Exercises Partial differential equation

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Sheet 7

1 Exercise

Let us consider $\Omega \subset \mathbb{R}^n$ an open set and let us define $\Omega_\varepsilon \doteq \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$. Let us now consider $u \in L^1_{loc}(\Omega)$ and its mollification $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$

$$u_\varepsilon(x) = (\rho_\varepsilon * u)(x)$$

we want to show that

a) If $u \in C^0(\Omega)$ than $u_\varepsilon \to u$ uniformly in compact subsets as $\varepsilon \to 0$. By definition of continuous we know that:

$$\forall \varepsilon > 0 \exists \delta : \forall N_\varepsilon \in \mathbb{N}, x, y \in \Omega |x - y| < \delta, |u(x) - u(y)| < \varepsilon$$

If we consider

$$|u_\varepsilon(x) - u(x)| = \left| \int_{\Omega'} \rho_\varepsilon(x - y) u(y) dy - u(x) \right|$$

Using the commutability of the convolution operator we have that

$$\left| \int_{\Omega'} \rho_\varepsilon(x - y) u(y) dy - u(x) \right| = \left| \int_{\Omega'} \rho_\varepsilon(y) u(x - y) dy - u(x) \int_{\Omega'} \rho_\varepsilon(y) dy \right|$$

where Ω' is the set where ρ_ϵ is defined.

$$\left| \int_{\Omega'} \rho_\varepsilon(y)(u(x - y) - u(x)) dy \right| \leq \int_{\Omega'} |\rho_\varepsilon(y)||u(x - y) - u(x)| dy < \int_{\Omega'} |\rho_\varepsilon(y)| \varepsilon dy$$

Let us consider a ball $\overline{B_\varepsilon(0)} \supset \Omega'$

$$\int_{\Omega'} |\rho_\varepsilon(y)| \varepsilon dy \leq \int_{\overline{B_\varepsilon(0)}} |\rho_\varepsilon(y)| \varepsilon dy$$

for $\varepsilon \to 0$ this integral goes to zero, meaning that $\sup_{x \in \Omega} |u_\varepsilon(x) - u(x)| \to 0$ and therefore u_ε converges uniformly to u.

b) We now want to show that if $u \in L^p(\Omega)$ than $u_\varepsilon \to u$ in L^p_{loc} as $\varepsilon \to 0$. Let us consider

$$|u_\varepsilon(x) - u(x)|^p = \left| \int_{\Omega'} \rho_\varepsilon(x - y) u(y) dy - u(x) \right|^p = \left| \int_{\overline{B_\varepsilon(0)}} \rho_\varepsilon(y)(u(x - y) - u(x)) dy \right|^p$$

by the same argument as before. Now this is bounded from above by

$$\left|\int_{B_\varepsilon(0)} \rho_\varepsilon(y) 2 \sup_{B_\varepsilon(0)} u(x)dy\right|^p$$

If we take a subset $\Omega' \subset\subset \Omega$ we want to show tha definition of L^p space

$$\int_{\Omega'} |u_\varepsilon(x) - u(x)|^p dx \leq \int_{\Omega'} 2 \sup_{B_\varepsilon(0)} u(x)dx = 2 \sup_{B_\varepsilon(0)} u(x)\mu(\Omega')$$

which goes to 0 as $\varepsilon \to 0$ since $\Omega' \subset\subset \Omega$.

c) We want to show that if $u \in W^{k,p}(\Omega)$ and $k > 0$ then $u_\varepsilon \to u$ for $\varepsilon \to 0$ in $W^{k,p}_{loc}$. By definition of $W^{k,p}(\Omega)$ we know that $D^\alpha u \in L^p$ for $|\alpha| < k$, then, using the previous result and the fact that $(D^\alpha u)_\varepsilon = D^\alpha u_\varepsilon$ we get the claim.

d) We want to show that if $u \in C^{0,1}$ with Lipschitz constant L then $u_\varepsilon \in C^{0,1}$ with the same constant. By definition of Lipschitz we want to prove that

$$|u_\varepsilon(x) - u_\varepsilon(z)| < L|x - z|$$

because of that let us consider

$$|u_\varepsilon(x) - u_\varepsilon(z) = \left|\int_\Omega \rho_\varepsilon(x-y)u(y)dy - \int_\Omega \rho_\varepsilon(z-y)u(y)\right| =$$

Using the commutation property for convolutions we have that:

$$= \left|\int_\Omega \rho_\varepsilon(y)(u(x-y) - u(z-y))dy\right| < \int_\Omega |\rho_\varepsilon(y)|L|x-z|dy$$

Since $u \in C^{0,1}(\Omega)$. Proving the statement.

2 Exercise

We want to find the weak derivative of

$$f(x) = \begin{cases} 2x - 1 & x \leq 0 \\ 1 - 3x & x \geq 0 \end{cases}$$

By its definition we have that:

$$\begin{aligned} -\int v\varphi dx &= \int uD\varphi dx \\ &= \int_{-infty}^{0} (2x-1)\varphi' dx + \int_0^\infty (1-3x)\varphi' dx = \\ &= \int_{-infty}^{0} 2\varphi dx + \int_0^\infty 3\varphi dx \end{aligned}$$

using integration by parts. We then have that the directional derivative of f is

$$v(x) = \begin{cases} 2 & x < 0 \\ -3 & x > 0 \end{cases}$$

Since we computed it using the definition we have that it is in fact the weak derivative of f.

3 Exercise

Let $\Omega \subset \mathbb{R}^n$ open and bounded such that $0 \in \Omega$. We want to show that a function $u(x) = |x|^{-\alpha}$ is in $W^{k,0}(\Omega)$ as long as $k + \alpha < n$.

This means that we want to show that

$$\int_\Omega v\varphi dx = -(-1)^k \int_\Omega D^\beta \varphi |x|^{-\alpha} dx$$

converges, where we used a multi index β such that $|\beta| = k$. If we move in spherical coordinates, we will have that:

$$\int_\Omega D_1^{k_1} \cdots D_n^{k_n} r^{-\alpha} r^{n-1} dr d\theta$$

where $d\theta$ contains all the angular part of the integral. Using integration by parts then we have that

$$(-1)^k \int_\Omega \varphi D^k r^{n-1-\alpha} dr d\theta \le \sup_\Omega |\varphi| \int_\Omega D^k r^{n-1-\alpha} dr d\theta$$

by the chain rule we know that the directional derivative in the direction of x_i or $r^{n-1-\alpha}$ will be less or equal than its directional derivative in the direction of r, because the function does not depend on the angular components. Then, let $\sup_\Omega |\varphi| = c$:

$$c \int_\Omega r^{n-1-\alpha-k} dr d\theta = \left[r^{n-\alpha-k} \right]_0^R$$

This will be non singular if, and only if, the exponent is not less than zero, that is $n - \alpha - k > 0$ which proves the statement.

4 Exercise

Let $\Omega \subset \mathbb{R}^n$ and an open and bounded subset $\Omega \subset\subset \Omega$ and $d \doteq \operatorname{dist}(\Omega', \partial\Omega)$, we want to show that there exists a function $\eta \in C_c^\infty(\Omega)$ and a constant $C = C(n)$ such that

$$0 \le \eta \le 1, \quad , \eta|_{\Omega'} \equiv 1, \quad |D\eta| \le \frac{C}{d}$$

What we want to do is construct a function which grows from zero to $1L$ before reaching Ω', and we want to bound this growth from above. Let us therefore define two subsets $A, B \subset \Omega$ such that $\Omega' \subset B \subset A \subset \Omega$. In particular, we want $\operatorname{dist}(A, \partial\Omega) = \operatorname{dist}(\Omega', \partial B) = \frac{d}{4}$, meaning that $\operatorname{dist}(B, \partial A) = \frac{d}{2}$. Then consider the function

$$f(x) = \begin{cases} 1 & x \in B \\ -\frac{2}{d}x + k & x \in A \setminus B \\ 0 & x \in \Omega \setminus A \end{cases}$$

Where k is just a constant which must be chosen to be such that the linear function has value 1 at ∂B and 0 at ∂A. However, this function is not smooth. To make it C_c^∞ we apply the mollification to f defining $0 < h < \operatorname{dist}(B, \partial A)$

$$\eta(x) = \int_\Omega \rho_h(x - y) f(y) dy$$