Logical Form

Argument: try to demonstrate truth value

Logical Connector

- operators
- $\sim$ not
- $\wedge$ and
- $\vee$ or

Order of Operations

1. not ($\sim$)
2. and, or ($\wedge$, $\vee$)
3. implies ($\longrightarrow$)

Tautology: statement that is always true

Contradiction: statement that is never true

"$p \rightarrow q$" : if p then q : p implies q

$p \rightarrow q$!= $q \rightarrow p$

↳ false only when $T \rightarrow F$ $\quad$ $p \rightarrow q \equiv \sim p \vee q$

base $p \rightarrow q$

contrapositive : $\sim q \rightarrow \sim p$

converse : $q \rightarrow p$

inverse : $\sim p \rightarrow \sim q$

p only if q

↳ $\sim q \rightarrow \sim p \equiv p \rightarrow q$

Biconditional

$p \to q \wedge q \to p \equiv p \leftrightarrow q$

iff = if and only if

Terminology:

Sufficient: if r is sufficent condition for s, $r \to s$

necessary: if r is necessary condition for s,

$s \to r$ or $\sim r \to \sim s$

if r is necessary & sufficient for s : $r \leftrightarrow s$

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Process for Nontrivial Examples

1. identify premise & conclusion
2. Construct truth table
3. find critical row → all premises are true
   - if conclusion is true in every critical row, then argument is valid
   - if one row has false conclusion, argument is invalid

Valid Forms

modus pollens : method of affirming : $p \to q$: p : $\therefore q$

modus tollens : method of denying ; $p \to q$: $\sim q$: $\therefore \sim p$

p : $\therefore p \vee q$

q : $\therefore p \vee q$

$p \wedge q$: $\therefore p$

$p \wedge q$: $\therefore q$

elimination $p \vee q$: $\sim q$: $\therefore p$

transitivity : $p \to q$: $q \to r$: $\therefore p \to r$

an argument form is valid if argument is valid for all substituted values

- an arguement is a series of statements
- an arguement form is a sequence of statement forms
- statements before final are premises/assumptions/hypotheses
- the final statement is called the conclusio ($\therefore$ means therefore)
- critical row: row of truth table where all premises are true
  - ↳ if every critical row true conclusion, arguement form is valid
    - ↳ otherwise false

- validity is a property of argument forms
- truth is a property of statement forms

- an argument is sound if and only if it is valid & all premise are true
  - ↳ otherwise, it is unsound

Proof by Contradiction

- if we can show that the assumption that p is false leads logically to a contradiction, then we can conclude that p is true.

division into cases: $p \vee q$: $p \to r$: $q \to r$: $\therefore r$

Fallacies

fallacy of affirming the consequent: $p \to q$ $\quad q \;!\!\to\; p$

fallacy of denying the antecedent: $p \to q$ $\quad \sim p \;!\!\to\; \sim q$

Contradiction rule

if you show that supposing p is false leads to a contradiction, then p must be true

Set Theory

$a \in A$

↳ "a belongs to A"

$a \notin A$

↳ not an element of

$\{a_1, \dots, a_n\}$ set containing $a_1, \dots, a_n$

$\{x \in D \mid P(x)\}$ all x in D s.t. (such that) $P(x)$ is true

symbols : $\mathbb{R}$: real $\mathbb{R}^+$ positive real $\mathbb{R}^-$ negative real

$\mathbb{Z}$: integer $\mathbb{Z}^+$ positive integer $\mathbb{Z}^-$ negative integer , $\mathbb{Z}^+ \cup \{0\}$

$\mathbb{Q}$: rational

$\mathbb{N}$: natural

$\mathbb{C}$: complex

Subset : $A \subset B$

superset : $A \subseteq B$

union : $A \cup B$

intersection : $A \cap B$

complement : A^c

Ordered n-tuple

- order matters

Cartesian Product

$\{1, 2\} \times \{1, 2, 3\} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

Special set : empty set $\emptyset = \{\} =$ null set

power set : all possible subsets of a given set $P(A)$

Predicates & Quantified Statements

- we have sentence w/ finite # of variables. It becomes statement when specific values are substituted, we can check if predicate is true.
- if $x \in D$ then $Q(x)$

variable domain predicate

universal quantifier : $\forall$ - for all

existential quantifier: $\exists$ - there exists

negate existential / universal

$\sim(\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x)$

$\sim(\exists x \in D, Q(x) \equiv \forall x \in D, \sim Q(x)$

$\sim(\forall x \in D, P(x) \to Q(x)) \equiv \exists x \in D \sim (P(x) \to Q(x)) \equiv \exists x \in D, P(x) \wedge \sim Q(x)$

$D = \{x_1, x_2, \ldots, x_n\}$

$\forall x \in D, Q(x) \equiv \forall x \in D, Q(x_1) \wedge Q(x_2) \wedge Q(x_3) \wedge \ldots \wedge Q(x_n)$

negation: $\exists x \in D, \sim Q(x_1) \vee \sim Q(x_2) \vee \ldots \vee \sim Q(x_n)$

$\exists x \in D, Q(x) \equiv \exists x \in D, Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$

negation: $\forall x \in D, \sim Q(x_1) \wedge \sim Q(x_2) \wedge \ldots \wedge \sim Q(x_n)$

Proof by Exhaustion: prove $\forall x \in D, Q(x)$ by showing $Q(x)$ is true for all x in D.

universal conditional statement : $\forall x (P(x) \to Q(x)) \equiv P(x) \Rightarrow Q(x)$

$\forall x (P(x) \leftrightarrow Q(x)) \equiv P(x) \Leftrightarrow Q(x)$

Equivalent forms of Universal & Existential statements

$A = \{x \in B \mid P(x)\}$

$\forall x \in B \,(P(x) \to Q(x)\}$: $\forall x \in A \,(\,Q(x)\,)$

Multiple Quantifier

$\forall x \in D \ \exists y x E$ so that x, y satisfy $P(x, y)$

↳ for all x in D, if there exists an y in E that satisfies $P(x, y)$

Arguments with Quantified Statements

- universal modus ponens
  - ↳ $\forall x$ if $P(x)$ then $Q(x)$

    $P(a)$ for a particular a

    $\therefore Q(a)$
- universal modus tonens
  - ↳ $\forall x$ if $P(x)$ then $Q(x)$

    not $Q(a)$ for a particular a

    $\therefore \sim P(a)$
- one way to check validity of argument is to draw venn diagram
  - ↳ set theory
- converse error
  - ↳ $\forall x$ if $P(x)$ then $Q(x)$

    $Q(x)$ for a particular a

    $\therefore P(a) \rightarrow$ invalid conclusion
- inverse error

- universal transitivity

  $\forall x \; P(x) \rightarrow Q(x)$

  $\forall x \; Q(x) \rightarrow R(x)$

  $\therefore \forall x \; P(x) \rightarrow R(x)$

General

Even

n is even iff $n=2k$ for $k \in \mathbb{Z}$

Odd

n is odd iff $n=2k+1$ for $k \in \mathbb{Z}$

Prime

n is prime iff $\forall\, r,s \in \mathbb{Z}^+$, if $n=rs$, $r=1$ and $s=n$, or $r=n$ and $s=1$

Composite

n is composite iff $\exists\, r,s \in \mathbb{Z}^+$ such that $n=rs$, $1<r<n$, $1<s<n$

Proving Existential Statements

- constructive proof of existance : give example
- nonconstructive ~ : (1) guarenteed by axiom or theorem
  (2) assumption of no such x leads to contradiction

$n \mid m$: n divides m : $n \cdot k = m$, $k \in \mathbb{Z}$

Quotient Remainder Theorem

- $\forall n \in \mathbb{Z}\ \forall d \in \mathbb{Z}^+\ \exists! q \in \mathbb{Z}\ \exists! r \in \mathbb{Z}\ (n=dq+r \ \wedge\ 0 \le r \le d)$
- given any integer n and positive integer d, there exists unique integers q and r such that $n=dq+r$ and $0 \le r \le d$.

Rational Number

- can be written as $\frac{m}{n}$, $m, n \in \mathbb{Z}$

$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \dots + a_n$

$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot \dots \cdot a_n$

$\binom{n}{k}$: n choose k : $\frac{n!}{k!(n-k)!}$

arithmetic series sum: $S_n = n\left(\frac{a_1 + a_n}{2}\right)$

geometric series sum: $S_n = \frac{a(1-r^n)}{(1-r)}$ a = first number

↳ if $0 < r < 1$, $S = \frac{a_1}{1-r}$

Sets & Subsets

- $A \subseteq B$: $\forall x (x \in A \rightarrow x \in B)$
- $A \nsubseteq B$: $\exists x (x \in A \land x \notin B)$
- $A \subsetneq B$: $A \subseteq B \land \exists x \in B (x \notin A)$
- $A \cup B$: $\{x \in U : x \in A \lor x \in B)$
- $A \cap B$: $\{x \in U : x \in A \land x \in B)$
- $A \setminus B$ || $A - B$: $\{x \in U : x \in A \land x \notin B)$
- $A^C = \{x \in U : x \notin A\}$
- $\bigcup_{i=1}^{n} A_i = \{x \in U : \exists i \in \{1, \dots, n\} (x \in A_i)\}$
- $\bigcap_{i=1}^{n} A_i = \{x \in U \mid \forall i \in \{1, \dots, n\} (x \in A_i)\}$
- $\emptyset = \{\}$; mutually disjoint is sets $A_1, A_2, \dots, A_n$
- A & B are disjoint iff $A \cap B = \emptyset$ have no elements in common

- $\{A_1, A_2, A_3, \dots\}$ is a partition of set A iff ① A is union of all A_i
  ② sets $A_1, A_2, \dots$ are mutually disjoint

- $P(A)$: power set: all subsets of A
  if $|A| = n$, $|P(A)| = 2^n$

Set Definitions

- $x \in X \cup Y \iff x \in X$ or $x \in Y$
- $x \in X \cap Y \iff x \in X$ and $x \in Y$
- $x \in X - Y \iff x \in X$ and $x \notin Y$
- $x \in X^c \iff x \notin X$
- $(x, y) \in X \times Y \iff x \in X$ and $y \in Y$

Set Identities

- commutative: $A \cup B = B \cup A$
- associative: $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$
- distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- identity: $A \cup \emptyset = A$; $A \cap \mathcal{U} = A$
- complement: $A \cup A^c = \mathcal{U}$; $A \cap A^c = \emptyset$
- double complement: $(A^c)^c = A$
- idempotent: $A \cup A = A$; $A \cap A = A$
- universal bound: $A \cup \mathcal{U} = \mathcal{U}$; $A \cap \emptyset = \emptyset$
- de Morgan: $(A \cup C)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$
- Absorption: $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$
- complement of $\mathcal{U}$ and $\emptyset$: $\mathcal{U}^c = \emptyset$, $\emptyset^c = \mathcal{U}$
- set difference: $A - B = A \cap B^c$

Functions

- $X \xrightarrow{f} Y$ or $f: X \to Y$ or $x \mapsto y$
- if $f: X \to Y$ then
  - every element in X is related to some element in Y
  - no single element in X is related to more than 1 element in Y
- Theorem 7.1.1: if $f: X \to Y$ and $g: X \to Y$ are functions,
  $f = g$ iff $f(x) = g(x)$ for all $x \in X$
- one to one (injective)
  - $f: X \to Y$ is 1-1 iff for any $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$
- onto (surjective)
  - $f: X \to Y$ is onto iff for any $y \in Y$, there exist $x \in X$ such that $f(x) = y$
- one to one correspondence (bijection)
  - $f: X \to Y$ is both 1-1 and onto
  - if $f: X \to Y$ is bijection, then $\exists$ $f^{-1}: Y \to X$ such that
    $f^{-1}(y) = x \iff y = f(x)$