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Monocentric analysis of restricting the budget share of housing alone or with transportation

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ABSTRACT

Considering the prolonged rise of energy price and the still elevated housing prices, the policy to limit the share of housing expenses in the households' budget, so as to secure their solvability, has been criticized. Supposedly, it induces people to get farther from the city center in search for cheaper housing prices, but with subsequent increased transportation costs that are often disregarded during the house search process. Therefore, to improve the well-being of households, it has been advocated to set a constraint on the share of both housing and transportation expenditure.

The paper is purported to analyze and compare the effects of the two policies in terms of:

1. Well-being of the households;
2. Land-use: city size and density curve;
3. Solvability of the households;
4. Transportation costs.

The analysis is carried out within the classical monocentric model of urban economics. After setting a general analysis, an applied model is specified to capture the effects of each policy in straightforward formulae.

It is shown that constraining housing expenses may increase the well-being of households. Besides, both policies prove effective in reducing urban sprawl and hereby energy consumption. Thus the choice of the optimal policy will depend on the local authority's objectives.

Keywords: monocentric model, urban economics, housing expenses, transportation expenses, housing policy

INTRODUCTION

Based on the sustained increase in the fuel price, and the still elevated housing prices, concerns increased about the solvability of households, especially in tight housing markets. Notably, because high housing prices and lack of housing supply have favored suburbanization (e.g. in the Paris metropolitan area), and because households living in the suburbs make the most extensive and expensive use of the car (*1*), these households are particularly sensitive to the price of energy.

Consequently, the relevance of bounding the housing expenditure up to a given share of the household income has become controversial. In several countries the household's housing budget share is capped so as to preserve its solvability. In France, this is ensured in two ways:

- Homebuyers' loans are based on monthly payments amounting to at most one third of the household's income.
- When applying for a dwelling, candidates for tenancy must earn at least three times the required rent (this ratio corresponds to widely spread practice).

While this policy seems to secure the solvability of households, it induces them to settle farther from the agglomeration center in search for affordable housing prices (*2*). Consequently, they expose themselves to significant transport costs, which jeopardize the households' budget (see (*3*) and (*4*) for U.S. figures on this topic). Therefore, the use of a joint housing plus transportation budget share constraint is widely supported. In the U.S. several initiatives have been taken in this direction (*5*,*6*).

Although there is an abundant economic literature on the analysis of land-use regulatory policies (e.g. *7*,*8*), to the best of our knowledge such is not the case for the policies we mentioned. Thus we propose to analyze and compare the impacts on the main features of the city of the policy limiting the share of the housing expenditure within the household's budget (which we call the Constrained Housing Expense (CHE) policy), and of that limiting the total share of transportation and housing expenditures (the Constrained Housing+Transportation (CHT) policy). We will scrutinize the issue of the well-being of the households, the city size and the related transport costs, and the global level of rents. As will be seen, both policies would reduce urban sprawl (thus energy consumption) while maintaining or even increasing the well-being of the households.

The analysis takes place within the classical framework of urban economics, the monocentric model (see (*9*) for detailed presentation of this model). In this model households, with income Y, maximize their utility $U(z,s)$ by trading-off between two goods, land (s representing land consumption or lot size), and a composite good denoted by z standing for all other goods, under a budget constraint. This is summed up in the following program:

$$\underset{z,s,r}{Max}\ U(z,s)\ \ s.t.\quad R(r)s + z + T(r) = Y$$

While $R(r)$ stands for the relative land rent, the z good is taken as the *numéraire*, and $T(r)$ represent transport costs. Variable r represents location: since locating farther from the central business district (CBD) implies higher transport costs, households typically trade-off between accessibility and housing prices when choosing their location. Note that in the simplified context of the monocentric model, land rents and housing prices are equivalent.

The plan is as follows: the first section being the present introduction, the second section analyzes the CHE policy while the third section scrutinizes the CHT policy. Section Four offers a comparative analysis of the two measures and policy recommendations.

Throughout the paper the theoretical properties are asserted in explicit statements; however, for the sake of brevity most of the proofs have been omitted (if so a mention “proof omitted” is introduced). Demonstrations are gracefully available on request to the author.

CAPPING THE HOUSING BUDGET SHARE

In this section we analyze the impacts brought about by the CHE policy in terms of:

- Household utility
- Land use: city size, density
- Composition of the household budget

To do so, we first present the constrained housing expenses (CHE) model and solve the household program. Then we characterize the equilibrium city and proceed to comparative statics in the general case. Lastly, we study in detail the different impacts of the CHE policy in the case of a linear city.

The Constrained Housing Expense (CHE) model

Let us consider the general case, where $U(z,s)$ and $T(r)$ are only assumed to comply with the classical hypotheses :

- $U(z,s)$ is concave, strictly increasing with z and s, and is well-behaved (according to the definition provided in Fujita (*9*)).
- $T(r)$ increases with distance r to the CBD.

Presentation of the CHE model

In order to study the CHE policy, we amend the monocentric model with the following constraint:

$$R(r)s \leq \alpha Y \qquad (E1)$$

Housing expenditure $R(r)s$ is capped to a fraction α of the household’s income Y. Given the budget constraint of the household, (*E1*) is equivalent to the following constraint, which will prove easier to handle:

$$z \geq (1-\alpha)Y - T(r) \qquad (E2)$$

Consequently, the household program becomes:

$$\max_{z,s,r} U(z,s) \quad s.t. \begin{cases} z + R(r)s + T(r) = Y \\ z \geq (1-\alpha)Y - T(r) \end{cases} \qquad (E3)$$

Note that $\alpha \geq 1$ yields the original unconstrained model.

Notations

We use the following notations:

- A tilde superscript (~) for the CHE model, no symbol for the original model
- We often omit the argument α when unnecessary.
- $E_A(u,\alpha) = \{r / z(r,u) < (1-\alpha)Y - T(r)\}$ is the strictly binding zone, defined as the set of locations r where the Lagrange multiplier associated to (*E2*) is strictly positive.
- $S(z,u)$ is the inverse function of $U(z,s)$ with respect to s.
- r_{max} is the farthest feasible location: $T(r_{\max}) = Y$.

The bid-max program

Bid rent function of the household Bid rent functions for the CHE and original models are defined as usual:

$$\begin{aligned} \tilde{\Psi}(r,u) &= \max_{z,s}\left\{ \frac{Y - T(r) - z}{s} \middle| \begin{array}{l} U(z,s) = u \\ z \geq (1-\alpha)Y - T(r) \end{array} \right\} \\ \Psi(r,u) &= \max_{z,s}\left\{ \frac{Y - T(r) - z}{s} \middle| U(z,s) = u \right\} \end{aligned} \qquad (E4)$$

Argmax of the unconstrained program are denoted $s(r,u)$ and $z(r,u)$. Let us recall classical properties:

- $s(r,u)$ increases with r and u
- $\Psi(r,u)$ decreases with r and u
- $z(r,u)$ decreases with r

Because $\tilde{\Psi}(r,u)$ is obtained by adding the HE constraint to the original program, we have the following property (proof omitted):

PROPERTY 1

$$\begin{cases} \tilde{z}(r,u) = \max[z(r,u), (1-\alpha)Y - T(r)] \\ \tilde{s}(r,u) = \min[s(r,u), S((1-\alpha)Y - T(r), u)] \\ \tilde{\Psi}(r,u) = \min[\Psi(r,u), \alpha Y / \tilde{s}(r,u)] \end{cases} \qquad (E5)$$

which implies that $\forall (r,u)$, $\tilde{z}(r,u) \geq z(r,u)$, $\tilde{s}(r,u) \leq s(r,u)$ and $\tilde{\Psi}(r,u) \leq \Psi(r,u)$

To sum up, for a given utility level, and inside the binding zone, capping housing expenditures reduces:

- the lot size which is bid for.
- the ability to pay for a unit of land.

Properties of the bid-max variables A binding HE constraint alters the solutions. Nevertheless, system (*E5*) ensures that:

- $\tilde{s}(r,u)$ increases with *r*, *u* and *α*
- $\tilde{\Psi}(r,u)$ decreases with *r*, *u* and increases with *α*
- $\tilde{z}(r,u)$ decreases with *r* and *α*

Conservation of the properties with respect to *r* and *u* is central to demonstrating the existence and uniqueness of the equilibrium land use. Regarding the role of *α*, relieving the constraint increases the maximum level of housing expenditures, which allows households to purchase bigger lots, increase their bid rent, and reduce their consumption of the *z* good.

The case of single household type

We investigate in this subsection the standard framework of a closed city with absentee landlords and inhabited by households of a given single type, with income *Y* and utility function *U(z,s)*. After demonstrating the existence and uniqueness of the equilibrium in the CHE model, we perform comparative statics in order to compare the CHE equilibrium to the original equilibrium.

As usual, we note *N* the number of households and we assume positive land supply $L(r)>0$ at all $r>0$.

Existence and uniqueness of the CHE equilibrium

As in Fujita (*9*) for the unconstrained model, demonstration of the existence and uniqueness of the equilibrium in the CHE model is equivalent to proving that there exists a unique couple $(\tilde{u},\tilde{r}_f)$ that complies with the following system:

$$\begin{cases} \tilde{\Psi}(\tilde{r}_f,\tilde{u}) = R_A \\ \displaystyle\int_0^{\tilde{r}_f} \frac{L(r)}{\tilde{s}(r,\tilde{u})} dr = N \end{cases} \qquad (E6)$$

The first equality is the boundary condition that determines the edge $\tilde{r}_f$ of the city: at $\tilde{r}_f$ bid rent equates the opportunity cost of land, R_A. The second equality corresponds to the population constraint: integration of the density function within the city gives *N*, the total number of households. Note that density *n*(*r*) is given by the available land supply divided by the land consumption per household, i.e. $n(r) = L(r) \div \tilde{s}(r,\tilde{u})$

PROPOSITION 1

The CHE monocentric model with single household type admits a unique equilibrium.

PROOF OF PROPOSITION 1

Similarly to Fujita (9), we consider the outer boundary function $\tilde{b}(u)$ characterized by $\int_0^{\tilde{b}(u)} \frac{L(r)}{\tilde{s}(r,u)} dr = N$. $\tilde{b}(u)$ determines the city size for a given target utility *u*. Since $\tilde{s}(r,u)$ exhibits the same required features as *s*(*r*,*u*), that is to say $\tilde{s}(r,u)$ is decreasing in *u*, tends toward +∞ when $u \to +\infty$ and tends toward 0 when $u \to -\infty$, we could proceed similarly to Fujita to show that $\tilde{b}(u)$ is well-defined on an interval]-∞,*a*[, where possibly *a*=+∞. Besides, $\tilde{b}(u)$ strictly increases with *u* and ranges from 0 to +∞ when *u* ranges from -∞ to *a*.

Then we consider $\tilde{R}_{Bound}(x) = \tilde{\Psi}(x, \tilde{U}(x))$ where $\tilde{U}(x) = \tilde{b}^{-1}(x)$ for $x \in [0, r_{max}[$. $\tilde{R}_{Bound}(x)$ is the land rent at the edge *x* of a city, the utility of which has been chosen so as to procure the required size *x*. Since $\tilde{b}(u)$ increases strictly with *u*, $\tilde{U}(x)$ also increases strictly with *x*, implying that $\tilde{R}_{Bound}(x)$ is strictly decreasing in *x* (remember that $\tilde{\Psi}(r,u)$ is decreasing in both *r* and *u*). Since $\tilde{R}_{Bound}(r_{\max}) = 0$ and $\tilde{R}_{Bound}(x) \underset{x \to 0}{\to} +\infty$, the equation $\tilde{R}_{Bound}(x) = R_A$ admits one and only one solution $\tilde{r}_f$. Eventually, by taking $\tilde{u} = \tilde{U}(\tilde{r}_f)$, it is trivial to check that $(\tilde{u}, \tilde{r}_f)$ satisfies system (*E6*).

Comparative statics in the general case

We determine here the influence of the constraint parameter *α* on the equilibrium city.

City Size Quite intuitively, the CHE policy reduces the city size:

PROPOSITION 2

For any set (N,Y,R_A) the size $\tilde{r}_f(\alpha)$ of the CHE city increases with *α*

PROOF OF PROPOSITION 2

Let us first show that the constrained boundary rent curve $\tilde{R}_{Bound}(x,\alpha_1)$ is below the second one, i.e.: $\tilde{R}_{Bound}(x,\alpha_1) \le \tilde{R}_{Bound}(x,\alpha_2)$

As $\forall (r,u)$, $\tilde{s}(r,u,\alpha_1) \le \tilde{s}(r,u,\alpha_2)$ then $\int_0^x \frac{L(r)}{\tilde{s}(r,u,\alpha_1)} dr \ge \int_0^x \frac{L(r)}{\tilde{s}(r,u,\alpha_2)} dr$.

Since $\int_0^{\tilde{b}(u,\alpha_1)} \frac{L(r)}{\tilde{s}(r,u,\alpha_1)} dr = \int_0^{\tilde{b}(u,\alpha_2)} \frac{L(r)}{\tilde{s}(r,u,\alpha_2)} dr = N$, this implies $\tilde{b}(u,\alpha_1) \le \tilde{b}(u,\alpha_2)$, which in turn implies that the inverse functions are in reversed order, that is to say $\tilde{U}(x,\alpha_1) \ge \tilde{U}(x,\alpha_2)$.

Using the inequality $\forall(r,u),\ \tilde{\Psi}(r,u,\alpha_1) \leq \tilde{\Psi}(r,u,\alpha_2)$, we have:

$$\tilde{\Psi}(x,\tilde{U}(x,\alpha_1),\alpha_1) \leq \tilde{\Psi}(x,\tilde{U}(x,\alpha_2),\alpha_1) \leq \tilde{\Psi}(x,\tilde{U}(x,\alpha_2),\alpha_2)$$

$$\Rightarrow\ \tilde{R}_{Bound}(x,\alpha_1) \leq \tilde{R}_{Bound}(x,\alpha_2)$$

which is the claimed property. Considering this, demonstration of proposition 2 is straightforward since $\tilde{R}_{Bound}(\tilde{r}_f(\alpha_1),\alpha_1) = \tilde{R}_{Bound}(\tilde{r}_f(\alpha_2),\alpha_2) = R_A$.

Because $\alpha \geq 1$ yields the original model, proposition 2 unveils that the CHE city is smaller than the original one.

Equilibrium utility While the analysis of equilibrium utility is more complex, the following proposition gives an insight:

PROPOSITION 3

For any couple $\alpha_1 < \alpha_2$, if the household located at the edge of the α_2 city spends less than $\alpha_1 Y$ on housing (i.e. $\tilde{r}_f(\alpha_2) \notin E_A(\tilde{u}(\alpha_2),\alpha_1)$), then the equilibrium utility $\tilde{u}(\alpha_1)$ of the α_1 city is superior to the equilibrium utility $\tilde{u}(\alpha_2)$ of the α_2 city.

PROOF

For a household located at $\tilde{r}_f(\alpha_2)$, we have the following relations:

- $\tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_1) = \tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_2)$ from $\tilde{r}_f(\alpha_2) \in \overline{E}_A(\tilde{u}(\alpha_2),\alpha_1)$
- $\tilde{\Psi}(\tilde{r}_f(\alpha_1),\tilde{u}(\alpha_1),\alpha_1) = R_A = \tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_2)$ (boundary conditions)
- $\tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_1),\alpha_1) \leq \tilde{\Psi}(\tilde{r}_f(\alpha_1),\tilde{u}(\alpha_1),\alpha_1)$ due to $\tilde{r}_f(\alpha_1) \leq \tilde{r}_f(\alpha_2)$ (proposition 2)

By combining these relations, we have $\tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_1),\alpha_1) \leq \tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_1)$, which implies $\tilde{u}(\alpha_1) \geq \tilde{u}(\alpha_2)$.

Proposition 3 shows specific conditions under which the equilibrium utility of the CHE city decreases with α. By choosing $\alpha_2=1$, it gives a simple condition, sufficient but not necessary, for the CHE city to display a higher equilibrium utility than the unconstrained city (with equilibrium utility u_{eq}). On the other hand, we will see in the application to come that the case $\tilde{u}(\alpha) < u_{eq}$ is possible when the constraint puts an excessive burden on the households.

Actually, the HE constraint induces two effects that alter the equilibrium utility level:

- Being constrained in their choices, households achieve a lower utility at a given location and land rent price
- But capping housing expenses has a depressing effect on bid prices, hence on land rents, which tends to increase the utility of the households

Depending on the relative magnitude of these two effects, the resulting utility level of the HE city is higher or lower than that of the original city.

Housing expenses Determining the influence of α on housing expenses proves not trivial, because tightening the HE constraint may result in a lower utility level, which in turn may increase the housing expenses of unconstrained households. Nonetheless, when the equilibrium utility rises, it is possible to show that tightening the constraint always diminishes the total land rent distributed to the landlords. The same goes when the constraint is binding for the whole city.

Application to a linear city

Considering the limitations of the general case analysis, we now provide a special case as an illustration, with a log-linear utility function $U(z,s) = 1/2 \log z + 1/2 \log s$, linear transport costs $T(r)=ar$ and a linear city: $L(r)=1$. We did not choose a disk-shaped city (i.e. $L(r)=2\pi r$) since calculations prove more complex, especially for deriving analytical results.

Derivation of the equilibrium city

After determining the binding zone, we derive the different variables of interest, that is to say bid-max variables, utility level and city size, which we use in the next subsection to analyze the equilibrium outcomes.

Determination of the binding zone The log-linear form utility function implying that $z(r,u) = 1/2(Y - ar)$, the housing expenditure constraint is strictly binding when:

$$r < r_{bind}(\alpha) = \frac{(1-2\alpha)Y}{a} \qquad (E7)$$

Thus:

- if $\alpha \geq 1/2$ the HE constraint is never binding, the CHE model is equivalent to the unconstrained model.
- if $\alpha < 1/2$, only households located closer than $r_{bind}(\alpha)$ are effectively submitted to the HE constraint.

Characterization of the equilibrium Resolution of the bid-max program brings about the following formulae:

$$r \leq r_{bind}(\alpha) \begin{cases} \tilde{z}(r,u) = (1-\alpha)Y - ar \\ \tilde{s}(r,u) = e^{2u} / \tilde{z}(r,u) \\ \tilde{\Psi}(r,u) = e^{-2u}\alpha Y\{(1-\alpha)Y - ar\} \end{cases} \qquad r \geq r_{bind}(\alpha) \begin{cases} \tilde{z}(r,u) = (Y - ar)/2 \\ \tilde{s}(r,u) = e^{2u} / \tilde{z}(r,u) \\ \tilde{\Psi}(r,u) = e^{-2u}(Y - ar)^2 / 4 \end{cases} \qquad (E8)$$

Figure 1 illustrates these solutions for the following settings (which will constitute our reference model): N=10, Y=80, a=8 and R_A=20. In addition to that we choose α=0.20 and u=21.21 (which corresponds to the equilibrium utility of the CHE model for the chosen settings). For these settings r_{max}=10 and r_{bind}=6.

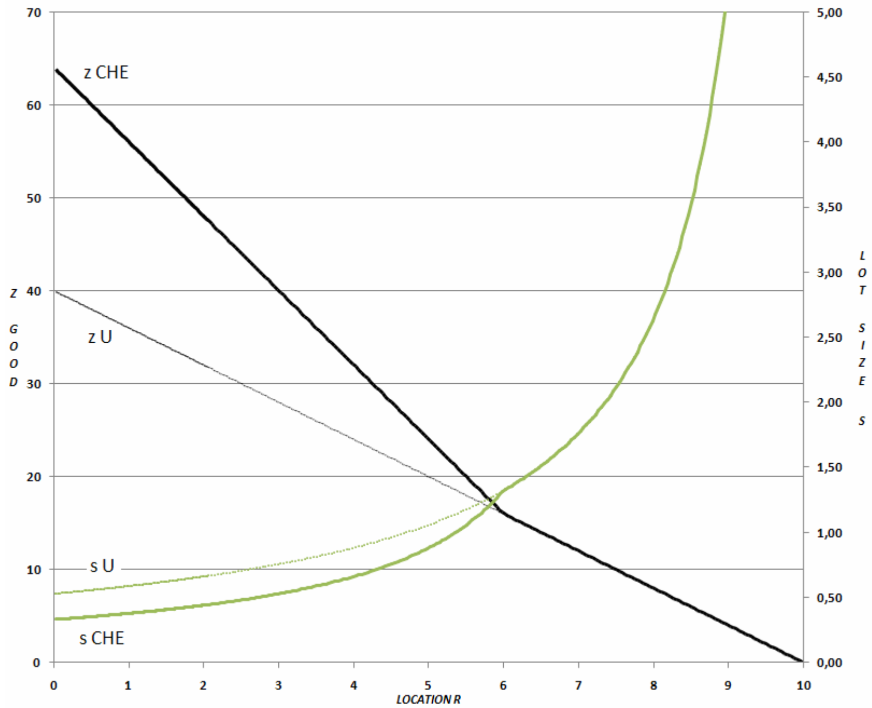

FIGURE 1a Lot size and z good consumption in the Unconstrained (U) and CHE models.

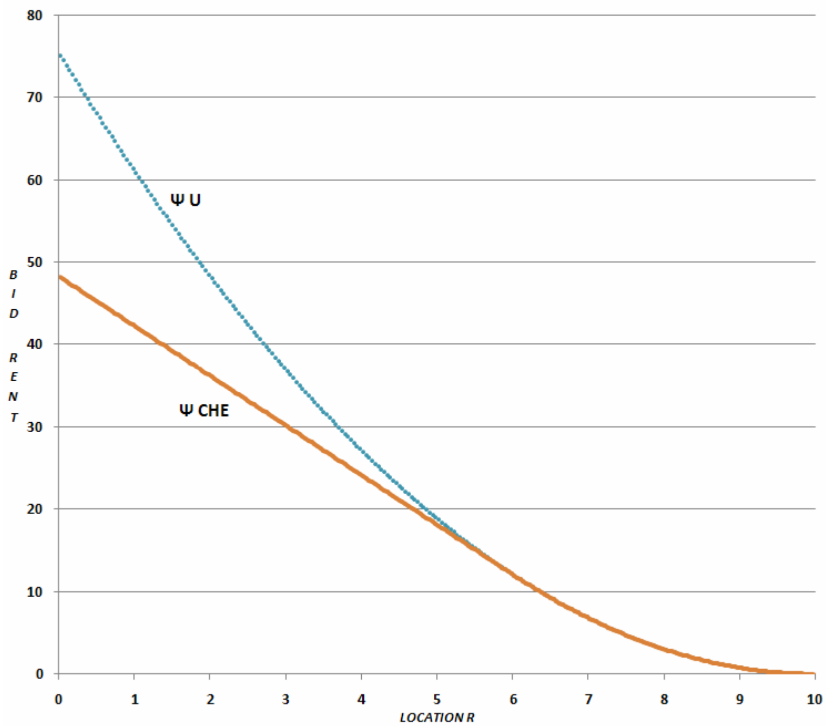

FIGURE 1b Bid rent functions.

As previously observed, for a given utility the HE constraint reduces both the lot size and the bid rent inside the binding zone, and increases the consumption of the composite good. Outside the binding zone, we find the same solutions for the CHE and unconstrained models.

We are now ready to characterize the equilibriums.

PROPOSITION 4

In the applied case, the equilibrium is characterized as follows:

	$\alpha \le \alpha_{cr}$	$\alpha \in [\alpha_{cr}, 1/2]$	$\alpha \ge 1/2$
$e^{2\tilde{u}}$	$\frac{\alpha^2 Y^2}{R_A^2}\left(\sqrt{a^2N^2+\left(\frac{1-\alpha}{\alpha}\right)^2 R_A^2}-aN\right)$	$\frac{Y^2\left(1-2\alpha+2\alpha^2\right)}{2(aN+R_A)}$	$\frac{Y^2}{4(aN+R_A)}$
$\tilde{r}_f$	$\frac{Y}{a}\left(1-\alpha\left(1+\sqrt{\left(\frac{aN}{R_A}\right)^2+\left(\frac{1-\alpha}{\alpha}\right)^2}-\frac{aN}{R_A}\right)\right)$	$\frac{Y}{a}\left(1-\sqrt{\frac{R_A}{aN+R_A}\left(2-4\alpha+4\alpha^2\right)}\right)$	$\frac{Y}{a}\left(1-\sqrt{\frac{R_A}{aN+R_A}}\right)$

where $\alpha_{cr} = \left(1+\sqrt{1+\frac{2aN}{R_A}}\right)^{-1}$

Calculations are based on the distinction of 3 cases:

- $\alpha \geq 1/2$ yields the unconstrained model
- If $\alpha \in [\alpha_{cr}, 1/2]$, the city fringe is beyond $r_{bind}(\alpha)$
- If $\alpha \leq \alpha_{cr}$, the HE constraint is active for the whole city

Comparative statics for the applied model

Thanks to the computation of the different equilibriums, we can proceed to a more precise analysis of the role of α.

Utility level In the applied model, while an appropriate choice of α increases the households' utility compared to the unconstrained city, setting α to too low a value usually decreases it.

PROPERTY 2

For any given set of parameters ($N,Y,R_A>0,a$), the equilibrium utility $\tilde{u}(\alpha)$ of the CHE city strictly decreases on $[\alpha_{cr}, 1/2]$ with $\tilde{u}(1/2) = u_{eq}$. It is maximal for $\alpha_{max} < \alpha_{cr}$, with $\tilde{u}(\alpha_{\max}) > u_{eq}$. Furthermore, $\tilde{u}(\alpha) \underset{\alpha \to 0}{\to} -\infty$

If $R_A=0$, $\tilde{u}(\alpha)$ strictly decreases on $]0,1/2]$ and therefore is maximal when α tends toward 0.

Demonstration (proof omitted) is carried out by using proposition 4.

Figure 2 depicts the variations of $e^{2\tilde{u}(\alpha)}$ for our reference model (corresponding to N=10, Y=80, a=8 and R_A=20); for these settings α_{cr}=0.25.

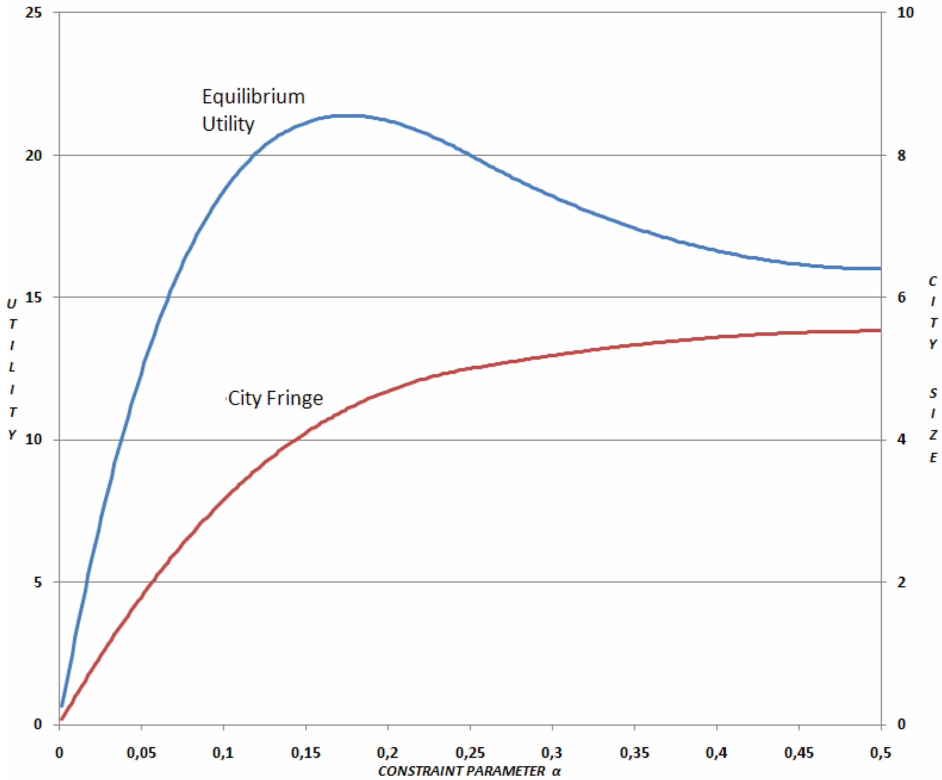

FIGURE 2 Utility level and city size of the CHE city.

We can check that 0.176=α_{max}<α_{cr} =0.25, which corroborates property 2.

Property 2 confirms proposition 3: whenever the city fringe is beyond the binding zone (i.e. $\tilde{r}_f(\alpha) \geq r_{bind}(\alpha)$ which is equivalent to $\alpha \geq \alpha_{cr}$), the CHE city displays a higher utility level than the unconstrained city. On the other hand, if the city is entirely constrained, reducing α proves worthwhile at first but quickly utility dwindles.

As a matter of fact, when the outside competition (the agricultural sector) for land is mild, the constraints put on households' choices are more than compensated by the drop of prices that results from a less fierce competition for land within the binding zone. This increases the utility of all households. Conversely, if competitiveness of the households is too weakened compared to the agricultural sector, reduction of the city size is exacerbated and leads to declining utility.

City Size and Density Contrary to the utility level, tightening the housing budget constraint always curtails the city (see proposition 2) as shown on Figure 2. When α is decreased from 0.5 to 0, city size shrinks, and this phenomenon is accentuated when α<α_{max}, i.e. when the HE constraint becomes too pregnant relatively to the need to compete for land with the agricultural sector.

Reduction of the city size is achieved in different ways according to the value of α:

- When the utility level increases, owing to higher densities near the CBD that overweigh lower densities in the suburban area
- When the utility level decreases, density uniformly rises throughout the city

Figure 3 illustrates the equilibrium densities for the original city, and two CHE cities with α=0.3 and α=0.15:

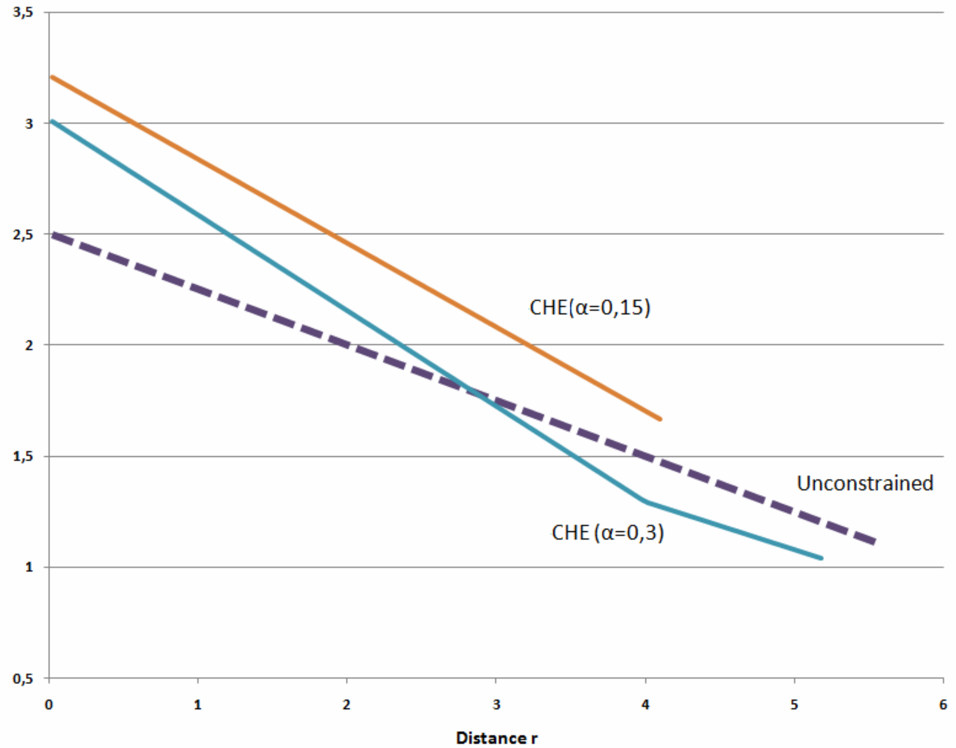

FIGURE 3 Influence of α on density in the reference model.

When α is chosen within $[\alpha_{max},1/2]$, we observe as predicted higher densities near the CBD, but lower densities in the suburbs. When α is chosen within $[0,\alpha_{max}]$, density rises throughout the whole town.

Average composition of the households' budgets Since the HE policy was designed to cap housing expenses so as to ensure the solvability of the households, one key issue pertains to the average composition of the household's budget at the equilibrium land use (proof omitted):

PROPERTY 3

For any given set (N,Y,R_A,a), both the housing and transportation average expenditures are rising with α, inducing a declining consumption of the z good

Figure 4 exemplifies Property 3 for the reference model. For high values of α (between 0.4 and 0.5), decreasing α only slightly reduces the housing and transportation budget shares, because a limited number of households is affected by the constraint. If α further decreases (approximately until α_{max}=0.176), housing expenses decrease more sharply while transport costs are moderately affected. On this interval, decreasing α has a more significant depressing effect on prices than on lot sizes. Below α_{max}, the constraint weighs more on the households' choices of lot size, resulting in the fall of the city size and lower transportation and housing expenditures.

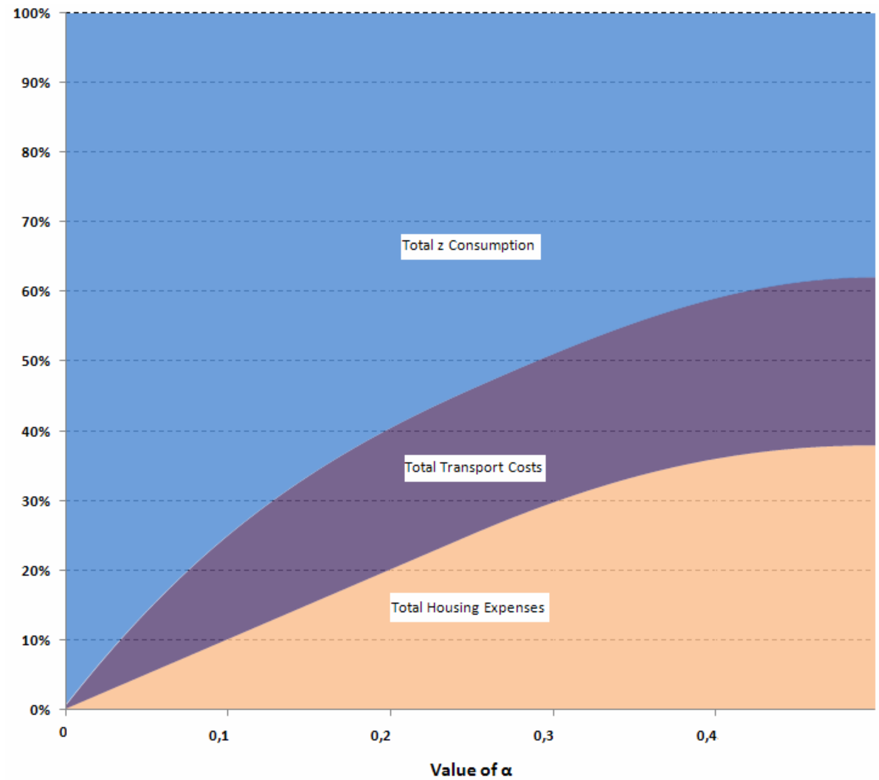

FIGURE 4 Influence of α on the average composition of the household's budget.

Concluding remarks for the CHE model To sum up, capping housing expenditures has the twofold effect of distorting households' residential choices (regarding lot size), and reducing equilibrium prices of the housing market. At first, the latter effect overweighs the former, leading to an increase in the utility level while the global structure of the city (size, use of transportation) is relatively unchanged. Nevertheless decreasing α further eventually tightens drastically the lot sizes, resulting in both dropping utility level and city size.

Of course, the utility rise generated by ad hoc values of α has a cost: the total housing expenses distributed to landlords (more precisely, the adequate notion would be the total differential land rent presented in (*9*) but for simplicity of the argument we keep with total housing expenses). By enforcing reduced prices, the CHE policy proceeds to a form of redistribution from the landlords to the households similar to the public ownership case described in Fujita (*9*), where rents are redistributed to the households. This redistribution is at the origin of the higher utility than in the unconstrained city with absentee landlords.

Given the analysis of the Herbert-Stevens model (*9*), we know that utility of the closed-city model is maximized in the case of public ownership. No other configuration of the city, and in particular the CHE city, can outperform this one in utility grounds. Yet, the CHE policy is widely enforced and accepted, while such is not the case for the public ownership of land. Thus it constitutes an interesting policy that can improve the solvability of the households and make them better off at the same time, though being detrimental to landlords.

CONSTRAINT ON THE BUDGET SHARE OF BOTH HOUSING AND TRANSPORTATION

Let us now turn our attention to an alternative policy, consisting in capping the share of housing and transportation expenditures. As previously, we scrutinize the impacts of such a policy on the equilibrium city, in particular the influence of the constraint parameter μ.

Considering the similarities borne by the CHT and CHE policies, we first present the main results, omitting the proofs, and then focus on the application to the linear city.

The Constrained Housing+Transportation (CHT) model

Overview of the CHT model

The CHT model is a monocentric model amended with the following constraint:

$$R(r)s + T(r) \leq \mu Y \qquad (E9)$$

The sum of housing and transportation expenditures is capped to a fraction μ of the household's income Y. The case $\mu \geq 1$ is consequently tantamount to the classic unconstrained model.

Enforcement of such a policy exerts the same effects as the CHE policy:

- curtailing lot size choices of the households (actually it sets *de facto* a minimal density)
- narrowing down prices

Yet this time it can be shown that the constraint concerns above all the households in the suburban area (starting from the edge of the city). The tighter it becomes, the more households it affects until covering the whole city.

Equilibrium features in the general case

The CHT land use equilibrium exists and is unique. The only specific property of the equilibrium in the general case is that city size increases with μ, which is the result of the minimal density enforcement. The HT constraint induces the two same economic forces that influence the equilibrium utility level:

- By dragooning the households to make sub-optimal choices, the latter achieve a lower utility level
- But capping HT expenses generates a “discount” on housing prices, which makes the households better-off

Nevertheless, contrary to the HE policy, there is no obvious case where we can predict the outcome. The same goes for housing expenses.

Application to a linear city

So as to compare the CHE and CHT policies, let us come back to the application that features: $U(z,s)=1/2\log z+1/2\log s$, $T(r)=ar$ and $L(r)=1$.

Derivation of the equilibrium city

Determination of the binding zone The HT constraint is strictly binding when:

$$r > r_{bind}(\mu)=\frac{(2\mu-1)Y}{a} \qquad (E10)$$

Hence the following cases:

- If $\mu<1/2$ the HT constraint is always binding.
- If $\mu\geq 1/2$, households located beyond $r_{bind}(\mu)$ are bound by the HT constraint.

Characterization of the equilibrium Resolution of the bid-max program brings about the following system of equations:

$$r\leq r_{bind}(\mu)\ \begin{cases}\hat{z}(r,u)=(Y-ar)/2\\ \hat{s}(r,u)=e^{2u}/\hat{z}(r,u)\\ \hat{\Psi}(r,u)=e^{-2u}(Y-ar)^2/4\end{cases} \qquad r\geq r_{bind}(\mu)\ \begin{cases}\hat{z}(r,u)=(1-\mu)Y\\ \hat{s}(r,u)=e^{2u}/\hat{z}(r,u)\\ \hat{\Psi}(r,u)=e^{-2u}(1-\mu)Y(\mu Y-ar)\end{cases} \qquad (E11)$$

Figure 5 illustrates (*E11*) for the settings of the reference model (N=10, Y=80, a=8, R_A=20). Moreover we choose μ=0.70 and u=16 (corresponding to the equilibrium utility of the CHT reference model for the selected value of μ), which yields r_{max}=7 and r_{bind}=4.

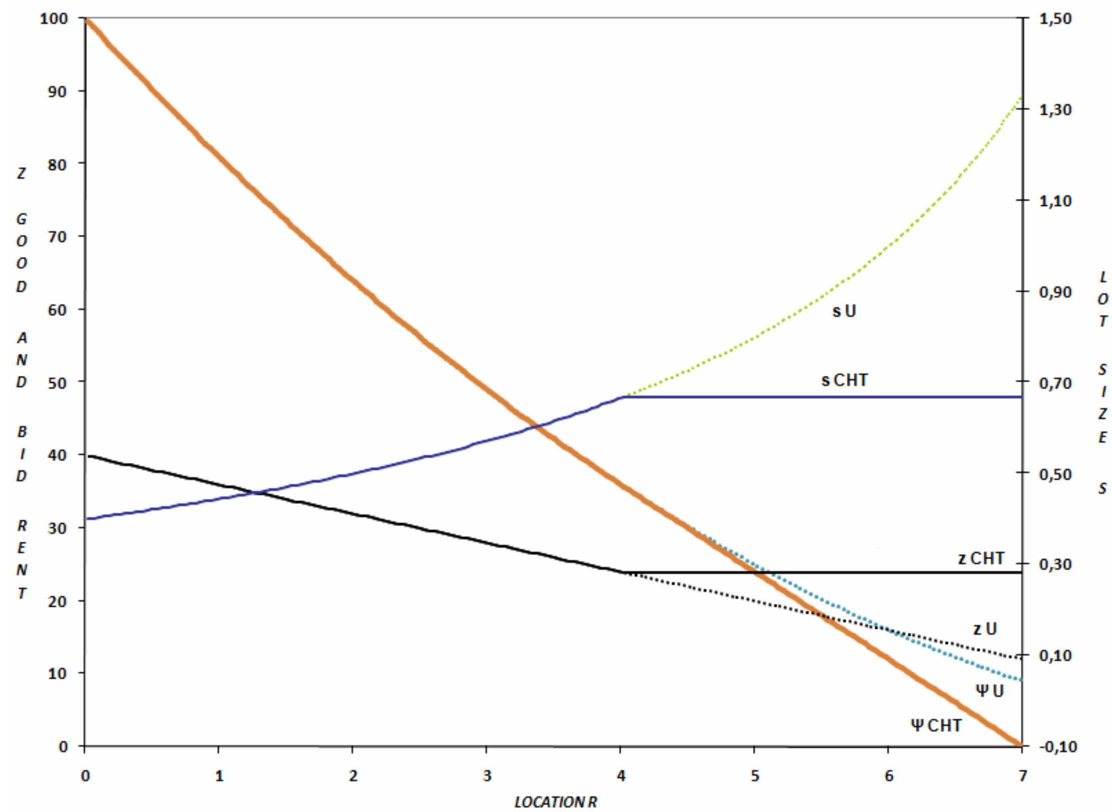

FIGURE 5 Bid Rent, Surfaces and z Good Consumptions in the Unconstrained (U) and CHT models.

As previously stated, for $r \leq r_{bind}$ the HT constraint is ineffective, leading to the same solution that in the original case. For $r \geq r_{bind}$, the constraint becomes active, leading to constant choices of surface and of the *z* good.

From (*E11*), we can derive the equilibrium utility and city size of the CHT city:

PROPOSITION 5

In the applied case, the equilibrium is characterized as follows:

	$\mu \leq 1/2$	$\mu \in [1/2, \mu_{cr}]$	$\mu \geq \mu_{cr}$
$e^{2\hat{u}}$	$\dfrac{\mu(1-\mu)Y^2}{aN + R_A}$	$\dfrac{Y^2}{4(aN + R_A)}$	$\dfrac{Y^2}{4(aN + R_A)}$
$\hat{r}_f$	$\dfrac{\mu Y}{a}\dfrac{aN}{aN + R_A}$	$\dfrac{Y}{a}\left\{\mu - \dfrac{1}{4(1-\mu)}\dfrac{R_A}{(aN + R_A)}\right\}$	$\dfrac{Y}{a}\left(1 - \sqrt{\dfrac{R_A}{aN + R_A}}\right)$

where $\mu_{cr} = 1 - \frac{1}{2}\sqrt{\frac{R_A}{aN + R_A}}$

Calculations are based on the distinction of 3 cases:

- $\mu \geq \mu_{cr}$ yields the unconstrained model
- If $\mu \in [1/2, \mu_{cr}]$, $r_{bind}(\mu) \geq 0$, thus households living in the central area of the city are unconstrained
- If $\mu \leq 1/2$, $r_{bind}(\mu) \leq 0$. The HT constraint is active for the whole city.

Comparative statics for the applied model

Utility level Starting from μ=1, while decreasing μ has no impact at first on the utility level of the households (compared to the unconstrained city), for $\mu \leq 1/2$ it decreases the utility level.

PROPERTY 4

For any given set of parameters (N,Y,R_A,a), the equilibrium utility $\hat{u}(\mu)$ of the CHT city strictly increases with μ on [0,1/2] and is constant for $\mu \geq 1/2$.

Considering Proposition 5, Property 4 is straightforward. Yet, this property proves enlightening for it states that, for $\mu \in [1/2, \mu_{cr}]$, the “discount” given to the households on housing prices is perfectly compensated by the capped lot sizes. If $\mu \leq 1/2$, the constraint becomes too strong, inducing a drop in the utility level.

Figure 6 depicts the variations of $e^{2\hat{u}(\mu)}$ for the reference model, where μ_{cr}=0.776:

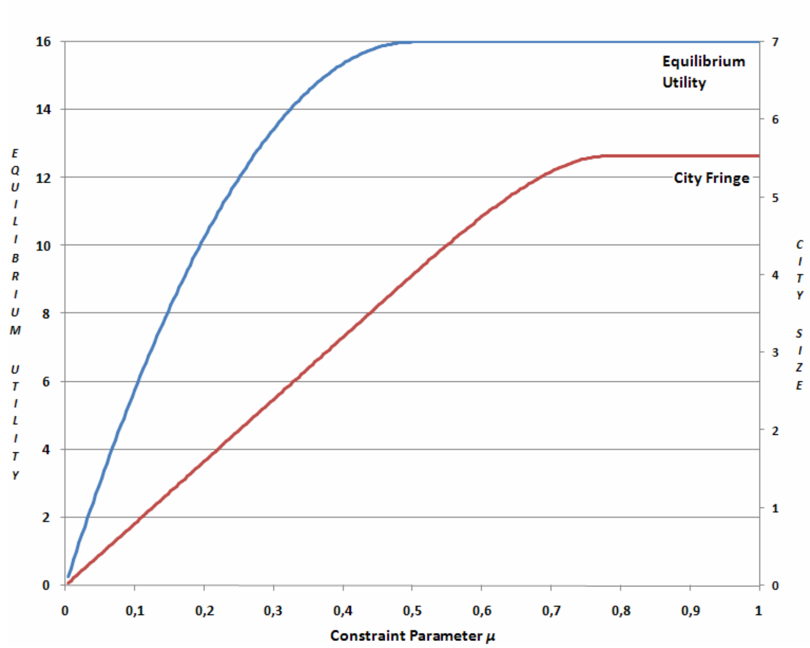

FIGURE 6 Utility level and city size of the CHT city.

City Size and Density As previously stated for any given set of parameters (N,Y,R_A,a), the city size increases with μ, which is illustrated for our reference model in Figure 6. On $[1/2,\mu_{cr}]$ the city size is fairly well approximated by a linear function, which underlines the efficiency of this policy in reducing the city size (relatively to the CHE policy).

Similarly to the CHE policy, the CHT policy alters the spatial distribution of density, but this time it sets a minimum density level that affects either the most remote part of the city ($\mu \in [1/2, \mu_{cr}]$), or the whole city ($\mu \leq 1/2$). This phenomenon is illustrated in Figure 7:

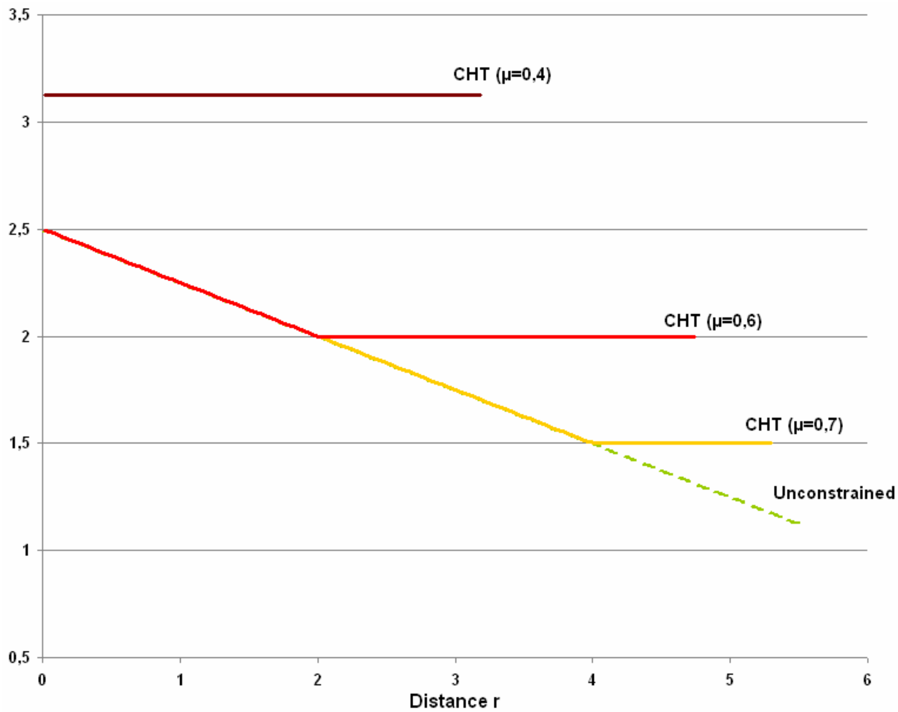

FIGURE 7 Influence of μ on equilibrium density in the reference model.

Average composition of the households' budgets Similarly to the HE policy, the HT policy brings about lower housing and transportation expenditures from the households, as illustrated in Figure 8 (proof of following property omitted):

PROPERTY 5

For any given set (N,Y,R_A,a), both the housing and transport average expenditures are rising with μ, while the average consumption of the composite good decreases with μ.

When the HT policy becomes active (starting from μ_{cr}), increasing the constraint results in decreasing transport costs and housing expenses. Contrary to the HE policy, the two items decrease simultaneously in similar proportions, which is indeed the result of capping housing and transportation expenses in place of solely housing expenditures. When μ gets lower than 1/2, the decrease steepens.

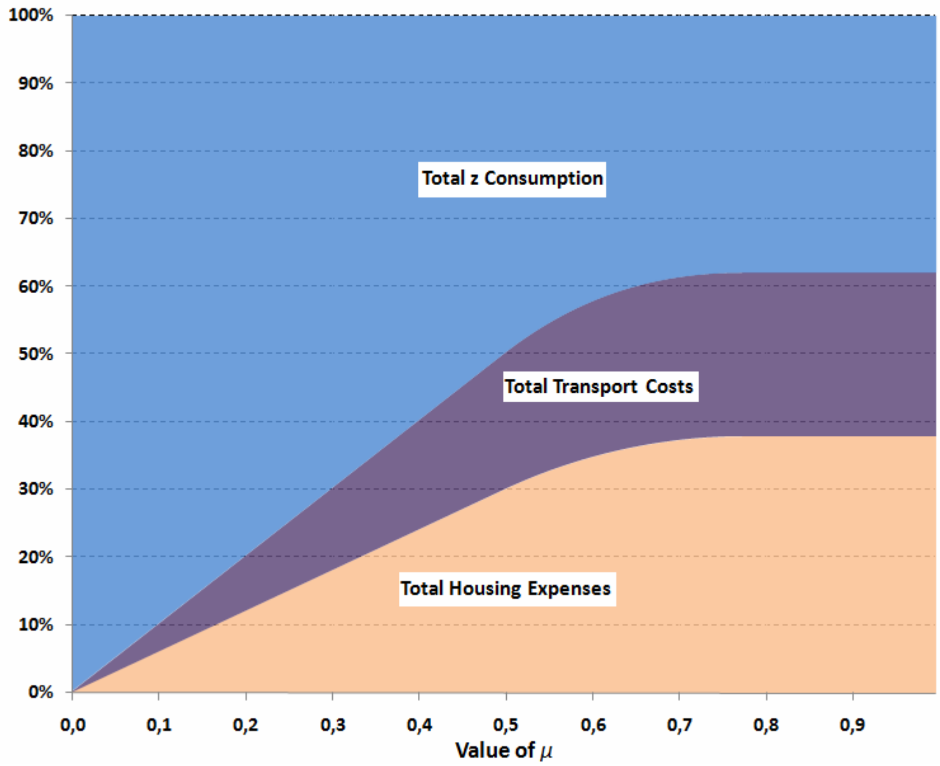

FIGURE 8 Influence of μ on the average composition of the household's budget.

CONCLUSIONS

To sum up, let us compare the main results about the linear city under the CHE and CHT policies. This is achieved by displaying the resulting equilibrium utility level and city size of the reference applied model for a target solvability level, defined as the fraction of income remaining after paying the housing and transportation costs (Figure 9).

In both models, increasing the solvability of households is done by tightening the corresponding constraint, until reaching the maximal solvability level of 100% for a value of the constraint parameter equal to zero. Since a constraint parameter of one yields the unconstrained model in both cases, each pair of curves starts at the same point.

Figure 9 unveils that the CHE policy provides a greater utility for any target level of solvability, but at the cost of a greater city size. As regards land use, while both policies induce shrinkage of the city, the CHE policy steepens the density curve when the utility rises, while the CHT policy always flattens the density curve. Moreover, the CHT policy is more efficient as regards reducing the city size, and equivalently transportation costs and energy consumption.

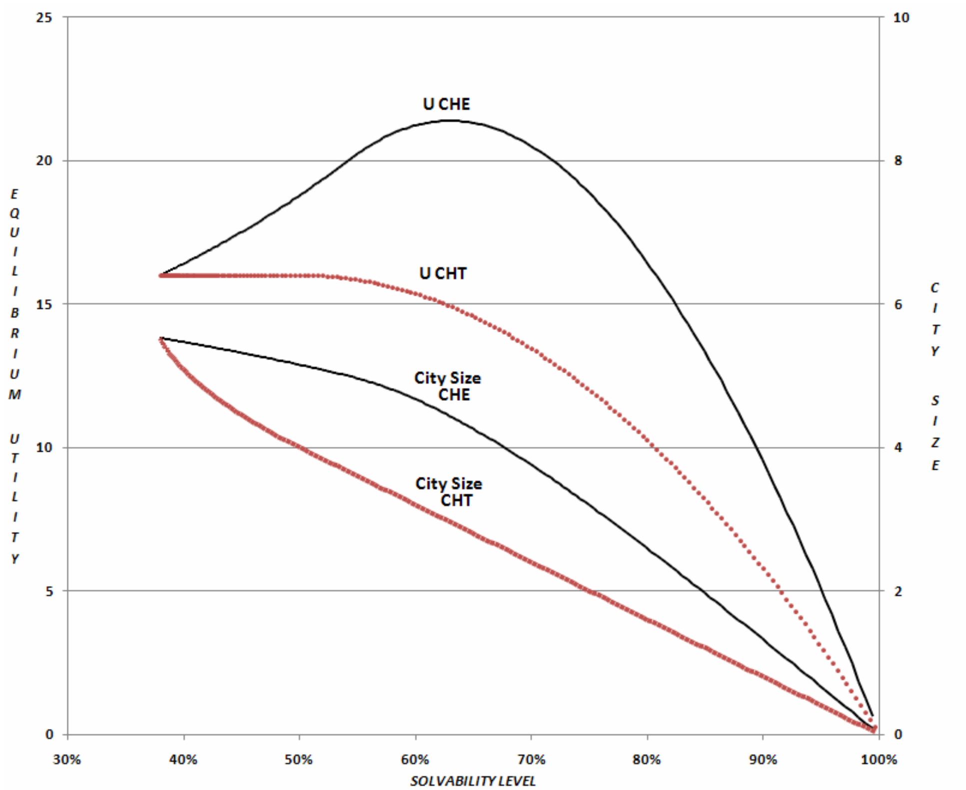

FIGURE 9 Comparison of the utility levels and the city sizes according to the solvability level

Consequently, our linear model suggests that while the CHE policy is beneficial to the households on utility grounds, and does improve their solvability while reducing city size and transport expenses at the same time, the CHT policy makes a better tool to struggle against urban sprawl and transportation costs. Because our model includes neither several externalities such as pollution or congestion, nor the scarcity of energy, the CHT policy might prove a better choice than the CHE policy, despite utility considerations, depending on the objectives of the local authorities. In all cases, both policies can be used to secure a target level of solvability for the households.

While our model was helpful in understanding the CHE and CHT policies, several improvements are planned so as to assess the policy effects in more realistic settings:

- Considering the case of a disk-shaped city, which will complicate the calculations.
- Calibrating the utility functions and the parameters on existing metropolitan areas.
- Considering the policy impacts in terms of car ownership decision and modal choice, especially for the CHT policy.

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