
Exercise 1. (Interpolation Inequality)

(i) Let $a, b \geq 0$. Show that for every $\varepsilon > 0$,

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}\,.$$

(ii) Let $u \in C^2(\bar{\Omega})$ and $u = 0$ on $\partial\Omega \in C^1$. Show that for every $\varepsilon > 0$ it holds that

$$\int_\Omega |Du|^2\,dx \leq \varepsilon \int_\Omega (\Delta u)^2\,dx + \frac{1}{4\varepsilon}\int_\Omega u^2\,dx\,.$$

Exercise 2. (Affine barriers) Let $B_1 \doteq B_1(0) \subset \mathbb{R}^n$ and $u \in C^2(\bar{B}_1)$ satisfy

$$\begin{cases} \Delta u &= 0 \quad \text{in } B_1 \\ u &= \varphi \quad \text{on } \partial B_1 \end{cases},$$

where $\varphi \in C^2(\bar{B}_1)$.

(i) Show that

$$\max_{\bar{B}_1} |Du|^2 \leq \max_{\partial B_1} |Du|^2\,.$$

(ii) Use affine barrier[1] functions to show that there is a constant C depending on n and $\|\varphi\|_{C^2(\bar{B}_1)}$ such that

$$\sup_{B_1} |Du|^2 \leq C + \sup_{\partial B_1} |D\varphi|^2\,.$$

(You may restrict to the $n = 2$ case for simplicity)

(iii) Does the estimate extend to arbitrary convex smooth domain?

[1] By affine barrier we mean a flat graphical surface of the form $W : \mathbb{R}^n \to \mathbb{R}$, $W(x) = C + \langle v, x - x_0\rangle$, where C is a constant and $v, x_0 \in \mathbb{R}^n$.

Exercise 3. Let $\Omega \in \mathbb{R}^n$ open, $n \geq 3$, $f \in C(\Omega)$ and $u \in C^2(\Omega)$ a solution of $-\Delta u = f$. Show that for every ball $B_r(x) \subset \Omega$ it holds that

$$u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, ds + \frac{1}{(n-2)n\omega_n} \int_{B_r(x)} \left(\frac{1}{|x-z|^{n-2}} - \frac{1}{r^{n-2}} \right) f(z) dz\,.$$

Hint: modify the proof of the Mean Value Formula.

Exercise 4. Let $0 \in \Omega \subset \mathbb{R}^n$ open. Show that there is no integrable function g in Ω such that

$$\int_\Omega g(x)\eta(x)\, dx = \eta(0)$$

for all $\eta \in C_c^0(\Omega)$.