_{\lambda\in\sigma(T)} |\lambda| = r(T)$$

(c) Gelfand's spectral radius formula says that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ we want to show that $r(T) = \|T\|$. We want to show that since $T$ is self-adjoint then $r(T) = \|T\|$.

We first want to show that $\|T^n\| = \|T\|^n$ when $T$ is self-adjoint. From the realtion $\|AB\| \leq \|A\|\|B\|$ we get that

$$\|T^2\| = \|TT\| = \|T^*T\| \leq \|T\|\|T\| = \|T\|^2$$

Then

$$\|T\|^2 = \sup_{\|\psi\|=1} \|T\psi\|^2 = \sup_{\|\psi\|=1} \langle T\psi, T\psi\rangle = \sup_{\|\psi\|=1} \langle TT\psi, \psi\rangle \overset{\text{C.S.}}{\leq}$$
$$\overset{\text{C.S.}}{\leq} \sup_{\|\psi\|=1} \|T\psi\|^2\|\psi\|^2 = \|T^2\|$$

Now using induction the statement follows. From the definition of limit we have that

$$|r(T) - \|T^n\|^{\frac{1}{n}}| < \varepsilon$$

from what we just proved:

$$|r(T) - \|T\|^{\frac{n}{n}}| = |r(T) - \|T\|| < \varepsilon$$

meaning that

$$r(T) = \lim_{n\to\infty} \|T\| = \|T\|$$