Throwing stones in Germany

0.1 Geocache problem

This problem is originally stated in German and can be found here: https://coord.info/GC251EZ

0.2 Question

Go from the parking lot to the coordinates N50 ° 46.255; E006 ° 21.980. From here it is only a 'stone's throw' to the monastery ruins, but the cache is outside the complex. The throwing parabola of a stone that is thrown at a height of 2 m in the direction of 30° (that is, facing north, to the east) at a speed of 69 m/s and a throwing angle (opposite the horizontal) of 32 ° ends directly at the cache. The terrain drops to the cache with an average of 4% (4m height difference on 100m path length). You should neglect the air resistance of the stone. The acceleration due to gravity is 9.81 m/s².
You need to find the right coordinates where the stone hits the ground.

0.3 Throwing the stone

0.3.1 Given

The equations for velocity v and distance s.
Remember v=ds/dt and a=dv/dt and so ∫dv = ∫a*dt and ∫ds = ∫v*dt.

$$\mathrm{f1}(v):=\int_{v_0}^{v} 1\,dv = \mathrm{f2}(t):=\int_{0}^{t} a\,dt$$

$$v-v_0=a\,t$$

$$v=v_0+a\,t$$

$$\mathrm{f3}(s):=\int_{s_0}^{s} 1\,ds = \mathrm{f4}(t):=\int_{0}^{t} v_0+a\,t\,dt$$

$$s-s_0=t\,v_0+\frac{a\,t^2}{2}$$

$$s=t\,v_0+\frac{a\,t^2}{2}+s_0$$

$$\alpha=\frac{8\,\pi}{45}$$
$$v=69$$
$$g=9.81$$
$$s_{y0}=2$$

0.3.2 Relevant formulas

Split the speed into x and y components.
The parametric equations for distance s in x and y direction.
We do not have acceleration in the x direction, only in the y direction, where it is equal to -g.

$$v_x = v\cos(\alpha)$$
$$v_y = v\sin(\alpha)$$
$$s_x(t) := v_x\, t$$
$$s_y(t) := s_{y0} + v_y\, t + \frac{-1}{2}\, g\, t^2$$
$$\mathrm{slope}(x) := \frac{-4}{100}\, x$$

To get the cartesian formula for the parabola, eliminate t.

$$t = \frac{x}{v_x}$$
$$\mathrm{f}(x) := s_y(t)$$

0.3.3 For fun: calculate the maximum height and where it occurs

To calculate a maximum, we need the derivative of the function to be equal to 0. So diff(f(x),x)=0, and then solve for x.

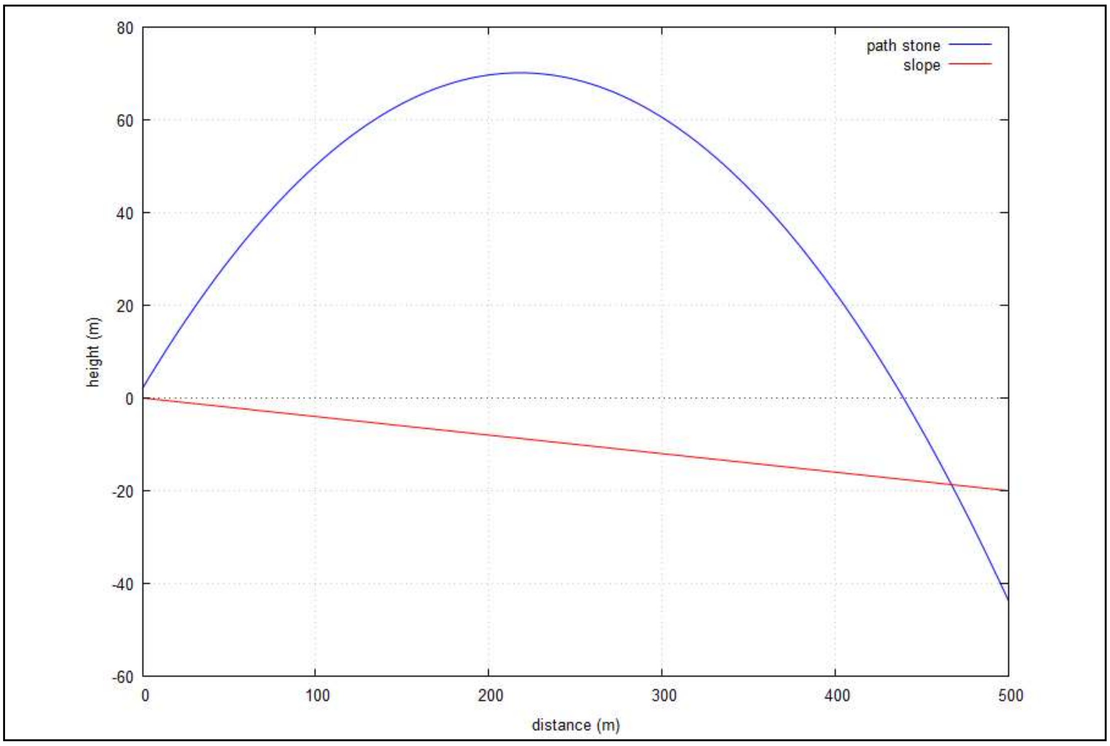
$$\frac{d}{d\,x}\,\mathrm{f}(x) = 0.6248693519093275 - 0.002865034519057818\, x$$
$$x_h = 218.1018580239721$$
$$t_h = 3.727260879927741$$
$$s_y(t_h) = 70.14258333682979$$

0.3.4 Some graphs to make things more clear

The path of the stone. Obviously, the path ends when the stone hits the slope.

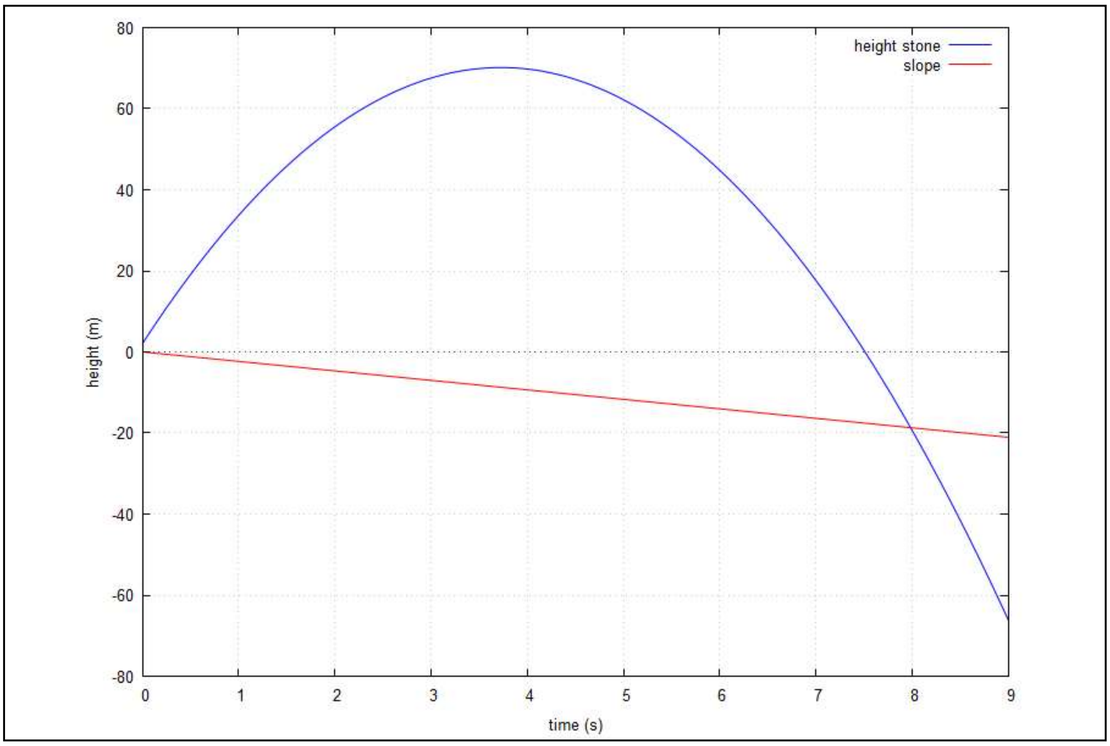
$$f(x)=-0.001432517259528909\,x^2+0.6248693519093275\,x+2.0$$

$$\mathrm{slope}(x)=-\frac{x}{25}$$



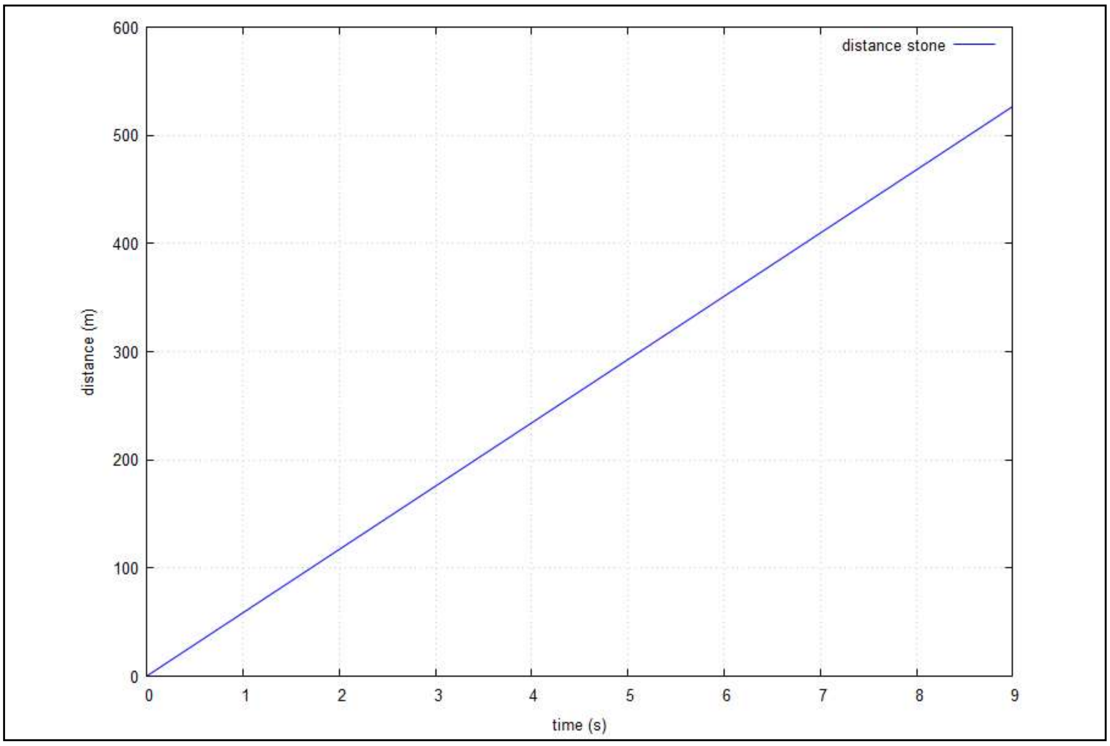
The height of the stone over time.

$$s_y(t)=-4.905\,t^2+36.56442923209114\,t+2.0$$



The distance of the stone over time.

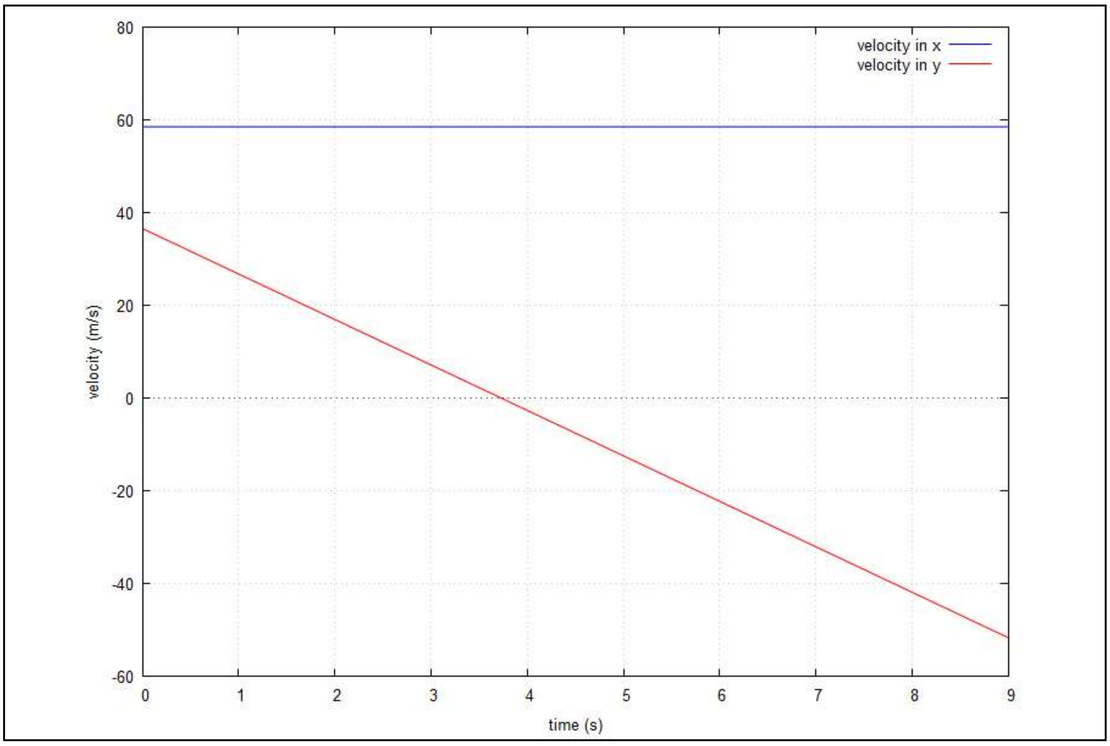
$$s_x(t)=58.51531863479339\,t$$



Velocity of the stone over t in the y direction. This graph intersects with the t axis when the stone reaches the highest point!

$$\frac{d}{d\,t}\,s_x(t)=58.51531863479339$$

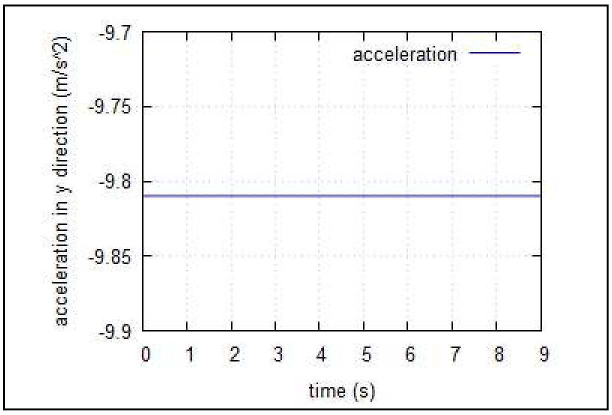
$$\frac{d}{d\,t}\,s_y(t)=36.56442923209114-9.81\,t$$



The acceleration of the stone over t in the x direction is zero. The acceleration of the stone over t in the y direction is constant and equal to -g.

$$\frac{d^2}{d\,t^2}\,s_x(t)=0$$

$$\frac{d^2}{d\,t^2}\,s_y(t)=-9.81$$



0.3.5 Where the stone hits the ground

Obviously, this is the intersection of the trajectory of the stone with the slope of the ground. So set f(x)=slope(x), then solve for x.
You'll get two values, one of them negative, which is of course not valid. Put the correct x value in slope(x) to get the height.

$$\mathrm{solve}(\mathrm{f}(x)=\mathrm{slope}(x),x)=$$
$$[x=-2.98886223510929,x=467.1154530225554]$$
$$x_{end}=467.1154530225554$$
$$y_{end}=-18.68461812090221$$

You can calculate the value of t.

$$t_{end}=7.982789189578249$$
$$s_x(t_{end})=467.1154530225554$$
$$s_y(t_{end})=-18.68461812090226$$

0.4 Getting the coordinates

0.4.1 Given

Coordinates N50 ° 46.255; E006 ° 21.980. Convert that to decimal by dividing the decimal minutes by 60 and adding that to the degrees.

$$\mathit{longitude}=6.366333333333333$$
$$\mathit{latitude}=50.77091666666666$$

"A stone that is thrown in the direction of 30°". So we need to split the distance in a longitude and latitude part, by multiplying the x-coordinate where the stone hits the ground with respectively sin(30°) and cos(30°).

$$\delta_x=\sin(30)\,x_{end}$$
$$\delta_x=233.5577265112777$$
$$\delta_y=\cos(30)\,x_{end}$$
$$\delta_y=404.5338488178095$$

0.4.2 Solution

Calculate how many meters we have per degree on the globe. This is simply the circuference dividend by 360. Since we use meters, we'll need to write the circuference as 40*10^6.

$$metersPerDegree = \frac{1000000}{9}$$

To convert a length to degrees, we need to divide by metersPerDegree. For the longitude, the lenght of a degree at a given latitude is shorter than the lenght of a degree at the equator, so we need to factor that in by multiplying the lenght with cos(latitude).
Add the calculated degrees for both directions to the respective starting points, and you have the solution.

$$N = \frac{\delta_y}{metersPerDegree} + latitude$$
$$E = \frac{\delta_x}{\cos(latitude)\, metersPerDegree} + longitude$$

$$N = 50.77455747130603$$
$$E = 6.369657091205277$$

0.5 Conclusion

Don't get into a rock throwing contest with a German!

0.6 Another way to solve this

0.6.1 Write the slope with parametric equations

Remember that sx = x = t * vx. So to get the parametric equations for the slope, sl, replace x by t*vx.

$sl_x(t) := s_x(t)$

$sl_y(t) := \mathrm{slope}(s_x(t))$

The parametric equations for the height of the slope and the height of the stone are equal when the stone hits the slope. So set them equal to each other and solve for the time. Once you get that, you can calculate the distance. From there on, the rest of the calculations are the same, of course.

$\mathrm{solve}(sl_y(z) = s_y(z), z) =$

$[z = -0.05107828693138276, z = 7.982789189578249]$

$t_p = 7.982789189578249$

$[s_x(t_p) = 467.1154530225554, s_y(t_p) = -18.68461812090226]$

$[sl_x(t_p) = 467.1154530225554, sl_y(t_p) = -18.68461812090221]$