$J[P_{eq}] = 0 \iff$ Reversibility.

[impossible to distinguish between a trajectory and its time reverse].

(explained next time).

Case of systems at equilibrium.

How related to that?

Steady state $P_{st}(x)$: $\rightarrow t \to \infty$: if exists will converge to it.

$$\partial_x \left[-F(x,t) P_{st}(x) + D \partial_x P_{st}(x) \right] = 0 .$$

Ex: [1] 1 particle on $\mathbb{R}$ $F=0$;

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{1}{2} \frac{x^2}{2Dt} \right) \xrightarrow{t \to \infty} 0$$

No steady state.

[2] 1 particle on $[a;b]$, isolated.

$$\partial_x^2 P_{st}(x) = 0 \rightarrow \boxed{P_{st}(x) = \frac{1}{b-a}} .$$

uniform distrib.

- Current of probabilities $J[P]$: FPeq:

$$\partial_t P = -\partial_x J[P]. \quad \text{with } J[P] = -\left[-FP + D \partial_x P \right].$$

- steady state: $\partial_x J[P_{st}] = 0$.
- by def. an equilibrium steady state P_{eq} verifies:

$$J[P_{eq}] = 0 \text{ , i.e. } \underset{\text{indep. of } t}{F(x,t)} P_{eq}(x) = + D \partial_x P_{eq}(x).$$

1D : $F(x) = -V'(x)$

$$V'(x) P_{eq}(x) = D \, \partial_x P_{eq} x ;$$

$$\boxed{P_{eq}(x) = \exp\left(-\frac{1}{D} V(x) \right)}$$

Boltzmann distribution in a potential V at "temperature" D.

$\rightarrow$ Thermodyn. results at long times.

$$= \underbrace{\int dx_1\, P(x_1, t)}_{\langle \varphi(x) \rangle_t} \Big[\varphi(x_1) + \delta_t \big(F(x,t)\varphi' + D\varphi'' \big) \Big]$$

So we get. F supposed regular to have mean law (2nd order are thanks to $\langle \eta(t) \rangle = 2D\delta t$.)

$$\frac{\langle \varphi(x) \rangle_{t+\delta t} - \langle \varphi(x) \rangle_t}{\delta t} = \langle F(x,t)\,\varphi'(x) + D\,\varphi''(x) \rangle_t$$

(if F not regular, more complicated: two Brownian motions. only solved recently ...)

eq:

$$\langle F(x,t)\,\varphi'(x) + \underbrace{D\varphi''(x)}_{\text{due to the noise}} \rangle_t = \partial_t \langle \varphi(x) \rangle_t$$

→ what's Ito here? ...

math: φ is a test function - (φ distrib!) (going back from observable to try ?)

3rd try: $\partial_t x_t = \eta + F$.

informally:

$$\partial_t \varphi(x_t) = \varphi'(x_t)\, \partial_t x_t$$

$$\partial_t \langle \varphi(x) \rangle = \underline{\langle F(x,t)\,\varphi'(x) \rangle} + \langle \eta \varphi'(x) \rangle$$

↳ PB to determine

$= D \langle \varphi''(x) \rangle$ from precedent calculation

NOT OBVIOUS -

Equation obtained by ITO Formula.

<u>Point of view of trajectories</u>

$$\partial_t x = F(x(t),t) + \eta(t) \quad \begin{bmatrix} m=0 \\ \gamma=1 \\ F\neq 0 \end{bmatrix}$$

other method without discretizing space.

(→ Fokker-Planck. Generalised F.P. from Langevin eq.)
[Kramers Eq]

<u>Let's discretise time & keep x continuous:</u>

$\int_t^{t+\delta t}$

$$x_{t+\delta_t} - x_t = \underbrace{\int_t^{t+\delta_t} dt' F(x(t'),t')}_{\simeq \delta_t F(x_t,t)} + \underbrace{\int_t^{t+\delta_t} dt' \eta(t')}_{\mathring{\eta}_t}$$

"Local mean field" approx. $= O(\delta_t)$ $\qquad O(\sqrt{\delta_t})$.

$$\mathring{\eta}_t = B(t+\delta_t) - B(t).$$

$$\to B(t) = \int_0^t dt' \eta(t') \quad : \text{Brownian motion}.$$

$$\langle \mathring{\eta}_t \rangle = 0.$$

$$\langle \mathring{\eta}_t \mathring{\eta}_t \rangle = \langle (B(t+\delta_t) - B(t))^2 \rangle = C(\delta_t) = 2D\,\delta_t^{>0}.$$

$\to \mathring{\eta}_t = O(\sqrt{\delta_t})$. : "Time derivatives of B scales as $\sqrt{t}$".

why important to have this scale property? → see after.

Evolution of $P(x,t)$ _ 1st try:

$$P(x,t+\delta t) = \int dx_1\, P(x_1,t)\, P(x,t+\delta t \mid x_1,t)$$

How depends on the discretisation we took?

in the exp:

$$\hookrightarrow \sum_{k=1}^{N} dt \left(\frac{x_k - x_{k-1}}{dt}\right)^2 \xrightarrow{dt \to 0} \int_0^t dt' (\partial_t x)^2 .$$

(Riemann Sum).

Prefactor? → both integral, role of normalisation

Continuous Time notations:

$$\int \frac{dx_0 \dots dx_N}{[4\pi D\, dt]^{N/2}} \overset{(\text{def})}{=} \int \mathcal{D}x .$$ ← path integral on all possible paths.

$$\boxed{\langle \mathcal{O}[x(t')] \rangle = \int \mathcal{D}x \cdot \mathcal{O}[x(t')]\, e^{-\frac{1}{2}\int_0^t dt' \frac{(\partial_t x)^2}{2D}} P(x_0)}.$$

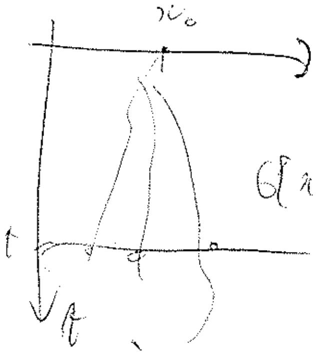
depends not of the discretisation. (if O reasonable!)

In a quantum view → "path integral".

↳ observable which depends on the full trajectory.

Wiener Integration...

e.g. $\mathcal{O}[x] = \int_0^t dt'\, x^2(t') \dots$



$\mathcal{O}[x(t)] = \int dt' E(x(t'), t')$

What about the white noise?

Goal: How can we describe systems with noise? What type of noise?

$$x_{t+\delta_t} = x_t + \mathring{\eta}_t \quad \text{with } \mathring{\eta}_t = \begin{cases} \delta_x & \mathring{D} \\ -\delta_x & \mathring{D} \\ 0 & 1-2\mathring{D} \end{cases}$$

$\delta_x \to 0$, $\delta_t \to 0$, $\frac{\mathring{D}\,\delta_x^2}{\delta t} \to D$

$$\boxed{\partial_t P = D\,\partial_x^2 P} \qquad \eta(t) = \frac{\mathring{\eta}_t}{\delta_t}$$

continuum space. $x(t)$: Brownian motion.

$\dot{x}(t) = \eta(t)$ → white noise.

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{1}{2}\frac{x^2}{2Dt}\right) \quad \text{with } x(0) = 0$$

Properties of Brownian and White Noise.

$$\langle [x(t_2) - x(t_1)]^2 \rangle = 2D|t_2 - t_1|$$
$$\langle \eta(t_2)\eta(t_1) \rangle = 2D\,\delta(t_2 - t_1)$$

Fully describe Brownian because Gaussian process.

Now more general tool to describe stochastic processes.

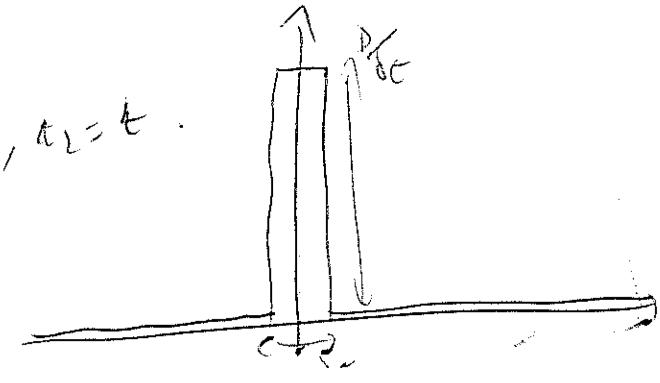
Remark: (from micro description).

$t_1 \neq t_2$: $\underbrace{\langle \eta(t_2)\eta(t_1) \rangle}_{\text{independent}} = 0$.

$t_1 = t_2$: $= \frac{1}{\delta_t^2}\langle \mathring{\eta}_{t_1}^2 \rangle = \frac{1}{\delta_t^2}\langle \delta_x^2 \mathring{D} + \delta_x^2 \mathring{D} + 0 \rangle$ probas

$= 2\mathring{D}\frac{\delta_x^2}{\delta_t^2} = \boxed{\frac{2D}{\delta t}}$

$t_1 = 0$, $t_2 = t$.



$\xrightarrow[\delta_t \to 0]{} 2D\,\delta(t_2 - t_1)$ is having width.

other way to obtain the result.