MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 6

All exercises are worth 5 points. Due on Friday, December 1, via URM.

- **Exercise 1** completes the theory of generators of momentum on the circle. The argument here is prototypical in mathematical quantum theory: we want to prove that some function is in a Sobolev space, which means a distributional derivative is representable as an L^2-function. This can be done via integration by parts provided the boundary terms vanish due to suitable boundary conditions.
- **Exercise 2** provides an example of a densely defined, symmetric that is not closed and therefore has uninteresting spectral theory. This is a good argument that symmetry is not enough and we should look for self-adjointness!
- **Exercise 3** is an application of our criterion for essential self-adjointness. This calculation shows more, that "momentum on the half-line" cannot be given physical meaning in quantum mechanics. (This is in contrast to momentum on an interval which as we showed in exercise 1 has self-adjoint realizations, i.e., physical meaning)
- **Exercise 4** shows that self-adjointness is preserved by unitary conjugation — a very convenient fact!

Exercise 1. For $\theta \in [0, \pi)$, let $T_t^{(\theta)} : L^2([0,1]) \to L^2([0,1])$ be the translation operator defined by

$$(T_t^{(\theta)}\psi)(x) = e^{i\theta k}\psi(x - t + k), \qquad \text{if } x - t + k \in [0,1] \text{ for } k \in \mathbb{Z}.$$

(a) Prove that the generator of the SCOPUG $\{T_t^{(\theta)} : t \in \mathbb{R}\}$, call it $(D(H_\theta), H_\theta)$, satisfies

$$D(H_\theta) = \left\{\psi \in H^1([0,1]) : e^{i\theta}\psi(1) = \psi(0)\right\}$$
$$H_\theta = -\, i\frac{\mathrm{d}}{\mathrm{d}x}$$

(b) Prove that $(D(H_\theta), H_\theta))$ is self-adjoint.

Hint. In part (a), you may use the following description of the domain of the generator proved in class:

$$D(H_\theta) = \{\psi \in L^2([0,1]) : T_t^{(\theta)}\psi \in H^1([0,1])\}.$$

In class, we already proved that $T_t^{(\theta)}\psi \in H^1([0,1])$ implies $e^{i\theta}\psi(1) = \psi(0)$. Here you need to show the converse, so you want to show that the distributional derivative of $T_t^{(\theta)}\psi$ is representable by an L^2 function.

Exercise 2. Consider the Hilbert space $\mathcal{H} = L^2([-1,1])$.

(a) Let $T = \frac{d}{dx}$ be defined on the domain
$$D(T) = C^1([-1,1]),$$
where $\psi \in C^1([-1,1])$ means that the one-sided limits of ψ and its difference quotients exist at the boundary points. Prove that the operator $(T, D(T))$ is not closed.
Hint. You may find it helpful to consider the sequence $\psi_n(x) = \sqrt{x^2 + n^{-1}}$.

(b) Adapt the argument from (a) to show that also the operator $S = -i\frac{d}{dx}$ with domain
$$D(S) = C_c^\infty(-1,1)$$
is not closed.

(c) Verify that S is a densely defined symmetric operator.

Exercise 3. Let $\mathbb{R}_+ = (0,\infty)$ be the half-line. On the Hilbert space $L^2(\mathbb{R}_+)$, consider the momentum operator $T = -i\frac{d}{dx}$ with domain $D(T) = C_c^\infty(\mathbb{R}_+)$.

(a) Find its adjoint T^*.
(b) Calculate $\ker(T^* \pm i)$.
(c) Is T essentially self-adjoint?

Remark: A fact that we shall not prove in this course says that an operator has self-adjoint extensions if and only if $\dim\ker(T^* + i) = \dim\ker(T^* - i)$. *An operator that does not have self-adjoint extensions cannot possibly be given physical meaning as a quantum observable.*

Exercise 4.

(a) Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary between two Hilbert spaces. Let $(H, D(H))$ be a self-adjoint operator on $\mathcal{H}_1$. Prove that $(UHU^*, UD(H))$ is a self-adjoint operator on $\mathcal{H}_2$.

(b) Let $g \in L^\infty(\mathbb{R}^d)$ be real-valued. Prove that the unbounded operator $(g(-i\nabla), L^2(\mathbb{R}^d))$ is self-adjoint. (Recall that $g(-i\nabla) = \mathcal{F}^{-1}\mathcal{M}_g\mathcal{F}$.)

(c) Suppose that $z \in \mathbb{C}$ is such that $H - z$ is injective. Prove the identity
$$U(H-z)^{-1}U^* = (UHU^* - z)^{-1}$$