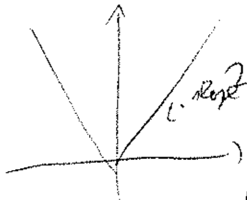
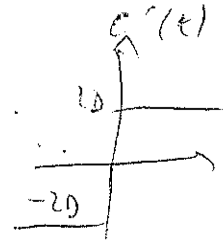
→ Think of random walk moments:

We've got $\boxed{C''(t) = 2R(t)}$



$C(t) = 2D|t|$



$C'(t)$ → $= 4D\delta_0$

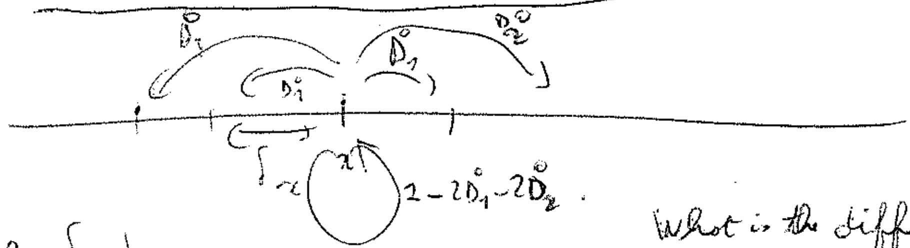
η is white noise.

$R(t) = 2D\,\delta(t)$ → $\boxed{\langle \eta(t_2)\,\eta(t_1)\rangle = 2D\,\delta(t_1 - t_2)}$

Exercises:

1 - Analytical part:

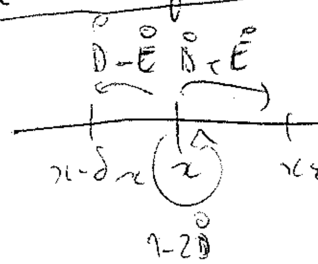
1.1: Consider a RW with next-to-neighbour jumps.



Take the $\left.\begin{matrix}\delta_x \\ \delta_t\end{matrix}\right) \to 0$ continuum limit.

What is the diffusion coefficient as a function of $\mathring{D}_1, \mathring{D}_2$

1.2: Consider the drifted RW.



Take the continuum limit for the master equation [→ order 2!].
How does $\mathring{E}$ have to scale with δ_x, δ_t so that the continuum limit is "well defined" [eq. which does not depend of δ_x, δ_t]
(define a ~~[illegible]~~ $E \leftrightarrow \mathring{E}, \delta_x, \delta_t$).

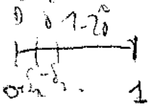
2 - Numerical simulations:

RW with $\mathring{D}, \delta_x, \delta_t$.

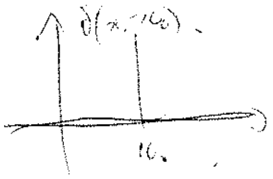
Determine the histogram of the arrival point for many trajectories.
Compare it with $P(x, t)$ → fixed. (Gaussian)

$t \gg \delta_t$. ⚠ $\mathring{D} \leq \frac{1}{2}$. [$\mathring{D} \ll \frac{1}{2}$] ↳ works better.

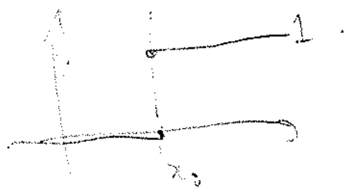
Trick: $x_t = \sum_{t' < t} \mathring{\eta}_{t'}$. Co draw η_t



"touch sampling".

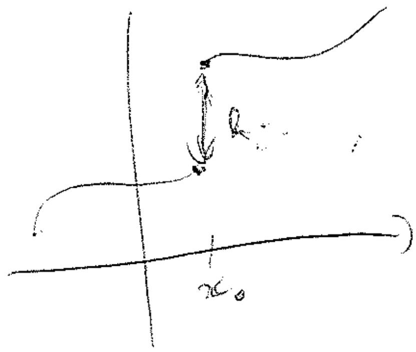


$$= \begin{cases} x < x_0 : 0 \\ x > x_0 : 1 \end{cases}$$ → Recover $\int dx\, \delta(x) = 1$



→ Heavyside step function in x_0.
$H(x-x_0)$

→ no meaning to value at x_0.



$$f(x) = \underbrace{f^{reg}(x)}_{\text{differentiable}} + \sum_i \underbrace{h_i H(x-x_0^i)}_{\text{jump of height } h}.$$

$$f'(x) = f^{reg\,\prime}(x) + \sum_i h_i\, \delta(x-x_i^0).$$

÷ Master by $\delta_x : \mathring{P} \to P$.

÷ by δ_t :

$$\frac{P(x, t+\delta_t) - P(x,t)}{\delta t} = \frac{\mathring{D}}{\delta t}\Big[\underbrace{P(x+\delta_x, t)}_{P(x,t)+\delta_x \partial_x P(x,t)} + P(x-\delta_x, t) - 2P(x,t)\Big] \quad \ldots \text{ Expansion.}$$

$\delta_t \to 0$

$$\partial_t P(x,t) = \Big[\underbrace{P + P - 2P}_{\substack{=0 \\ \text{conservation of prob.}}} + \delta_x \underbrace{\big[\partial_x P - \partial_x P\big]}_{\substack{=0 \\ \text{symmetry of random walk.}}} + 2 \cdot \tfrac{1}{2}\delta_x^2 \big[\partial_x^2 P\big]\Big]$$

We obtain Diffusion Equation:

$$\boxed{\partial_t P(x,t) = D \cdot \partial_x^2 P(x,t).} \quad \circledast$$ with $$\boxed{D = \frac{\delta_x^2}{\delta_t}\mathring{D}}$$ Diffusion coefficient.

→ well defined ($D \neq 0$, $D \neq \infty$)
if $\delta_x \to 0$, $\delta_t \to 0$ with $\frac{\delta_x^2}{\delta_t} \to$ const.

if go further in δt, other non-linear. other object ≠ Brownian.()

<u>Diffusive scaling</u> for $\frac{\delta_x^2}{\delta_t}$. Other processes with other scaling exponents.

∘ <u>Solution by scaling</u> : Let's assume that $P(x,t) = t^{-\xi}\hat{P}(\underbrace{t^{-\xi}x}_{\hat{x}})$

Substituting into $\circledast$

$$-\xi t^{-1-\xi}\big[\hat{P}(\hat{x}) + \hat{x}\hat{P}'(\hat{x})\big] = D\, t^{-3\xi}\, \hat{P}''(\hat{x}).$$

• 2^{nd} solut°: mesoscopic point of view

evolut° of distribution  time t 

$P(\{\vec{x}_i\}, t=0)$ $\qquad$ $P(\{\vec{x}_i\}, t)$.

→ Effective degrees of freedom.

Follow small scales of particles.

Thermodynamics gives only evolution of mean of distrib. on the long time.

effective degrees of freedom: $\{\vec{x}_i\}$: $m_i \ddot{\vec{x}}_i = \underbrace{-\gamma_i \dot{\vec{x}}_i}_{\text{dissipation}} + \vec{F}(\{\vec{x}_j\}, t) + \underbrace{\vec{\eta}_i}_{\text{"noise"}}$

coordinates of robot or still more

after integration of small particles. molecules of a liquid.

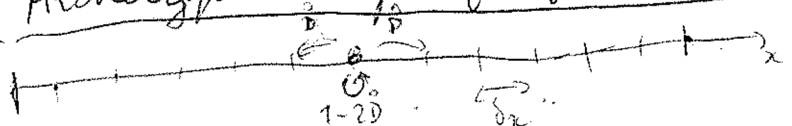
large particles (cells).

Robert BROWN 1828.
Suspension of big cells in Ficus trees.
Always fluctuation.

microscopic effects translated into noise and dissipative effect.

Very difficult to justify mathematically the jump micro → meso in that case.

2 - Archetypal example of diffusion of one particle in 1D.



$0 < \mathring{D} \leq \frac{1}{2}$

(system in free also asymmetric field of gravity, etc.)

x_t: position. $\mathring{D}$: discrete param.

discrete time evolution between $t \to t+\delta_t$

$$x_{t+\delta_t} = x_t + \boxed{\begin{cases} +\delta_x & \text{prob } \mathring{D} \\ -\delta_x & \text{prob } \mathring{D} \\ 0 & \text{prob } 1-2\mathring{D} \end{cases}}$$

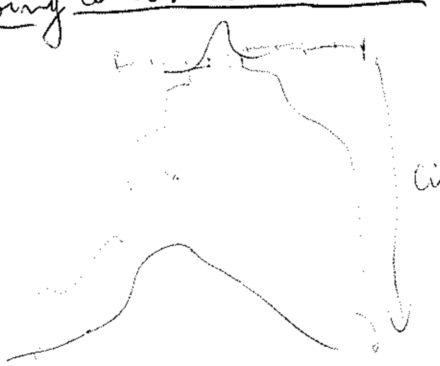
$\mathring{\eta}_t$ → discrete noise.

all the $\mathring{\eta}_t$ are independent from the previous ones (noise is memory less).

We will build Brownian Motion...
↳ continuous but never differentiable...

• Going to continuum limit $\delta_x \to 0$, $\delta_t \to 0$. instantaneous velocity:

$$\frac{x_{t+\delta_t} - x_t}{\delta_t} = \frac{\mathring{\eta}_t \to 0}{\delta_t \to 0}$$



time.

→ Go to probas: more easy to handle