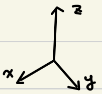
3D Coordinate System

Distance Formula

$d = \sqrt{x^2 + y^2 + z^2 \dots}$

Coordinate Plane



$\rightarrow$ right hand rule

Vectors

Properties

① $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

② $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

③ $\vec{u} + (-\vec{u}) = \vec{0}$

④ $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

⑤ $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$

⑥ $(r+s)\vec{u} = r\vec{u} + s\vec{u}$

⑦ $(rs)\vec{u} = r(s\vec{u})$

⑧ $1\vec{u} = \vec{u}$

Basis Vectors

$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

unit vector $\|\vec{u}\| = 1$

nonzero $\vec{v}$: $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Dot Product

Dot Product

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = v_1u_1 + v_2u_2 + v_3u_3$

also called scalar product

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$

Properties

① $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

② $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

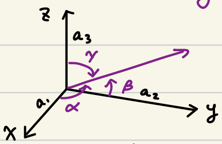
③ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

④ $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$

⑤ $\vec{0} \cdot \vec{a} = 0$

two vectors are perpendicular if dot product = 0

Direction Angles & Cosines



angles relative to x, y, z axis

$$\cos(\alpha) = \frac{\vec{a} \cdot \hat{\imath}}{|\vec{a}||\hat{\imath}|} = \frac{a_1}{|\vec{a}|} \leftarrow \boxed{\frac{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{1^2}} \Rightarrow \frac{a_1}{||a||}}$$

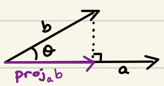
$$\cos(\beta) = \frac{a_2}{|\vec{a}|}$$

$$\cos(\gamma) = \frac{a_3}{|\vec{a}|}$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$$

$$\frac{a}{|a|} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

Projection



↳ projection of b on to a

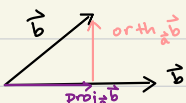
scalar projection = component

component of $\vec{b}$ along $\vec{a}$ → $comp_{\vec{a}} \vec{b} = |\vec{b}| \cos(\theta)$

$$comp_a b = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||}$$

$$proj_a b = \left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} \right) \frac{\vec{a}}{||\vec{a}||} = \frac{a \cdot b}{||\vec{a}||} \frac{a}{||\vec{a}||}$$ ← component unit vector

$$orth_a \vec{b} = \vec{b} - proj_{\vec{a}} \vec{b}$$

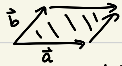


↳ $||proj_{\vec{a}} \vec{b}|| = comp_{\vec{a}} \vec{b}$

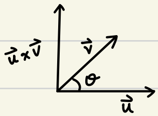
Cross Product

- only for vectors with 3 components (or 7)
- $\vec{u} \times \vec{v}$ = vector $\perp$ $\vec{u}$ & $\vec{v}$
- $\vec{u} \parallel \vec{v}$ if $\vec{v} = \lambda \vec{u}$
- $\vec{u} \times \vec{u} = \vec{0}$, $\vec{u} \times \lambda \vec{u} = 0$
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta)$

geometric interpretation



area = $\|\vec{a} \times \vec{b}\|$



"right hand rule"

- properties
  - $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u}\vec{v} + \vec{u}\vec{w}$
  - $c(\vec{u} \times \vec{w}) = (c\vec{u}) \times \vec{w} = \vec{u} \times (c\vec{w})$
  - $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u}\vec{w} + \vec{v}\vec{w}$
  - $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
  - $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
- parallelpiped volume : $|\vec{u} \cdot (\vec{v} \times \vec{w})|$
  - ↳ $\vec{u}, \vec{v}, \vec{w}$ start from same point

Equations of Lines & Planes

- vector(?) $\vec{r}(t) = \vec{r}_0 + t\vec{v}$
- parametric $\begin{cases} x = a + bt \\ y = c + dt \\ z = e + ft \end{cases}$
- symmetric $\dfrac{x-a}{b} = \dfrac{y-c}{d} = \dfrac{z-e}{f}$
- given $ax + by + cz = d$, vector $\perp$ to plane is $\langle a, b, c \rangle$
- integration prefer $0 \sim 1$
  - ↳ $r(t) = (1-t)\langle x_1, y_1, z_1 \rangle + t\langle x_2, y_2, z_2 \rangle$
  - ↳ find equation between two points
- plane equation: point (x_1, y_1, z_1) normal vector $\langle A, B, C \rangle$

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

- distance between point (x_1, y_1, z_1) and plane $aX + bY + cZ + d = 0$

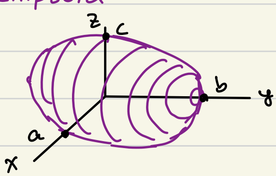
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

- distance between 2 skew lines ① cross product find normal vector
  ② find plane & point
  ③ use distance between plane & point

Quadratic Surfaces & Cylinders

- a *cylinder* is a surface that consists of all lines (rulings) that are parallel to a given line and pass through a given plane curve
- a *quadratic surface* is the graph of a second degree equation with 3 variables
  ↳ general form: $Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$
  → standard forms: $Ax^2+By^2+Cz^2+J=0$
  $Ax^2+By^2+Iz=0$

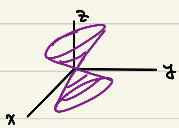
- types: ▷ ellipsoid



equation: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

trace: all ellipses

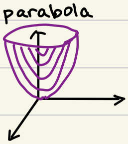
▷ Cone



equation: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

trace: $\begin{cases} \text{horizontal: ellipses} \\ \text{vertical: hyperbola} \end{cases}$

▷ elliptic parabola

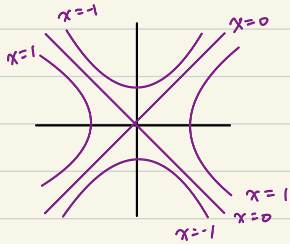


equation: $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

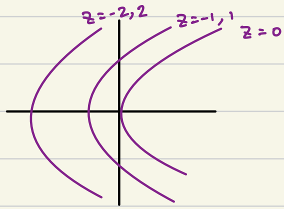
Traces

eg. $x = y^2 - z^2$

fix		
fix	x	$x = y^2 - z^2$
	-1	$-1 = y^2 - z^2$
	0	$0 = y^2 - z^2$
	1	$1 = y^2 - z^2$



fix	z	$x = y^2 - z^2$
	-1	$x = y^2 - 1$
	0	$x = y^2$
	1	$x = y^2 - 1$



fix one variable to find cross section

Space Curve & Derivatives/Integral

Arclength

- $s(t) = \int_a^t ||\vec{r}\,'(u)||\ du$

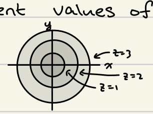
  $L = \int_a^b ||\vec{r}\,'(t)||\ dt$

- if range of integration has negative $\sqrt{\sim\sim}$, then must split
- $s = f(t) \longleftrightarrow t = g(s)$

- normal component: $\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$
- curvature: $k = \frac{||\vec{T}\,'||}{||\vec{r}\,'||}$
- $\vec{N} = \frac{\vec{T}\,'(t)}{||\vec{T}\,'(t)||}$
- acceleration: $a = a_T\vec{T} + a_n\vec{N}$

Function of Several Variables

- eg. $z = f(x, y)$ or $w = g(x, y, z)$
- method: tables to test values
- level curves: - fix one variable, plot graph of other variable. Have multiple lines for different values of the fixed variable
  - eg



  - this can also be done for x,y,z plane

Partial Derivative

- treat other variables as a constant and take derivative with respect to one variable
- $\boxed{f_x(a,b) = \frac{\partial}{\partial x} f(x,y) \Big|_{(x,y)=(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}}$
- $\boxed{f_y(a,b) = \frac{\partial}{\partial y} f(x,y) \Big|_{(x,y)=(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}}$
- $f_x(a,b) > 0$ f increase in x direction $f_x(a,b) < 0$ f decrease in x direction
- $f_{xy} = f_{yx}$, $f_{xyxy} = f_{xxyy}$

Tangent Plane

- draw plane tangent to point on surface
- if $f(x,y)$, point (x_0, y_0, z_0), plane is: $\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$
  - ↳ justification: find change in x direction & change in y direction at point
    - ↳ use Δx and Δy
- *linear approximation*: if want to find $f(x,y)$ at (a,b)
  - ↳ $\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$
- $w = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$

Directional Derivative

- purpose: find how f changes in the direction of a vector
- suppose $\vec{u}$ is a unit vector $\langle a, b \rangle$, directional derivative of f in direction $\vec{u}$ is:

$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y) \cdot a + f_y(x,y) \cdot b}$$

↳ a, b should be substituted into $f_x(x,y)$ and $f_y(x,y)$

↳ result is often x, y

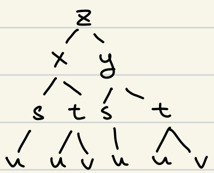
Gradient Vector

- $\vec{\nabla} f = \langle f_x, f_y \rangle$
- direction of greatest change
- max value of $D_{\vec{u}} f$
- tangent plane to level surface: $F(x,y,z) = k$

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

Chain Rule

- if $z = f(x,y)$: $\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$
- eg.



$$\frac{dz}{du} = \frac{dz}{dx}\frac{dx}{ds}\frac{ds}{du} + \frac{dz}{dx}\frac{dx}{dt}\frac{dt}{du} + \ldots$$

Implicit Differentiation

- if $F(x,y) = 0$, assume $y = f(x)$
- $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Critical Point

- in $f(x,y)$, $f_x(a,b) = f_y(a,b) = 0$ is critical point
- can use second derivative test
  - ↳ $D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

    $D > 0$ $f_{xx}(a,b) > 0$ local min

    $D > 0$ $f_{xx}(a,b) < 0$ local max

    $D < 0$ saddle point

    $D = 0$ inconclusive
- extreme value theorem: if f is continuous on closed & bounded set D in $\mathbb{R}$, then f has max and min in D

Lagrange Multipliers

- used to optimize constraints
- used to find min/max values of $f(x,y)$ in constraint $g(x,y) = k$
- find x, y, λ such that:

$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= k \end{aligned}\right\} \text{general: } \nabla f = \lambda \nabla g$$

↳ may result in multiple critical points

- if two constraints: $\nabla f = \lambda \nabla g + \mu \nabla h$
- if constraint has inequality: (1) use lagrange multiplier on =

  (2) use regular optimization of inequality

Polar Coordinates

- $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$
- $dA = r\,dr\,d\theta$

Triple & Double Integrals (definite)

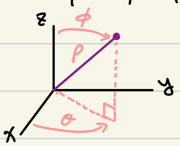
- try to arrange constants on outer integrals
- order integrals are assigned generally doesn't matter

Cylindrical Coordinates

- $x = r\sin(\theta)$ $y = r\sin(\theta)$ $r^2 = x^2 + y^2$ $z = z$
- $dV = r\,dr\,d\theta\,dz$

Spherical Coordinates

- $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\cos\theta$, $z = \rho\cos\phi$, $\rho^2 = x^2 + y^2 + z^2$
- $dV = \rho^2\sin\phi\; d\rho\, d\phi\, d\theta$



- generally: $a \leq \rho \leq b$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$

Vector Fields

- $\vec{F}(x,y)$ assigns a vector to each point $(x,y) \in D$
- $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath} = \langle P(x,y), Q(x,y) \rangle$
- can be extended to higher dimensions
- gradient field - given $f(x,y)$, $\vec{\nabla} f = f_x(x,y)\hat{\imath} + f_y(x,y)\hat{\jmath}$
  - gradient field is perpendicular to level curves
    - ↳ if $\vec{F}$ comes from gradient of some $f(x,y)$, $\vec{F}$ is conservative

Line Integrals

- imagine little line segments at each point
  - ↳ sum of line segments is length of line → $\sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i$
- limit → $\int_C f(x,y)\,ds$ line integral along C
- $\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
- $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C \vec{F} \cdot d\vec{r}$
- $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$

Fundamental Theorem of Line Integrals

- $\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$
- integral of line is equal to difference of end points
  - ↳ only true if $\vec{F} = \vec{\nabla} f$ is conservative, $\int_C \vec{F}\,d\vec{r}$ is path independent

$$\iint_D Q_x - P_y \, dA = \int_{\partial D} P\,dx + Q\,dy$$

Curl

- $\text{curl } \vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$
- $= \vec{\nabla} \times \vec{F}$ $\vec{\nabla} = \frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}$
- result is scalar field
- how much something is rotating

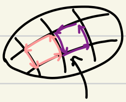
Divergence

- $\text{div } \vec{F} = P_x + Q_y + R_z$

  $= \vec{\nabla} \cdot \vec{F}$
- result is vector field

Parametric Surfaces and Areas

- $\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$
- $u, v \in D$
- $A(s) = \iint_D \| \vec{r}_u \times \vec{r}_v \| \, dA$



▷ sometimes picks new surface that shares boundary

Final Formula Sheet

- $\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$
  - parameterization of function w/ respect to "t"

- $D_{\vec{u}} f = \nabla f \cdot \vec{u}$
  - $\vec{u} = \langle a, b \rangle$, $\|\vec{u}\| = 1$: $D_{\vec{u}} f(x,y) = f_x(a,b) \cdot a + f_y(a,b) \cdot b$
  - $\nabla f = \langle f_x, f_y, f_z, \dots \rangle$
  - find how much f changes in direction of $\vec{u}$

- $D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$
  - second derivative test
  - $D \xrightarrow{<0}$ saddle point
    $D \xrightarrow{=0}$ inconclusive
    $D \xrightarrow{>0}$ $f_{xx} > 0$: local min $f_{xx} < 0$: local max

- $A(S) = \iint_D \sqrt{1 + z_x^2 + z_y^2}\,dA$
  - special case of next equation if $x = x$, $y = y$, $z = f(x,y)$

- $A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\|\,dA$, $(u,v) \in D$
  - find surface area of parametric surface
  - method: taking small cross products of tangential vectors

- $\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
  - find length of curve

- $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\,dA$
  - flux of vector field across parametric surface
  - find normal vector of surface at point & use dot product to find magnitude of $\vec{F}$ in normal direction
  - $\vec{F}$ needs to be parameterized with u, v

- $\iint_S \vec{F} \cdot dS = \iint_S \vec{F} \cdot \vec{n}\, dS$
  - ▹ find flux of vector field through parametric surface
- $\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R)\, dA$
- $\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}$
  - ▹ if $F(x, z) = 0$
- $\int_C \vec{F} \cdot d\vec{r} = \int_a^b F(r(t)) \cdot r'(t)\, dt = \int_C F \cdot T\, ds = \int_C P\,dx + Q\,dy + R\,dz$
  - ▹ force of vector field on curve
  - ▹ can use parameterization of curve
  - ▹ or dot product of field w/ tangential component of curve
- $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$
  - ▹ fundamental theorem of line integrals
- $\iint_S f(x,y,z)\, dS = \iint_D f(r(u,v))\, \| r_u \times r_v \|\, dA$
  - ▹ parameterization w/ respect to u, v
- $\text{curl}\,\vec{F} = (R_y - Q_z)\hat{\imath} + (P_z - R_x)\hat{\jmath} + (Q_x - P_y)\hat{k}$
  - ▹ alternate: $\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F}$
- BONUS: $\text{div}\,\vec{F} = \vec{\nabla} \cdot \vec{F}$

- $\int_C F \cdot dr = \iint_D \text{curl}(\vec{F} \cdot \hat{k}\, dA$
  - Green's Theorem, special case of Stoke's Theorem
  - alt. $\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\, dA$
- $\int_C F \cdot dr = \iint_S \text{curl}(\vec{F} \cdot d\vec{S}$
  - stoke's theorem: the line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface
  - note that $d\vec{S} = \vec{n} \cdot dA$
- $\iint_S \vec{F} \cdot d\vec{s} = \iiint_E \text{div}\vec{F}\, d\vec{V}$

Curl Test

- $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$
  - flux of vector field across parametric surface
  - find normal vector of surface at point & use dot product to find magnitude of $\vec{F}$ in normal direction
  - $\vec{F}$ needs to be parameterized with u, v

- $\int_C F \cdot dr = \iint_D \operatorname{curl} \vec{F} \cdot \hat{k}\, dA$
  - Green's Theorem, special case of Stoke's Theorem
  - alt. $\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\, dA$

- $\int_C F \cdot dr = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$
  - stoke's theorem: the line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface
  - note that $d\vec{S} = \vec{n} \cdot dA$

- $\operatorname{curl} \vec{F} = (R_y - Q_z)\hat{\imath} + (P_z - R_x)\hat{\jmath} + (Q_x - P_y)\hat{k}$
  - alternate: $\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$
    $= \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$

$\iint_D \operatorname{curl} \vec{F} \cdot \hat{k}\, dA \rightarrow$ when $\hat{k} : \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$