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A Method for Exact Calculation of the Discrepancy of Low-dimensional Finite Point Sets I

By P. Bundschuh and Y. Zhu*

Abstract. In the present paper the formulas of exactly calculating the discrepancy of 2- and 3-dimensional finite point sets are explicitly given only in terms of the components of points.

1 Introduction

Let $d \geq 1$, $G_d = [0,1)^d$, and $S_d = \{u_k\ (1 \leq k \leq n)\}$ be a finite sequence of points in G_d. We define the discrepancy of the sequence S_d by

$$D_n = D_n(S_d) = \sup_J \left| \frac{A(J;n)}{n} - V(J) \right| ,$$

where $J = [0,\alpha_1) \times \cdots \times [0,\alpha_d)$, $0 < \alpha_i \leq 1\ (1 \leq i \leq d)$, runs through all d-dimensional subrectangles of G_d, and $V(J) = \alpha_1 \cdots \alpha_d$ is the volume of J, and $A(J;n)$ is the number of $u_k\ (1 \leq k \leq n)$ such that $u_k \in J$. (Exactly speaking, according to [3], the matter defined here is the star-discrepancy of S_d and it is often denoted by D_n^*.) It is well-known that in the quasi-Monte Carlo method of calculating the multiple integrals for a wide class of functions $f(x)$ defined on G_d the error $R(f;n)$ of the quadrature formula

$$\int_{G_d} f(x)\, dx = \frac{1}{n} \sum_{k=1}^{n} f(u_k) + R(f;n)$$

satisfies

$$|R(f;n)| \leq c_d(f) D_n(S_d) ,$$

where $c_d(f)$ is a constant depending at most on the dimension d and, of course, on the function f (Ref. e.g. [2] and [6]). Thereby low-discrepancy sequences play an important role. In general it is difficult to calculate exactly the discrepancy of a sequence of points. Usually we may only estimate the order of magnitude of discrepancy in terms of n. For the case $d = 1$, H. Niederreiter [4] gave the following exact formula for the discrepancy of $S_1 = \{x_i\ (1 \leq i \leq n)\}$

$$\begin{aligned} D_n(S_1) &= \max_{1\leq i\leq n} \max\left(|x_i - \frac{i}{n}|, |x_i - \frac{i-1}{n}|\right) \\ &= \frac{1}{2n} + \max_{1\leq i\leq n} |x_i - \frac{2i-1}{2n}| , \end{aligned}$$

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where $x_1 \leq x_2 \leq \ldots \leq x_n$ are n numbers in $[0, 1)$. Due to the lack of a suitable ordering in E^d $(d \geq 2)$, a complete multidimensional analogue of this formula can not be given, but the discrepancy of multidimensional sequences is still representable as a maximum of finitely many numbers (See H. NIEDERREITER [4] or [5]). Since the NIEDERREITER's multidimensional analogue has the recursive character essentially and is not given explicitly in terms of the components of points, his formula has the theoretic value but is unsuitable for practically calculating the discrepancy. In 1986 L. DE CLERCK [1] considered the 2-dimensional set $S_2 = \{(x_k, y_k)\ (1 \leq k \leq n)\}$ satisfying

$$x_i < x_j \quad \text{and} \quad y_i \neq y_j \quad \text{for any} \quad i < j. \tag{1}$$

She gave the following formula:

$$D_n(S_2) = \max_{1\leq i\leq n}\Big(x_i - \frac{i-1}{n}, g_{n,i} - \frac{i-1}{n}, \frac{t_i}{n} - x_i y_i, \\ \max_{t_i\leq k\leq i-1} \max(x_i g_{i-1,k} - \frac{k-1}{n}, \frac{k+1}{n} - x_i g_{i-1,k})\Big), \tag{2}$$

where the numbers $g_{i,k}$ $(1 \leq k \leq i)$ are the y-coordinates y_k $(1 \leq k \leq i)$ arranged from smaller to bigger, and the parameters t_i $(i = 1, \ldots, n)$ are as follows: $t_1 = t_2 = 1$, and for $i \geq 3$

$$t_i = \begin{cases} 1 & \text{if } y_i \in [0, g_{i-1,1}) \cup (g_{i-1,i-1}, 1), \\ k+1 & \text{if } y_i \in (g_{i-1,k}, g_{i-1,k+1}),\ 1 \leq k \leq i-2. \end{cases}$$

By use of the formula (2) DE CLERCK deduced some explicit and exact calculating formulas of the discrepancy for the 2-dimensional Hammersley's sequence. Generally speaking, when we use the DE CLERCK's formula for other 2-dimensional sequences, the parameters t_i $(1 \leq i \leq n)$ must be calculated in advance. Furthermore, if we want to generalize her formula to the case $d \geq 3$, the analogue of parameters t_i $(1 \leq i \leq n)$ will be quite complex. Recently, in his work about numerical integration in Bayesian statistics J.E.H. SHAW [7] considered certain low-dimensional sequences and computed discrepancies of some 2-dimensional finite sequences, but did not give the computational method. He also pointed out the importance of the exact calculation of discrepancies of low-dimensional finite sequences for the application of the quasi-Monte Carlo method to certain statistical problems (also refer to [8]).

The aim of the present paper is to develop a method for exactly calculating the discrepancy of low-dimensional finite point sets. We will remove the restriction (1) and the parameters t_i in DE CLERCK's formula (2). Only the coordinates of points appear in our formula, and the amount of the elementary operations in our method is $cn^d/d!$, where $c > 0$ is an absolute constant. (The NIEDERREITER's recursive structure [4] will require cn^d elementary operations). In particular, under the condition (1), using the definition of t_i, we deduce easily the formula (2) from our Theorem 1.

In section **2** and **3** of this paper we discuss the case of the dimension $d = 2$ and 3 by means of quite elementary arguments, and the general formulas for $d \geq 2$ will be given in another paper.

2 The 2-dimensional Case

Theorem 1. *Suppose that* $S_2 = \{u_k = (x_k, y_k)\ (1 \leq k \leq n)\} \subset G_2$ *satisfies*

$$x_1 \leq x_2 \leq \ldots \leq x_n . \tag{3}$$

Let $u_0 = (x_0, y_0) = (0,0)$ *and* $u_{n+1} = (x_{n+1}, y_{n+1}) = (1,1)$. *For every* ℓ $(\ell = 0, 1, \ldots, n)$ *rearrange* y_i $(i = 0, 1, \ldots, \ell, n+1)$ *in increasing order and rewrite them as*

$$0 = \xi_{\ell,0} \leq \xi_{\ell,1} \leq \ldots \leq \xi_{\ell,\ell} < \xi_{\ell,\ell+1} = 1 . \tag{4}$$

Then the discrepancy of S_2 *is given by the expression*

$$D_n(S_2) = \max_{0 \leq \ell \leq n} \max_{0 \leq k \leq \ell} \max\Big(\Big|\frac{k}{n} - x_\ell \xi_{\ell,k}\Big|, \Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}\Big|\Big) . \tag{5}$$

Remark 1. We easily see that (5) can be replaced by

$$D_n(S_2) = \max\Big(\max_{1 \leq \ell \leq n} \Big(\max_{1 \leq k \leq \ell} \max\Big(\Big|\frac{k}{n} - x_\ell \xi_{\ell,k}\Big|, \Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}\Big|\Big), x_{\ell+1}\xi_{\ell,1}\Big), x_1 \Big) .$$

In particular, if $u_1 = (0,0)$, then (5) becomes

$$D_n(S_2) = \max_{1 \leq \ell \leq n} \max_{1 \leq k \leq \ell} \max\Big(\Big|\frac{k}{n} - x_\ell \xi_{\ell,k}\Big|, \Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}\Big|\Big) .$$

Remark 2. If $\frac{k}{n} - x_\ell \xi_{\ell,k} < 0$, then

$$\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} \leq \frac{k}{n} - x_\ell \xi_{\ell,k} < 0 ,$$

and so

$$\Big|\frac{k}{n} - x_\ell \xi_{\ell,k}\Big| \leq \Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}\Big| = x_{\ell+1}\xi_{\ell,k+1} - \frac{k}{n} .$$

Similarly, if $\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} > 0$, then

$$0 < \frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} \leq \frac{k}{n} - x_\ell \xi_{\ell,k} ,$$

and so

$$\Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}\Big| \leq \Big|\frac{k}{n} - x_\ell \xi_{\ell,k}\Big| = \frac{k}{n} - x_\ell \xi_{\ell,k} .$$

Thus, formula (5) can be replaced by

$$D_n(S_2) = \max_{0 \leq \ell \leq n} \max_{0 \leq k \leq \ell} \max\Big(\frac{k}{n} - x_\ell \xi_{\ell,k} , x_{\ell+1}\xi_{\ell,k+1} - \frac{k}{n}\Big) .$$

Example 1. Let $u_k = \left(\frac{k-1}{n}, \{(k-1)\frac{g}{n}\}\right)$ $(k = 1, \ldots, n)$, where g and n are coprime (see [7]). For every ℓ $(\ell = 1, 2, \ldots, n)$ rearrange the numbers $\{(k-1)\frac{g}{n}\}$ $(k = 1, \ldots, \ell)$ and 1 in increasing order and write them as

$$0 = \xi_{\ell,1} < \xi_{\ell,2} < \ldots < \xi_{\ell,\ell} < \xi_{\ell,\ell+1} = 1 .$$

Then

$$D_n(S_2) = \max_{1 \le \ell \le n} \tilde{d}_n(\ell) ,$$

where

$$\begin{aligned} \tilde{d}_n(\ell) = \frac{1}{n} \max\bigl(1, |1 - \ell\xi_{\ell,2}|, \quad & |2 - (\ell - 1)\xi_{\ell,2}|, \\ |2 - \ell\xi_{\ell,3}|, \quad & |3 - (\ell - 1)\xi_{\ell,3}|, \\ \ldots \quad & \ldots \\ |(\ell - 1) - \ell\xi_{\ell,\ell}|, & |\ell - (\ell - 1)\xi_{\ell,\ell}|\bigr) . \end{aligned}$$

For the point set $S_2 = \bigl\{(k/32, \{7k/32\})\ (k = 0, 1, \ldots, 31)\bigr\}$, we compute the discrepancy by this formula and obtain

$$D_{32}(S_2) = \tilde{d}_{32}(25) = 0.0849609 ,$$

and the point where this value is reached is $(7/32, \{7 \cdot 7/32\}) = (7/32, 17/32)$.

We now prove Theorem 1. Adding the numbers $x_0 = 0$ and $x_{n+1} = 1$ to (3), we rewrite them as

$$\begin{aligned} (6) \qquad 0 = x_0 = \ldots = x_{h_1} < x_{h_1+1} = \ldots = x_{h_2} < \ldots < x_{h_{m-1}+1} \\ = \ldots = x_{h_m} < x_{h_{m+1}} = 1 , \end{aligned}$$

where $m \ge 1$, $h_1 \ge 0$, $h_{\rho+1} - h_\rho \ge 1$ $(\rho = 1, \ldots, m)$, $h_m = n$ and $x_{h_{m+1}} = x_{n+1} = 1$. We also let $x_{h_m+1} = x_{h_{m+1}}$ for convenience. In the case $m > 1$ we assume that

$$(7) \qquad y_{h_\rho+1} \le y_{h_\rho+2} \le \cdots \le y_{h_{\rho+1}} \quad (\rho = 1, \ldots, m-1) .$$

For every ℓ $(0 \le \ell \le n)$ we rewrite (4) as

$$\begin{aligned} (8) \qquad 0 = \xi_{\ell,0} = \ldots = \xi_{\ell,r_1} < \xi_{\ell,r_1+1} = \ldots = \xi_{\ell,r_2} < \ldots < \xi_{\ell,r_{u-1}+1} \\ = \ldots = \xi_{\ell,r_u} < \xi_{\ell,r_{u+1}} = 1 , \end{aligned}$$

where $u = u(\ell) \ge 1$, $r_1 \ge 0$, $r_{\tau+1} - r_\tau \ge 1$ $(\tau = 1, \ldots, u)$, $r_u = \ell$ and $\xi_{\ell,r_{u+1}} = \xi_{\ell,\ell+1} = 1$. We also let $\xi_{\ell,r_u+1} = \xi_{\ell,r_{u+1}}$ for convenience. We define a rectangle

$$S_{\rho,\tau} = (x_{h_\rho}, x_{h_{\rho+1}}] \times (\xi_{h_\rho,r_\tau}, \xi_{h_\rho,r_{\tau+1}}] \qquad (1 \le \rho \le m, 1 \le \tau \le u(h_\rho)) .$$

Thus, if $(\alpha, \beta) \in S_{\rho,\tau}$, then $A\bigl([0,\alpha) \times [0,\beta); n\bigr) = r_\tau$. We define

$$d_n^*(\rho, \tau) = \sup_{(\alpha,\beta) \in S_{\rho,\tau}} \left|\frac{r_\tau}{n} - \alpha\beta\right| \qquad (1 \le \rho \le m, 1 \le \tau \le u(h_\rho)) .$$

Lemma 1. *We have for* $1 \le \rho \le m$, $1 \le \tau \le u(h_\rho)$

$$d_n^*(\rho,\tau) = \max\left(\left|\frac{r_\tau}{n} - x_{h_\rho}\xi_{h_\rho,r_\tau}\right|, \left|\frac{r_\tau}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\right|\right).$$

Proof. Obvious, since $|r_\tau/n - t|$ is a convex function of t. □

Lemma 2. *We have*

$$D_n(S_2) = \max_{1\le\rho\le m}\max_{1\le\tau\le u(h_\rho)} d_n^*(\rho,\tau).$$

Proof. It is clear that

$$D_n(S_2) \ge \max_{1\le\rho\le m}\max_{1\le\tau\le u(h_\rho)} d_n^*(\rho,\tau).$$

Conversely, for any $(\alpha,\beta) \in (0,1]^2$ there exists a pair (ρ,τ) such that $(\alpha,\beta) \in S_{\rho,\tau}$. Since $A\big([0,\alpha)\times[0,\beta);n\big) = r_\tau$, we have

$$\left|\frac{A\big([0,\alpha)\times[0,\beta);n\big)}{n} - \alpha\beta\right| \le d_n^*(\rho,\tau) \le \max_{1\le\rho\le m}\max_{1\le\tau\le u(h_\rho)} d_n^*(\rho,\tau),$$

and so

$$D_n(S_2) \le \max_{1\le\rho\le m}\max_{1\le\tau\le u(h_\rho)} d_n^*(\rho,\tau).$$

Thus the lemma is proved. □

In what follows, let for $0 \le \ell \le n$ and $0 \le k \le \ell$

$$d_n(\ell,k) = \max\left(\left|\frac{k}{n} - x_\ell\xi_{\ell,k}\right|, \left|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}\right|\right).$$

In order to prove Theorem 1, we only need to prove the following

Proposition 1. *We have*

$$\max_{1\le\rho\le m}\max_{1\le\tau\le u(h_\rho)} d_n^*(\rho,\tau) = \max_{0\le\ell\le n}\max_{0\le k\le\ell} d_n(\ell,k).$$

Lemma 3. *For every* ρ $(1 \le \rho \le m)$ *let*

$$\delta_n^*(\rho) = \max_{0\le k\le h_\rho}\max\left(\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k}\right|, \left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1}\right|\right),$$

where $\xi_{h_\rho,k}$ $(k = 0,\dots,h_\rho+1)$ *are defined by* (4) *with* $\ell = h_\rho$. *Then*

$$\max_{1\le\tau\le u(h_\rho)} d_n^*(\rho,\tau) = \delta_n^*(\rho) \qquad (1 \le \rho \le m).$$

Proof. If $0 \le k \le r_1 - 1$, then $\xi_{h_\rho,k} = \xi_{h_\rho,k+1} = 0$ according to (8), and so we deduce from Lemma 1

$$\max_{0\le k\le r_1-1} \max\left(\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k}\right|, \left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1}\right|\right) = \max_{0\le k\le r_1-1} \frac{k}{n} \le d_n^*(\rho,1)\,. \tag{9}$$

Next, if in (8) there is a suffix τ, $1 \le \tau \le u(h_\rho) - 1$, such that $r_{\tau+1} - r_\tau > 1$, then we can prove that

$$\begin{aligned} &\max_{r_\tau<k<r_{\tau+1}} \max\left(\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k}\right|, \left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1}\right|\right) \\ &< \max\left(\left|\frac{r_{\tau+1}}{n} - x_{h_\rho}\xi_{h_\rho,r_{\tau+1}}\right|, \left|\frac{r_\tau}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\right|\right) \\ &\le \max\left(d_n^*(\rho,\tau), d_n^*(\rho,\tau+1)\right). \end{aligned} \tag{10}$$

In fact, noticing $\xi_{h_\rho,k} = \xi_{h_\rho,k+1} = \xi_{h_\rho,r_{\tau+1}}$ for $k = r_\tau + 1, \ldots, r_{\tau+1} - 1$, if $\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k} \ge 0$, then

$$\begin{aligned} &\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k}\right| = \frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k} < \frac{r_{\tau+1}}{n} - x_{h_\rho}\xi_{h_\rho,r_{\tau+1}} \\ &= \left|\frac{r_{\tau+1}}{n} - x_{h_\rho}\xi_{h_\rho,r_{\tau+1}}\right| \le d_n^*(\rho,\tau+1)\,; \end{aligned}$$

and if $\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k} < 0$, then

$$\begin{aligned} &\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k}\right| = x_{h_\rho}\xi_{h_\rho,k} - \frac{k}{n} < x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}} - \frac{r_\tau}{n} \\ &= \left|\frac{r_\tau}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\right| \le d_n^*(\rho,\tau)\,. \end{aligned}$$

Similarly, if $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1} \ge 0$, then

$$\left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1}\right| < \left|\frac{r_{\tau+1}}{n} - x_{h_\rho}\xi_{h_\rho,r_{\tau+1}}\right| \le d_n^*(\rho,\tau+1)\,;$$

and if $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1} < 0$, then

$$\left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1}\right| < \left|\frac{r_\tau}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\right| \le d_n^*(\rho,\tau)\,.$$

Thus inequality (10) follows.

Finally, using Lemma 1, we deduce from (9) and (10)

$$\max_{0\le k\le r_1} \max\left(\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k}\right|, \left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1}\right|\right) = d_n^*(\rho,1)\,,$$

and for $\tau = 1, \ldots, u(h_\rho) - 1$

$$\max_{r_\tau\le k\le r_{\tau+1}} \max\left(\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,k}\right|, \left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,k+1}\right|\right) = \max\left(d_n^*(\rho,\tau+1), d_n^*(\rho,\tau)\right).$$

Therefore the lemma follows. □

Lemma 4. *For any ℓ $(0 \le \ell \le n)$ let*

$$\delta_n(\ell) = \max_{0\le k\le \ell} \max\left(|\frac{k}{n} - x_\ell \xi_{\ell,k}|, |\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}|\right). \tag{11}$$

Then

$$\max_{1\le\rho\le m} \delta_n^*(\rho) = \max_{0\le\ell\le n} \delta_n(\ell).$$

Proof. If $0 \le \ell \le h_1 - 1$, then $x_\ell = x_{\ell+1} = 0$, and so

$$\delta_n(\ell) = \max_{0\le k\le\ell} \frac{k}{n} < \frac{h_1}{n} \le \delta_n^*(1) \qquad (0 \le \ell \le h_1 - 1). \tag{12}$$

If in (6) there is a suffix ρ such that $1 \le \rho \le m-1$ and $h_{\rho+1} - h_\rho > 1$, then we can prove that

$$\delta_n(\ell) \le \max\big(\delta_n(h_\rho), \delta_n(h_{\rho+1}), \delta_n(\ell+1)\big) \qquad (h_\rho < \ell < h_{\rho+1}). \tag{13}$$

In fact, we have $x_\ell = x_{\ell+1} = x_{h_{\rho+1}}$ for $h_\rho < \ell < h_{\rho+1}$. For any fixed $\ell \in \{h_\rho + 1, \ldots, h_{\rho+1} - 1\}$ we distinguish two cases.

Case 1. Let $y_{\ell+1} \ge \xi_{\ell,\ell}$. Then we see $\xi_{\ell,k} = \xi_{\ell+1,k}$ $(k = 0, 1, \ldots, \ell)$, and so

$$|\frac{k}{n} - x_\ell\xi_{\ell,k}| = |\frac{k}{n} - x_{\ell+1}\xi_{\ell+1,k}| \le \delta_n(\ell+1) \qquad (k = 0, 1, \ldots, \ell).$$

Next, let $k \in \{0, 1, \ldots, \ell - 1\}$. If $\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} \ge 0$, then

$$\begin{aligned}&|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| = \frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} \le \frac{k}{n} - x_{\ell+1}\xi_{\ell+1,k+1}\\ &< \frac{k+1}{n} - x_{\ell+1}\xi_{\ell+1,k+1} \le \delta_n(\ell+1);\end{aligned}$$

and if $\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} < 0$, then

$$\begin{aligned}&|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| = x_{\ell+1}\xi_{\ell,k+1} - \frac{k}{n} = x_{\ell+1}\xi_{\ell+1,k+1} - \frac{k}{n}\\ &\le x_{\ell+2}\xi_{\ell+1,k+1} - \frac{k}{n} \le \delta_n(\ell+1).\end{aligned}$$

Moreover, if $k = \ell$, then

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| = |\frac{\ell}{n} - x_{\ell+1}|.$$

Thus, we obtain

$$\delta_n(\ell) \le \max\big(\delta_n(\ell+1), \max_{h_\rho<\ell<h_{\rho+1}} |\frac{\ell}{n} - x_{\ell+1}|\big) \qquad (h_\rho < \ell < h_{\rho+1}). \tag{14}$$

Notice that

$$\max_{h_\rho<\ell<h_{\rho+1}} |\frac{\ell}{n} - x_{\ell+1}| = \max_{h_\rho<\ell<h_{\rho+1}} |\frac{\ell}{n} - x_{h_{\rho+1}}| \le \max_{h_\rho\le\ell\le h_{\rho+1}} |\frac{\ell}{n} - x_{h_{\rho+1}}|$$
$$= \max\big(|\frac{h_\rho}{n} - x_{h_{\rho+1}}|, |\frac{h_{\rho+1}}{n} - x_{h_{\rho+1}}|\big)\,,$$

and

$$|\frac{h_\rho}{n} - x_{h_{\rho+1}}| = |\frac{h_\rho}{n} - x_{h_\rho+1}\xi_{h_\rho,h_\rho+1}| \le \delta_n(h_\rho)\,.$$

Furthermore, if $\frac{h_{\rho+1}}{n} - x_{h_{\rho+1}} \ge 0$, then

$$|\frac{h_{\rho+1}}{n} - x_{h_{\rho+1}}| = \frac{h_{\rho+1}}{n} - x_{h_{\rho+1}} < \frac{h_{\rho+1}}{n} - x_{h_{\rho+1}}\xi_{h_{\rho+1},h_{\rho+1}} \le \delta_n(h_{\rho+1})\,;$$

and if $\frac{h_{\rho+1}}{n} - x_{h_{\rho+1}} < 0$, then

$$|\frac{h_{\rho+1}}{n} - x_{h_{\rho+1}}| = x_{h_{\rho+1}} - \frac{h_{\rho+1}}{n} < x_{h_{\rho+1}+1}\xi_{h_{\rho+1},h_{\rho+1}+1} - \frac{h_{\rho+1}}{n} \le \delta_n(h_{\rho+1})\,.$$

Thereby from (14) we obtain

$$\delta_n(\ell) \le \max\big(\delta_n(\ell+1), \delta_n(h_\rho), \delta_n(h_{\rho+1})\big) \qquad (h_\rho < \ell < h_{\rho+1})\,. \tag{15}$$

Case 2. Let $y_{\ell+1} < \xi_{\ell,\ell}$. Then there is a suffix $k_0 < \ell$ such that $y_{\ell+1} \in (\xi_{\ell,k_0}, \xi_{\ell,k_0+1}]$. Thus we easily see

$$\begin{aligned} \xi_{\ell+1,k} &= \xi_{\ell,k} \qquad (0 \le k \le k_0)\,,\\ \xi_{\ell+1,k_0+1} &= y_{\ell+1} \le \xi_{\ell,k_0+1}\,,\\ \xi_{\ell+1,k} &= \xi_{\ell,k-1} \qquad (k_0+2 \le k \le \ell+1)\,. \end{aligned} \tag{16}$$

Thereby we immediately obtain from (16)

$$|\frac{k}{n} - x_\ell\xi_{\ell,k}| = |\frac{k}{n} - x_{\ell+1}\xi_{\ell+1,k}| \le \delta_n(\ell+1) \qquad (k = 0, 1, \ldots, k_0)\,.$$

Furthermore, noticing (16), for $k_0 + 1 \le k \le \ell$, if $\frac{k}{n} - x_\ell\xi_{\ell,k} \ge 0$, then

$$|\frac{k}{n} - x_\ell\xi_{\ell,k}| = \frac{k}{n} - x_\ell\xi_{\ell,k} \le \frac{k+1}{n} - x_{\ell+1}\xi_{\ell+1,k+1} \le \delta_n(\ell+1)\,;$$

and if $\frac{k}{n} - x_\ell\xi_{\ell,k} < 0$, then

$$|\frac{k}{n} - x_\ell\xi_{\ell,k}| = x_\ell\xi_{\ell,k} - \frac{k}{n} \le x_{\ell+2}\xi_{\ell+1,k+1} - \frac{k}{n} \le \delta_n(\ell+1)\,.$$

Therefore we obtain

$$|\frac{k}{n} - x_\ell\xi_{\ell,k}| \le \delta_n(\ell+1) \qquad (k = 0, \ldots, \ell) \tag{17}$$

Next, we easily see from (16) that

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| = |\frac{k}{n} - x_{\ell+1}\xi_{\ell+1,k}| \leq \delta_n(\ell+1) \qquad (k = 0,\ldots,k_0-1).$$

Now let $k_0 \leq k \leq \ell$. If $\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} \geq 0$, then, using (16), we have for $k = k_0,\ldots,\ell-1$

$$\begin{aligned}|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| &= \frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} = \frac{k}{n} - x_{\ell+1}\xi_{\ell+1,k+2} \\ &< \frac{k+1}{n} - x_{\ell+1}\xi_{\ell+1,k+1} \leq \delta_n(\ell+1)\,;\end{aligned}$$

and this inequality is also valid for $k = \ell$, because $\xi_{\ell,\ell+1} = \xi_{\ell+1,\ell+2} = 1$. Moreover, since $k \geq k_0$, we have $\xi_{\ell,k+1} \geq \xi_{\ell,k_0+1} \geq y_{\ell+1}$. From (7) we deduce that $\xi_{\ell,k+1} \in \{y_0, y_1, \ldots, y_{h_\rho}\}$ for all $k \geq k_0$. Thereby there is a suffix k' such that $\xi_{\ell,k+1} = \xi_{h_\rho,k'}$, where $1 \leq k' \leq h_\rho$ and $k' \leq k$. Thus if $\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} < 0$, then we have

$$\begin{aligned}|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| &= x_{\ell+1}\xi_{\ell,k+1} - \frac{k}{n} = x_{h_\rho+1}\xi_{h_\rho,k'} - \frac{k}{n} \\ &< x_{h_\rho+1}\xi_{h_\rho,k'} - \frac{k'-1}{n} \leq \delta_n(h_\rho) \qquad (k = k_0,\ldots,\ell).\end{aligned}$$

Thereby we obtain

$$(18) \qquad |\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| < \max\big(\delta_n(\ell+1), \delta_n(h_\rho)\big) \qquad (k = 0,1,\ldots,\ell).$$

From (17) and (18) we have

$$(19) \qquad \delta_n(\ell) \leq \max\big(\delta_n(\ell+1), \delta_n(h_\rho)\big) \qquad (h_\rho < \ell < h_{\rho+1}).$$

To sum up both cases, from (15) and (19) we obtain (13).
Now we easily see from (13) that

$$\delta_n(\ell) \leq \max\big(\delta_n(h_\rho), \delta_n(h_{\rho+1})\big) \qquad (h_\rho < \ell < h_{\rho+1}).$$

Noticing $\delta_n(h_\rho) = \delta_n^*(\rho)$, from this inequality and (12) we have

$$\max_{0\leq\ell\leq n} \delta_n(\ell) \leq \max_{1\leq\rho\leq m} \delta_n^*(\rho).$$

Since the opposite inequality holds clearly, the lemma follows. □

We now can conclude the proof of Theorem 1, because Lemma 3 and Lemma 4 imply Proposition 1.

Remark 3. From the above proof we see that if $\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1} < 0$ $(0 \leq k \leq \ell$, $h_\rho < \ell < h_{\rho+1})$, then

$$\begin{aligned}&|\frac{k}{n} - x_{\ell+1}\xi_{\ell,k+1}| \\ &\leq \max(x_{\ell+2}\xi_{\ell+1,k+1} - \frac{k}{n}, x_{\ell+1} - \frac{\ell}{n}, x_{\ell+1}\xi_{\ell+1,k} - \frac{k}{n}, x_{h_\rho+1}\xi_{h_\rho,k^*} - \frac{k^*-1}{n}),\end{aligned}$$

where the suffix k^* satisfies $1 \leq k^* \leq k$.

3 The 3-dimensional Case

Theorem 2. *Suppose that $S_3 = \{u_k = (x_k, y_k, z_k)\ (1 \le k \le n)\} \subset G_3$ satisfies (3). Let $u_0 = (x_0, y_0, z_0) = (0,0,0)$ and $u_{n+1} = (x_{n+1}, y_{n+1}, z_{n+1}) = (1,1,1)$. For every ℓ ($\ell = 0,1,\dots,n$) rearrange y_i ($i = 0,1,\dots,\ell,n+1$) in increasing order and rewrite them as (4) and the corresponding z-coordinates z_i ($i = 0,1,\dots,\ell,n+1$) as $z_{\ell,i_0}, z_{\ell,i_1},\dots,z_{\ell,i_\ell}, z_{\ell,i_{\ell+1}}$. For every fixed ℓ ($0 \le \ell \le n$), for any t ($t = 0,1,\dots,\ell$) rearrange z_{ℓ,i_μ} ($\mu = 0,1,\dots,t,\ell+1$) in increasing order and rewrite them as*

(20) $$0 = \eta_{\ell,t,0} \le \eta_{\ell,t,1} \le \dots \le \eta_{\ell,t,t} < \eta_{\ell,t,t+1} = 1\,.$$

Then the discrepancy of S_3 is

(21) $$D_n(S_3) = \max_{0\le\ell\le n}\max_{0\le t\le\ell}\max_{0\le k\le t}\max\Big(\Big|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}\Big|, \Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}\Big|\Big)\,.$$

Remark 4. Formula (21) can be rewritten as

$$D_n(S_3) = \max\Big(\max_{1\le\ell\le n}\Big(\max_{1\le t\le\ell}\Big(\max_{1\le k\le t}\max\Big(\Big|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}\Big|,$$
$$\Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}\Big|\Big), x_{\ell+1}\xi_{\ell,t+1}\xi_{\ell,t,1}\Big), x_{\ell+1}\xi_{\ell,1}\Big), x_1\Big)\,.$$

In particular, if $u_1 = (0,0,0)$, then (21) becomes

$$D_n(S_3) = \max_{1\le\ell\le n}\max_{1\le t\le\ell}\max_{1\le k\le t}\max\Big(\Big|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}\Big|, \Big|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}\Big|\Big)\,.$$

Example 2. Let $u_k = \big(\frac{k-1}{n}, \{(k-1)\frac{g}{n}\}, \{(k-1)\frac{g^2}{n}\}\big)$ ($k = 1,\dots,n$), where g and n are coprime (see [7]). For every ℓ ($\ell = 0,1,\dots,n$) rearrange the numbers $\{(k-1)\frac{g}{n}\}$ ($k = 1,\dots,\ell$) and 1 in increasing order and rewrite them as

$$0 = \xi_{\ell,1} < \xi_{\ell,2} < \dots < \xi_{\ell,\ell} < \xi_{\ell,\ell+1} = 1\,.$$

For every fixed ℓ, for any t ($t = 0,1,\dots,\ell$) rewrite the z-coordinates corresponding to $\xi_{\ell,1},\dots,\xi_{\ell,t},\xi_{\ell,\ell+1}$ as

$$0 = \eta_{\ell,t,1} < \eta_{\ell,t,2} < \dots < \eta_{\ell,t,t} < \eta_{\ell,t,t+1} = 1\,.$$

Then

$$D_n(S_3) = \max_{1\le\ell\le n}\max_{1\le t\le\ell}\tilde{d}_n(\ell,t)\,,$$

where

$$\begin{aligned}\tilde{d}_n(\ell,t) = \frac{1}{n}\max\big(1, &|1-(\ell+1)\xi_{\ell,t+1}\eta_{\ell,t,2}|, && |2-\ell\xi_{\ell,t}\eta_{\ell,t,2}|,\\ &|2-(\ell+1)\xi_{\ell,t+1}\eta_{\ell,t,3}|, && |3-\ell\xi_{\ell,t}\eta_{\ell,t,3}|,\\ &\dots && \dots\\ &|(t-1)-(\ell+1)\xi_{\ell,t+1}\eta_{\ell,t,t}|, && |t-\ell\xi_{\ell,t}\eta_{\ell,t,t}|,\\ &|t-(\ell+1)\xi_{\ell,t+1}|\big)\,.\end{aligned}$$

We now prove Theorem 2 by the method used in the proof of Theorem 1, and adopt the geometric language for easier understanding. We again write (6) and assume that (7) holds. For every ℓ ($\ell = 0, 1, \ldots, n$) the projections of u_i ($i = 0, \ldots, \ell, n+1$) onto the plane $x = x_\ell$ are $P_\ell(u_i) = (x_l, y_i, z_i)$ ($i = 0, \ldots, \ell, n+1$). We rewrite them as $u^*_{\ell,\mu} = (x_\ell, \xi_{\ell,\mu}, z_{\ell,i_\mu})$ ($\mu = 0, \ldots, \ell, \ell+1$) according to (4). In particular, $u^*_{\ell,0} = (x_\ell, 0, 0)$ and $u^*_{\ell,\ell+1} = (x_\ell, 1, 1)$. Moreover, we again rewrite (4) as (8), and also assume in the case $u > 1$ that

$$z_{\ell,i_{r_\tau+1}} \leq z_{\ell,i_{r_\tau+2}} \leq \ldots \leq z_{\ell,i_{r_{\tau+1}}} \qquad (\tau = 1, \ldots, u-1)\,. \tag{22}$$

Finally, fixing ℓ ($0 \leq \ell \leq n$), for every t ($t = 0, 1, \ldots, \ell$) the projections of $u^*_{\ell,\mu}$ ($\mu = 0, \ldots, t, \ell+1$) onto the line

$$\begin{cases} x = x_\ell \\ y = \xi_{\ell,t} \end{cases}$$

are $P_{\ell,t}(u^*_{\ell,\mu}) = (x_\ell, \xi_{\ell,t}, z_{\ell,i_\mu})$ ($\mu = 0, \ldots, t, \ell+1$). According to (20), we rewrite them as $u^*_{\ell,t,k} = (x_\ell, \xi_{\ell,t}, \eta_{\ell,t,k})$ ($k = 0, \ldots, t, t+1$). In particular, $u^*_{\ell,t,0} = (x_\ell, \xi_{\ell,t}, 0)$ and $u^*_{\ell,t,t+1} = (x_\ell, \xi_{\ell,t}, 1)$. Furthermore, we rewrite (20) as

$$\begin{aligned} 0 = \eta_{\ell,t,0} = \ldots = \eta_{\ell,t,s_1} < \eta_{\ell,t,s_1+1} = \ldots = \eta_{\ell,t,s_2} \\ < \ldots < \eta_{\ell,t,s_{v-1}+1} = \ldots = \eta_{\ell,t,s_v} < \eta_{\ell,t,s_{v+1}} = 1 \end{aligned} \tag{23}$$

where $v = v(\ell, t) \geq 1$, $s_1 \geq 0$, $s_{\sigma+1} - s_\sigma \geq 1$ ($\sigma = 1, \ldots, v$), $s_v = t$, $\eta_{\ell,t,s_{v+1}} = \eta_{\ell,t,t+1} = 1$. We also let $\eta_{\ell,t,s_v+1} = \eta_{\ell,t,s_{v+1}}$ for convenience.
Construct a rectangular parallelepiped

$$V_{\rho,\tau,\sigma} = (x_{h_\rho}, x_{h_{\rho+1}}] \times (\xi_{h_\rho,r_\tau}, \xi_{h_\rho,r_{\tau+1}}] \times (\eta_{h_\rho,r_\tau,s_\sigma}, \eta_{h_\rho,r_\tau,s_{\sigma+1}}] \\ (1 \leq \rho \leq m, 1 \leq \tau \leq u, 1 \leq \sigma \leq v)\,,$$

which has a pair of adjacent vertices $u^*_{h_\rho,r_\tau,s_\sigma}$ and $u^*_{h_\rho,r_\tau,s_{\sigma+1}}$. It is clear that if $(\alpha, \beta, \gamma) \in V_{\rho,\tau,\sigma}$, then $A\big([0,\alpha) \times [0,\beta) \times [0,\gamma)\big) = s_\sigma$.
We define

$$d^*_n(\rho,\tau,\sigma) = \sup_{(\alpha,\beta,\gamma)\in V_{\rho,\tau,\sigma}} \left|\frac{s_\sigma}{n} - \alpha\beta\gamma\right| \qquad (1 \leq \rho \leq m, 1 \leq \tau \leq u, 1 \leq \sigma \leq v)\,.$$

Similar to above, we easily prove the following two lemmas.

Lemma 5. *We have*

$$d^*_n(\rho,\tau,\sigma) = \max\big(\big|\frac{s_\sigma}{n} - x_{h_\rho}\xi_{h_\rho,r_\tau}\eta_{h_\rho,r_\tau,s_\sigma}\big|, \big|\frac{s_\sigma}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\eta_{h_\rho,r_\tau,s_{\sigma+1}}\big|\big)\,.$$

Lemma 6. *We have*

$$D_n(S_3) = \max_{1\leq\rho\leq m}\max_{1\leq\tau\leq u}\max_{1\leq\sigma\leq v} d^*_n(\rho,\tau,\sigma)\,.$$

Now let for $0 \le \ell \le n$, $0 \le t \le \ell$ and $0 \le k \le t$

$$d_n(\ell, t, k) = \max\bigl(|\frac{k}{n} - x_\ell \xi_{\ell,t} \eta_{\ell,t,k}|, |\frac{k}{n} - x_{\ell+1} \xi_{\ell,t+1} \eta_{\ell,t,k+1}|\bigr).$$

Therefore, to prove Theorem 2, we only need to prove the following

Proposition 2. *We have*

$$\max_{1\le\rho\le m} \max_{1\le\tau\le u} \max_{1\le\sigma\le v} d_n^*(\rho,\tau,\sigma) = \max_{0\le\ell\le n} \max_{0\le t\le\ell} \max_{0\le k\le t} d_n(\ell,t,k).$$

In the sequel we are going to prove this proposition.

Lemma 7. *For any ρ $(1 \le \rho \le m)$ and τ $(1 \le \tau \le u(h_\rho))$ let*

$$\theta_n^*(\rho,\tau) = \max_{0\le k\le r_\tau} \max\bigl(|\frac{k}{n} - x_{h_\rho} \xi_{h_\rho,r_\tau} \eta_{h_\rho,r_\tau,k}|, |\frac{k}{n} - x_{h_{\rho+1}} \xi_{h_\rho,r_{\tau+1}} \eta_{h_\rho,r_\tau,k+1}|\bigr),$$

where $\eta_{h_\rho,r_\tau,k}$ $(k = 0,\dots,r_\tau + 1)$ are defined by (20) *with $\ell = h_\rho$ and $t = r_\tau$. Then*

$$\max_{1\le\sigma\le v} d_n^*(\rho,\tau,\sigma) = \theta_n^*(\rho,\tau) \qquad (1 \le \rho \le m, 1 \le \tau \le u(h_\rho)).$$

Proof. Similar to the proof of Lemma 3, we easily prove the following.

1. If $s_1 > 0$ then

$$\max_{0\le k\le s_1-1} \max\bigl(|\frac{k}{n} - x_{h_\rho} \xi_{h_\rho,r_\tau} \eta_{h_\rho,r_\tau,k}|, |\frac{k}{n} - x_{h_{\rho+1}} \xi_{h_\rho,r_{\tau+1}} \eta_{h_\rho,r_\tau,k+1}|\bigr)$$
$$= \max_{0\le k\le s_1-1} \frac{k}{n} < \frac{s_1}{n} \le d_n^*(\rho,\tau,1).$$

2. If in (23) there is a suffix σ, $1 \le \sigma \le v(h_\rho, r_\tau) - 1$ such that $s_{\sigma+1} - s_\sigma > 1$ then

$$\max_{s_\sigma<k<s_{\sigma+1}} \max\bigl(|\frac{k}{n} - x_{h_\rho} \xi_{h_\rho,r_\tau} \eta_{h_\rho,r_\tau,k}|, |\frac{k}{n} - x_{h_{\rho+1}} \xi_{h_\rho,r_{\tau+1}} \eta_{h_\rho,r_\tau,k+1}|\bigr)$$
$$\le \max\bigl(d_n^*(\rho,\tau,\sigma), d_n^*(\rho,\tau,\sigma+1)\bigr).$$

Thereby the lemma follows from Lemma 5. □

Lemma 8. *For any ρ $(1 \le \rho \le m)$ let*

$$\theta_n^*(\rho) = \max_{0\le t\le h_\rho} \max_{0\le k\le t} \max\bigl(|\frac{k}{n} - x_{h_\rho} \xi_{h_\rho,t} \eta_{h_\rho,t,k}|, |\frac{k}{n} - x_{h_{\rho+1}} \xi_{h_\rho,t+1} \eta_{h_\rho,t,k+1}|\bigr),$$

where $\xi_{h_\rho,t}$ $(t = 0,\dots,h_\rho + 1)$ and $\eta_{h_\rho,t,k}$ $(t = 0,\dots,h_\rho;\ k = 0,\dots,t+1)$ are defined by (4) *and* (20) *with $\ell = h_\rho$, respectively. Then*

$$\max_{1\le\tau\le u} \theta_n^*(\rho,\tau) = \theta_n^*(\rho) \qquad (1 \le \rho \le m). \tag{24}$$

Proof. This is very similar to the proof of Lemma 4. For $1 \le \rho \le m, 0 \le t \le h_\rho$ we denote

$$\tilde{\theta}_n^*(\rho, t) = \max_{0\le k\le t} \max\left(\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k}\right|, \left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}\right|\right).$$

First, if $0 \le t \le r_1 - 1$, then $\xi_{h_\rho,t} = \xi_{h_\rho,t+1} = 0$, and so for any k $(0 \le k \le t)$,

$$\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k}\right| = \left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}\right| = \frac{k}{n} < \frac{r_1}{n},$$

therefore

(25) $$\tilde{\theta}_n^*(\rho, t) \le \theta_n^*(\rho, 1) \qquad (0 \le t \le r_1 - 1).$$

Next, let τ be a suffix in (8) such that $1 \le \tau \le u - 1$, $r_{\tau+1} - r_\tau > 1$. Then we have

(26) $$\tilde{\theta}_n^*(\rho, t) \le \max\left(\tilde{\theta}_n^*(\rho, r_\tau), \tilde{\theta}_n^*(\rho, r_{\tau+1})\right) \qquad (r_\tau < t < r_{\tau+1}).$$

In fact, we have $\xi_{h_\rho,t} = \xi_{h_\rho,t+1} = \xi_{h_\rho,r_{\tau+1}}$ for any $t \in \{r_\tau + 1, \dots, r_{\tau+1} - 1\}$.
In the sequel we fix this suffix t and consider two cases.

Case 1. Let $z_{h_\rho,i_{t+1}} \ge \eta_{h_\rho,t,t}$. Then $\eta_{h_\rho,t,k} = \eta_{h_\rho,t+1,k}$ $(k = 0, 1, \dots, t)$, and so for $k = 0, 1, \dots, t$

$$\left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k}\right| = \left|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k}\right| \le \tilde{\theta}_n^*(\rho, t+1).$$

Next, let $k \in \{0, \dots, t-1\}$. If $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1} \ge 0$, then

$$\left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}\right| < \frac{k+1}{n} - x_{h_\rho}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k+1} \le \tilde{\theta}_n^*(\rho, t+1);$$

and if $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1} < 0$, then

$$\left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}\right| \le x_{h_{\rho+1}}\xi_{h_\rho,t+2}\eta_{h_\rho,t+1,k+1} - \frac{k}{n} \le \tilde{\theta}_n^*(\rho, t+1).$$

Moreover, if $k = t$, then

$$\left|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}\right| = \left|\frac{t}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\right|.$$

Thus we obtain

(27) $$\tilde{\theta}_n^*(\rho, t) \le \max\left(\tilde{\theta}_n^*(\rho, t+1), \max_{r_\tau<t<r_{\tau+1}} \left|\frac{t}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\right|\right) \quad (r_\tau < t < r_{\tau+1}).$$

Notice that

$$\max_{r_\tau<t<r_{\tau+1}} \left|\frac{t}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\right| \le \max_{r_\tau\le t\le r_{\tau+1}} \left|\frac{t}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\right|$$
$$\le \max\left(\left|\frac{r_\tau}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\right|, \left|\frac{r_{\tau+1}}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}\right|\right),$$

and

$$|\frac{r_\tau}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}| = |\frac{r_\tau}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_\tau+1}\eta_{h_\rho,r_\tau,r_\tau+1}| \le \tilde{\theta}_n^*(\rho, r_\tau).$$

Furthermore, if $\frac{r_{\tau+1}}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}} \ge 0$, then

$$|\frac{r_{\tau+1}}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}| < \frac{r_{\tau+1}}{n} - x_{h_\rho}\xi_{h_\rho,r_{\tau+1}}\eta_{h_\rho,r_{\tau+1},r_{\tau+1}} \le \tilde{\theta}_n^*(\rho, r_{\tau+1});$$

and if $\frac{r_{\tau+1}}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}} < 0$, then

$$|\frac{r_{\tau+1}}{n} - x_{h_{\rho+1}}\xi_{h_\rho,r_{\tau+1}}| < x_{h_{\rho+1}}\xi_{h_\rho,r_\tau+1}\eta_{h_\rho,r_\tau,r_\tau+1} - \frac{r_\tau}{n} \le \tilde{\theta}_n^*(\rho, r_\tau).$$

Thus we obtain from (27)

$$\tilde{\theta}_n^*(\rho, t) \le \max\left(\tilde{\theta}_n^*(\rho, t+1), \tilde{\theta}_n^*(\rho, r_\tau), \tilde{\theta}_n^*(\rho, r_{\tau+1})\right) \qquad (r_\tau < t < r_{\tau+1}). \tag{28}$$

Case 2. Let $z_{h_\rho,i_{t+1}} < \eta_{h_\rho,t,t}$. Then there is a suffix $k_0 < t$ such that $z_{h_\rho,i_{t+1}} \in (\eta_{h_\rho,t,k_0}, \eta_{h_\rho,t,k_0+1}]$. Thus we have

$$\begin{aligned} \eta_{h_\rho,t+1,k} &= \eta_{h_\rho,t,k} \qquad (0 \le k \le k_0), \\ \eta_{h_\rho,t+1,k_0+1} &= z_{h_\rho,i_{t+1}} \le \eta_{h_\rho,t,k_0+1}, \\ \eta_{h_\rho,t+1,k} &= \eta_{h_\rho,t,k-1} \qquad (k_0+2 \le k \le t+1). \end{aligned} \tag{29}$$

Thereby we easily see by use of (29) that for $k = 0, 1, \ldots, k_0$

$$|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k}| = |\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k}| \le \tilde{\theta}_n^*(\rho, t+1).$$

Furthermore, noticing (29), for $k = k_0+1, \ldots, t$, if $\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k} \ge 0$, then

$$|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k}| \le \frac{k+1}{n} - x_{h_\rho}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k+1} \le \tilde{\theta}_n^*(\rho, t+1);$$

and if $\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k} < 0$, then

$$|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k}| \le x_{h_{\rho+1}}\xi_{h_\rho,t+2}\eta_{h_\rho,t+1,k+1} - \frac{k}{n} \le \tilde{\theta}_n^*(\rho, t+1).$$

Therefore we obtain

$$|\frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t}\eta_{h_\rho,t,k}| \le \tilde{\theta}_n^*(\rho, t+1) \qquad (k = 0, 1, \ldots, t). \tag{30}$$

Next, let $0 \le k \le k_0 - 1$. Using (29), if $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1} \ge 0$, then

$$\begin{aligned} |\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}| &< \frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k+1} \\ &\le \frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k} \le \tilde{\theta}_n^*(\rho, t+1); \end{aligned}$$

and if $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1} < 0$, then

$$|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}| \le x_{h_{\rho+1}}\xi_{h_\rho,t+2}\eta_{h_\rho,t+1,k+1} - \frac{k}{n} \le \tilde{\theta}_n^*(\rho, t+1).$$

Thus we have

$$(31) \qquad |\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}| \le \tilde{\theta}_n^*(\rho, t+1) \qquad (k = 0, 1, \ldots, k_0 - 1).$$

Now we let $k_0 \le k \le t$. If $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1} \ge 0$, then we have by means of (29) for $k_0 \le k < t$

$$\begin{aligned}|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}| &= \frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k+2} \\ < \frac{k}{n} - x_{h_\rho}\xi_{h_\rho,t+1}\eta_{h_\rho,t+1,k} &\le \tilde{\theta}_n^*(\rho, t+1);\end{aligned}$$

and if $k = t$, then this inequality also holds, because $\eta_{h_\rho,t,t+1} = \eta_{h_\rho,t+1,t+2} = 1$. Moreover, since $k \ge k_0$, we have $\eta_{h_\rho,t,k+1} \ge \eta_{h_\rho,t,k_0+1} \ge z_{h_\rho,i_{l+1}}$. From (22) we deduce that $\eta_{h_\rho,t,k+1} \in \{z_{h_\rho,i_\mu} \ (\mu = 0, \ldots, r_\tau)\}$ for all $k \ge k_0$, and so there is a suffix k' such that $\eta_{h_\rho,t,k+1} = \eta_{h_\rho,r_\tau,k'}$, $1 \le k' \le r_\tau$ and $k' \le k$. Thus if $\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1} < 0$, then for $k_0 \le k \le t$

$$|\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}| < x_{h_{\rho+1}}\xi_{h_\rho,r_\tau+1}\eta_{h_\rho,r_\tau,k'} - \frac{k'-1}{n} \le \tilde{\theta}_n^*(\rho, r_\tau).$$

Therefore we have

$$(32)\ |\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}| \le \max\left(\tilde{\theta}_n^*(\rho, t+1), \tilde{\theta}_n^*(\rho, r_\tau)\right) \quad (k = k_0, \ldots, t).$$

Combining (31) and (32) we obtain

$$(33)\ |\frac{k}{n} - x_{h_{\rho+1}}\xi_{h_\rho,t+1}\eta_{h_\rho,t,k+1}| \le \max\left(\tilde{\theta}_n^*(\rho, t+1), \tilde{\theta}_n^*(\rho, r_\tau)\right) \quad (k = 0, 1, \ldots, t).$$

From (30) and (33) we see that

$$(34) \qquad \tilde{\theta}_n^*(\rho, t) \le \max\left(\tilde{\theta}_n^*(\rho, t+1), \tilde{\theta}_n^*(\rho, r_\tau)\right) \qquad (r_\tau < t < r_{\tau+1}).$$

To sum up the two cases, we deduce (26) from (28) and (34).
Now we easily obtain, by using (25) and (26) and noticing $\tilde{\theta}_n^*(\rho, r_\tau) = \theta_n^*(\rho, \tau)$,

$$\max_{0\le t\le h_\rho} \tilde{\theta}_n^*(\rho, t) \le \max_{1\le\tau\le u} \theta_n^*(\rho, \tau),$$

or

$$\theta_n^*(\rho) \le \max_{1\le\tau\le u} \theta_n^*(\rho, \tau).$$

Since the opposite inequality holds clearly, the equality (24) follows. The lemma is proved. □

Lemma 9. *For any ℓ $(0 \le \ell \le n)$ let*

$$\theta_n(\ell) = \max_{0\le t\le \ell} \max_{0\le k\le t} \max\big(|\frac{k}{n} - x_\ell \xi_{\ell,t}\eta_{\ell,t,k}|, |\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}|\big).$$

Then

$$\max_{1\le\rho\le m} \theta_n^*(\rho) = \max_{0\le\ell\le n} \theta_n(\ell).$$

Proof. This runs essentially along the same lines as that of Lemma 8 (or, more exactly speaking, as that of Lemma 4). First, we easily prove that,

$$\theta_n(\ell) \le \theta_n^*(1) \qquad (0 \le \ell \le h_1 - 1). \tag{35}$$

Next, we prove that if in (6) there is a suffix ρ such that $1 \le \rho \le m$ and $h_{\rho+1} - h_\rho > 1$, then

$$\theta_n(\ell) \le \max\big(\theta_n(h_\rho), \theta_n(h_{\rho+1})\big) \qquad (h_\rho < \ell < h_{\rho+1}). \tag{36}$$

Notice that $x_\ell = x_{\ell+1} = x_{h_{\rho+1}}$ for any $\ell \in \{h_\rho + 1, \dots, h_{\rho+1} - 1\}$. Furthermore, since the positions of $P_\ell(u_i)$ $(i = 0, \dots, \ell)$ are the same as that of $P_{\ell+1}(u_i)$ $(i = 0, \dots, \ell)$, we have $\xi_{\ell,t} \ge \xi_{\ell+1,t}$ $(0 \le t \le \ell,\ h_\rho < \ell < h_{\rho+1})$.
We first consider $|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}|$ for $0 \le t \le \ell, 0 \le k \le t$. Let $\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k} \ge 0$. If $\xi_{\ell,t} = \xi_{\ell+1,t}$, then also $\eta_{\ell,t,k} = \eta_{\ell+1,t,k}$, and so we have

$$|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}| = \frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k} = \frac{k}{n} - x_{\ell+1}\xi_{\ell+1,t}\eta_{\ell+1,t,k} \le \theta_n(\ell+1);$$

and if $\xi_{\ell,t} > \xi_{\ell+1,t}$ then $\xi_{\ell,t} = \xi_{\ell+1,t+1}$ and $\eta_{\ell,t,k} \ge \eta_{\ell+1,t+1,k}$, and so we have

$$|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}| \le \frac{k}{n} - x_{\ell+1}\xi_{\ell+1,t+1}\eta_{\ell+1,t+1,k} \le \theta_n(\ell+1).$$

Now let $\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k} < 0$. If $\xi_{\ell,t} = \xi_{\ell+1,t}$, then similarly we have

$$|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}| = x_{\ell+1}\xi_{\ell+1,t}\eta_{\ell+1,t,k} - \frac{k}{n} \le \theta_n(\ell+1);$$

and if $\xi_{\ell,t} > \xi_{\ell+1,t}$, then $\xi_{\ell,t} = \xi_{\ell+1,t+1} \le \xi_{\ell+1,t+2}$ and $\eta_{\ell,t,k} \le \eta_{\ell+1,t+1,k+1}$, and so we have

$$|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}| \le x_{\ell+2}\xi_{\ell+1,t+2}\eta_{\ell+1,t+1,k+1} - \frac{k}{n} \le \theta_n(\ell+1).$$

To sum up, we obtain

$$|\frac{k}{n} - x_\ell\xi_{\ell,t}\eta_{\ell,t,k}| \le \theta_n(\ell+1) \qquad (h_\rho < \ell < h_{\rho+1}). \tag{37}$$

Next, we consider $|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}|$ for $0 \le t \le \ell$, $0 \le k \le t$. Let $\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1} \ge 0$. If $\xi_{\ell,t+1} = \xi_{\ell+1,t+1}$ then also $\eta_{\ell+1,t+1,k+1} = \eta_{\ell,t+1,k+1} \le \eta_{\ell,t,k+1}$, thereby

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \le \frac{k+1}{n} - x_{\ell+1}\xi_{\ell+1,t+1}\eta_{\ell+1,t+1,k+1} \le \theta_n(\ell+1); \tag{38}$$

and if $\xi_{\ell,t+1} > \xi_{\ell+1,t+1}$, then $\xi_{\ell,t+1} = \xi_{\ell+1,t+2}$ and $\eta_{\ell,t,k+1} \geq \eta_{\ell,t+1,k+1} \geq \eta_{\ell+1,t+2,k+1}$, thereby

$$(39)\ |\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \leq \frac{k+1}{n} - x_{\ell+1}\xi_{\ell+1,t+2}\eta_{\ell+1,t+2,k+1} \leq \theta_n(\ell+1).$$

Now let $\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1} < 0$. We have to distinguish three cases.

Case 1. Suppose that there is a suffix $i \in \{h_\rho+1,\dots,\ell\}$ such that $P_\ell(u_i) = (x_\ell, \xi_{\ell,t+1}, z_{\ell,i_{t+1}})$, i.e. $\xi_{\ell,t+1}$ is the y-coordinate of the projection of one of the points u_i $(i = h_\rho+1,\dots,\ell)$ onto the plane $x = x_\ell$. Due to (7), we have $\xi_{\ell,t+1} = \xi_{\ell+1,t+1}$ and $\eta_{\ell,t,k+1} = \eta_{\ell+1,t,k+1}$, and so

$$(40)\qquad |\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \leq x_{\ell+2}\xi_{\ell+1,t+1}\eta_{\ell+1,t,k+1} - \frac{k}{n} \leq \theta_n(\ell+1).$$

Case 2. Suppose that there are i, $j \in \{0,\dots,h_\rho\}$ such that $P_\ell(u_i) = (x_\ell, \xi_{\ell,t+1}, z_{\ell,i_{t+1}})$, $P_\ell(u_j) = (x_\ell, \xi_{\ell,t}, z_{\ell,i_t})$. Then we have $\xi_{\ell,t+1} = \xi_{h_\rho,t'}$, $\eta_{\ell,t,k+1} = \eta_{h_\rho,t'-1,k'}$, where $1 \leq t' \leq t+1$, $1 \leq k' \leq k+1$, and so

$$(41)\qquad |\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \leq x_{h_\rho+1}\xi_{h_\rho,t'}\eta_{h_\rho,t'-1,k'} - \frac{k'-1}{n} \leq \theta_n(h_\rho).$$

Case 3. Suppose that there are $i \in \{0,\dots,h_\rho\}$ and $j \in \{h_\rho+1,\dots,\ell\}$ such that $P_\ell(u_i) = (x_\ell, \xi_{\ell,t+1}, z_{\ell,i_{t+1}})$ and $P_\ell(u_j) = (x_\ell, \xi_{\ell,t}, z_{\ell,i_t})$. According to (7), we have $y_{\ell+1} \geq \xi_{\ell,t}$, where $y_{\ell+1}$ is the y-coordinate of $u_{\ell+1}$.

a) Let $y_{\ell+1} \geq \xi_{\ell,t+1}$. Then $\xi_{\ell,t+1} = \xi_{\ell+1,t+1}$, $\eta_{\ell,t,k+1} = \eta_{\ell+1,t,k+1}$, and so

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \leq x_{\ell+2}\xi_{\ell+1,t+1}\eta_{\ell+1,t,k+1} - \frac{k}{n} \leq \theta_n(\ell+1).$$

b) Let $y_{\ell+1} < \xi_{\ell,t+1}$. For the moment we assume that $t \neq \ell$, thus $\xi_{\ell,t+2}$ is well defined. Due to (7), we have $P_\ell(u_w) = (x_\ell, \xi_{\ell,t+2}, z_{\ell,i_{t+2}})$, where $w \in \{0,\dots,h_\rho\}$. Thereby we have $\xi_{\ell,t+1} = \xi_{h_\rho,t''} \leq \xi_{h_\rho,t''+1}$, $\xi_{\ell,t+2} = \xi_{h_\rho,t''+1}$, $\eta_{\ell,t,k+1} \leq \eta_{\ell,t+1,k+2} = \eta_{h_\rho,t'',k''}$, where $1 \leq t'' \leq t$, $1 \leq k'' \leq k+1$. Thus we have

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \leq x_{h_\rho+1}\xi_{h_\rho,t''+1}\eta_{h_\rho,t'',k''} - \frac{k''-1}{n} \leq \theta_n(h_\rho) \qquad (0 \leq t < \ell).$$

For the case $t = \ell$, we define the 2-dimensional set

$$\tilde{S}_2 = \{\tilde{u}_j = (\tilde{x}_j, \tilde{y}_j) = (x_j, z_j) \quad (j = 1,\dots,n)\}.$$

Then from Remark 3 we easily deduce that

$$|\frac{k}{n} - x_{\ell+1}\eta_{\ell,\ell,k+1}| = |\frac{k}{n} - \tilde{x}_{\ell+1}\tilde{\xi}_{\ell,k+1}| \leq$$

$$\max(x_{\ell+2}\eta_{\ell+1,\ell+1,k+1} - \frac{k}{n}, x_{\ell+1} - \frac{\ell}{n}, x_{\ell+1}\eta_{\ell+1,\ell+1,k} - \frac{k}{n}, x_{h_\rho+1}\eta_{h_\rho,h_\rho,k^*} - \frac{k^*-1}{n}),$$

where $1 \le k^* \le k$. Since

$$x_{\ell+2}\eta_{\ell+1,\ell+1,k+1} - \frac{k}{n} = x_{\ell+2}\xi_{\ell+1,\ell+2}\eta_{\ell+1,\ell+1,k+1} - \frac{k}{n} \le \theta_n(\ell+1),$$

$$x_{\ell+1} - \frac{\ell}{n} < x_{h_\rho+1}\xi_{h_\rho,h_\rho+1}\eta_{h_\rho,h_\rho,h_\rho+1} - \frac{h_\rho}{n} \le \theta_n(h_\rho),$$

$$x_{\ell+1}\eta_{\ell+1,\ell+1,k} - \frac{k}{n} \le x_{\ell+2}\xi_{\ell+1,\ell+2}\eta_{\ell+1,\ell+1,k} - \frac{k}{n} \le \theta_n(\ell+1),$$

$$x_{h_\rho+1}\eta_{h_\rho,h_\rho,k^*} - \frac{k^*-1}{n} = x_{h_\rho+1}\xi_{h_\rho,h_\rho+1}\eta_{h_\rho,h_\rho,k^*} - \frac{k^*-1}{n} \le \theta_n(h_\rho),$$

we obtain

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \le \max\big(\theta_n(\ell+1), \theta_n(h_\rho)\big) \qquad (t=\ell).$$

To sum up, we have in Case 3

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \le \max\big(\theta_n(\ell+1), \theta_n(h_\rho)\big). \tag{42}$$

Combining from (38) through (42), we have

$$|\frac{k}{n} - x_{\ell+1}\xi_{\ell,t+1}\eta_{\ell,t,k+1}| \le \max\big(\theta_n(\ell+1), \theta_n(h_\rho)\big) \qquad (h_\rho < \ell < h_{\rho+1}). \tag{43}$$

From (37) and (43) we deduce (36). Since $\theta_n(h_\rho) = \theta_n^*(\rho)$, the lemma follows from (35) and (36). □

Lemma 7, Lemma 8 and Lemma 9 imply Proposition 2, thereby Theorem 2 is proved.

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Authors addresses: Peter Bundschuh, Mathematisches Institut der Universität zu Köln, Weyertal 86–90, W-5000 Köln 41, Germany. Zhu Yaochen, Institute of Applied Mathematics, Academia Sinica, Beijing 100080, China.