Exercise sheet Geometry in Physics - 1

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1 Solution

a)

Let $\mathbf{x}, \mathbf{h} \in V$ then by definition of differentiability for $f(\mathbf{x}) = A(\mathbf{h})$

$$A(\mathbf{x} + \mathbf{h}) = A(\mathbf{x}) + A'(\mathbf{h}) + o(|\mathbf{h}|),$$

such that $||\mathbf{h}||$ is small and $A' \in L(V, W)$. Since $A \in L(V, W)$, then it is linear and,

$$A(\mathbf{x}) + A(\mathbf{h}) = A(\mathbf{x}) + A'(\mathbf{h}) + o(|\mathbf{h}|).$$

From which $Df_{|W}(\mathbf{h}) = A(\mathbf{h})$. In the same way we can calculate the second derivative:

$$Df_{|W}(\mathbf{x} + \mathbf{h}) = A(\mathbf{x} + \mathbf{h}) = A(\mathbf{x}) + A''(\mathbf{h}) + o(|\mathbf{h}|),$$

meaning that $D^2 f_{|W}(\mathbf{h}) = A(\mathbf{h})$ amd $D^3 f_{|W}(\mathbf{h}) = A(\mathbf{h})$.

For a bilinear map $g(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$, with $b \in L(V, V; W)$ the definition of differentiability implies:

$$b(\mathbf{x} + \mathbf{h}, \mathbf{x} + \mathbf{h}') = b(\mathbf{x}, \mathbf{x}) + b'(\mathbf{h}, \mathbf{h}') + o(||\mathbf{h}|^2 + |\mathbf{h}'|^2|),$$

where $\mathbf{h} \in \mathbb{V}$. Because of the bilinearity

$$\begin{aligned} b(\mathbf{x} + \mathbf{h}, \mathbf{x} + \mathbf{h}) &= b(\mathbf{x}, \mathbf{x}) + b(\mathbf{h}, \mathbf{x}) + b(\mathbf{x}, \mathbf{h}) + b(\mathbf{h}, \mathbf{h}) \\ &= b(\mathbf{x}, \mathbf{x}) + b'(\mathbf{h}, \mathbf{h}) + o(|\mathbf{h}|), \end{aligned}$$

meaning that $Dg_{|W}(\mathbf{h}, \mathbf{h}) = 2b(\mathbf{h}, \mathbf{x})$. Iterating this process we still get

$$D^2 = 4b(x, h)$$

$$D^3 = 8b(x, h)$$

b)

Since $\mathbf{x} \in V = \mathbb{R}^2$ then it is a two-dimensional vector, meaning that $f \in V^*$ so that:

$$f(\mathbf{x}) = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

where $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ and with $a, b, c, d \in \mathbb{R}$.

Since $g(\mathbf{x}) = b(\mathbf{x}, \mathbf{x})$ this has to be of the form

$$g(\mathbf{x}) = b(\mathbf{x}, \mathbf{x}) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}$.

2 Solution

Let us consider a map $f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ such that $A \to \det A$, then by definition of differentiability, we have that:

$$f(A + H) = \det(A + H) = f(A) + f'(H) + o(||H||).$$

The definition of the determinant for an $n \times n$ matrix is:

$$T = \sum_{\in S_n} \mathrm{sg}(\sigma)\, a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

If we calculate the derivative of our map in $A = I$, where I is the identity, we get that:

$$\begin{aligned} Df_{|I}(H) = &\begin{vmatrix} h+1 & h & \cdots & h \\ h & h+1 & \cdots & h \\ \vdots & \vdots & \ddots & \vdots \\ h & h & \cdots & h+1 \end{vmatrix} - 1 = \\ &= 1 + h + h + \cdots + h + h^2 + \cdots - 1 \end{aligned}$$

There are just n terms of the sum that have a linear dependence from h that is $Df_{|I}(H) = nh$ which is the trace of the H matrix. therefore

$$Df_{|I}(H) = \mathrm{tr}(H)$$

3 Solution

Let us consider an implicit representation of the curve $\gamma(t)$, that is a function $h(x(t), y(t)) = 0$ where $t \in I$. If we assume this function to regular then

$$Dh_{|t_0}(t) = \frac{\partial h}{\partial x}\frac{\partial x}{\partial t}\bigg|_{t_0} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t}\bigg|_{t_0} \neq 0\,,$$

since by hypothesis $x'(t_0) \neq 0$ and $y'(t_0) \neq 0$. By the implicit function theorem:

$$\exists\, J \subset I \;:\; t_0 \in J$$

and such that either

$$\mathrm{graph}(g_1) = \{(x(t), g_1(x(t))),\ t \in J\} \stackrel{\text{def}}{=} \gamma_{|J}(t)\,,$$

$$\mathrm{graph}(g_2) = \{(g_2(y(t)), y(t)),\ t \in J\} \stackrel{\text{def}}{=} \gamma_{|J}(t)\,.$$

So that $y(t) = g_1(x(t))$ and $x(t) = g_2(y(t))$. The two functions are therefore one the inverse of the other.

4 Solution

a)

To prove the orthogonality of the two vectors we just need to show that their (standard) scalar product is zero. Let ∂_θ and ∂_t be:

$$\partial_\theta f = \begin{pmatrix} -r(t)\sin(\theta) \\ r(t)\cos(\theta) \\ 0 \end{pmatrix}\,,\ \partial_t = \begin{pmatrix} r'(t)\cos(\theta) \\ r'(t)\sin(\theta) \\ h'(t) \end{pmatrix}\,,$$

then:

$$\langle \partial_\theta, \partial_t \rangle = -r(t)r'(t)\sin(\theta)\cos(\theta) + r(t)r'(t)\sin(\theta)\cos(\theta) + 0 = 0$$

b)

If we assume $r(t) = 1/\cosh(t)$ and $h(t) = tanh(t)$, then we have that:

$$\left\| \frac{\partial f}{\partial \theta} \right\| = \sqrt{(1+t^2)(\cos^2\theta + \sin^2\theta)} = \sqrt{(1+t^2)}$$

$$\left\| \frac{\partial f}{\partial t} \right\| = \sqrt{\frac{t^2}{1-t^2} + 1} = \sqrt{\frac{t^2+1-t^2}{1-t^2}} = \frac{1}{\sqrt{1+t^2}}$$

Which proves that:

$$\left\| \frac{\partial f}{\partial \theta} \right\| \left\| \frac{\partial f}{\partial t} \right\| = 1$$

c)

If we assume $r(t) = 1/\cosh(t)$ and $h(t) = \tanh(t)$, then we have that:

$$\left\| \frac{\partial f}{\partial \theta} \right\| = \frac{1}{\cosh^2(t)}(\sin^2\theta + \cos^2\theta) = \frac{1}{\cosh^2(t)}$$

$$
\begin{aligned}
\left\|\frac{\partial f}{\partial t}\right\| &= \frac{\sinh^2(t)}{\cosh^4(t)}\,(\sin^2\theta + \cos^2\theta) + \frac{1}{\cosh^4(t)} = \\
&= \frac{\sinh^2(t) + \cosh^2(t) - \sinh^2(t)}{\cosh^4(t)} = \\
&= \frac{\cosh^2(t)}{\cosh^4(t)} = \frac{1}{\cosh^2(t)}
\end{aligned}
$$

which proves that, given these definitions of $r(t)$ and $h(t)$:

$$
\left\|\frac{\partial f}{\partial \theta}\right\| = \left\|\frac{\partial f}{\partial t}\right\|
$$