S2 Text : Semi-analytical analysis of the simplified model

Partial Differential Equation

We propose to derive the PDE in a simplified setting. To recall the configuration given in main text, the system has one dimension, such that $x \in \mathbb{R}$ with $1/\delta x$ cells of size δx, and we use the expected values of cell population $p(x,t) = \mathbb{E}[P(x,t)]$. We furthermore take $n_d = 1$. Larger values would imply derivatives at an order higher than 2 but the following results on the existence of a stationary solution should still hold.

Denoting $\tilde{p}(x,t)$ the intermediate populations obtained after the aggregation stage, we have

$$\tilde{p}(x,t) = p(x,t) + N_g \cdot \frac{p(x,t)^\alpha}{\sum_x p(x,t)^\alpha}$$

since all populations units are added independently. If $\delta x \ll 1$ then $\sum_x p^\alpha \simeq \int_x p(x,t)^\alpha dx$ and we write this quantity $P_\alpha(t)$. We furthermore write $p = p(x,t)$ and $\tilde{p} = \tilde{p}(x,t)$ in the following for readability.

The diffusion step is then deterministic, and for any cell not on the border ($0 < x < 1$), if δt is the interval between two time steps, we have

$$\begin{aligned} p(x, t+\delta t) &= (1-\beta) \cdot \tilde{p} + \frac{\beta}{2}\left[\tilde{p}(x-\delta x, t) + \tilde{p}(x+\delta x, t)\right] \\ &= \tilde{p} + \frac{\beta}{2}\left[(\tilde{p}(x+\delta x,t) - \tilde{p}) - (\tilde{p} - \tilde{p}(x-\delta x, t))\right] \end{aligned}$$

Assuming the partial derivatives exist, and as $\delta x \ll 1$, we make the approximation $\tilde{p}(x+\delta x, t) - \tilde{p} \simeq \delta x \cdot \frac{\partial \tilde{p}}{\partial x}(x,t)$, what gives

$$(\tilde{p}(x+\delta x,t) - \tilde{p}) - (\tilde{p} - \tilde{p}(x-\delta x,t)) = \delta x \cdot \left(\frac{\partial \tilde{p}}{\partial x}(x,t) - \frac{\partial \tilde{p}}{\partial x}(x-\delta x, t)\right)$$

and therefore at the second order

$$p(x, t + \delta t) = \tilde{p} + \frac{\beta \delta x^2}{2} \cdot \frac{\partial^2 \tilde{p}}{\partial x^2}$$

Substituting $\tilde{p}$ gives

$$\begin{aligned}
\frac{\partial^2 \tilde{p}}{\partial x^2} &= \frac{\partial^2 p}{\partial x^2} + \frac{N_G}{P_\alpha} \cdot \frac{\partial}{\partial x}\left[\alpha \frac{\partial p}{\partial x} p^{\alpha-1}\right] \\
&= \frac{\partial^2 p}{\partial x^2} + \alpha \frac{N_G}{P_\alpha}\left[\frac{\partial^2 p}{\partial x^2} p^{\alpha-1} + (\alpha-1)\left(\frac{\partial p}{\partial x}\right)^2 p^{\alpha-2}\right]
\end{aligned}$$

By supposing that $\frac{\partial p}{\partial t}$ exists and that δt is small, we have $p(x, t+\delta t) - p(x,t) \simeq \delta t \frac{\partial p}{\partial t}$, what finally yields , by combining the results above, the partial differential equation

$$\delta t \cdot \frac{\partial p}{\partial t} = \frac{N_G \cdot p^\alpha}{P_\alpha(t)} + \frac{\alpha\beta(\alpha-1)\delta x^2}{2} \cdot \frac{N_G \cdot p^{\alpha-2}}{P_\alpha(t)} \cdot \left(\frac{\partial p}{\partial x}\right)^2 + \frac{\beta \delta x^2}{2} \cdot \frac{\partial^2 p}{\partial x^2} \cdot \left[1 + \alpha \frac{N_G p^{\alpha-1}}{P_{\alpha(t)}}\right] \quad (1)$$

Properties Initial conditions should be specified as $p_0(x) = p(x, t_0)$. To have a well-posed problem similar to more classical PDE problems, we need to assume a domain and boundary conditions. A finite support is expressed by $p(x,t) = 0$ for all t and x such that $|x| > x_m$.

An infinite domain implies that density, in the sense of population proportion $d(x,t) = \frac{p(x,t)}{P_1(t)}$, goes to zero anywhere when time goes to infinity. Indeed, $P_1(t) = N_G \cdot t$. If $d(x,t)$ does not vanish, there exist t_1 such that

Stationary solution for density

Randomness and bifurcations

The previous analyses were done on a deterministic version of the system. How can randomness influence the final configuration ?

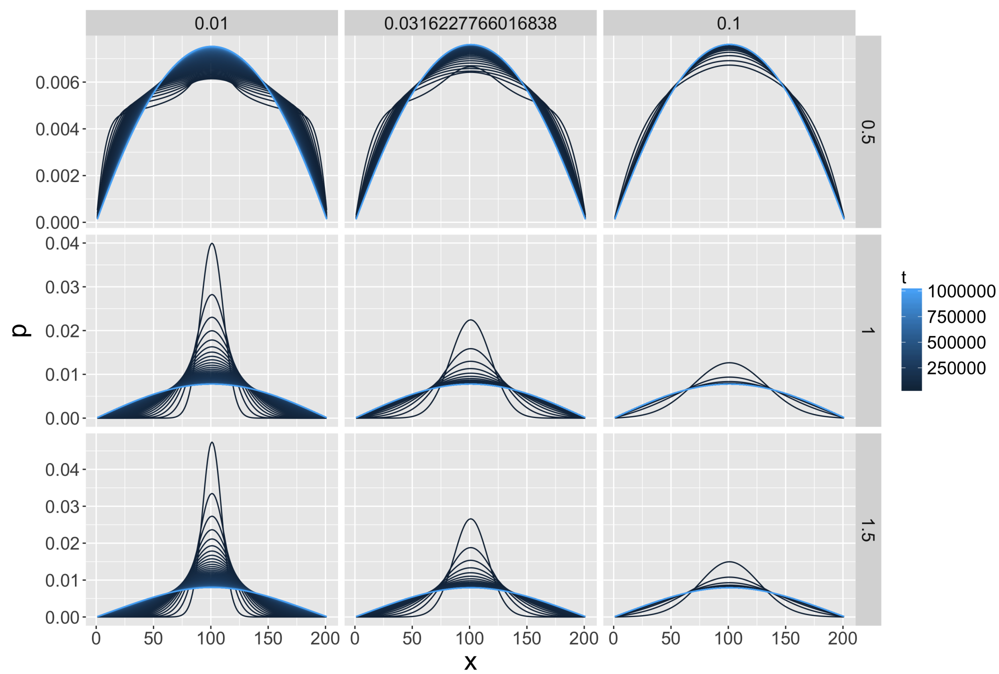
0.01
0.0316227766016838
0.1
0.5
1
1.5
p
x
t
1000000
750000
500000
250000
0.000
0.002
0.004
0.006
0.00
0.01
0.02
0.03
0.04
0
50
100
150
200

Figure 1