Exercises - Geometry in Physics

Simone Coli - 6771371
Noah Migoski - 6694829

Sheet 4

1 Exercise

Use Lemma 3.3 to define the tangent space as the kernel of the derivative of a submersion.

$$T_pM = \mathrm{Ker}(Dh\big|_p))$$

1.1 G = O(n)

Let $h(Q) = Q^TQ - I$, then $O(n) = h^{-1}(\{0\})$

$$h(Q + \delta H) = (Q + \delta H)^T(Q + \delta H) - I = h(Q) + \delta(H^TQ + Q^TH) + \mathcal{O}(\delta^2)$$

$$\implies Dh\big|_g(Q) = Q^Tg + g^TQ$$

At the identity we have

$$T_eO(n) = \ker(Dh\big|_I(Q)) = \{Q \in M_{n\times n}(\mathbb{R})\big|Q^T + Q = 0\}$$

So the tangent space of $O(n)$ at the identity is the set of $n \times n$ skew-symmetric matrices.
Consider $g \in O(n)$, recall $g^{-1} = g^T$

$$T_gO(n) = \{Q \in M_{n\times n}(\mathbb{R})\big|Q^Tg + g^TQ = 0\}$$

$$\begin{aligned} g \cdot T_eO(n) &= \{gQ \in M_{n\times n}(\mathbb{R})\big|Q^T + Q = 0\} \\ &= \{Q \in M_{n\times n}(\mathbb{R})\big|(g^{-1}Q)^T + g^{-1}Q = 0\} \\ &= \{Q \in M_{n\times n}(\mathbb{R})\big|Q^Tg + g^TQ = 0\} \\ &= T_gO(n) \end{aligned}$$

$$
\begin{aligned}
T_eO(n)\cdot g &= \{Qg \in M_{n\times n}(\mathbb{R})\big|Q^T+Q=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{R})\big|(Qg^{-1})^T+Qg^{-1}=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{R})\big|gQ^T+Qg^T=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{R})\big|g^{-1}gQ^Tg+g^{-1}Qg^Tg=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{R})\big|Q^Tg+g^TQ=0\}\\
&= T_gO(n)
\end{aligned}
$$

1.2 G=U(n)

Let $h(Q) = Q^*Q - I$, by a similar calculation as before

$$Dh\big|_g(Q) = Q^*g + g^*Q$$

And

$$T_eU(n) = \ker(Dh\big|_I(Q)) = \{Q \in M_{n\times n}(\mathbb{C})\big|Q^*+Q=0\}$$

These are the skew-Hermitian matrices. Generally we have

$$T_gU(n) = \{Q \in M_{n\times n}(\mathbb{C})\big|Q^*g+g^*Q=0\}$$

Again using $g^{-1} = g^*$ we can show

$$
\begin{aligned}
g\cdot T_eU &= \{gQ \in M_{n\times n}(\mathbb{C})\big|Q^*+Q=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{C})\big|(g^{-1}Q)^*+g^{-1}Q=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{C})\big|Q^*g+g^*Q=0\}\\
&= T_gU(n)
\end{aligned}
$$

$$
\begin{aligned}
T_eU\cdot g &= \{Qg \in M_{n\times n}(\mathbb{C})\big|Q^*+Q=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{C})\big|(Qg^{-1})^*+Qg^{-1}=0\}\\
&= \{Q \in M_{n\times n}()\big|gQ^*+Qg^*=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{C})\big|g^{-1}gQ^*g+g^{-1}Qg^*g=0\}\\
&= \{Q \in M_{n\times n}(\mathbb{C})\big|Q^*g+g^*Q=0\}\\
&= T_gU(n)
\end{aligned}
$$

1.3 $G = SL(n, \mathbb{K})$

Let $h(Q) = \det(Q) - 1$. As we showed in a previous exercise set, the derivative of this map at the identity is the trace.

$$Dh\big|_I(Q) = \mathrm{Tr}(Q)$$

generally we have

$$\det(Q + \delta H) = \det(Q) + \det(Q)\mathrm{Tr}(Q^{-1}H)\delta + \mathcal{O}(\delta^2)$$

$$\implies Dh\big|_g(Q) = \det(g)\mathrm{Tr}(g^{-1}Q) = \mathrm{Tr}(g^{-1}Q)$$

$$T_eSL(n,\mathbb{K}) = \ker(Dh\big|_I(Q)) = \{Q \in M_{n\times n}(\mathbb{K})\big|\mathrm{Tr}(Q) = 0\}$$

$$T_gSL(n,\mathbb{K}) = \ker(Dh\big|_g(Q)) = \{Q \in M_{n\times n}(\mathbb{K})\big|\mathrm{Tr}(g^{-1}Q) = 0\}$$

$$\begin{aligned} g \cdot T_eSL(n,\mathbb{K}) &= \{gQ \in M_{n\times n}(\mathbb{K})\big|\mathrm{Tr}(Q) = 0\} \\ &= \{Q \in M_{n\times n}(\mathbb{K})\big|\mathrm{Tr}(g^{-1}Q) = 0\} \\ &= T_gSL(n,\mathbb{K}) \end{aligned}$$

Recall $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\begin{aligned} T_eSL(n,\mathbb{K}) \cdot g &= \{Qg \in M_{n\times n}(\mathbb{K})\big|\mathrm{Tr}(Q) = 0\} \\ &= \{Q \in M_{n\times n}(\mathbb{K})\big|\mathrm{Tr}(Qg^{-1}) = 0\} \\ &= \{Q \in M_{n\times n}(\mathbb{K})\big|\mathrm{Tr}(g^{-1}Q) = 0\} \\ &= T_gSL(n,\mathbb{K}) \end{aligned}$$

2 Exercise

Following **Remark3.5** from the lecture notes, we notice that the submersion $h = (h_1, \ldots, h_k)$ yields a basis of the normal space $N_pM = (T_pM)^\perp$

$$N_pM = \mathrm{span}(\{\mathrm{grad}(h_1)|_p, \ldots, \mathrm{grad}(h_k)|_p\})$$

Thus writing $\mathrm{grad}(f)|_p$ in terms of lagrange multipliers is equivalent to showing $\mathrm{grad}(f)|_p$ lies in the normal space at point p

$$\mathrm{grad}(f)|_p = \sum_{i=1}^{k} \lambda_i \mathrm{grad}(h_i)|_p \iff \mathrm{grad}(f)|_p \in N_pM$$

To see that $\mathrm{grad}(f)|_p \in N_pM$ we use that p is a local extremum of f restricted to M.

$$\exists B_\varepsilon(p) \text{ such that either } f(p) > f(x) \text{ or } f(p) < f(x) \,\forall x \in B_\varepsilon(p) \cap M \tag{1}$$

Now suppose there is some $V \in T_pM$ such that $\langle \mathrm{grad}(f)|_p, v\rangle \neq 0$, and let $0 < \delta < \varepsilon$, where δ is chosen small enough that $\mathcal{O}(\delta^2) < \delta\langle \mathrm{grad}(f)|_p, v\rangle$. Then we can find points $p \pm \delta v$ close to p such that

$$\begin{aligned} f(p+\delta v) &= f(p) + \delta\langle \mathrm{grad}(f)|_p, v\rangle + \mathcal{O}(\delta^2) \\ f(p-\delta v) &= f(p) - \delta\langle \mathrm{grad}(f)|_p, v\rangle + \mathcal{O}(\delta^2) \end{aligned}$$

Case 1: $\langle \mathrm{grad}(f)|_p, v\rangle > 0$

$$\implies \begin{cases} f(p+\delta v) > f(p) \\ f(p-\delta v) < f(p) \end{cases}, \text{ which contradicts (1)}$$

Case 2: $\langle \text{grad}(f)|_p, v\rangle < 0$

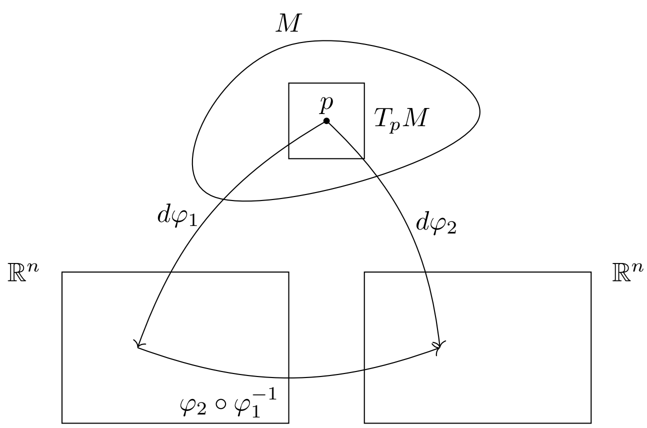
$$\implies \begin{cases} f(p+\delta v) < f(p) \\ f(p-\delta v) > f(p) \end{cases}, \text{ which contradicts (1)}$$

Aside: $\delta < \varepsilon$ guarantees that $p \pm \delta \in B_\varepsilon(p)$, but these points are not necessarily in M. However, since $v \in T_pM$ these points must be within some distance of M controlled by δ^2, so we can still find a point in M that contradicts (1).

Thus $\langle \text{grad}(f)|_p, v\rangle = 0$ for all $v \in T_pM$, and so $\text{grad}(f)|_p \in N_pM$.

3 Exercise

Let us consider two charts from a manifold M to $\mathbb{R}^n$: φ_1 and φ_2. The first maps the tangent vector at a point p into into the canonical basis in $\mathbb{R}^n$, while the second maps it into the cylindrical basis.



We define the transformation of coordinates as

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = D(\varphi_2 \circ \varphi_1^{-1})|_p \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

In which we define v^i and u^i to be the components of the vectors in T_pM expressed with respect to basis defined by φ_2 and φ_1, respectively. While D is just the Jacobi matix of that parametrization.

If $\varphi_2 \circ \varphi_1^{-1}$ is given as:

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = z \end{cases}$$

then the Jacobi matrix, i.e. the coordinate change matrix, will be

$$D(\varphi_2 \circ \varphi_1^{-1}) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4 Exercise

Let us define $TM \stackrel{\text{def}}{=} \dot\bigcup_{p\in M} T_pM$ to be the disjoint union of all tangent spaces of M. That is, there exist the canonical injection $\psi_p : T_pM \to TM$.

i) We want to show that every atlas $\mathcal{A}$ of M defines an atlas of TM such that for $(U, \varphi(x_1, \dots, x_n)) \in \mathcal{A}$ there is

$$\begin{aligned} \Phi : & TM_{|U} \to \varphi(U) \times \mathbb{R}^n \\ & v^i\partial_i(p) \mapsto (\varphi(p), v_1, \dots, v_n) \end{aligned}$$

where $TM_{|U} = \dot\bigcup_{p\in U} T_p$. To do that we want to show that the charts defined using this map together with the subsets T_pM are actually an atlas.
We first notice that by definition of TM

$$TM = \dot\bigcup_{p\in M} T_p.$$

We just need to show that, considering two maps Φ_1 and Φ_2 such that

$$\Phi_1 : TM_{|U_1} \to \varphi(U_1) \times \mathbb{R}^n$$

$$\Phi_2 : TM_{|U_2} \to \varphi(U_2) \times \mathbb{R}^n,$$

their composition $\Phi_2 \circ \Phi_1^{-1}$ is differentiable. It is obvious that Φ^{-1} is invertible, since φ are injective by hypothesis and $v_1, \dots, v_n$ are unique, give the choice of chart. Φ is also trivially differentiable given the definition we gave. Proving it is an atlas.

ii) To show that it is actually and manifold, we need to show that the set is topologically second countable and Hausdorff. Let us start proving the second by considering two points (x_1, v_1) and (x_2, v_2) so that either $x_1 \neq x_2$ or $x_1 = x_2 = x, v_1 \neq v_2$. In the first case we have that there exists two U_1 and U_2 containing x_1 and x_2 respectively disjoint. Applying $\Phi_1^{-1}(U_1)$ and $\Phi_2^{-1}(U_2)$ we still get two disjoint set containing respectively (x_1, v_1) and (x_2, v_2). In the second case, we have that there exists two sets $U_1, U_2 \subset \mathbb{R}^n$ containing $\Phi_1|_x(v_1)$ and $\Phi_2|_x(v_2)$, then we have that preimages are still open disjoint sets containing (x_1, v_1) and (x_2, v_2). Meaning it is Hausdorff To prove the second countable property we just need to consider a countable basis for M (which is second countable by hypothesis), i.e. $\{U_i\}_i$ and a countable basis for $\mathbb{R}^n$ (which is second countable since it is a differentiable manifold), i.e. $\{V_j\}_j$ then taking the preimage of the Cartesian product between these sets $\Phi_p^{-1}(U_i \times V_j)$ we get a countable number of sets which is the basis for TM, proving it is second countable and therefore a manifold.

iii) Let us consider the map $\pi : TM \to M$ which maps a element $(x, v) \mapsto x$. Let us consider $U \subset TM, V \subset M$ then from part i) of this exercise we also know that there exists a map

$$\Phi_{|U} : TM \to \mathbb{R}^{2n}$$

, such that $\Phi(x, v) = (x_1, \dots, x_n, v_1, \dots, v_n)$ which was built using the map

$$\varphi_{|V} : M \to \mathbb{R}^n$$

We can finally define a map

$$\nu : \mathbb{R}^{2n} \to \mathbb{R}^n$$

which is a canonical submersion from $\mathbb{R}^{2n}$ into $\mathbb{R}^n$ mapping $(x_1, \dots, x_n, v_1, \dots, v_n) \mapsto (x_1, \dots, x_n)$. We can therefore construct the following diagram

$$\begin{tikzcd} U \arrow[r, "\pi_{|U}"] \arrow[d, "\Phi_{|U}"'] & V \arrow[d, "\varphi_{|V}"] \\ \Phi(U) \arrow[r, "\nu"] & \varphi(V) \end{tikzcd}$$

which allows us to see that $\nu = \varphi_{|V} \circ \pi_{|U} \circ \Phi^{-1}_{|U}$. From the local submersion theorem it follows that π must be a submersion.