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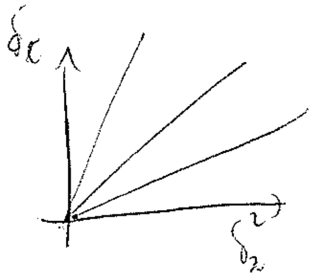
Theorical Analysis of Complex Systems

Course 2:

* Consulting hours.
* Seen course.
* Search Algo ...
* Courses overview
MAT 33L, MAP Malért,
PHY Complex Systems. F Blm
* Ref, wiener, gener.

20th Sept, Friday
9h → 12h.
web of knowledge.



if not on line
↳ bad macro scale to see sg.

Universality of description → superposition.

2. Assymetric. Equation is $\partial_t P + E \partial_x P = D \partial_x^2 P$
drift → field of force?

$\frac{2 \hat{E} \delta_x}{\delta_t} \Rightarrow$ cte.
"Drift" term.

if going → 0, neglect the drift.

ex $\hat{D} \sim \varepsilon,\ \delta_x \sim \varepsilon,\ \delta_t \sim \varepsilon^2$.

$\hat{D}$ and $\hat{E}$ do not scale in the same way.

Are the two described by same scaling function?

$\tilde{x} = x_0 + V \cdot t$.

$P(x,t) = P(x_0 + vt, t) = P_0(x_0, t) = P_0(x - V \cdot t, t)$

$\partial_t P(x,t) = \partial_t P_0 - V \cdot \partial_{x_0} P_0$

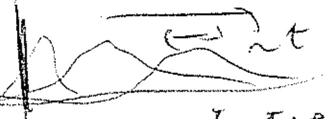
eq. verified by P_0:

$$\partial_t P_0 - V \partial_{x_0} P_0 + E \partial_{x_0} P_0 = D \partial_{x_0}^2 P_0.$$

take $V = E$.

$$\boxed{\partial_t P_0 = D \partial_{x_0}^2 P_0}$$

In general, such a transformation is not possible.



constant velocity $v = E$.

if limit don't depend on δ_x, δ_t
↳ forget about micro-scale.

Goal: How can we describe systems with noise? What type of noise?

$$x_{t+\delta_t} = x_t + \mathring{\eta}_t \quad \text{with} \quad \mathring{\eta}_t = \begin{cases} \delta_x & \mathring{D} \\ -\delta_x & \mathring{D} \\ 0 & 1-2\mathring{D} \end{cases}$$

$$\delta_x \to 0, \quad \delta_t \to 0, \quad \frac{\mathring{D}\,\delta_x^2}{\delta t} \to D$$

$$\boxed{\partial_t P = D\,\partial_x^2 P} \qquad \eta(t) = \frac{\mathring{\eta}_t}{\delta_t}$$

continuum space. $x(t)$: Brownian motion.

$\dot{x}(t) = \eta(t) \to$ white noise.

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{1}{2}\frac{x^2}{2Dt}\right) \quad \text{with } x(0) = 0.$$

Properties of Brownian and White Noise.

$$\langle [x(t_2) - x(t_2)]^2 \rangle = 2D|t_2 - t_1|$$
$$\langle \eta(t_2)\eta(t_1) \rangle = 2D\,\delta(t_2 - t_1)$$

Fully describe Brownian because Gaussian process.

Nono more general tool to describe stochastic processes.

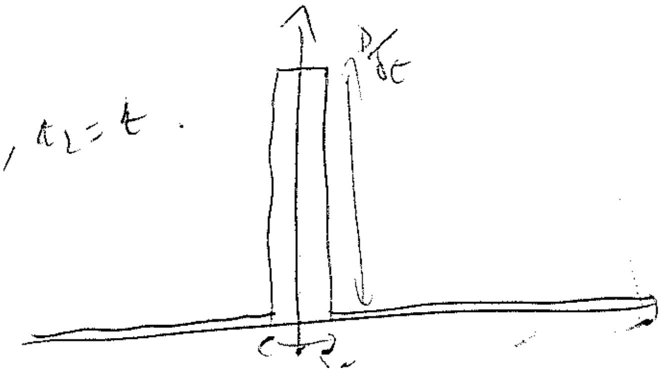
Remark: (from micro description).

$t_1 \neq t_1$: $\underbrace{\langle \eta(t_2)\eta(t_1) \rangle}_{\text{independent}} = 0$.

$$t_1 = t_2 \quad = \frac{1}{\delta_t^2}\langle \mathring{\eta}_{t_1}^2 \rangle = \frac{1}{\delta_t^2}\langle \delta_x^2 \overset{\text{probas}}{\mathring{D}} + \delta_x^2 \mathring{D} + 0 \rangle$$

$$= 2\mathring{D}\frac{\delta_x^2}{\delta t^2} = \boxed{\frac{2D}{\delta t}}$$

$t_1 = 0$, $t_2 = t$.



$\xrightarrow{\delta_t \to 0} 2D\,\delta(t_2 - t_1)$ is having width.

other way to obtain the result.

et see how related to irregularity of Brownian?

if $x(t)$ was a regular function of t.
$$x(t_2)-x(t_1) \overset{t_2 \simeq t_1}{\sim} (t_2-t_1)\, x'(t).$$
then $\langle [x(t_2)-x(t_1)]^2 \rangle \sim (t_2-t_1)^2$.

BUT in fact: "$|x(t_2)-x(t_1)| \sim \sqrt{t_2-t_1} \gg |t_2-t_1|$.

→ Use of Brownian in a physical way.

<u>Distribution of the full trajectories?</u>
$$[x(t')]_{0\leq t'\leq t} \ , \ [\eta(t')]_{0\leq t'\leq t}.$$

"$\dot{x}(t)=\eta(t)$" → $x(t)=\int_0^t dt'\,\eta(t')$ / Itô Integral.

$(dX = \beta(X) + dB_s)$

N slices of time. $dt=\frac{t}{N}$, $dt \gg \delta_t$. so as to keep a continuous space. (% discrete description).

$$\left[P(x_k,t_k \mid x_{k-1},t_{k-1}) = \frac{1}{\sqrt{4\pi D(\underbrace{t_k-t_{k-1}}_{\to dt})}}\, e^{-\frac{1}{2}\frac{(x_k-x_{k-1})^2}{2D\underbrace{(t_k-t_{k-1})}_{\to dt}}} \right.$$
" Gaussian growth rates of the Brownian".

→ Full trajectory.
$$P(x_N t_N \mid x_{N-1} t_{N-1} \mid \dots \mid x_0 t_0) \overset{\text{independence}}{=} [4D\pi dt]^{-N/2} \cdot \exp\left[-\frac{1}{2}\cdot\frac{1}{2D}\sum_{k=1}^{N}\frac{(x_k-x_{k-1})^2}{dt}\right]$$

<u>observable</u>
$$\langle \mathcal{O}[(x_0,t_0);\dots;x_N t_N] \rangle = \int dx_0 \dots dx_N\, P(x_N t_N \mid \dots \mid x_0 t_0)\, P(x_0)\, \mathcal{O}(\dots).$$

prob. density of the trajectory

of the initial position.

How depends on the discretisation we took?

in the exp:

$$\sum_{k=1}^{N} dt \left(\frac{x_k - x_{k-1}}{dt}\right)^2 \xrightarrow{dt \to 0} \int_0^t dt' \left(\partial_t x\right)^2.$$

(Riemann Sum).

Prefactor? → both integral, role of normalisation

Continuous Time notations:

$$\int \frac{dx_0 \dots dx_N}{[4\pi D\, dt]^{N/2}} \overset{(def)}{=} \int \mathcal{D}x .$$ ← path integral on all possible paths.

$$\boxed{\langle O[x(t')] \rangle = \int \mathcal{D}x \cdot O[x(t')]\, e^{-\frac{1}{2}\int_0^t dt' \frac{(\partial_t x)^2}{2D}} P(x_0)}.$$

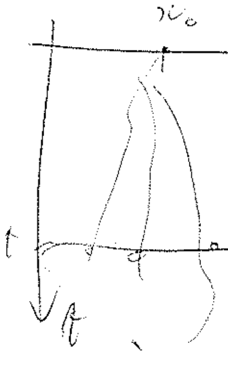
depends not of the discretisation. (if O reasonable!)

In a quantum view → "path integral".

observable which depends on the full trajectory.

Wiener Integration...

e.g. $O[x] = \int_0^t dt'\, x^2(t') \dots$



$O[x(t)] = \int dt'\, E(x(t'), t')$.

What about the white noise?

$\eta(t_k) = \eta_k$

$$P\big(\underbrace{\eta_N, t_N, \dots, \eta_0, t_0}_{N+1}\big) = \frac{1}{(4\pi D\, dt)^{\frac{N+1}{2}}}\, e^{\overbrace{-\frac{1}{2}\sum_{k=0}^{N}\frac{\eta_k^2}{2D}\,dt}}$$

$$-\frac{1}{2}\int_0^t dt'\,\frac{\eta^2(t')}{2D}$$

because $(x_0, \dots, x_N) \Leftrightarrow (x_0, \eta_0, \dots, \eta_{N-1})$ because $x_k = x_{k-1} + dt\,\eta_{k-1}$

In the continuous limit,

$$P(\dots) = \prod_{k=0}^{N}\left[\frac{1}{\sqrt{(4\pi D\, dt)}}\cdot \exp\left(-\frac{1}{2}\,\frac{\eta_k^2}{2D}\cdot dt\right)\right]$$

quadratic form on the coordinate. for x_k, η_k.

↳ Generalization possible → Gaussian Integrals.

- Use of Brownian with forces and interactions ...

In many systems, phases transitions between ≠ kinds of trajectories.
phase coexistence.

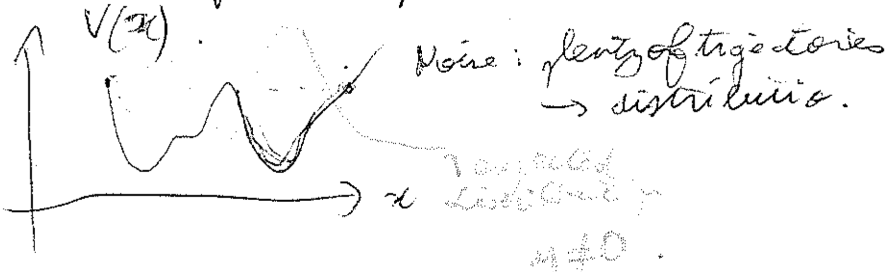
Use of white noise and brownian motion to describe the evolution of a particle in a force:

The equation $\partial_t x = \eta(t)$ can be seen as a particular case. $m=0$, $F=0$

$$\underset{\text{mass}}{m}\,\partial_t^2 x = \underbrace{-\gamma\cdot\partial_t x}_{\text{friction}} + \underbrace{F(x(t), t)}_{\text{deterministic force}} + \underbrace{\eta(t)}_{\text{noise}}.$$

e.g. $F(x,t) = -V'(x)$.

forces derive from a potential.



Noise: plenty of trajectories
→ distribution.

Tunneled ...

$\eta \neq 0$.

Point of view of trajectories

$$\partial_t x = F(x(t),t) + \eta(t) . \quad \begin{bmatrix} m=0 \\ \gamma=1 \\ F\neq 0 \end{bmatrix}$$

other method without discretising space.

(→ Fokker-Planck . Generalised F.P. from Langevin eq.) .
[Kramers Eq]

Let's discretise time & keep x continuous :

$\int_t^{t+\delta t}$

$$x_{t+\delta_t} - x_t = \underbrace{\int_t^{t+\delta_t} dt' F(x(t'),t')}_{\simeq \delta_t F(x_t,t)} + \underbrace{\int_t^{t+\delta_t} dt' \eta(t')}_{\mathring{\eta}_t} .$$

"local mean field approx" $= O(\delta_t)$; $\mathring{\eta}_t$: $O(\sqrt{\delta_t})$.

$$\mathring{\eta}_t = B(t+\delta_t) - B(t) .$$

$$\rightarrow B(t) = \int_0^t dt' \eta(t') \quad : \text{Brownian motion} .$$

$$\langle \mathring{\eta}_t \rangle = 0 .$$

$$\langle \mathring{\eta}_t \mathring{\eta}_t \rangle = \langle (B(t+\delta_t) - B(t))^2 \rangle = C(\delta t) = 2D\,\delta_t^{\,>0} .$$

$\rightarrow \mathring{\eta}_t = O(\sqrt{\delta_t})$. : "Time derivatives of B scales as $\sqrt{t}$" .

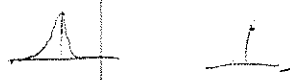
why important to have this scale property? → see after .

Evolution of $P(x,t)$ – 1st try :

$$P(x,t+\delta t) = \int dx_1\, P(x_1,t)\, P(x,t+\delta t \mid x_1,t)$$

So we get:

$$\frac{P(x,t+\delta t)-P(x,t)}{\delta t} = \frac{1}{\delta t}\int dx_1\, P(x_1,t)\,\Big[\underbrace{P(x,t+\delta_t|x_1,t)-\delta(x_1-}_{}$$



small δ_t: expansion around a Dirac δ.

difficult → singularities - (small & large values)

From trajectories to distrib for more regularity -
However, still singul. → average of an observable to more reg

• 2nd try: evolut° of an observable $\varphi(x)$.

$$\underbrace{\langle \varphi(x)\rangle_{t+\delta_t}}_{\text{mean value of } \varphi \text{ at time } t+\delta_t} = \int dx_1\, P(x_1,t)\,\varphi(x_1)$$

possibilities at time t.

$$\underset{t+\delta t}{x} = \underset{t}{x_1} + \delta_t F(x,t) + \underbrace{\mathring{\eta}_t}_{O(\sqrt{\delta_t})} + O\big(\cancel{\delta_t^{3/2}}\big)$$

$$\Updownarrow$$

$$\langle\varphi(x)\rangle_{t+\delta_t} = \int dx_1\, P(x_1,t)\int d\mathring{\eta}_t\, \mathbb{P}(\mathring{\eta}_t)\,\varphi\big(x_1+\delta_t F(x,t)+\mathring{\eta}_t\big)$$

→ expansion of φ.

small δt:

$$\varphi(\dots) = \varphi(x_1) + \delta_t F(x,t)\varphi'(x_1) + \overbrace{\mathring{\eta}_t}^{O(\sqrt{\delta_t})}\varphi'(x_1) + \underbrace{\frac{1}{2}\mathring{\eta}_t^2}_{O(\delta t)}\varphi''(x_1) + O\big(\delta_t^{3/2}\big)$$

$\langle \mathring{\eta}_t^2\rangle = 2D\delta t$

→ apparition of the 2nd derivative thanks to order of $\mathring{\eta}_t$ in $\sqrt{\delta_t}$.

$$= \underbrace{\int dx_1\, P(x_1,t)}_{\langle \varphi(x) \rangle_t} \left[\varphi(x_1) + \delta_t \left(F(x,t)\varphi' + D\varphi'' \right) \right]$$

So we get. F supposed regular to have mean loop/ 2nd order are there to $\langle \dot{x}(t)\rangle = 2\delta t$.

$$\frac{\langle \varphi(x) \rangle_{t+\delta t} - \langle \varphi(x) \rangle_t}{\delta t} = \langle F(x,t)\,\varphi'(x) + D\,\varphi''(x) \rangle_t$$

(if F not regular, more complicated: two Brownian motions. only solved recently ...).

eq:

$$\langle F(x,t)\,\varphi'(x) + \underbrace{D\varphi''(x)}_{\text{due to the noise}} \rangle_t = \partial_t \langle \varphi(x) \rangle_t$$

→ what does it do here? ...

math: φ is a test function (φ distrib.!) (going back from observable to try ?)

3rd try: $\partial_t x_t = \eta + F$.

informally:

$$\partial_t \varphi(x_t) = \varphi'(x_t)\, \partial_t x_t .$$

$$\partial_t \langle \varphi(x) \rangle = \underline{\langle F(x,t)\,\varphi'(x) \rangle} + \langle \eta\, \varphi'(x) \rangle$$

↳ PB to determine

$= D \langle \varphi''(x) \rangle$ from precedent calculation

NOT OBVIOUS.

Equation obtained by ITO Formula.

Équation eq to diffusion with Force? → Fokker-Planck.

Rq: ∂_t not along trajectory, but as time evol?

<u>Time derivative of averages:</u>

Correspondence between – trajectories
– distributions.

$$\left\langle \frac{\varphi(x_{t+\delta_t}) - \varphi(x_t)}{\delta_t} \right\rangle = \int dx \, \frac{P(x, t+\delta_t) - P(x,t)}{\delta_t} \varphi(x)$$

$\delta_t \to 0$. x as a mute variable: not along trajectory, <u>but the distribution</u>.

$$\boxed{\partial_t \langle \varphi(x) \rangle_t = \int dx \, \partial_t P(x,t) \varphi(x)}.$$

Hence.

$$\int dx \, \partial_t P(x,t)\varphi(x) = \int dx \left[P(x,t) F(x,t) \varphi'(x) + P(x,t) D \varphi''(x) \right]$$

[+ done for $\varphi \to$ distributions?].

Integration by part. $\varphi \in \mathcal{D}(\mathbb{R})$

$$= \int dx \left\{ -\varphi(x) \partial_x \left[P(x,t) F(x,t) \right] + D \varphi(x) \partial_x^2 P(x,t) \right\}$$

$$= \int dx \, \varphi(x) \left\{ \underline{\partial_x \left[-F(x,t) P(x,t) + D \, \partial_x P \right]} \right\}.$$

$\forall \varphi$.

So $(\dots)$ is $0_{\mathcal{D}'(\Omega)}$, we obtain the F. Planck eq.:

<u>Fokker-Planck eq. of evolution</u>.

$$\boxed{\partial_t P(x,t) = \partial_x \Big[\underbrace{-F(x,t) P(x,t)}_{\text{force term}} + \underbrace{D \, \partial_x P(x,t)}_{\text{diffusion term}} \Big]}$$

[Physical interpretation for φ?].
$\varphi \in \mathcal{D}(\Omega)$?
→ System finite, proba conserved: boundaries conditions.

Case of systems at equilibrium.

How related to that?

Steady state $P_{st}(x)$: $\to t\to\infty$: if exists will converge to it.

$$\partial_x\left[-F(x,t)P_{st}(x) + D\,\partial_x P_{st}(x)\right]=0.$$

Ex: [1] 1 particle on $\mathbb{R}$ $F=0$;

$$P(x,t)=\frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{1}{2}\frac{x^2}{2Dt}\right)\xrightarrow{t\to\infty}0$$

No steady state.

[2] 1 particle on $[a;b]$, isolated.

$$\partial_x^2 P_{st}(x)=0 \longrightarrow \boxed{P_{st}(x)=\frac{1}{b-a}}.$$

uniform distrib.

- Current of probabilities $J[P]$: FPeq:

$$\partial_t P=-\partial_x J[P]. \quad \text{with } J[P]=-\left[-FP+D\partial_x P\right].$$

- steady state: $\partial_x J[P_{st}]=0$.
- by def. an equilibrium steady state P_{eq} verifies:

$$J[P_{eq}]=0, \text{ i.e. } F(x,t)P_{eq}(x)=+D\partial_x P_{eq}(x).$$

indep. of t.

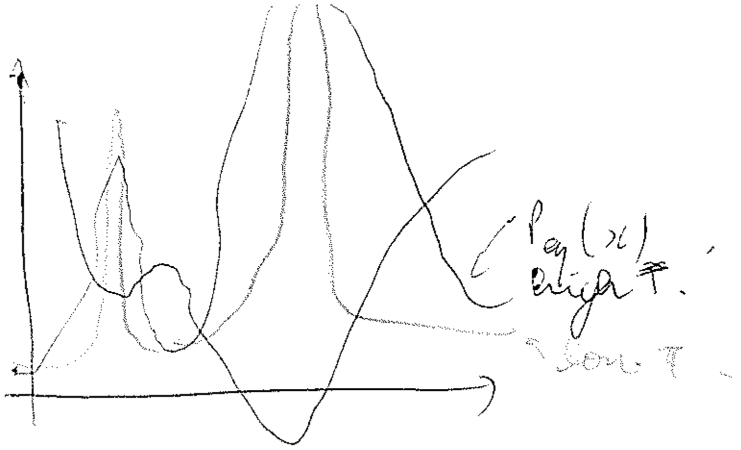
1D : $F(x)=-V'(x)$

$$V'(x)P_{eq}(x)=D\,\partial_x P_{eq}(x);$$

$$\boxed{P_{eq}(x)=\exp\left(-\frac{1}{D}V(x)\right)}$$

Boltzmann distribution in a potential V at "temperature" D.

$\to$ Thermodyn. results at long times.



Temperature: How distributed around minima.

Case with a current? → Project.

<u>Meaning of the current?</u>

<u>example out of equil.</u>

$F(x) = f$

$\partial_t x = f + \eta$

average: $\partial_t \langle x \rangle = f \Rightarrow$ mean velocity v is $\boxed{v = f}$.

<u>space is periodic</u>: $[0;1]$.

$J(P) = fP + D\partial_x P$.

$v = \partial_t \langle x \rangle = \langle \partial_t x \rangle = \int dx\, \partial_t P(x,t)\, x$

$= \int dx (-\partial_x J)\, x = \int dx J[P]$

long t. $\quad v = \underbrace{\int_0^1 dx\, J[P_{st}]}\, = \int_0^1 dx\, j = j$.

$\partial_x J[P_{st}] = 0 \Rightarrow \boxed{J[P_{st}] = j}$ 1D.

$\boxed{j = v}$ in that particular case!

$v = f \Rightarrow$ to have eq. steady state: no force!

$J[P_{eq}] = 0$ $\Leftrightarrow$ Reversibility.

[impossible to distinguish between a trajectory and its time reverse].

(explained next time).