INTRODUCTION TO PDE **Assignment 7**

Due date: Thursday, December 7th WS 2023-2024

Exercise 1. Let $\Omega \subset \mathbb{R}^n$ open and define $\Omega_\varepsilon \doteq \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$ for $\varepsilon > 0$. Consider $u \in L^1_{loc}(\Omega)$ and its mollification $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$

$$u_\varepsilon(x) \doteq (\rho_\varepsilon * u)(x) = \int_\Omega \rho_\varepsilon(x-y)u(y)\,dy$$

with respect to the standard mollifier[1]. Show that

(i) if $u \in C^0(\Omega)$, then $u_\varepsilon \to u$ uniformly in compact subsets as $\varepsilon \to 0$.

(ii) if $u \in L^p(\Omega)$ for $1 \le p < \infty$, then $u_\varepsilon \to u$ in L^p_{loc} as $\varepsilon \to 0$.

(iii) if $u \in W^{k,p}(\Omega)$ for $1 \le p < \infty$ and $k \ge 0$, then $u_\varepsilon \to u$ in $W^{k,p}_{loc}$ as $\varepsilon \to 0$.

(iv) if $u \in C^{0,1}(\Omega)$ with Lipschitz constant L, then all $u_\varepsilon \in C^{0,1}(\Omega)$ with Lipschitz constant L.

Exercise 2. Find the weak derivative of the function

$$f(x) = \begin{cases} 2x+1, & x \le 0 \\ 1-3x, & x \ge 0 \end{cases}$$

and actually show that it is the weak derivative of f.

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ open and bounded with $0 \in \Omega$. Show that the function $u(x) = |x|^{-\alpha}$ is in $W^k(\Omega)$ as long as $k + \alpha < n$[2].

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ and consider an open subset $\Omega' \subset\subset \Omega$ with $d \doteq \mathrm{dist}(\Omega', \partial\Omega)$. Show that there exists a function $\eta \in C^\infty_c(\Omega)$ and a constant $C = C(n)$ such that

$$0 \le \eta \le 1, \qquad \eta|_{\Omega'} \equiv 1, \qquad |D\eta| \le \frac{C}{d}\,.$$

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[1] The rotationally symmetric mollifier we have seen in the lecture.

[2] By W^k we refer to the space of k-times weakly differentiable functions.