MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 9

All exercises are worth 5 points. Due on Friday, December 22, at 12:00 noon via URM.

- **Exercise 1** establishes useful basic properties of projection-valued measures (PVMs).
- **Exercise 2** provides a broad class of examples of PVMs — those arising from multiplication operators. It generalizes the considerations from class about the PVM associated to the position operator.
- **Exercise 3** rephrases the fact that we know how to "diagonalize" the Laplacian on H^2, now in terms of projection-valued measures.
- **Exercise 4** studies the spectral radius, a number capturing the extent of the spectrum of a bounded operator.

Exercise 1. Let $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be a projection-valued measure. Let $\Omega, \Omega_1, \Omega_2$ be Borel sets. Prove the following properties.

(a) $P(\emptyset) = 0$, $P(\Omega^{\mathrm{c}}) = \mathbb{1} - P(\Omega)$ (where $\mathbb{1}$ is the identity on $\mathcal{H}$)
(b) $P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2) - P(\Omega_1 \cap \Omega_2)$
(c) $P(\Omega_1 \cap \Omega_2) = P(\Omega_1)P(\Omega_2)$
(d) If $\Omega_1 \subseteq \Omega_2$, then $P(\Omega_1) \leq P(\Omega_2)$ (meaning that $\langle \psi, P(\Omega_1)\psi \rangle \leq \langle \psi, P(\Omega_2)\psi \rangle$ holds for all $\psi \in \mathcal{H}$).

Exercise 2. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called Borel measurable, if for any $\Omega \in \mathcal{B}(\mathbb{R})$, it holds that $f^{-1}(\Omega) \in \mathcal{B}(\mathbb{R})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Prove that

$$P(\Omega) = \chi_{f^{-1}(\Omega)}$$

defines a projection-valued measure.

Remark. *Once we construct for any self-adjoint operator an associated PVM, we will see that the PVM's constructed here are precisely those associated to the self-adjoint multiplication operator $\mathcal{M}_f$. (Recall that the multiplication operator $\mathcal{M}_f\psi(x) = f(x)\psi(x)$ is indeed self-adjoint on its natural domain $D(\mathcal{M}_f)$ whenever f is real-valued.)*

Exercise 3. Consider the self-adjoint operator $T = -\Delta$ with $D(T) = H^2(\mathbb{R}^d)$. Find a PVM so that

$$D(T) = \left\{ \psi \in L^2(\mathbb{R}^d) \,:\, \int_{\mathbb{R}} \lambda^2 d\mu_\psi(\lambda) < \infty \right\},$$
$$T = \int_{\mathbb{R}} \lambda dp(\lambda)$$

where the second identity is interpreted as usual as

$$\langle \varphi, T\psi \rangle = \int_{\mathbb{R}} \lambda \, d\mu_{\varphi,\psi}(\lambda), \qquad \varphi, \psi \in D(T).$$

Remark. The fact that this type of representation exists for any self-adjoint $(T, D(T))$ *is precisely the content of the general spectral theorem that we will prove soon.*

Exercise 4. Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator.

(a) Liouville's theorem from complex analysis says that any bounded analytic function $\mathbb{C} \to \mathbb{C}$ is constant. Use this fact to prove that $\sigma(T) \neq \emptyset$.

(b) Define the spectral radius of T as

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Prove that $r(T) \leq \|T\|$.

(c) Gelfand's spectral radius formula says that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. Use this fact to prove that if T is a self-adjoint operator, then

$$r(T) = \|T\|.$$

Hint. The power series representation of the resolvent is helpful.