INTRODUCTION TO PDE **Assignment 6**

Due date: Thursday, November 30th WS 2023-2024

Exercise 1. Let $u \in C^1(\mathbb{R})$ be such that

$$\int_{\mathbb{R}} u'(x)\, \varphi'(x)\, dx = 0$$

for every $\varphi \in C_c^\infty(\mathbb{R})$. Show that u is a linear function.

Hint: For each interval $[a,b]$, carefully choose a sequence of functions $\varphi_k \in C^\infty(\mathbb{R})$ which approximate the characteristic function on $[a,b]$ and send $k \to \infty$.

Exercise 2. Let $1 \le p, q \le \infty$. Show that given $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |fg|^k \right)^{\frac{1}{k}} dx \le \left(\int_{\mathbb{R}^n} |f|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g|^q dx \right)^{\frac{1}{q}}$$

can only hold if

$$\frac{1}{k} = \frac{1}{p} + \frac{1}{q}\,.$$

Exercise 3. Show that for each $u \in C_c^1(\mathbb{R})$ it holds that

$$\sup_{\mathbb{R}} |u| \le \frac{1}{2} \|Du\|_{L^1(\mathbb{R})}\,.$$

Exercise 4. Let $\Omega \in \mathbb{R}^n$ open and bounded.

(i) Show that

$$L^p(\Omega) \subset L^q(\Omega)$$

for all $1 \le q \le p \le \infty$.

(ii) Let $(f_n)_n \subset L^p(\Omega)$. Show that if $f_n \to f$ uniformly, then $f \in L^p(\Omega)$ and $f_n \to f$ in L^p.

(iii) Let $(f_n)_n \subset L^p(\Omega)$. Show that $f_n \to f$ pointwise does not imply $f_n \to f$ in L^p.