MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 8

All exercises are worth 5 points unless otherwise indicated. Due on Friday, December 17, at 12:00 noon via URM.

- **Exercise 1** clarifies that the formulation of the spectral theorem for matrices that we will be using in lecture is indeed equivalent to the familiar formulation from linear algebra.
- **Exercise 2** gives an example of a spectral decomposition of an operator with infinitely many eigenvalues.
- **Exercise 3** examines the discrete cousin of the Laplacian. This example shows that also the spectral theory of *bounded* operator on infinite-dimensional Hilbert spaces can be of interest.
- **Exercise 4** introduces the influential *Anderson model* (which led to Anderson's Nobel prize in 1958) which describes electrons in an imperfect wire. The imperfect material is modeled by a random electric potential. This exercise proves an important fact — the spectrum is not random with probability 1, so no matter what wire we are looking at, the spectrum will be almost surely the same!

Exercise 1. Let H be a self-adjoint $n \times n$ matrix on the Hilbert space $\mathbb{C}^n$ with spectrum $\lambda_1, \dots, \lambda_m$ where $m < n$ if some eigenvalues are equal. Let $\mathcal{E}_1, \dots, \mathcal{E}_m$ denote the corresponding eigenspaces and let $P_{\mathcal{E}_j}$ denote the projectors onto $\mathcal{E}_j$.

(a) From linear algebra, it is known that H can be written as $H = U\Lambda U^*$ with U unitary and Λ a diagonal matrix comprised of the eigenvalues$\lambda_1, \dots, \lambda_m$ repeated according to their multiplicity.

Show that we can also write

$$H = \sum_{j=1}^{m} \lambda_j P_{\mathcal{E}_j}. \tag{1}$$

Hint: Start from the eigenvalue equation $Hv_j = \lambda_j v_j$ to connect the eigenvectors v_j to the unitary U.

(b) Prove the converse, i.e., that the representation (1) implies that H can be written as $H = U\Lambda U^*$ with U unitary and Λ diagonal.

Exercise 2. Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ denote the one-dimensional torus. Consider a function $g \in L^2(\mathbb{T})$ satisfying $g(x) = \overline{g(-x)}$ for a.e. $x \in \mathbb{T}$. Consider the operator T defined by

$$T\psi = g * \psi.$$

(a) Show that T is defined on all of $L^2(\mathbb{T})$ and bounded.

(b) Prove that T is self-adjoint.

(c) Find an orthonormal system $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ and real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that
$$T = \sum_{n\in\mathbb{Z}} \lambda_n |e_n\rangle\langle e_n|$$
where $|e_n\rangle\langle e_n|$ denotes the projector onto the one-dimensional subspace spanned by e_n ("Dirac notation").
Hint. Remember that convolution behaves well in Fourier space. What is the analog of the Fourier transform for functions defined on the torus $\mathbb{T}$?

(d) Show that the spectrum is $\sigma(T) = \{\lambda_n\}_{n\in\mathbb{Z}} \cup \{0\}$. What is $\sigma_p(T)$?

Exercise 3. On the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$, consider the discrete Laplacian $\Delta_{\mathbb{Z}}$ which acts on a sequence $(\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ by
$$(\Delta_{\mathbb{Z}}\psi)_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$$

(a) Prove that $\Delta_{\mathbb{Z}}$ is bounded with $\|\Delta_{\mathbb{Z}}\| \leq 4$.
(b) Prove that $\Delta_{\mathbb{Z}}$ is self-adjoint.
(c) Use Fourier theory to show that $\Delta_{\mathbb{Z}}$ is unitarily equivalent to a multiplication operator.
(d) Prove that $\sigma(\Delta_{\mathbb{Z}}) = [-4, 0]$.

Remark: The discrete Laplacian describes "hopping" between neighboring sites and arises in the study of tight-binding models in condensed matter-physics, where "hopping" amounts to quantum tunneling through potential barriers.

Exercise 4. Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence of independent fair coin flips with outcomes 0 and $v \in \mathbb{R} \setminus \{0\}$ occurring with probability $\frac{1}{2}$ each.

On the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$, consider the discrete Schrödinger operator H which acts on a sequence $(\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ by
$$(H\psi)_n = (-\Delta_{\mathbb{Z}}\psi)_n + V_n\psi_n.$$

(a) Prove that H is bounded with $\|H\| \leq 4 + |v|$ and self-adjoint.
(b) Prove that $\sigma(H) = [0, 4] \cup [v, v + 4]$ holds with probability equal to 1.
Hint: This relies on the principle that for infinitely many random trials "whatever can happen locally, will happen somewhere". Use the existence of certain rare local events to construct appropriate Weyl sequences.