MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 4

All exercises are worth 5 points. Due on Friday, November 17, at 12:00 noon via URM.

- **Exercise 1** practices distributional derivatives for tempered distributions.
- **Exercise 2** solves the free Schrödinger equation for initial data that we could not handle with the Schwartz-based theory before: a quantum particle uniformly localized in a box.
- **Exercise 3** explores the concepts of SCOPUG and its generator on a finite-dimensional quantum state space. There, linear operators are just matrices and many subtleties are absent.The last part shows that, since there is "not much room" in finite dimensions, the system keeps (almost) returning to the same state as time passes.
- **Exercise 4** develops an important example of a linear unbounded operator — the Laplacian. Keep this in mind as it provides a strong motivation for studying the spectral theory of unbounded linear operators in Unit 2 of the course: they are everywhere in quantum mechanics! (Unbounded operators only exist in infinite dimensions, the simplest of which are sequence spaces like ℓ^2. While the example in Part (b) may seem artificial, Part (c) shows that it is the same mechanism underlying the unboundedness of the Laplacian.)

Exercise 1. Define $\mathrm{a} : \mathbb{R} \to \mathbb{R}$ as $\mathrm{a}(x) = |x|$.

(a) Prove that $T_\mathrm{a} \in \mathcal{S}'(\mathbb{R})$.
(b) Compute $\frac{d^n}{dx^n} T_\mathrm{a}(f)$ for all $n \in \mathbb{N}$.
(c) For all $n \in \mathbb{N}$, prove that there does not exist a function $g_n \in C^\infty_{\text{poly}}(\mathbb{R})$ such that $\frac{d^n}{dx^n} T_\mathrm{a}(f) = T_{g_n}$.

Exercise 2. Let $\chi_{[-1,1]} : \mathbb{R} \to \mathbb{R}$ be the following indicator function

$$\chi_{[-1,1]}(x) = \begin{cases} 1, & \text{if } x \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that there exists a distributional solution $\psi_t \in \mathcal{S}'(\mathbb{R})$ to the free Schrödinger equation with initial datum $\psi_0 = T_{\frac{\chi_{[-1,1]}}{\sqrt{2}}} \in \mathcal{S}'(\mathbb{R})$.
(b) Show that the distributional solution ψ_t can be identified with a function in $\tilde{\psi}_t \in L^2(\mathbb{R})$. For any $t > 0$, compute the large-distance asymptotics of $\tilde{\psi}_t(x)$, namely its behavior as $|x| \to \pm\infty$.

Date: November 13. Added clarifications to exercises 3 and 4.

(c) Interpret the result of (b) quantum-mechanically: What happens to a free particle that is initially localized inside of a box after long time?

Exercise 3.

Let $H \in \mathbb{C}^{d\times d}$ be a Hermitian matrix.

(a) Prove that the matrix sum
$$\sum_{n=0}^{\infty} \frac{1}{n!}(-itH)^n$$
is convergent. Denote it by e^{-itH}.

(b) Prove that $U(t) = e^{-itH}$ forms a strongly continuous one-parameter unitary group (SCOPUG).

(c) Prove that for every $\psi_0 \in \mathbb{C}^d$, it holds that $i\partial_t U(t)\psi_0 = HU(t)\psi_0$.

(d) Prove that the associated quantum dynamics is *recurrent* (or almost-periodic): For any $\epsilon > 0$, there exists a sequence of times $\tau_n \to \infty$ such that
$$\|\psi_{\tau_n} - \psi_0\| < \epsilon$$
(This is the quantum analog of the Poincaré recurrence theorem from classical mechanics.)

Hint. Use that, by Dirichlet's approximation theorem, for any real numbers $\lambda_1, \dots, \lambda_n$ and for any natural number K there exists integers $p_1, \dots, p_n$, $1 \le q \le K$ such that
$$\left|\lambda_i - \frac{p_i}{q}\right| \le \frac{1}{qK^{1/n}} \ .$$

Exercise 4.

(a) Consider the following sequence space
$$\ell^2 = \left\{ (a_n)_{n\ge1} \subset \mathbb{C} \ : \ \sum_{n\ge1} |a_n|^2 < \infty \right\}.$$
Check that
$$\|(a_n)_{n\ge1}\|_{\ell^2} = \sqrt{\sum_{n\ge1} |a_n|^2}.$$
defines a norm on ℓ^2.

(b) Consider the subspace
$$X = \left\{ (a_n)_{n\ge1} \subset \mathbb{C} \ : \ \sum_{n\ge1} (1+n^2)|a_n|^2 < \infty \right\} \subset \ell^2.$$
Define the operator $\mathcal{N}$ on X by setting $\mathcal{N} a_n = n a_n$ for every $n \ge 1$. Prove that $\mathcal{N} : X \to \ell^2$ is a linear unbounded operator between the normed spaces $(X, \|\cdot\|_{\ell^2})$ and $(\ell^2, \|\cdot\|_{\ell^2})$.

(c) Consider the Sobolev space
$$H^2(\mathbb{R}^d) = \left\{ \psi \in L^2(\mathbb{R}^d) \ : \ \sqrt{1+k^2}\hat{\psi} \in L^2(\mathbb{R}^d) \right\}$$
Prove that the Laplacian $-\Delta$ is a linear unbounded operator between the normed space $(H^2, \|\cdot\|_2)$ and $(L^2, \|\cdot\|_2)$.

Remarks. (i) We say that a linear operator A is unbounded, if it is not a bounded operator, i.e., if $\|A\| = \infty$.
(ii) For part (c), we interpret $-\Delta\psi$ for $\psi \in H^2$ as the distributional derivative $-\Delta T_\psi \in \mathcal{S}'$.