Exercises - Geometry in Physics

Simone Coli - 6771371

Sheet 3

1 Exercise

Let us consider the set of the n-sphere S^n

$$S^n = \{(x_0, \ldots, x_1) \in \mathbb{R}^{n*1} | \, x_0^2 + \cdots + x_n^2 = 1\},$$

and define two stereographic projections ψ_S and ψ_N defined as the two maps

$$\psi_N : S^n \setminus \{N\} \to \mathbb{R}^n, \quad (x_0, \ldots, x_n, x_{n+1}) \mapsto \left(\frac{x_0}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right)$$

$$\psi_S : S^n \setminus \{S\} \to \mathbb{R}^n, \quad (x_0, \ldots, x_n, x_{n+1}) \mapsto \left(\frac{x_0}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}} \right)$$

where x_{n+1} is is one of the poles. We want to show that these define a differentiable atlas, which makes S^n a differentiable manifold.

Let us define two charts (ψ_S, U), (ψ_N, V) where $U \overset{\text{def}}{=} S^n \setminus \{S\}$, $V \overset{\text{def}}{=} S^n \setminus \{N\}$ we have that

$$S^n = U \cup V$$

Moreover by a direct calculation we find that the inverse map of ψ_N we get that it maps the i element of the map $X_i \mapsto 2X_i/(1+\sum X_i^2)$, which is a differentiable function. Since ψ_S is differentiable by hypothesis, then

$$\psi_S \circ \psi_N^{-1}|_{\psi_N(U \cap V)} : \psi_N(U \cap V) \to \psi_S(U \cap V)$$

is the composition of two differentiable maps, making it differentiable. This allows us to prove that this is a differentiable atlas, and that S^n is a manifold.

Now let us consider the map $\psi_S \circ \psi_N^{-1}$ which goes from $\mathbb{R}^n \to \mathbb{R}^n$ and define another map $\psi : s^n \subset R^{n+1} \to \mathbb{R}^{n+1}$. If,

$$\psi|_{\mathbb{R}} \overset{\text{def}}{=} \psi_S \circ \psi_N^{-1}$$

then, because $\psi_S \circ \psi_N^{-1}$ is a differentiable function, $\psi|_{\mathbb{R}}$ is an homeomorphism, which proves that the two typologies are equal.

2 Exercise

Let us consider a topological space (X, τ_1) with support $X = \mathbb{R} \cup \{0^*\}$ in which we suppose $U \subset \mathbb{R}$ to be open if either $U \subset X$ or, given $0 \in U$, $U \cup \{0^*\} = U' \subset X$ as well as $(U \setminus \{0\} \cup \{0^*\}) = U'' \subset X$. We want to show that the this topology is equivalent to the one induced by the atlas defined on $\mathbb{R} \cup \{0^*\}$ quipped with the differentiable maps:

$$\psi_1 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x$$

$$\psi_2 : (\mathbb{R} \setminus \{0\}) \cup \{0^*\} \to \mathbb{R}, \quad x \mapsto \begin{cases} x & , x \neq 0^* \\ 0 & , x = 0^* \end{cases}$$

To show the equivalency we need to prove that given an open set in the first topological space, it remains open through ψ_1, ψ_2.

– U' though ψ_1: the pre-image of this map at U' is given by the intersection

$$\mathbb{R} \cap (U \cup \{0^*\}) = U \setminus \{0^*\},$$

since $\{0^*\}$ is not contained in $\mathbb{R}$. While the image will be $U \setminus \{0^*\}$, since ψ_1 is just the identity map, which is open in τ_1.

– U' though ψ_2 the pre-image of this map at U' will be

$$((\mathbb{R} \setminus \{0\}) \cup \{0^*\}) \cap (U \cup \{0^*\}) = \{0^*\} \cup U \setminus \{0\}$$

While the image will be $U \cup \{0\}$ since in ψ_2 the only role of $\{0^*\}$ is to decide whether it is equal to identity or zero. This set is open in τ_1 since it is possible to write it as the union of two open intervals not containing zero.

– U'' though ψ_1: the pre-image of this map at U'' will be

$$((U \setminus \{0\}) \cup \{0^*\}) \cap \mathbb{R} = U \setminus \{0, 0^*\}$$

while the image will be $U \setminus \{0, 0^*\}$, which is open using the same reasoning used in the previous bullet point.

– U'' though ψ_2: the pre-image of this map at U'' will be

$$((U \setminus \{0\}) \cup \{0^*\}) \cap ((\mathbb{R} \setminus \{0\}) \cup \{0^*\}) = \{0^*\} \cup U \setminus \{0\}$$

while the image will be $U \setminus \{0\}$, which again is open for the same reason the previous image was open.

Proving that the two topology are actuallx equivalent.

Now, considering the definition of the limit of a sequence converging to 0, that is $\forall U_i(0), \exists N > 0 : \forall i > N, x_i \in U_i(0)$, we see that from the definition of open set in τ_1 all the open neighborhoods of 0 must also contain 0^*, showing that there exists a sequence that approaches the two values.

The definition of Hausdorff space says that, given tow distinct points x, y there exists two open neighborhood U, V such that $x \in U$, $y \in V$ and $U \cap V \emptyset$. However since we have a sequence that approaches two different limits, there does not exist two different open neighborhood of $0 \in O$ and $0^* \in O'$ such that $O \cap O' = \emptyset$. This show that X cannot be an Hausdorff space.