Exercises -

Simone Coli - 6771371

Sheet

1 Exercise

Use the function $f(x) = \ln\ln\left(\frac{1}{|x|}\right)$ to construct a function which is in $W^{1,n}(\mathbb{R}^n)$ unbounded on every open domain of $\mathbb{R}^n$.

Because we require a function unbounded over every open domain of $\mathbb{R}^n$ we want a function with singularities on countable many points of $\mathbb{R}^n$ which, for example, could be the set of the rational numbers $\mathbb{Q}^n$. Let $\{z_k\}_{k\in N} \subset \mathbb{R}^n$ be the sequence of the singular points of our new function. Then we can take

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{B_{1/e}(z_k)} f(x - z_k)$$

Then we need to show that such a function is indeed in $L^n(\mathbb{R}^n)$, and that its weak derivative is also in $L^n(\mathbb{R}^n)$.
Let us begin by showing that $g \in L^n(\mathbb{R}^n)$:

$$\|g\|_n^n = \int_{\mathbb{R}^n} \left| \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{B_{1/2}(z_k)} f(x - z_k) \right|^n dx \leq$$
$$\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{\mathbb{R}^n} \left| \chi_{B_{1/e}(z_k)} f(x - z_k) \right|^n dx =$$

by the triangular inequality. Now since it is zero outside the ball

$$= \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{B_{1/e}(z_k)} |f(x - z_k)|^n dx = \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{B_{1/e}(z_k)} \left| \ln\ln\left(\frac{1}{|x - z_k|}\right) \right|^n dx =$$

let us consider a translation so that $y = x - z_k$ then

$$= \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{B_{1/e}(0)} \left| \ln\ln\left(\frac{1}{|y|}\right) \right|^n dy$$

2 Exercise

Consider two points $x, y \in \mathbb{R}^n$. We want to show that the shortest smooth curve $\gamma : [0,1] \to \mathbb{R}^n$ between x and y is a straight line segment.

Let us define $\gamma(1) = y, \gamma(0) = x$, then if we define the length of the curve $\gamma(t)$ as:

$$L[\gamma] = \int_0^1 |D\gamma| dt,$$

it is possible to minimize it using the variational method. Let $\eta \in C_c^\infty$ and let s be small enough, we want to find γ which minimize the previous integral, that is

$$\delta d[\gamma + s\eta] = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 |D(\gamma + s\eta)| dx = 0$$

from hypothesis γ and η are smooth functions, and therefore it is possible to bring the differential operator inside the integral

$$\delta d[\gamma + s\eta] = \left.\int_0^1 \frac{\partial}{\partial s} |D(\gamma + s\eta)| dx \right|_{s=0} = 0$$

by the chain rule:

$$\begin{aligned}
\delta d[\gamma + s\eta] &= \left.\int_0^1 \frac{D(\gamma^i + s\eta^i)}{|D(\gamma + s\eta)|} D\eta^i \, dt \right|_{s=0} = \\
&= \int_0^1 \frac{D\gamma^i}{|D\gamma|} D\eta^i \, dt = \int_0^1 \frac{1}{|\frac{\mathrm{d}\gamma}{\mathrm{d}t}|} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\eta^i}{\mathrm{d}t} \, dt = 0
\end{aligned}$$

Note that we are considering only the variation of the i-th component of the parametrization using $|D\gamma| = \sqrt{D\gamma^{1^2} + \cdots + D\gamma^{n^2}}$. Moreover, the "speed" along the curve should not depend on the choice of parametrization, then we can choose our to be such that $\frac{\mathrm{d}\gamma}{\mathrm{d}t} = 1$.

$$\int_0^1 \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\eta^i}{\mathrm{d}t} \, dt = -\int_0^1 \frac{\mathrm{d}^2\gamma^i}{\mathrm{d}t^2} \eta^i \, dt = 0$$

integrating by parts. Now, assuming $\eta^i \neq 0$ we have that this is true only if $d^2\gamma/dt^2 = 0$ almost everywhere. Proving that the curve $\gamma(t)$ which minimize the length of the segment is the segment of a line.

3 Exercise

Let $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ be a smooth solution to the problem

$$\Delta u = \frac{\partial u}{\partial t}, \quad u(\cdot, 0) = u_0, \quad u_{\partial\Omega \times [0,\infty)} = 0$$

(a) Let us show that the $L^2(\Omega)$ norm of u decays to 0 exponentially as $t \to \infty$.

First we let us define the functions

$$E(t) = \int_\Omega |u|^2 dx$$

Then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 dx$$

since u is at least $C^2(\Omega)$, we can bring the derivative inside the integral

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega \frac{\partial}{\partial t}(u \cdot u)\, dx = \int_\Omega 2\frac{\partial u}{\partial t} \cdot u\, dx = 2\int_\Omega \Delta u \cdot u\, dx = -2\int_\Omega |Du|^2 dx$$

Then by Poincaré inequality

$$-2\int_\Omega |Du|^2 dx \leq -2\, C(n;\Omega)\int_\Omega |u|^2 dx$$

we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -2\, C(n,\Omega)E(t)$$

From Grönwall's inequality we have that given two real valued function $f(t), \beta(t)$ on $[a, \infty)$, then if f is differentiable in the interior of such interval and such that $f'(t) \leq \beta(t)f(t)$ with $t \in (a, \infty)$ then f is bounded by the solution of the corresponding ODE. In our case the ODE has the form:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2\, C(n,\Omega)E(t)$$

which is solved by $E(t) = \alpha e^{-2\, C(n;\Omega)\, t}$. Our function E is therefore bounded as

$$\sqrt{E(t)} = \left(\int_\Omega |u|^2 dx\right)^{1/2} = \|u\|_2 \leq \alpha e^{-\, C(n;\Omega)\, t}$$

4 Exercise