General

- $i = \sqrt{-1}$, $i^2 = -1$
- $z = a + ib = (a, b)$ $a, b \in \mathbb{R}$, $b \neq 0$
- $Re(z) = a$, $Im(z) = b$

Algebra

- let $z_1 = a_1 + ib_1$ $z_2 = a_2 + ib_2$
- $z_1 \pm z_2 = a_1 \pm a_2 + i(b_1 \pm b_2)$
- $z_1 z_2 = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1)$
- complex numbers closed under $+, -, \times, \div$
- one method of solving $\frac{z_1}{z_2}$ is set $z_1 = z_2(x + iy)$, and find Re = Re, Im = Im find x & y

Conjugate

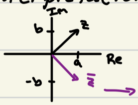
- $z = a + ib$, then $\bar{z} = a - ib$
- properties
  - $\bar{\bar{z}} = z$
  - $\overline{\sum_{i=1}^{N} z_i} = \sum_{i=1}^{N} \bar{z_i}$
  - $\overline{\prod_{i=1}^{N} z_i} = \prod_{i=1}^{N} \bar{z_i}$
  - $\overline{z_1 / z_2} = \bar{z_1} / \bar{z_2}$

Modulus

- $|z| = \sqrt{a^2 + b^2}$
- $z\bar{z} = a^2 + b^2 = |z|^2$
- $\left| \prod_{i=1}^{N} z_i \right| = \prod_{i=1}^{N} |z_i|$
- $|z_1 / z_2| = |z_1| / |z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$ (Δ inequality)
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1 \bar{z_2})$
- $Re(z) \leq |z|$
- for inequalities, squaring can remove radical signs

Geometric Interpretation

- $z = a + ib$



$\longrightarrow$ reflection across x-axis

- $\bar{z} = a - ib$
- $|z|$ is length of vector
- can be added & subtracted like vectors

Polar Coordinates

- $\cos\theta + i\sin\theta$ $\quad\theta$ is radian
  - ↳ $\text{cis}\,\theta$, $e^{i\theta}$
- properties: 1. $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ on unit circle
  2. $\overline{e^{i\theta}} = e^{-i\theta}$
  3. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$
- for any $z = a + ib$, the exists $re^{i\theta}$
  - ↳ θ is called argument of z , $-\pi < \theta \leq \pi$

Roots of Unity

- $\sqrt[n]{z_0} = w$

$\sqrt[n]{r_0 e^{i\theta_0}} = re^{i\theta}$

$$\begin{cases} r = \sqrt[n]{r_0} \\ \theta = \dfrac{\theta_0 + 2\pi k}{n}, \quad k \in \mathbb{Z} \end{cases}$$

Functions of Complex Numbers

- $z = x + iy$
- $u(x, y) = \text{Re}(f)$
- $v(x, y) = \text{Im}(f)$

Cauchy Riemann Eqs

u: real v: imaginary

- Let $f(z) = u(z) + i\,v(z)$
- f is differentiable at $z = z_0$ iff $\left.\begin{array}{l} u_y(z_0) = -v_x(z_0) \\ u_x(z_0) = v_y(z_0) \end{array}\right\}$ Cauchy Riemann Eq
- if f is diff at $z = z_0$

  $f'(z_0) = u_x(z_0) + i\,v_x(z_0)$

  $f'(z_0) = -iu(z_0) + v_y(z_0)$

- let $f(r,\theta) = u(r,\theta) + i\,v(r,\theta)$

  f is differentiable at (r_0, θ_0) where $r_0 \neq 0$ iff

  $\left.\begin{array}{l} ru_r = v_\theta \\ u_\theta = -rv_r \quad \text{at } r_0, \theta_0 \end{array}\right\}$ cauchy riemann in polar form

- Let D be an open connected region in $\mathbb{C}$ plane, f is analytic in D iff f is differentiable $\forall z \in D$.
- f is entire iff f is differentiable $\forall z$

Harmonic Functions

- $g(x,y)$ is harmonic in D iff $g_{xx} + g_{yy} = 0$ $\forall (x,y) \in D$
- if $f(z)$ is analytic in D, the $u(x,y)$ and $v(x,y)$ are harmonic in D
- let $f = u + iv$, f is analytic in D, then curves $u = c$, $v = d$, $c, d \in \mathbb{R}$ intersect at $90°$ if intersection occurs in D

Logarithm

- precalc intuition: $y = \log_e x$ iff $e^x = y$
- let $z \in \mathbb{C}$: $w = \log z$ iff $e^w = z$
  - ↳ infinite values for w
- $\log z = \ln|z| + i(\theta + 2\pi k) \quad \forall k \in \mathbb{Z}$
- principle branch of logarithm: $\mathrm{Log}(z) = \ln|z| + i(\mathrm{Arg}(z))$
- arctan output always $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\frac{d}{dz} \log z = \frac{1}{z}$

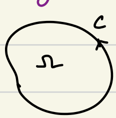
Contour Integration

- $f(z) = u(z) + i v(z)$
- $\int_C f(z)\, dz$

parameterize C : $\gamma(t)$ $a \leq t \leq b$

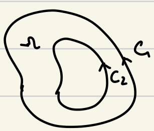
$$\int_C f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt$$

Cauchy Goursat



Ω, C analytic

$$\int_C f(z)\, dz = 0$$



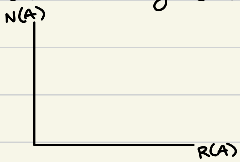
Ω, C_1, C_2 analytic

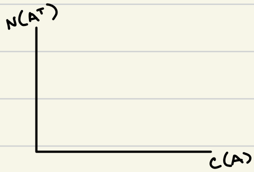
$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

Cauchy Integral

- $\int_C \frac{f(z)}{z - z_0} = 2\pi i\, f(z_0)$

Basic

- $A_{m \times n}$: m = rows , n = columns
- $C(A)$: column space $N(A)$: nullspace
- $n = \dim(N(A)) + \dim(C(A))$
- $C(A) = \text{range}(A)$
- 



(nullspace of transpose)

rowspace $\perp$ nullspace columnspace $\perp$ left nullspace

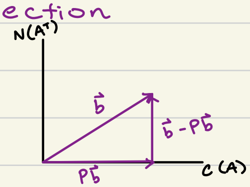
- $e^A = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$

Q^TQ

$Q^{-1}Q$

inverse 3x3 or 4x4

Projection



if $A\vec{x} = \vec{b}$

$\text{proj}\,\vec{b} = A(A^TA)^{-1}A^T\vec{b}$

$\vec{x} = (A^TA)^{-1}A^T\vec{b}$

Gram Schmidt

- apply GS to $\{x_1, x_2, x_3, \dots, x_n\}$
  - $v_1 = x_1$

$$v_2 = x_2 - \frac{x_2^Tv_1}{v_1^Tv_1}v_1$$

$$v_3 = x_3 - \frac{x_3^Tv_2}{v_2^Tv_2}v_2 - \frac{x_3^Tv_1}{v_1^Tv_1}v_1 \qquad \text{etc...}$$

QR Decomposition

- $\begin{cases} A = QR \\ Q \text{ is orthonormal matrix (all columns othogonal \& normalized)} \\ R \text{ is upper triangular invertible matrix} \end{cases}$
- methodology: ① apply GS to A
  ② normalize all columns → result to Q
  ③ $R = Q^T A$ because $Q^T Q = I$
- can be used to find best value: $A\vec{x} = \vec{b}$ → $\vec{x} = R^{-1} Q^T \vec{b}$

Eigenstuff

- eigenvalue λ iff $Ax = \lambda x$
- method: ① find λ s.t. $\det(A - \lambda I) = 0$
  ② find $(A - \lambda I)\vec{v} = \vec{0}$
- algebraic multiplicity: number of roots of det=0 of eigenvalue (hard to explain
- geometric multiplicity: number of eigenvectors corresponding to a eigenvalue
- complex eigenvalues appear in conjugate pairs

Diagonalization

- if A has a full set of eigenvectors, A is diagonalizable
- $A = MDM^{-1}$
- D: diagonal matrix of eigenvalues $\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
- M: matrix of eigenvectors: $[v_1 \ v_2 \ \dots \ v_n]$
- can be used to find matrix powers
- if A is symmetric & Q is orthogonal: $A = QDQ^T$
  ↳ note that only evector w/ same evalue need to be GS
- if $Ax = \lambda x$, e^A eigenvalue is e^λ, correspond to same evector

Generalized Eigenvector

- $(A-\lambda I)^k x = 0$ $\quad k \geq 1$
- if missing eigenvector (AM > GM)
  - ↳ $(A-\lambda I)v_2 = v_1$, to use evector v_1 to find v_2
- for generalized: $e^A v = e^\lambda \left(I + (A-\lambda I) + \frac{1}{2}(A-\lambda I)^2 + \frac{1}{3!}(A-\lambda I)^3 \right) v$

LU decomposition

- $A = LU$ • L : upper triangular matrix
  - • U : lower triangular matrix
- method of finding : • row echelon matrix $\quad$ ☆ do not swap row
  - ↳ result is U $\quad$ ☆ always R - R before
  - • to find L, get transformation matrix

SVD Decomposition

- singular value decomposition
- $A_{m\times n} = U \Sigma V^T$ $\quad (m\times m)(m\times n)(n\times n)$
- diagonal entries of Σ are the singular values of A
- U othogonal evectors of AA^T
- V orthogonal evectors of A^TA
- method of solving: find $A^TA \rightarrow$ evalues $\lambda_1, \lambda_2, \ldots$

  $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$ ...

  evectors (normal) $\rightarrow v_1, v_2$

  $u_1 = \frac{Av_1}{\sigma_1}$, $u_2 = \frac{Av_2}{\sigma_2}$ ...

Complex Vectors

- $||\vec{x}|| = \sqrt{||x_1||^2 + ||x_2||^2 + \cdots}$
- A is Hermetian iff $A^H = A$ $\quad A^H = (\bar{A})^T$
  - ↳ if A is Hermetian, evalues are real
- A is unitary iff columns of A are $\perp$ and all have magnitude 1

$\hat{x}(k) = \sum_{m=0}^{N-1} x_m w^{km}$ where $w = e^{-2\pi i/N}$