Exercises - MQT

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Sheet 8

1 Exercise

Let H be a self-adjoint $n \times n$ matrix on the Hilbert space $\mathbb{C}^n$ with spectrum $\lambda_1, \ldots, \lambda_m$ where $m < n$ if some eigenvalues are equal. Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be the corresponding eigenspaces with $P_{\mathcal{E}_j}$ the projection onto $\mathcal{E}_j$.

(a) From linear algebra, it is known that H can be written as $H = U\Lambda U^*$ with U unitary and Λ a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$ repeated according to their multiplicity: we want to prove that it is also possible to write H as

$$H = \sum_{j=1}^{m} \lambda_j P_{\mathcal{E}_j}$$

If we have that $\widetilde{P}_{\mathcal{E}_j}$ as the matrix which is 1 on the diagonal on the positions $j, j+1, \ldots, j+n_j$ (where n_j is the multiplicity of the λ_j eigenvalue) and zero everywhere else. We can write

$$\Lambda = \sum_{m}^{j=1} \lambda_j \widetilde{P}_{\mathcal{E}_j}$$

multiplying both sides by U, U^*:

$$U\Lambda U^* = \sum_{m}^{j=1} \lambda_j U \widetilde{P}_{\mathcal{E}_j} U^* = H$$

Now defining $P_{\mathcal{E}_j} = U\widetilde{P}_{\mathcal{E}_j} U^*$ we get that the two expression are equivalent.

(b) Now let us prove that the converse:

$$H = \sum_{j=1}^{m} \lambda_j P_{\mathcal{E}_j}$$

using the previous definition of $U^* P_{\mathcal{E}_j} U = \widetilde{P}_{\mathcal{E}_j}$ we get

$$U^* H U = \sum_{m}^{j=1} \lambda_j \widetilde{P}_{\mathcal{E}_j} = \Lambda$$

2 Exercise

Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be the one dimensional torus. Consider the function $g \in L^2(\mathbb{T})$ satisfying $g(x) = \overline{g(-x)}$ for almost every $x \in \mathbb{T}$. Taking the operator T defined as:

$$T\psi = g * \psi$$

(a) We want to show T to be bounded and defined on all $L^2(\mathbb{T})$. Let us take a generic test function $\psi \in L^2(\mathbb{T})$, and consider

$$\left| \int_{\mathbb{T}} T\psi(x)dx \right|^2 \leq \int_{\mathbb{T}} |T\psi(x)|^2 dx = \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \psi(x-y)g(y)dy \right|^2 dx \leq$$
$$\leq \int_{\mathbb{T}} \int_{\mathbb{T}} |\psi(x-y)g(y)|^2 dydx =$$

By Fubini's theorem we can interchange the order of integration

$$= \int_{\mathbb{T}} |g(y)|^2 \int_{\mathbb{T}} |\psi(x-y)|^2 dxdy =$$

from the fact that ψ is periodic since $x \in \mathbb{T}$

$$= \int_{\mathbb{T}} |(y)|^2 \int_{\mathbb{T}} |\psi(x)|^2 dxdy = \|g\|_2^2 \cdot \|\psi\|_2^2$$

meaning it is well-defined.

$$\|T\psi\|_2 = \|g\|_2 \|\psi\|_2 \leq C\|\psi\|_2$$

since $g \in L^2(\mathbb{T})$.

(b) We want to see that T is self-adjoint. Since it is bounded it is also densly compact, and we are left to show that it is symmetric. Let $\psi, \varphi \in L^2$

$$\langle \psi, T\varphi \rangle = \int_{\mathbb{T}} \overline{\psi}(x)\, g * \varphi(x)dx = \int_{\mathbb{T}} \overline{\psi}(x) \int_{\mathbb{T}} g(y)\varphi(x-y)dydx =$$

let us change variable $u = x - y$ Then

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} \overline{\psi}(u+y)g(y)\varphi(u)dydu = \int_{\mathbb{T}} \int_{\mathbb{T}} \overline{\psi}(u+y)\overline{g}(-y)\varphi(u)dydu =$$

changing the direction over which we are integrating around the torus

$$= -\int_{\mathbb{T}} \int_{-\mathbb{T}} \overline{\psi}(u-y)\overline{g}(y) = \varphi(u)dydu = \int_{\mathbb{T}} \int_{\mathbb{T}} \overline{\psi}(u-y)\overline{g}(y)\varphi(u)dydu =$$
$$= \int_{\mathbb{T}} \overline{g * \psi}(x)\varphi(x)dx = \langle \mathbb{T}\psi, \varphi \rangle$$

showing tha it is symmetric and therefore self-adjoint.

(c) We now want ro find am orthonormal system $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ and real numbers $\{\lambda_n\}_{\in\mathbb{Z}}$ such that

$$T = \sum_{n\in\mathbb{Z}} \lambda_n \left|e_n\right\rangle \left\langle e_n\right|$$

Taking

$$Te_n(x) = (g * e_n)(x)$$

let us assume that the solution of the eigenvalue problem $Te_n = \lambda_n e_n$ is given by $e_n = e^{inx}$. Then

$$\begin{aligned} Te_n &= \int_{\mathbb{T}} g(y) e^{in(x-y)} dy = e^{inx} \int_{\mathbb{T}} g(y) e^{-iny} dy = \\ &= e^{inx} \sqrt{2\pi} \hat{g}(n) = \lambda_n e_n \end{aligned}$$

meaning that $\lambda_n = \sqrt{2\pi}\hat{g}(n)$.

(d) Let us now show that the spectrum of this operator is $\sigma(T) = \{\lambda_n\}_{n\in\mathbb{Z}} \cup \{0\}$. Since we found that the eigenvalues of the operator are $\{\lambda_n\}_{n\in\mathbb{Z}}$ we have that

$$\sigma_P(T) = \{\lambda_n\}_{n\in\mathbb{Z}}$$

We know that

$$\sigma(T) \supset \overline{\sigma_p(T)} = \{\lambda_n\}_{n\in\mathbb{Z}} \cup \{0\}$$

from the fact that

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \int_{\mathbb{T}} g(y) e^{-iny} dy = 0$$

by the Riemann lemma. We are left to show that $\sigma(T) = \overline{\sigma_p(T)}$.

3 Exercise

On a Hilbert space $\mathbb{H} = \ell^2(\mathbb{Z})$ we consider the Laplacian which acts on a sequence $(\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ By

$$(\Delta_{\mathbb{Z}}\psi)_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$$

(a) Let us show that $\Delta_{\mathbb{Z}}$ is bounded. If we take

$$\|(\Delta_{\mathbb{Z}}\psi)_n\|_{\ell^2} = \|\psi_{n+1} + \psi_{n-1} - 2\psi_n\|_{\ell^2}$$

using the triangular inequality:

$$\|(\Delta_{\mathbb{Z}}\psi)_n\|_{\ell^2} \leq \|\psi_{n+1}\|_{\ell^2} + \|\psi_{n-1}\|_{\ell^2} + 2\|\psi_n\|_{\ell^2}$$

using the definition of ℓ^2 norm

$$\|(\Delta_{\mathbb{Z}}\psi)_n\|_{\ell^2} \leq \left(\sum_{n\in\mathbb{Z}} |\psi_{n+1}|\right)^{1/2} + \left(\sum_{n\in\mathbb{Z}} |\psi_{n-1}|\right)^{1/2} + 2\left(\sum_{n\in\mathbb{Z}} |\psi_n|\right)^{1/2}$$

doing a redefinition of the indices

$$\|(\Delta_{\mathbb{Z}}\psi)_n\|_{\ell^2} \leq \left(\sum_{n\in\mathbb{Z}} |\psi_n|\right)^{1/2} + \left(\sum_{n\in\mathbb{Z}} |\psi_n|\right)^{1/2} + 2\left(\sum_{n\in\mathbb{Z}} |\psi_n|\right)^{1/2} = 4\|\psi_n\|_{\ell^2}$$

(b) We want to show $\Delta_{\mathbb{Z}}$ self-adjoint. Since it is bounded we know it is densly-defined, we are therefore left to show it is symmetric.

$$\begin{aligned}
\langle \psi_n, \Delta_{\mathbb{Z}}\varphi_n\rangle &= \sum_{n\in\mathbb{Z}} \overline{\psi_n}(\varphi_{n+1} + \varphi_{n-1} - 2\varphi_n) = \\
&= \sum_{n\in\mathbb{Z}} \overline{\psi_n}\varphi_{n+1} + \sum_{n\in\mathbb{Z}} \overline{\psi_n}\varphi_{n-1} - 2\sum_{n\in\mathbb{Z}} \overline{\psi_n}\varphi_n = \\
&= \sum_{n\in\mathbb{Z}} \overline{\psi_{n-1}}\varphi_n + \sum_{n\in\mathbb{Z}} \overline{\psi_{n+1}}\varphi_n - 2\sum_{n\in\mathbb{Z}} \overline{\psi_n}\varphi_n = \langle \Delta_{\mathbb{Z}}\psi_n, \varphi_n\rangle
\end{aligned}$$

where in we renamed the indices from the second line to the third. Proving that it is symmetric and therefore self-adjoint.

(c) Using the Fourier theory let us show that $\Delta_{\mathbb{Z}}$ is unitarly equivalent to a multiplication operator. Let us consider the Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z})$, which for an ℓ^2 function is defined as

$$\mathcal{F}A_n = \sum_{n=-\infty}^{\infty} A_n e^{-i2\pi nx}$$

Then we want to find $\mathcal{M}_g$ such that

$$\begin{aligned}
\mathcal{F}^{-1}\mathcal{M}_g\mathcal{F}\psi_n &= \Delta_{\mathbb{Z}}\psi_n \\
\mathcal{M}_g\mathcal{F}\psi_n &= \mathcal{F}\Delta_{\mathbb{Z}}\psi_n
\end{aligned}$$

Using the definition of $\Delta_{\mathbb{Z}}$ and the definition of Fourier transform

$$\begin{aligned}
\mathcal{F}\Delta_{\mathbb{Z}}\psi_n &= \mathcal{F}(\psi_{n+1} + \psi_{n-1} - 2\psi_n) = \\
&= \sum_{n=-\infty}^{\infty} (\psi_{n+1} + \psi_{n-1} - 2\psi_n)e^{i2\pi nx} = \\
&= \sum_{n=-\infty}^{\infty} \psi_{n+1}e^{i2\pi nx} + \sum_{n=-\infty}^{\infty} \psi_{n-1}e^{i2\pi nx} - \sum_{n=-\infty}^{\infty} 2\psi_n e^{-i2\pi nx} = \\
&= \sum_{n=-\infty}^{\infty} \psi_n e^{-i2\pi(n-1)x} + \sum_{n=-\infty}^{\infty} \psi_{n-1}e^{-i2\pi(n+1)x} - \sum_{n=-\infty}^{\infty} 2\psi_n e^{-i2\pi nx} =
\end{aligned}$$

where in the last step we renamed the indices

$$\begin{aligned}
\mathcal{F}\Delta_{\mathbb{Z}}\psi_n &= (e^{i2\pi x} + e^{-i2\pi x} - 2)\sum_{n=-\infty}^{\infty} \psi_n e^{-i2\pi nx} = \\
&= (2\cos(2\pi x) - 2)\mathcal{F}\psi_n.
\end{aligned}$$

Using both the Euler identity and the fact that the last factor was equivalent to the Fourier transform of ψ_n. Now

$$\mathcal{M}_{(2\cos(2\pi x)-2)} = \mathcal{F}\Delta_{\mathbb{Z}}\mathcal{F}^{-1}$$

(d) We want to show that $\sigma(\Delta_{\mathbb{Z}}) = [-4, 0]$. Let us use a property proved in the previous sheet to say that $\sigma(T) = \sigma(UTU^*)$ if U is unitary, which works since $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{R}/\mathbb{Z})$. Then

$$\sigma(\Delta_{\mathbb{Z}}) = \sigma\left(\mathcal{F}\Delta_{\mathbb{Z}}\mathcal{F}^{-1}\right) = \sigma\left(\mathcal{M}_{(2\cos(2\pi x)-2)}\right)$$

again, from the previous sheet we proved that $\sigma(\mathcal{M}_f) = \text{ran}\ (f)$ which is enough to prove that

$$\sigma(\Delta_{\mathbb{Z}}) = [-4, 0]$$

4 Exercise

Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence of independent coin flip with outcomes 0 and $v \in \mathbb{R}\setminus\{0\}$ occurring with probability 1/2. On the Hilber space $\mathcal{H} = \ell^2(\mathbb{Z})$, consider the discrete Schrödinger equation operator H acting on $\{\psi_n\}_{n\in\mathbb{Z}}$ by

$$(H\psi)_n = (-\Delta_{\mathbb{Z}}\psi)_n + V_n\psi_n$$

(a) We want to show that H is bounded by $\|H\| \leq 4 + |v|$, and it is self adjoint. The definition of bound operator requires that

$$\|H\| = \sup_{\psi\in\mathbb{H}} \frac{\|(H\psi)_n\|_{\ell^2}}{\|\psi_n\|_{\ell^2}} \leq C$$

Let us start by considering

$$\|(H\psi)_n\|_{\ell^2} = \| - \Delta_{\mathbb{Z}}\psi_n + V_n\psi_n\|_{\ell^2} \leq \|\Delta_{\mathbb{Z}}\psi_n\|_{\ell^2} + \|V_n\psi_n\|_{\ell^2}$$

using the triangular inequality. Moreover, from part (a) of exercise 3, we have that $\|\Delta_{\mathbb{Z}}\| \leq 4$ from which

$$\|\Delta_{\mathbb{Z}}\psi_n\|_{\ell^2} + \|V_n\psi_n\|_{\ell^2} \leq 4\|\psi_n\| + \|V_n\psi_n\|_{\ell^2}$$

Now we want to bound from above $\|V_n\psi_n\|_{\ell^2}$. By definition of $\ell^2(\mathbb{Z})$ we have that:

$$\|V_n\psi_n\|_{\ell^2} = \left(\sum_{n=-\infty}^{\infty} |V_n\psi_n|^2\right)^{1/2}$$

We can construct a new sequence $W_n = V_{\sigma(n)}$ such that

$$W_n = \begin{cases} v & n \geq 0 \\ 0 & n < 0 \end{cases}$$

So that we did not change the result of the sum:

$$\sum_{n=-\infty}^{\infty} |V_n\psi_n|^2 = \sum_{n=-\infty}^{\infty} |W_n\psi_n|^2 = \sum_{n=-\infty}^{-1} |0\psi_n|^2 + \sum_{n=0}^{\infty} |v\psi_n|^2 = |v|^2 \sum_{n=-\infty}^{\infty} |\psi_n|^2$$

Then we have that

$$\|V_n\psi_n\|_{\ell^2} = |v|\|\psi_n\|_{\ell^2}$$

and

$$\|(H\psi)_n\|_{\ell^2} \leq 4\|\psi_n\|_{\ell^2} + |v|\|\psi_n\|_{\ell^2}$$

Using the definition we prove $\|H\|_{\ell^2}$ is bounded

$$\|H\| = \sup_{\psi\in\mathcal{H}} \frac{\|(H\psi)_n\|_{\ell^2}}{\|\psi_n\|_{\ell^2}} \leq \frac{(4+|v|)\|\psi_n\|_{\ell^2}}{\|\psi_n\|_{\ell^2}} = 4 + |v|$$

We now want to show that it is self-adjoint, i.e. symmetric and densly defined. The second is guaranteed by the fact that it is bounded, let us show that it is symmetric to complete the proof.

$$\langle\psi_n, (H\varphi)_n\rangle = \langle\psi_n, -\Delta_Z\varphi_n + V_n\varphi\rangle = \langle-\Delta_{\mathbb{Z}}\psi_n, \varphi_n\rangle + \langle\psi_n, V_n\varphi_n\rangle$$

since by (b) of exercise 3 $\Delta_{\mathbb{Z}}$ is self-adjoint. Now using the definition of ar product in $\ell^2(\mathbb{Z})$:

$$\langle\psi_n, V_n\varphi_n\rangle = \sum_{n=-\infty}^{\infty} \overline{\psi_n} V_n\varphi_n = \sum_{n=-\infty}^{\infty} \overline{V_n\psi_n}\varphi_n = \langle V_n\psi_n, \varphi_n\rangle$$

since V_n is real-valued. Showing it is symmetric and therefore self-adjoint.

(b) We now want to prove that $\sigma(H) = [0,4]\cup[v, v+4]$ holds with probability 1. Using a procedure similar to part (d) of exercise 3 we look for a multiplication operator $\mathcal{M}_g$ such that

$$\mathcal{F}^{-1}\mathcal{M}\mathcal{F}\psi_n = (H\psi)_n = (-\Delta_{\mathbb{Z}} + V_n)\psi_n$$

which means finding the Fourier transform of:

$$\begin{aligned}\mathcal{F}(-\Delta_{\mathbb{Z}} + V_n)\psi_n = \mathcal{F}(-\psi_{n+1} - \psi_{n-1} + 2\psi_n + V_n) = \\ = \sum_{n=-\infty}^{\infty} (-\psi_{n+1} - \psi_{n-1} + 2\psi_n + V_n)e^{i2\pi nx} =\end{aligned}$$

using the same strategy as before, we arrive at a point in which we need to consider the two cases $V_n = 0$, in which the result is the same as in part (d) of exercise 3, but with inverted signs, and $V_n = v$ which just adds a scalar factor v to the multiplier $(\mathcal{M}_{2+2\cos(2\pi x)+v})$ whose range becomes $[v, v+4]$. The probability of having $[0,4]$ and $[v, v+4]$ is both 1/2. The union of the two cases gives us the range of the total Schrödinger operator H, $\sigma(H) = [0,4]\cup[v, v+4]$ with probability 1.