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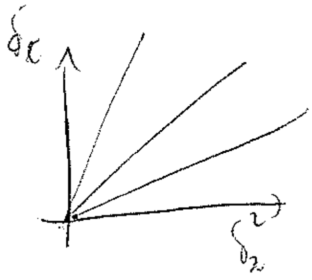
Theorical Analysis of Complex Systems

Course 2:

* Coming King Posts.
* Sion courses.
* Search Algo ...
* Courses overview MAT 432 MAP Mélant, PHY Complex Systems. F Plan
* Ref. invitation, gener.

20th Sept, Friday
9h → 12h.
web of knowledge.



if not on line
↳ bad macro scale to see sg.

Universality of description → superposition.

2. Assymetric. Equation is $\partial_t P + E \partial_x P = D \partial_x^2 P$
drift → field of force?

$\dfrac{2E\delta_x}{\delta_t} \Rightarrow$ cte.
"Drift" term.

if going → 0, neglect the drift.

ex $\hat{D} \sim \varepsilon$, $\delta_x \sim \varepsilon$, $\delta_t \sim \varepsilon^2$.

$\hat{D}$ and $\hat{E}$ do not scale in the same way.

Are the two described by same scaling function?

$\tilde{x} = x_0 + V \cdot t$.

$P(x,t) = P(x_0 + v t, t) = P_0(x_0, t) = P_0(x - V \cdot t, t)$

$\partial_t P(x,t) = \partial_t P_0 - V \cdot \partial_{x_0} P_0$

eq. verified by P_0:

$$\partial_t P_0 - V \partial_{x_0} P_0 + E \partial_{x_0} P_0 = D \partial_{x_0}^2 P_0.$$

take $V = E$.

$$\boxed{\partial_t P_0 = D \partial_{x_0}^2 P_0}$$

In general, such transformation is not possible.

constant velocity $v = E$.

if limit don't depend on δ_x, δ_t
↳ forget about micro-scale.

let see how related to irregularity of Brownian?

if $x(t)$ was a regular function of t. $\to$ ~

$$x(t_2) - x(t_1) \overset{t_2 \simeq t_1}{\sim} (t_2 - t_1)\, x'(t).$$

then $< [x(t_2) - x(t_1)]^2 > \sim (t_2 - t_1)^2$.

BUT in fact: "$|x(t_2) - x(t_1)| \sim \sqrt{t_2 - t_1} \gg |t_2 - t_1|$.

$\to$ Use of Brownian in a physical way.

Distribution of the full trajectories?

$$[x(t')]_{0 \le t' \le t} \quad , \quad [\eta(t')]_{0 \le t' \le t} .$$

" $\dot{x}(t) = \eta(t)$ " $\to$ $x(t) = \int_0^t dt' \eta(t')$ / Itô Integral.

$(dX = \beta(X) + dB_s)$

$x_0 = 0$ x_1 x_k d δ x_N

$t_0 = 0$ $t_1 = dt$ $t_k = k\,dt$ $t = t_N$

N slices of time. $dt = \frac{t}{N}$, $dt \gg \delta_t$. so as to keep a continuous space. (% discrete description).

$$\boxed{P(x_k, t_k \,|\, x_{k-1}, t_{k-1}) = \frac{1}{\sqrt{4\pi D(t_k - t_{k-1})}} \, e^{-\frac{1}{2}\frac{(x_k - x_{k-1})^2}{2D(t_k - t_{k-1})}}}$$

$(t_k - t_{k-1}) \to dt$

"Gaussian growth rates of the Brownian".

$\to$ Full trajectory.

$$P(x_N t_N | x_{N-1} t_{N-1} | \dots | x_0 t_0) \overset{\text{independence}}{=} [4D\pi dt]^{-N/2} \cdot \exp\left[-\frac{1}{2} \cdot \frac{1}{2D} \sum_{k=1}^{N} \frac{(x_k - x_{k-1})^2}{dt}\right]$$

observable

$$< \mathcal{O}[(x_0, t_0); \dots ; x_N t_N] > = \int dx_0 \dots dx_N \, P(x_N t_N | \dots | x_0 t_0) \, P(x_0) \, \mathcal{O}(\dots).$$

prob. density of the trajectory

of the initial position.

$\eta(t_k) = \eta_k$

$$P(\underbrace{\eta_N, t_N, \dots, \eta_0, t_0}_{N+1}) = \frac{1}{(4\pi D\,dt)^{\frac{N+1}{2}}}\, e^{-\frac{1}{2}\sum_{k=0}^{N} \frac{\eta_k^2}{2D}\,dt}$$

because $(x_0,\dots,x_N) \Leftrightarrow (x_0, \eta_0, \dots, \eta_{N-1})$ because $x_k = x_{k-1} + dt\,\eta_{k-1}$

$$\to -\frac{1}{2}\int_0^t dt' \, \frac{\eta^2(t')}{2D}$$

In the continuous limit,

$$P(\dots) = \prod_{k=0}^{N}\left[\frac{1}{\sqrt{4\pi D\,dt}}\cdot \exp\left(-\frac{1}{2}\,\frac{\eta_k^2}{2D}\cdot dt\right)\right]$$

quadratic form on the coordinate. for x_k, η_k.

↳ Generalization possible → Gaussian Integrals.

- Use of Brownian with forces and interactions [illegible] ...

[...]

In many system, phases transition between ≠ kinds of trajectories.
phase coexistence.

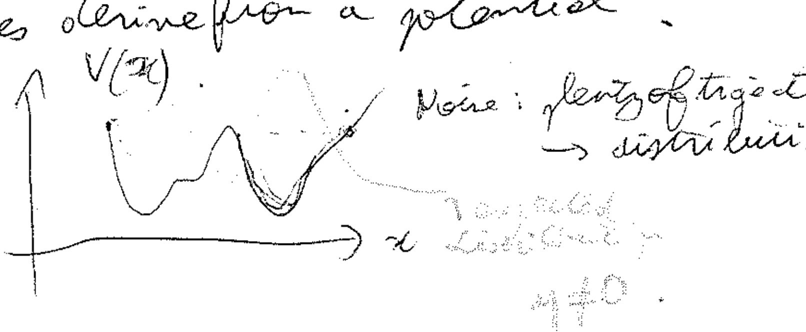
Use of white noise and brownian motion to describe the evolution of a particle in a force:

The equation $\partial_t x = \eta(t)$ can be seen as a particular case. ($m=0$, $F=0$)

$$\underset{\text{mass}}{m}\,\partial_t^2 x = \underbrace{-\gamma\cdot\partial_t x}_{\text{friction}} + \underbrace{F(x(t),t)}_{\text{deterministic force}} + \underbrace{\eta(t)}_{\text{noise}}$$

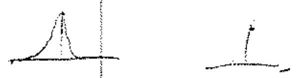
e.g. $F(x,t) = -V'(x)$.

forces derive from a potential.



Noise: plenty of trajectories
→ distribution.

So we get :



$$\frac{P(x,t+\delta t)-P(x,t)}{\delta t} = \frac{1}{\delta t}\int dx_1\, P(x_1,t)\Big[\underbrace{P(x,t+\delta_t|x_1,t)-\delta(x_1-}_{\text{small } \delta_t:\ \text{expansion around a Dirac } \delta}$$

difficult → singularities - (small & large values)

From trajectories to distrib for more regularity - However, still singul. → average of an observable to more reg

• 2^{nd} <u>try</u> : evolut° of an observable $\varphi(x)$.

$$\underbrace{\langle \varphi(x)\rangle_{t+\delta_t}}_{\text{mean value of } \varphi \text{ at time } t+\delta_t} = \int dx_1\, \underbrace{P(x_1,t)}_{\text{probabilities at time } t}\varphi(x_1)$$

$$\left[\ \underset{t+\delta t}{x} = \underset{t}{x_1} + \delta_t F(x,t) + \underbrace{\mathring{\eta}_t}_{O(\sqrt{\delta t})} + O(\cancel{\delta_t^{3/2}})\ \right]$$

$$\Updownarrow$$

$$\langle \varphi(x)\rangle_{t+\delta_t} = \int dx_1\, P(x_1,t)\int d\mathring{\eta}_t\, \mathbb{P}(\mathring{\eta}_t)\,\varphi\big(x_1+\delta_t F(x,t)+\mathring{\eta}_t\big)$$

→ expansion of φ.

small δt :

$$\varphi(\dots) = \varphi(x_1) + \delta_t F(x,t)\varphi'(x_1) + \overbrace{\mathring{\eta}_t}^{O(\sqrt{\delta t})}\varphi'(x_1)$$
$$+ \underbrace{\frac{1}{2}\mathring{\eta}_t^2}_{O(\delta t)}\varphi''(x_1) + O(\delta_t^{3/2}).$$

$\langle \mathring{\eta}_t^2\rangle = 2D\delta t$

→ apparition of the 2^{nd} derivative thanks to order of $\mathring{\eta}_t$ in $\sqrt{\delta_t}$:

Équation eq to diffusion with Force ? → Fokker-Plank.

Rq : ∂_t not along trajectory, but as time end ?

<u>Time derivative of averages :</u>

Correspondence between – trajectories
– distributions.

$$\left\langle \frac{\varphi(x_{t+\delta_t}) - \varphi(x_t)}{\delta_t} \right\rangle = \int dx \, \frac{P(x, t+\delta_t) - P(x,t)}{\delta_t} \varphi(x)$$

$\delta_t \to 0$. x as a mute variable : not along trajectory, <u>but all distribution</u>.

$$\boxed{\partial_t \langle \varphi(x) \rangle_t = \int dx \, \partial_t P(x,t) \varphi(x)}.$$

Hence .

$$\int dx \, \partial_t P(x,t) \varphi(x) = \int dx \left[P(x,t) F(x,t) \varphi'(x) + P(x,t) D \varphi''(x) \right]$$

[+ done for $\varphi \to$ distributions ?].

Integration by part . $\varphi \in \mathcal{D}(\mathbb{R})$

$$= \int dx \left\{ -\varphi(x) \partial_x \left[P(x,t) F(x,t) \right] + D \varphi(x) \partial_x^2 P(x,t) \right\}$$

$$= \int dx \, \varphi(x) \underline{\left\{ \partial_x \left[-F(x,t) P(x,t) + D \, \partial_x P \right] \right\}}.$$

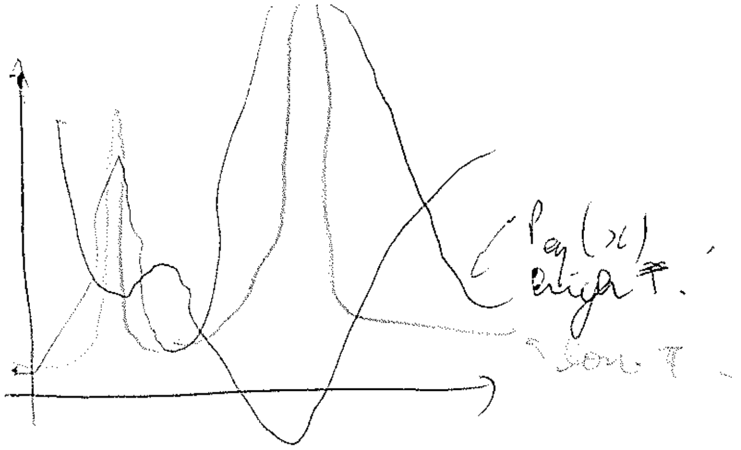
$\forall \varphi$.

So $(\cdots)$ is $0_{\mathcal{D}'(\Omega)}$, we obtain the F. Planck eq. :

Fokker-Planck eq. of evolution .

$$\boxed{\partial_t P(x,t) = \partial_x \Big[\underbrace{-F(x,t) P(x,t)}_{\text{force term}} + \underbrace{D \, \partial_x P(x,t)}_{\text{diffusion term}} \Big]}$$

[Physical interpretation for φ ?]
$\varphi \in \mathcal{D}(\Omega)$?
→ System finite, proba conserved : boundaries conditions.



Temperature: How distributed around minima.

Case with a current? → Project.

<u>Meaning of the current</u>?

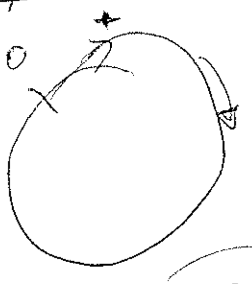
<u>example out of equil</u>.

$F(x) = f$

$\partial_t x = f + \eta$

average: $\partial_t \langle x \rangle = f \Rightarrow$ mean velocity V is $\boxed{V = f}$.

<u>space is periodic</u>: $[0;1]$.



$J(P) = f P + D \partial_x P$.

$V = \partial_t \langle x \rangle = \langle \partial_t x \rangle = \int dx\, \partial_t P(x,t)\, x$

$= \int dx (-\partial_x J)\, x = \int dx\, J[P]$

long t. $\quad V = \int_0^1 dx\, \underbrace{J[P_{st}]}\, = \int_0^1 dx\, j = j$.

$\partial_x J[P_{st}] = 0 \Rightarrow \boxed{J[P_{st}] = j}$ 1D.

$\boxed{j = V}$ in that particular case!

$V = f \Rightarrow$ to have eq. steady state: no force!