Math3283W study guiding

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Contents

1 Midterm one

1.1 Vocabulary

1.1.1 Logical Connectives

Definition 1 (statement)**.** A sentence classified as something either true or false is a **statement**.

Definition 2 (sentential connectives)**.** **not**, **and**, **or**, **if ... then**, **if and only if**.

- negation $\neg p$ represents the logical opposite of p.
- conjunction $p \wedge q$
- disjunction $p \vee q$
- implication/conditional $p \rightarrow q$, antecedant p statement, consequent q statement
- biconditional $p \leftrightarrow q$

Definition 3 (tautology)**.** A statement which is true in all cases. examples:

$$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$$
$$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$
$$\neg(p \rightarrow q) \leftrightarrow (p \vee \neg q)$$

1.1.2 Quantifiers

Definition 4 (Quantifier)**.** Given a statement $p(x)$, **universal quantifier** is $\forall x, p(x)$. **existential quantifier** is $\exists x \ni p(x)$

1.1.3 Techniques of Proof

Definition 5 (Inductive Reasoning)**.** Making a general conclusion on the basis of looking at individual cases.

Definition 6 (Counterexample)**.** Finding an example such that the statement is false.

Definition 7 (Deductive Reasoning)**.** Applying a general principle to a particular case to make a conclusion. Most of the proofs encountered in mathematics are based on this type of reasoning.

Definition 8 (Hypothesis $\Rightarrow$ Conclusion)**.** When an implication is identified as a theorem, it is customary to refer to p as the **hypothesis** and q as the **conclusion**.

Definition 9 (Contrapositive)**.** The **converse** of $p \rightarrow q$ is $q \rightarrow p$, but is not equivalent to the implication.

The **inverse** of $p \rightarrow q$ is $\neg p \Rightarrow \neg q$, but is not equivalent to the implication.

The **contrapositive** is both the converse and the inverse at once, and is tautologically equivalent to the implication.

$$(p \to q) \leftrightarrow (\neg q \to \neg p)$$

Definition 10 (Contradiction)**.** The letter c is used to represent a statement that is always false. Suc ha statement is called a **contradiction**.

1.1.4 Set Operations

Subset $A \subseteq B$. A is a **subset** of B (or A is **contained** in B). If we want to prove $A \subseteq B$, then we must prove "if $x \in A$, then $x \in B$"

- **Proper Subset** $A \subset B$. $(\forall a \in A, a \in B) \wedge (\exists b \in B \ni b \notin A)$. That is, all elements of A are in B, but some elements of B are not in A

- **Equal** A set A is equal to a set B provided that $A \subseteq B$ and $A \supseteq B$ ($A \subseteq B$ and $B \subseteq A$)

Closed Interval $[a, b]$. $a, b \in [a, b]$

Open Interval (a, b). $a, b \notin (a, b)$

Half–Open (Half–Closed) Interval $[a, b)$, or $(a, b]$

Union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- If $A \cap B = \varnothing$, then A and B are said to be **disjoint**

Complement $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

1.1.5 Relations

Definition 11 (Ordered Pairs)**.** ordered pairs :: $(a, b) = \{\{a\}, \{a, b\}\}$

Theorem 1. $(a, b) = (c, d) \iff a = c$ and $b = d$

Definition 12 (Cartesian Product (Cross Product))**.** $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Definition 13 (Relation)**.** A **relation** between A and B is any subset $\mathcal{R}$ of $A \times B$. We say that an element a in A is related by $\mathcal{R}$ to an element in b in B if $(a, b) \in \mathcal{R}$. The first set A is referred to as the *domain*, of the relation and denoted dom $\mathcal{R}$. If $B = A$, then we speak of a relation $\mathcal{R} \subseteq A \times A$ being a relation on A.

Definition 14 (Equivalence Relation)**.** A relation $\mathcal{R}$ is an **equivalence relation** if:

1. $x\mathcal{R}x$ (reflexive property)
2. If $x\mathcal{R}y$, then $y\mathcal{R}x$ (symmetric property)
3. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$ (transitive property)

An **equivalence class** (with respect to $\mathcal{R}$) of $x \in S$ is defined to be the set

$$E_x = \{y \in S \mid y\mathcal{R}x\}$$

Definition 15 (Partition)**.** We see that an equivalence relation $\mathcal{R}$ on a set S breaks S into **disjoint** pieces in a natural way. A **partition** of a set S is a collection $\mathscr{P}$ of nonempty subsets of S such that

1. Each $x \in S$ belongs to some subset $A \in \mathscr{P}$.
2. For all $A, B \in \mathscr{P}$, if $A \neq B$, then $A \cap B = \varnothing$.

A member of $\mathscr{P}$ is called a **piece** of the partition.

1.1.6 Examples

- This example shows a *direct proof.*
  - For every $\varepsilon > 0$ there exists a $\delta > 0$ such that
$$1 - \delta < x < 1 + \delta \text{ implies that } 5 - \varepsilon < 2x + 3 < 5 + \varepsilon.$$
  - Begin by letting ε be an arbitrary positive number, i.e. $\varepsilon > 0$. We need to use this ε to find a positive δ with the property that
$$1 - \delta < x < 1 + \delta \text{ implies that } 5 - \epsilon < 2x + 3 < 5 + \epsilon.$$
  - Given any $\varepsilon > 0$, let $\delta = \varepsilon/2$. $\delta > 0$, and whenever
$$1 - \delta < x < 1 + \delta$$
we have
$$1 - \frac{\epsilon}{2} < x < 1 + \frac{\epsilon}{2}$$
so that
$$2 - \epsilon < 2x < 2 + \epsilon$$
and
$$5 - \epsilon < 2x + 3 < 5 + \epsilon$$
thus
$$1 - \delta < x < 1 + \delta \text{ implies that } 5 - \epsilon < 2x + 3 < 5 + \epsilon.$$
- This example shows a *indirect proof.*
  - Let f be an integrable function, so that

If $\int_0^1 f(x)dx \neq 0$, then there exists a point x in the interval $[0, 1]$ such that $f(x) \neq 0$.

1. Symbolically, we have $p \Rightarrow q$, where

$$p : \int_0^1 f(x)dx \neq 0,$$
$$q : \exists x \text{ in } [0,1] \ni f(x) \neq 0.$$

The contrapositive implication, $\neg q \Rightarrow \neg p$, can be written

$$\text{If for every } x \text{ in } [0,1],\ f(x) = 0, \text{ then } \int_0^1 f(x)dx = 0.$$

2. This is obviously true. The integral of all 0 integrands is obviously 0.

- This example shows a *proof by contradiction.*
  - Let x be a real number.

$$\text{If } x > 0, \text{ then } 1/x > 0.$$

1. Symbolically, we have $p \Rightarrow q$, where

$$p : x > 0$$
$$q : 1/x > 0$$

so that, $(p \Rightarrow q) \Leftrightarrow ((p \wedge \neg q) \Rightarrow c)$, where c represents a contradiction.

2. Begin by supposing $x > 0$ and $1/x \leq 0$. Since $x > 0$, we can multiply both sides of the inequality $1/x \leq 0$ by x to obtain

$$(x)\left(\frac{1}{x}\right) \leq (x)(0)$$

But $(x)(1/x) = 1$ and $(x)(0) = 0$, so we have $1 \leq 0$, a contradiction to the (presumably known) fact that $1 > 0$. Having show that $p \wedge \neg q$ leads to a contradiction, we conclude that $p \Rightarrow q$.

- This example shows a *proof with absolute value.*
  - If x is a real number, then $x \leq |x|$

$$s : x \text{ is a real number}$$
$$r : x \leq |x|$$

The definition of statement r can be rewritten as:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

1. Since the definition is divided into two parts, it is natural to divide our proof into two cases. Thus statement s is replaced by the equivalent disjunction $p \vee q$, where

$$p : x \geq 0 \text{ and } q : x < 0.$$

2. The case to prove now is $(p \vee q) \Rightarrow r$, which is the same as $(p \Rightarrow r) \wedge (q \Rightarrow r)$.
3. If $x \geq 0$, then $x = |x|$. If $x < 0$, then $-x > 0$, so that $x < 0 < -x = |x|$. Or, $x \leq |x|$. Thus, $(p \Rightarrow r) \wedge (q \Rightarrow r)$. Hence, if x is a real number, then $x \leq |x|$

2 Midterm two

2.1 Functions

Definition 16 (Function)**.** Let A and B be sets. A **function** from A to B is nonempty relation $f \subseteq A \times B$ that satisfies the following two conditions:

1. *Existence*: For all a in A, there exists a b in B such that $(a, b) \in f$.

2. *Uniqueness*: If $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

That is, given any element a in A, there is one and only one element b in B such that $(a, b) \in f$. Set A is called the domain of f and is denoted by dom f. Set B is referred to as the codomain of f. We may write $f : A \to B$ to indicate f has domain A and codomain B. The range of f, denoted rng f, is the set of all second elements of members of f. That is,

$$\{b \in B : \exists a \in A \ni (a, b) \in f\}.$$

Definition 17 (Identity)**.** A function defined on a set A that maps each element in A onto itself is called the **identity function** on A, and is denoted $f^{-1} \circ f = i_A$. Furthermore, if $f(x) = y$, then $x = f^{-1}(y)$, so that

$$f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y$$

.

Thus, $f \circ f^{-1} = i_B$.

Definition 18 (Surjective)**.** A function $f : A \to B$ is called **surjective** (or is said to map A **onto** B) if $B = \text{rng } f$. A surjective function is also referred to as a **surjection**

Definition 19 (Injective)**.** A function $f : A \to B$ is called **injective** (or **one-to-one**) if, for all a and a' in A, $f(a) = f(a')$. An injective function is also referred to as an **injection**.

Definition 20 (Bijective)**.** A function $f : A \to B$ is called **bijective** or a **bijection** if it is both surjective and injective.

Definition 21 (Composition)**.** If f and g are functions with $f : A \to B$ and $g : B \to C$, then for any $a \in A$, $f(a) \in B$. But B is the domain of g, so g can be applied to $f(a)$. This yields $g(f(a))$, an element of C. Thus we have established a correspondance between a in A and $g(f(a))$ in C. This correspondance is called the composition function of f and g and is denoted by $g \circ f$ (read "g of f"). It defines a function $g \circ f : A \to C$ given by

$$(g \circ f)(a) = g(f(a)) \text{ for all } a \in A.$$

Definition 22 (Inverse)**.** Let $f : A \to B$ be bijective. The inverse function of f is the function f^{-1} given by

$$f^{-1} = \{(y, x) \in B \times A : (x, y) \in f\}.$$

2.2 Cardinality

Definition 23 (Equinumerous)**.** Two sets S and T are called **equinumerous**, written $S \sim T$, if there exists a bijective function from S onto T.

Definition 24 (Finite or Infinite)**.** A set S is said to be **finite** if $S = \varnothing$ or if there exists $n \in \mathbb{N}$ and a bijection $f : \{1, 2, ..., n\} \to S$. If a set is not finite, it is said to be **infinite**.

Definition 25 (Cardinal number)**.** The **cardinal number** of I_n is n, and if $S \sim I_n$, then we say that S **has *n* elements**. The cardinal number of $\varnothing$ is taken to be 0. If a cardinal number is not finite, it is called **transfinite**.

Theorem 2. Let S be a countable set and let $T \subseteq S$. Then T is countable.

Definition 26 (Power Set)**.** Given any set S, let $\mathscr{P}(S)$ denote the collection of all the subsects of S. The set $\mathscr{P}(S)$ is called the **power set** of S.

Theorem 3. For any set S, we have $|S| < |\mathscr{P}(S)|$.

2.3 The Real Numbers

Axiom. (Well-ordering property of $\mathbb{N}$) If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \leq k$ for all $k \in S$.

Theorem 4. (Mathematical Induction) A technique of mathematical proof. Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$, provided that

1. $P(1)$ is true
2. Whenever $P(k)$ is true, for some number k, then $P(k+1)$ is true.

Proof. (By contradiction) Given statements $P(n)$, $n \in \mathbb{N}$. Show if we have properties 1) and 2), then $P(n)$ holds for all n. Suppose $P(n)$ false for some n. Let $F = \{n \in \mathbb{N} : P(n) \text{ false}\}$. F is non-empty by assumption, so by well-ordering principle it has a least element, say $n_0 \neq 1$. Consider $n_0 - 1$, a natural number, so that $P(n_0 - 1)$ is true, since otherwise n_0 wasn't the smallest element of F. ■

Remark (*Slight Generalization*)**.** For proving $p(n)$ for all $n \geq n_0$, then prove:

1. $p(n_0)$ is true.
2. if $p(k)$ is true for k, then $p(k+1)$ is true with $k \geq n_0$

2.4 Ordered Fields

Axiom (Axioms of an Ordered Field)**.** We begin by assuming the existence of a set R, called the set of real numbers, and two operations "+" and "·", called addition and multiplication, such that the following properties apply :—

A1. For all $x, y \in \mathbb{R}$, $x + y \in \mathbb{R}$ and if $x = w$ and $y = z$, then $x + y = w + z$.

A2. For all $x, y \in \mathbb{R}$, $x + y = y + x$.

A3. For all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.

A4. There is a unique real number 0 such that $x + 0 = x$, for all $x \in \mathbb{R}$.

A5. For each $x \in \mathbb{R}$ there is a unique real number $-x$ such that $x + (-x) = 0$.

M1. For all $x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$ and if $x = w$ and $y = z$, then $x \cdot y = w \cdot z$.

M2. For all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.

M3. For all $x, y, z \in \mathbb{R}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

M4. There is a unique real number 1 such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

M5. For each $x, y \in \mathbb{R}$ with $x \neq 0$, there is a unique real number $1/x$ such that $x \cdot (1/x) = 1$. We also write x^{-1} or $\frac{1}{x}$ in place of $1/x$.

DL. For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Remark. These first 11 axioms are called the field axioms because they describe a system know as a field in the study of abstract algebra. Axioms A2 and M2 are called the ***commutative laws*** and axioms A3 and M3 are the ***associative laws***. Axiom DL is the ***distributive law*** that shows how addition and multiplication relate to each other. Because of A1 and M1, we can think of addition and multiplication as functions that map $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. When writing multiplication we often omit the raised dot and write xy instead of $x \cdot y$.

In addition to the field axioms, the real numbers also satisfy four order axioms.

O1. For all $x, y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, or $x < y$ holds (***trichotomy law***).

O2. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

O3. For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.

O4. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $xz < yz$.

Definition 27 (Ordering of Rational Functions)**.** If p/q and f/g are rational functions, then we say that

$$\begin{aligned} \frac{p}{q} > \frac{f}{g} &\iff \frac{p}{q} - \frac{f}{g} > 0 \\ &\iff \frac{pg - fq}{qg} > 0 \end{aligned}$$

Definition 28 (Absolute Value)**.** If $x \in \mathbb{R}$, then the absolute value of x, denoted by $|x|$, is defined by:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then

(a) $|x| \geq 0$,

(b) $|x| \leq a \iff -a \leq x \leq a$,

(c) $|xy| = |x| \cdot |y|$,

(d) $|x + y| \leq |x| + |y|$.

Remark. Part (d) is referred to as the ***triangle inequality***, and has other forms. For example, letting $x = a - c$ and $y = c - b$, we obtain

$$|a - b| \leq |a - c| + |c - b| \, .$$

Theorem 5. *Let $x, y \in \mathbb{R}$ such that $x \leq y + \varepsilon$ for every $\varepsilon > 0$. Then $x \leq y$.*

2.5 Completeness Axiom

Definition 29 (Irrational)**.** Let p be a prime number. Then $\sqrt{p}$ is not a rational number.

Definition 30 (Bounds)**.** Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s$ for all $s \in S$, then m is called an ***upper bound*** of S, and we say that S is bounded above. If $m \leq s$ for all $s \in S$, then m is a ***lower bound*** of S and S is bounded below. The set S is said to be ***bounded*** if it is bounded above and bounded below.

Definition 31 (Maximum and Minimum)**.** If an upper bound m of S is a member of S, then m is called the **maximum** (or largest element) of S, and we write

$$m = \max S.$$

Similarly, if a lower bound of S is a member of S, then it is called the **minimum** (or least element) of S, denoted by

$$m = \min S.$$

Definition 32 (Supremum and Infimum)**.** Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the least upper bound of S is called its ***supremum*** and is denoted by $\sup S$. Thus $m = \sup S$ iff

1. $m \geq s$, for all $s \in S$, and
2. if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

If S is bounded below, then the greatest lower bound of S is called its **infimum** and is denoted by $\inf S$, so $m = \inf S$.

Definition 33 (Completeness Axiom)**.** Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem 6 (Archimedean Property of $\mathbb{R}$)**.** The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Theorem 7 (Alternative Archimedean Properties)**.** Each of the following is equivalent to the Archimedean Property.

1. For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$.
2. For each $x > 0$ and for each $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$
3. For each $x > 0$, there exists an $n \in \mathbb{N}$ such that $0 < 1/n < x$.

Theorem 8 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** If x and y are real numbers with $x < y$, then there exists a rational number r such that $x < r < y$.

Theorem 9. If x and y are real numbers with $x < y$, then there exists an irrational number w such that $x < w < y$.

2.6 Examples

- Example: For which n is $n! > n^n$?

  Expect $n! > 2^n$ if $n \geq 4$. (i.e. for all $n \geq 4$).

  Base case: True if $n_0 = 4$ since $24 > 16$. Suppose $k! > 2^k$.

  Inductive step: Show: $(k+1)! > 2^{k+1}$. Easy:

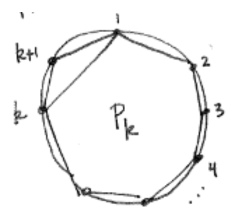
  $$k! > 2^k \Rightarrow (k+1)! > 2^k(k+1) > 2^{k+1}$$

  since $k + 1 > 2$ if $k \geq 4$.

- Example: For $n \geq 3$, if we connect n points on circle w with straight line segments, the interior angles of the resulting polygon add up to $(n-2) \cdot 180$.

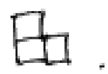
  Base case: $n = 3$. Angles of triangle add up to $180°$.

  Inductive Step: Suppose true for k. Prove true for k+1. By hypothesis, P_k has interior angles $(k-2) \cdot 180\,\text{deg}$. Triangle P_{k+1} has interior angles defined to be the sum of the p_k angles and the triangle with vertices $k, k+1, 1$. That is $(k-2) \cdot 180\,\text{deg} + 180\,\text{deg} = ((k+1)-2) \cdot 180\,\text{deg}$. ✓.

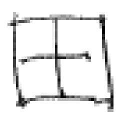


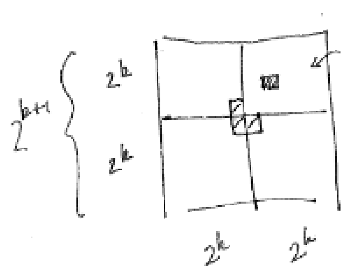
Imagine, any number of edges k, where the first edge is named 1. One could simply add another $k+1$ edge to the list.

- Example: Prove that any $2^n \times 2^n$ grid of squares with any one square removed can be tiled with L-shaped tiles.

 $\Leftarrow$ *L-shaped tiles*

Removing any box results in the remaining boxes being an L-shaped tile.

 $\Leftarrow$ *box*



A large block can be covered by L-shapes by hypothesis. What about other 3 blocks?

Inductive hypothesis doesn't apply to grid $2^k \times 2^k$ without removing a square. Meaning without removing one atomic box.

Solution: consider removing a single L instead, taking away a box from three quadrants. It is an equivalent procedure.

3 Midterm three

3.1 Boundaries

Definition 34 (Interior Point and Boundary Point)**.** Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an ***interior point*** of S if there exists a neighborhood N of x such that $N \subseteq S$. If for every neighborhood N of x, $N \cap S \neq \varnothing$ and $N \cap (\mathbb{R} \setminus S) \neq \varnothing$, then x is called a ***boundary point*** of S. The set of all interior points of S is denoted by int S, and the set of all boundary points of S is denoted by bd S.

Definition 35 (Neighborhood)**.** Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A ***neightborhood*** of x (or an ε***-neightborhood*** of x) is a set of the form $N(x;\varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\}$.

Remark. The professor uses the notation:

$$N_\varepsilon(x) = \{y \in \mathbb{R} : |x - y| < \varepsilon\},$$

which is probably nicer.

Definition 36 (Deleted Neighborhood)**.** Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A ***deleted neighborhood*** of x is a set of the form $N^*(x;\varepsilon) = \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\}$. Clearly, $N^*(x;\varepsilon) = N(x;\varepsilon) \setminus \{x\}$

Remark. The professor uses the notation:

$$N^*_\varepsilon(x) = \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\},$$

which is probably nicer.

Definition 37 (Open and Closed Sets)**.** Let $S \subseteq \mathbb{R}$. If bd $S \subseteq S$, then S is said to be ***closed***. If bd $S \subseteq \mathbb{R} \setminus S$, then S is said to be ***open***.

Theorem 10.

- A set S is open $\iff$ $S = \text{int } S$. Equivalently, S is open $\iff$ every point in S is an interior point of S.
- A set S is closed $\iff$ its complement $\mathbb{R} \setminus S$ is open.
- The union of any collection of open sets is an open set.
- The intersection of any finite collection of open sets is an open set.

3.2 Topology of the Real Numbers

Every bounded sequence has a convergent subsequence.

- If $\{s_n\}$ bounded, then
  1. for every ε, $\exists N \in \mathbb{N} \ni s_n < m + \varepsilon$ when $n \geq N$. (Else there are infinitely many $s_n \geq m + \varepsilon$, so there can't be a lim sup.)
  2. for every $\varepsilon > 0$, $\forall i \in \mathbb{N}, \exists k > i$ with $s_k > m - \varepsilon$. (There are infinitely many $s_k \in (m - \varepsilon, m + \varepsilon)$, else $m - \varepsilon$ is upper bound for all limits of subsequences.)

Definition 38 (Accumulation Points)**.** Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an **accumulation point** of S if every deleted neighborhood of x contains a point of S. That is, for every $\varepsilon > 0$, $N^*(x,\varepsilon) \cup S \neq \varnothing$. The set of all accumulation points of S is denoted by S'. If $x \in S$ and $x \notin S'$, then x is called an **isolated point** of S.

Definition 39 (Closure)**.** Let $S \subseteq \mathbb{R}$. Then the closure of S, denoted cl S, is defined by

$$\text{cl } S = S \cup S',$$

where S' is the set of all accumulation points of S.

Also,

$$\text{cl } S = S \cup \text{bd } S.$$

3.3 Compact Sets

Definition 40 (Compact, Open Cover, and Subcover)**.** A set S is said to be ***compact*** if whenever it is contained in the union of a family $\mathscr{F}$ of open sets, it is contained in the union of some finite number of the sets in $\mathscr{F}$. If $\mathscr{F}$ is a family of open sets whose union contains S, then $\mathscr{F}$ is called an ***open cover*** of S. If $\mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{G}$ is also an open cover of S, then $\mathscr{G}$ is called a ***subcover*** of S.

Corollary. S is compact $\overset{Heine-Borel}{\Longleftrightarrow}$ S is closed and bounded $\iff$ every infinite subset of S has an accumulation point in S.

S is a nonempty closed bounded subset of $\mathbb{R} \Rightarrow S$ has a maximum and a minimum.

Definition 41 (Heine–Borel)**.** A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.

Definition 42 (Bolzano–Weierstrass)**.** If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.

3.4 Sequences

Definition 43 (Sequence)**.** A sequence S is a function whose domain is the set $\mathbb{N}$ of natural numbers. Denoted by its value of n at s_n instead of $S(n)$ or by listing its values $(s_1, s_2, s_3, ...)$. s_n is the n^{th} term of the sequence.

Definition 44 (Convergence, Divergence, Limit)**.** A sequence (s_n) is said to ***converge*** to the real number s provided that

for every $\varepsilon > 0$ there exists a natural number N such that for all $n \in \mathbb{N}$, $n \geq N$ implies that $|s_n - s| < \varepsilon$.

If (s_n) converges to s, then s is called the ***limit*** of the sequence (s_n), and we write $\lim\limits_{n\to\infty} s_n = s$, $\lim s_n = s$, or $s_n \to s$. If a sequence does not converge to a real number, it is said to ***diverge***.

Definition 45 (Subsequence)**.** Let $(s_n)_{n=1}^{\infty}$ be a sequence and let $(n_k)_{k=1}^{\infty}$ be any sequence of natural numbers such that $n_1 < n_2 < ....$ The sequence $(s_{n_k})_{k=1}^{\infty}$ is called a ***subsequence*** of $(s_n)_{n=1}^{\infty}$.

Definition 46 (Limit Superior and Limit Inferior)**.** Let (s_n) be a bounded sequence. A ***subsequential limit*** of (s_n) is any real number that is the limit of some subsequence of (s_n). If S is the set of all subsequential limits of (s_n), then we define the ***limit superior*** (or ***upper limit***) of (s_n) to be

$$\limsup s_n = \sup S.$$

Similarly, we define the ***limit inferior*** (or ***lower limit***) of (s_n) to be

$$\lim\inf s_n = \inf S.$$

Definition 47 (Bounded Sequence)**.** A sequence (s_n) is said to be ***bounded*** if the range $\{s_n : n \in \mathbb{N}\}$ is a bounded set, that is, if there exists an $M \geq 0$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$

Every convergent sequence is bounded.

If a sequence converges, its limit is unique.

Every bounded sequence has a convergent subsequence.

3.5 Limit Theorem

Definition 48 (Limit Theorems)**.**

1. $\lim(s_n + t_n) = s + t$
2. $\lim(ks_n) = ks$ and $\lim(k + s_n) = k + s$, for any $k \in \mathbb{R}$
3. $\lim(s_n t_n) = st$
4. $\lim(s_n/t_n) = s/t$, provided that $t_n \neq 0$ for all n and $t \neq 0$

Definition 49 (Lesser Convergence)**.** Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$, and $\lim t_n = t$. If $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $s \leq t$.

Corollary. If (t_n) converges to t and $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$.

Definition 50 (Ratio Convergence)**.** Suppose that (s_n) is a sequence of positive terms and that the sequence of rations (s_{n+1}/s_n) converges to L. If $L < 1$, then $\lim s_n = 0$

Definition 51 (Divergence)**.** A sequence (s_n) is said to ***diverge to*** $+\infty$, and we write $\lim s_n = +\infty$ provided that

$$\text{for every } M \in \mathbb{R} \text{ there exists a natural number } N \text{ such that } n \geq N \text{ implies that } s_n > M.$$

A sequence (s_n) is said to ***diverge to*** $-\infty$, and we write $\lim s_n = +\infty$ provided that

$$\text{for every } M \in \mathbb{R} \text{ there exists a natural number } N \text{ such that } n \geq N \text{ implies that } s_n < M.$$

Definition 52 (Greater Divergence)**.** Suppose that (s_n) and (t_n) are sequences such that $s_n \leq t_n$ for all $n \in \mathbb{N}$.

1. If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.
2. If $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

Definition 53 (Inverse of Divergence)**.** Let (s_n) be a sequence of positive numbers. Then $\lim s_n = +\infty \iff \lim(1/s_n) = 0$.

3.6 Monotone Sequences and Cauchy Sequences

Definition 54 (Monotone Sequences)**.** A sequence (s_n) of real numbers is ***increasing*** if $s_n \leq s_{s_n+1}$ for all $n \in \mathbb{N}$ and is ***decreasing*** if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. A sequence is ***monotone*** if it is increasing or decreasing.

Definition 55 (Monotone Convergence Theorem)**.** A monotone sequence is convergent $\iff$ it is bounded.

Definition 56 (Cauchy Sequence)**.** If, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then $|s_n - s_m| < \varepsilon$.

Every convergent sequence is ***Cauchy***.

If (s_n) is a ***Cauchy*** sequence, then (s_n) converges.

Proof. Given any $\varepsilon > 0$, choose N such that $|s_n - s| < \frac{\epsilon}{2}$ if $n \geq N$ (which is possible to do since $s_n \to s$). Then $|s_n - s_m| = |s_n - s + s - s_m|$ because adding and subtracting by the limit is the same as doing nothing, and, by the triangle inequality, $|s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. ■

4 Final

4.1 Limits of Functions

Definition 57 (Limit of f at c)**.** Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that a real number L is a limit of f at c, if

for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L)| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$.

Theorem 11. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L \iff$ for each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.

Theorem 12. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L \iff$ for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequences $(f(s_n))$ converges to L.

Theorem 13. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then the following are equivalent:

1. f does not have a limit at c.
2. There exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c, but $(f(s_n))$ is not convergent in $\mathbb{R}$.

4.2 Continuity

Definition 58 (Continuous)**.** Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that f is continuous at c, if

$$\text{for each } \varepsilon > 0 \text{ there exists a } \delta > 0 \text{ such that } |f(x) - L)| < \varepsilon \text{ whenever } x \in D \text{ and } |x - c| < \delta.$$

If f is continuous at each point of a subset S of D, then f is said to be **continuous on** S. If f is continuous on its domain D, then f is said to be a **continuous function**.

Theorem 14. Let $f : D \to \mathbb{R}$ and let $c \in D$. Then the following three conditions are equivalent:

1. f is continuous at c.
2. If (x_n) is any sequence in D such that (x_n) converges to c, then $\lim_{n\to\infty} f(x) = f(c)$.
3. For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$.
4. If c is an accumulation point of D, then f has a limit at c and $\lim_{x\to c} f(x) = f(c)$.

Theorem 15. Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at $c \iff$ there exists a sequence (x_n) in D such that (x_n) converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.

Theorem 16. Let $f : D \to \mathbb{R}$ and let $c \in D$. Suppose that f and g are continuous at c. Then

1. $f + g$ and fg are continuous at c, and
2. f/g is continuous at c if $g(c) \neq 0$

Theorem 17. Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at a point $c \in D$ and g is continuous at $f(c)$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at c.

Theorem 18. A function $f : D \to \mathbb{R}$ is continuous on $D \iff$ for every open set $G \subseteq \mathbb{R}$ there exists an open set $H \subseteq \mathbb{R}$ such that $H \cap D = f^{-1}(G)$.

Corollary. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous $\iff f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$.

4.3 Properties of Continuous Functions

Definition 59 (Bounded)**.** A function $f : D \to \mathbb{R}$ is said to be bounded if its range $f(D)$ is a bounded subset of $\mathbb{R}$. That is, f is bounded if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$

Theorem 19. Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Corollary. Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then f assumes minimum and maximum values on D. That is, there exist points x_1 and x_2 in D such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in D$.

Lemma. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(c) = 0$.

Definition 60 (Intermediate Value Theorem)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f has the intermediate value property on $[a, b]$. That is, if k is any value between $f(a)$ and $f(b)$ [*i.e.*, $f(a) < k < f(b)$ or $f(b) < k < f(a)$], then there exists $c \in (a, b)$ such that $f(c) = k$.

Theorem 20. Let I be a compact interval and suppose that $f : I \to \mathbb{R}$ is a continuous function. Then the set $f(I)$ is a compact interval.

4.4 Uniform Continuity

Definition 61 (Uniform Continuity)**.** Let $f : D \to \mathbb{R}$. We say that f is **uniformly continuous** on D if

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in D$.

A function is continuous at a point, but uniform continuity is a property of a that applies to a function *on a set*. We never speak of a function being uniformly continuous at a point.

4.5 Differentiation

Definition 62 (Derivative)**.** Let f be a real-valued function defined on an interval I containing the point c. (We allow the possibility that c is an endpoint of I.) We say that f is differentiable at c (or has a derivative at c) if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. We denote the derivative of f at c by $f'(c)$ so that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

whenever the limit exists and is finite. If the function f is differentiable at each point of the set $S \subseteq I$, then f is said to be differentiable on S, and the function $f' : S \to \mathbb{R}$ is called the derivative of f on S.

Theorem 21. Let I be an interval containing the point c and suppose that $f : I \to \mathbb{R}$. Then f is differentiable at $c \iff$ for every sequence (x_n) in I that converges to c with $x_n \neq c$ for all n, the sequence

$$\left(\frac{f(x_n) - f(c)}{x_n - c}\right)$$

converges. Furthermore, if f is differentiable at c, then the sequence of quotients above will converge to $f'(c)$.

Theorem 22. If $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$, then f is continuous at c.

Theorem 23. Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then

1. If $k \in \mathbb{R}$, then the function kf is differentiable at c and

$$(kf)'(c) = k \cdot f'(c).$$

2. The function $f + g$ is differentiable at c and

$$(f + g)'(c) = f'(c) + g'(c)$$

3. (Product Rule) The function fg is differentiable at c and

$$(fg)'(c) = f(c)g'(c) + g(c)f'(c)$$

4. (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$$

Theorem 24. (Chain Rule) Let I and J be intervals in $\mathbb{R}$, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where $f(I) \subseteq J$, and let $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

4.6 The Mean Value Theorem

Theorem 25. If f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

Theorem 26. (Rolle's Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and such that $f(a) = f(b)$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Theorem 27. (Mean Value Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 28. Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x)=0$ for all $x\in(a,b)$, then f is constant on $[a,b]$.

Corollary. Let f and g be continuous an $[a,b]$ and differentiable on (a,b). Suppose that $f'(x)=g'(x)$ for all $x\in(a,b)$. Then there exists a constant C such that $f=g+C$ on $[a,b]$.

Theorem 29. Let f be differentiable on an interval I. Then

1. if $f'(x)>0$ for all $x\in I$, then f is strictly increasing on I, and
2. if $f'(x)<0$ for all $x\in I$, then f is strictly decreasing on I.

Theorem 30. (Intermediate Value Theorem for Derivatives) Let f be differentiable on $[a,b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then there exists a point $c\in(a,b)$ such that $f'(c)=k$.

Theorem 31. (Inverse Function Theorem) Let f be differentiable on an interval I and $f'(x)\neq 0$ for all $x\in I$. Then f is injective, f^{-1} is differentiable on $f(I)$, and

$$(f^{-1})'(y)=\frac{1}{f'(x)},$$

where $y=f(x)$

4.7 Taylor's Theorem

Theorem 32. (Taylor's Theorem) Let f and its first n derivatives be continuous on $[a,b]$ and differentiable on (a,b), and let $x_0\in[a,b]$. Then for each $x\in[a,b]$ with $x\neq x_0$ there exists a point c between x and x_0 such that

$$\begin{aligned} f(x) = f(x_0) + f'(x)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + ... \\ + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

4.8 Integration

Definition 63 (Partition)**.** Let $[a,b]$ be an interval in $\mathbb{R}$. A **partition** P of $[a,b]$ is a finite set of points $\{x_0,x_1,...,x_n\}$ such that

$$a=x_0<x_1<...<x_n=b.$$

If P and Q are two partitions of $[a,b]$ with $P\subseteq Q$, then Q is called a **refinement** of P.

Definition 64 (Upper and Lower Sum)**.** Suppose that f is a bounded function defined on $[a,b]$ and that $P=\{x_0,...,x_n\}$ is a partition of $[a,b]$. For each $i=1,...,n$ we let

$$M_i(f)=\sup\{f(x):x\in[x_{i-1},x_i]\}$$

and

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

When only one function is under consideration, we may abbreviate thes to M_i and m_i, respectively. Letting $\Delta x_i = x_i - x_{i-1}(i = 1, ..., n)$, we define the **upper sum** of f with respect to P to be

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

and the **lower sum** of f with respect to P to be

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Since we are assuming that f is a bounded function on $[a, b]$, there exist numbers m and M such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Thus for any partition P of [a,b] we have

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a).$$

This implies that the uuper and lower sums for f form a bounded set, and it guarantees the existence of the following upper and lower integrals of f.

Remark (History)**.** (Sometimes $U(f, P)$ and $L(f, P)$ are called the upper and lower Darboux sums in honor of Gaston Darboux (1842–1917), who first developed this approach to the Riemann integral.)

Definition 65 (Upper and Lower Integral)**.** Let f be a bounded function defined on $[a, b]$. Then

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

is called the **upper integral** of f on $[a, b]$. Similarly,

$$L(f) = \subset \{L(f, P) : P \text{ is a partition of } [a, b]\}$$

is called the **lower integrals** of f on $[a, b]$, and we denote their common value by $\int_a^b f$ or by $\int_a^b f(x)dx$. That is, if L(f) = U(f), then

$$\int_a^b f = \int_a^b f(x)dx = L(f) = U(f)$$

is the **Riemann integral** of f on $[a, b]$.

When convenient, refer to the function f as being **integrable** on $[a, b]$ and call $\int_a^b f$ the **integral** of f on $[a, b]$.

Theorem 33. Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and Q is a refinement of P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Theorem 34. Let f be a bounded function on $[a, b]$. Then $L(f) \leq U(f)$.

Theorem 35. Let f be a bounded function on $[a, b]$. Then f is integrable $\iff$ for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Theorem 36. (The Fundamental Theorem of Calculus I) Let f be integrable on $[a, b]$. For each $x \in [a, b]$, let

$$F(x) = \int_a^x f(t)dt.$$

Then F is uniformly continuous on $[a, b]$. Furthermore, if f is continuous on $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

4.9 Infinite Series