MATHEMATICAL QUANTUM THEORY
WINTER TERM 2023/2024

EXERCISE SET 3

All exercises are worth 5 points. Due on Wednesday, November 9, at 12:00 noon via URM.

- **Exercise 1** studies the long-time asymptotics of the position of a free particle on different time scales. The upshot is that the velocity scale $\alpha = 1$ is the interesting one.
- **Exercise 2** practices basic operations with convolution and Fourier transform.
- **Exercise 3** shows that the Schrödinger equation on a finite interval leads to periodicity in time, just like the wave equation describing a vibrating string. So the complex nature of the Schrödinger equation makes it behave to some extent like a wave equation. As the exercise also shows, this is in stark contrast to the equilibration behavior one finds for the heat equation which is the Schrödinger equation without the i.
- **Exercise 4** is our first taste of abstract argumentation — in a Hilbert space, understanding the norm is sometimes sufficient for understanding the inner product.

Exercise 1. Consider the free Schrödinger equation:

$$i\partial_t\psi_t = -\frac{1}{2}\Delta\psi_t \ ,$$

with initial data $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Let $B_r(x_0) \subset \mathbb{R}^d$ be a ball of radius $r > 0$ and center x_0. For $\alpha > 0$, compute

$$\lim_{t\to\infty} \mathrm{Prob}(\text{position} \in t^\alpha B_r(x_0) \text{ in state } \psi_t) \ ,$$

distinguishing the cases $\alpha < 1$, $\alpha = 1$, $\alpha > 1$.

Exercise 2. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Define their convolution by

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\mathrm{d}y.$$

(a) Prove that $f * g = g * f$.
(b) Prove that $\widehat{(f * g)} = (2\pi)^{d/2}\hat{f}\hat{g}$.

Exercise 3.

(a) Consider the free Schrödinger equation on a finite interval $[0, L]$,

$$i\partial_t\psi_t = -\frac{1}{2}\Delta\psi_t \tag{1}$$

with periodic boundary conditions $\psi_t(0) = \psi_t(L)$. Notice that $\psi_t(x) = \psi_0(x) = L^{-1/2}$ is a solution that is constant in time. Are there any other ones?

(b) Let $\psi_t(x) = f(t)g(x)$ be a separable solution of (1) with periodic boundary conditions $\psi_t(0) = \psi_t(L)$. Prove that it is also periodic in time: there exists a time T such that $\psi_t = \psi_{t+T}$ for all t. Then find T for either Dirichlet ($\psi_t(0) = \psi_t(L) = 0$) or Neumann ($\partial_x \psi_t(0) = \partial_x \psi_t(L) = 0$) boundary conditions.

(c) Consider now the heat equation

$$\partial_t \psi_t = \frac{1}{2} \Delta \psi_t$$

with $\psi_t(0) = \psi_t(L) = c$. Let $\psi_t(x) = c + f(t)g(x)$. Compute $\lim_{t\to\infty} \psi_t(x)$.

Remark. *We study separable solutions for this exercise only for simplicity. The results can be extended to all solutions by using completeness of the eigenbasis of $-\Delta$, which follows from the spectral theorem for unbounded operators that we will prove later.*

Exercise 4. Let $\mathcal{H}$ be a Hilbert space over the complex numbers with induced norm $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$.

(a) Prove the polarization identity

$$\langle \psi, \phi \rangle = \frac{1}{4} \left(\|\psi + \phi\|^2 - \|\psi - \phi\|^2 - i\|\psi + i\phi\|^2 + i\|\psi - i\phi\|^2 \right).$$

(b) Use part (a) to conclude that any linear $T : \mathcal{H} \to \mathcal{H}$ that is an isometry (meaning $\|T\psi\| = \|\psi\|$ for all $\psi \in \mathcal{H}$) also preserves the inner product.