Math3283W study guiding

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May 3, 2017

Contents

1 Midterm one

1.1 Vocabulary

1.1.1 Section 1.1

statement A sentence classified as something either true or false is a *statement*.

sentential connectives *not*, *and*, *or*, *if... then*, *if and only if*.

- **conjunction** $p \wedge q$
- **disjunction** $p \vee q$
- **implication/conditional** $p \Rightarrow q$
  - **antecedant** p statement
  - **consequent** q statement
- **biconditional** $p \Leftrightarrow q$
- **negation** $\neg p$ represents the logical opposite of p.

tautology A statement which is true in all cases.

- examples:

$$\neg(p \wedge q) \Leftrightarrow [(\neg p) \vee (\neg q))]$$
$$\neg(p \vee q) \Leftrightarrow [(\neg p) \wedge (\neg q))]$$
$$\neg(p \Rightarrow q) \Leftrightarrow [p \vee (\neg q)]$$

1.1.2 Section 1.2

universal quantifier $\forall x, p(x)$

existential quantifier $\exists x \ni p(x)$

1.1.3 Section 1.3

deductive reasoning Applying a general principle to a particular case.

$p \Rightarrow q$ **as a theorem** When an implication is identified as a theorem, it is customary to refer to p as the **hypothesis** and q as the **conclusion**.

converse $p \Rightarrow q$ has the converse $q \Rightarrow p$. Not tautologically equivalent to implication.

inverse $p \Rightarrow q$ has the inverse $(\neg p) \Rightarrow (\neg q)$. Not tautologically equivalent to implication.

contrapositive Tautologically equivalent to the implication.

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

1.1.4 Section 2.1

subset $A \subseteq B$. A is a **subset** of B (or A is **contained** in B). If we want to prove $A \subseteq B$, then we must prove "if $x \in A$, then $x \in B$".

proper subset AB. $\forall a \in A, a \in B$, but $\exists b \in B \ni b \notin A$. That is, all elements of A are in B, but some elements of B are not in A.

equal A set A is equal to a set B provided that $A \subseteq B$ and $A \supseteq B$ (or $A \subseteq B$ and $B \subseteq A$).

closed interval $[a, b]$

open interval (a, b)

half–open (half–closed) interval $[a, b)$, or $(a, b]$.

union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- If $A \cap B = \varnothing$, then A and B are said to be **disjoint**

complement $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

1.1.5 Section 2.2

ordered pairs $(a, b) = \{\{a\}, \{a, b\}\}$

Cartesian product (cross product) $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

relation A **relation** between A and B is any subset $\mathcal{R}$ of $A \times B$. We say that an element a in A is related by $\mathcal{R}$ to an element in b in B if $(a, b) \in \mathcal{R}$. The first set A is referred to as the **domain**, of the relation and denoted dom $\mathcal{R}$. If $B = A$, then we speak of a relation $\mathcal{R} \subseteq A \times A$ being a **relation on** A.

equivalence relation A relation $\mathcal{R}$ is an **equivalence relation** if:

1. $x\mathcal{R}x$ (reflexive property)
2. If $x\mathcal{R}y$, then $y\mathcal{R}x$ (symmetric property)
3. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$ (transitive property)

equivalence class An equivalence class (with respect to $\mathcal{R}$) of $x \in S$ is defined to be the set

$$E_x = \{y \in S \mid y\mathcal{R}x\}$$

partition Also, we see that an equivalence relation $\mathcal{R}$ on a set S breaks S into **disjoint** pieces in a natural way. A partition of a set S is a collection $\mathscr{P}$ of nonempty subsets of S such that

1. Each $x \in S$ belongs to some subset $A \in \mathscr{P}$.
2. For all $A, B \in \mathscr{P}$, if $A \neq B$, then $A \cap B = \varnothing$.

A member of $\mathscr{P}$ is called a **piece** of the partition.

1.1.6 Section 2.3

identity function A function defined on a set A that maps each element in A onto itself is called the **identity function** on A, and is denoted $f^{-1} \circ f = i_A$. Furthermore, if $f(x) = y$, then $x = f^{-1}(y)$, so that

$$f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y.$$

Thus, $f \circ f^{-1} = i_B$.

1.2 Theorem

1.2.1 Section 1.4

- This example shows a *direct proof.*
  - For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$1 - \delta < x < 1 + \delta \text{ implies that } 5 - \epsilon < 2x + 3 < 5 + \epsilon.$$

  1. Begin by letting ϵ be an arbitrary positive number, i.e. $\epsilon > 0$. We need to use this ϵ to find a positive δ with the property that
  $$1 - \delta < x < 1 + \delta \text{ implies that } 5 - \epsilon < 2x + 3 < 5 + \epsilon.$$
  2. Given any $\epsilon > 0$, let $\delta = \epsilon/2$. $\delta > 0$, and whenever
  $$1 - \delta < x < 1 + \delta$$
  we have
  $$1 - \frac{\epsilon}{2} < x < 1 + \frac{\epsilon}{2}$$
  so that
  $$2 - \epsilon < 2x < 2 + \epsilon$$
  and
  $$5 - \epsilon < 2x + 3 < 5 + \epsilon$$
  thus

$$1 - \delta < x < 1 + \delta \text{ implies that } 5 - \epsilon < 2x + 3 < 5 + \epsilon.$$

- This example shows a *indirect proof.*
  - Let f be an integrable function, so that

If $\int_0^1 f(x)dx \neq 0$, then there exists a point x in the interval $[0, 1]$ such that $f(x) \neq 0$.

  1. Symbolically, we have $p \Rightarrow q$, where

$$p : \int_0^1 f(x)dx \neq 0,$$
$$q : \exists x \text{ in } [0,1] \ni f(x) \neq 0.$$

The contrapositive implication, $\neg q \Rightarrow \neg p$, can be written

If for every x in $[0,1]$, $f(x) = 0$, then $\int_0^1 f(x)dx = 0$.

2. This is obviously true. The integral of all 0 integrands is obviously 0.

- This example shows a *proof by contradiction.*
  - Let x be a real number.

If $x > 0$, then $1/x > 0$.

1. Symbolically, we have $p \Rightarrow q$, where

$$p : x > 0$$
$$q : 1/x > 0$$

so that, $(p \Rightarrow q) \Leftrightarrow ((p \wedge \neg q) \Rightarrow c)$, where c represents a contradiction.

2. Begin by supposing $x > 0$ and $1/x \leq 0$. Since $x > 0$, we can multiply both sides of the inequality $1/x \leq 0$ by x to obtain

$$(x)\left(\frac{1}{x}\right) \leq (x)(0)$$

But $(x)(1/x) = 1$ and $(x)(0) = 0$, so we have $1 \leq 0$, a contradiction to the (presumably known) fact that $1 > 0$. Having show that $p \wedge \neg q$ leads to a contradiction, we conclude that $p \Rightarrow q$.

- This example shows a *proof with absolute value.*
  - If x is a real number, then $x \leq |x|$

$$s : x \text{ is a real number}$$
$$r : x \leq |x|$$

The definition of statement r can be rewritten as:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

1. Since the definition is divided into two parts, it is natural to divide our proof into two cases. Thus statement s is replaced by the equivalent disjunction $p \vee q$, where

$$p : x \geq 0 \text{ and } q : x < 0.$$

2. The case to prove now is $(p \vee q) \Rightarrow r$, which is the same as $(p \Rightarrow r) \wedge (q \Rightarrow r)$.
3. If $x \geq 0$, then $x = |x|$. If $x < 0$, then $-x > 0$, so that $x < 0 < -x = |x|$. Or, $x \leq |x|$. Thus, $(p \Rightarrow r) \wedge (q \Rightarrow r)$. Hence, if x is a real number, then $x \leq |x|$

1.2.2 Section 2.3

- Let $f : A \longrightarrow B$. Then

  1. $f^{-1} : B \longrightarrow A$ is bijective.
  2. $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

1.3 Practice test

- question (1, d)

  – Define $A_n = (3, 4+\frac{1}{n})$, an open interval in $\mathbb{R}$, for each natural number n. Without writing a proof, determine
  $$\bigcap_{n=1}^{\infty} A_n$$
  .

answer: When $n = 1$, $A_1 = (3, 4 + 1) = (3, 5)$. When $n = 2$, the $A_1 = (3, 4 + \frac{1}{2}) = (3, 4.5)$. Here, consider $f(n) = 4 + \frac{2}{n}$, the function for the upper bound of A_n for each n. Here, the forward difference quotient $\Delta f(1) = f(2) - f(1) = 4.5 - 5 < 0$. Considering the function $f(n)$, this difference quotient will always be negative. Therefore, the highest upper bound over all the parts of the union will be $f(1) = 5$. Since the lower bound function is constant, the lowest lower bound over all the parts of the union will be 3. Therefore, $A_n \subseteq (3, 5)$, and $A_n = \{(3, 4 + \frac{1}{n}) \mid n \in \mathbb{N}\}$.

- question (2, a) For all x, there exists y such that for all z, if $y < x$ then $z < y$.

  1. write the negation:
  2. Determine whether the original statement is true or false. Write "true" or "false", and then justify your answer by proving the original statment or the negation that you wrote in (a):

answer 1: There exists x such that for all y, there exists z such that $y < x$ and $z \geq y$.
answer 2: Let x be some constant x_0, such that $y < x_0$ for all y. Since this y can be any of all the numbers in $\mathbb{R}$, say $y \geq x_0$, the statement $y < x_0$ is not true. Given a conjunction of a false statement and any other statement, the conjunction is false. Since this statement is the negation of the original statement and false, the original statement must be true.

- question (3,a)

  1. Suppose that $A = \{1, 2, 3\}$, $B = \{4, 5\}$, and $C = \{6, 7, 8\}$. Let R be the relation on $A \times C$ given by $\{(1, 7), (3, 6), (3, 7)\}$ and S by the relation on $B \times C$ given by $\{(4, 7), (4, 8), (5, 6)\}$. Find $S^{-1} \circ R$.
  2. Given R a relation on $A \times B$ and S a relation on $B \times C$, prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

3. Suppose that A and B are non-empty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.

answer 1: Since S is defined, and $S^{-1} = \{(c, b) \in C \times B \mid (b, c) \in S\}$, $S^{-1} = \{(7, 4), (8, 4), (6, 5)\}$. Then, $S^{-1} \circ R = \{(7, 4), (8, 4), (6, 5)\} \circ \{(1, 7), (3, 6), (3, 7)\}$. This can be simplified to $S^{-1} \circ R = \{(1, 4), (3, 5), (3, 4)\}$.
answer 2: Consider $a \in A$, $b \in B$, and $c \in C$. $S \circ R$ is a relation such that $(a, c) \in S \circ R \subseteq A \times C$. Thus, $(S \circ R)^{-1} = \{(c, a) \in C \times A \mid (a, c) \in S \circ R\}$.

Given $(a, b) \in R$, by definition of Inverse $R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}$. Given $(b, c) \in S$, by definition of Inverse $S^{-1} = \{(c, b) \in C \times B \mid (b, c) \in S\}$. Then, $R^{-1} \circ S^{-1} = \{(c, a) \in C \times A \mid \exists b \in B \ni (a, b) \in R \wedge (b, c) \in S\}$. However, this is the same as $R^{-1} \circ S^{-1} = \{(c, a) \in C \times A \mid (a, c) \in S \circ R\}$, by the definition of Cartesian Product.

Thus, since $(S \circ R)^{-1}$ and $R^{-1} \circ S^{-1}$ have the same definitions, it must be that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

answer 3: ($\Leftarrow$) Let $A = B$. It must be that $A \times B = B \times A$, as having both leads to $B \times B = B \times B$ or $A \times A = A \times A$ which are true.
($\Rightarrow$) Let $A \times B = B \times A$, and remember A and B are non-empty sets. By definition, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. Just as well, by definition, $B \times A = \{(b, a) \mid a \in A \wedge b \in B\}$. Having this implies that for every $(a, b) \in A \times B$ and the corresponding $(b, a) \in B \times A$, $(a, b) = (b, a)$. Thus, $A = B$.
In sum, this means $A \times B = B \times A$ if and only if $A = B$.

- question (4, a) Prove which one is true and which one is false.
  - For all subsets S and T of a universal set U, we have $U \setminus (S \setminus T) \subseteq (U \setminus S) \cup T$.
  - For all subsets S and T of a universal set U, we have $U \setminus (S \setminus T) \subseteq (U \setminus S) \cap T$.

answer: The first statement to consider, $U \setminus S$, has definition $U \setminus S = \{u \in U \mid u \notin S\}$. Then $(U \setminus S) \cup T = \{u \in U \mid u \notin S\} \cup \{u \in T\}$. Which is to say that $u \in U$, $u \notin S$, and $u \in T$, so that $(U \setminus S) \cup T = \{u \mid u \in U \wedge u \notin S \wedge u \in T\}$.
Consider the next statement A, in particular $U \setminus (S \setminus T)$. By definition $S \setminus T = \{u \in S \mid u \notin T\}$. By definition, again, $U \setminus (S \setminus T) = \{u \in U \mid u \notin (S \setminus T)\}$, that is $U \setminus (S \setminus T) = \{u \in U \mid u \notin S \wedge u \in T)\}$. Which is to say that $u \in U$, $u \notin S$, and $u \in T$, so that $U \setminus (S \setminus T) = \{u \mid u \in U \wedge u \notin S \wedge u \in T\}$.
Thus, statement A, $U \setminus (S \setminus T) \subseteq (U \setminus S) \cup T$, is the true statement.

- question (4, b) Give a counterexample to the false statement, where $U = \mathbb{R}$, and S and T are open interval in $\mathbb{R}$.

answer: Let subset $S = \{x \in \mathbb{R} \mid x > 1\}$, and $T = \{x \in \mathbb{R} \mid x < 1\}$. S and T are disjoint, meaning $U \setminus (S \setminus T) = U \setminus S$. So, we have $U \setminus S \subseteq (U \setminus S) \cap T$. By definition of set intersection, $(U \setminus S) \cap T = \{x \in \mathbb{R} \mid x \in (U \setminus S) \wedge x \in T\}$, where $U \setminus S = \{x \in \mathbb{R} \mid x \in U \wedge x \notin S\}$. Here, we have $U \setminus S = \{x \in \mathbb{R} \mid x \leq 1\}$, and $(U \setminus S) \cap T = \{x \in \mathbb{R} \mid x \leq 1 \wedge x < 1\}$ which is $(U \setminus S) \cap T = \{x \in \mathbb{R} \mid x < 1\}$. Just as well, it can be seen that $U \setminus (S \setminus T) \supseteq (U \setminus S) \cap T$, which is sort of the opposite of the statement B.

- question (5) Prove that, for all integers p and q, if pq is an even integer, then p is an even integer or q is an even integer.

answer: Consider the contrapositive of the predicate in the statement: For all integers p and q, if p is an odd integer and q is an odd integer, then pq is an odd integer. Let $p = 2k-1$ and $q = 2k-1$, for integers k. Then, it is straightforward that $pq = (2k-1)(2k-1) = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1 = 2j + 1$ for some integer $j = 2k^2 - 2k$ and is therefore odd. Thus, "for all integers p and q, if p and q are odd integers, then pq is an odd integer" being true means that its contrapositive is true, i.e. "for all integers p and q, if pq is an even integer, then p is an even integer or q is an even integer" is true.

2 Midterm two

2.1 Functions

Definition 1 (Function)**.** Let A and B be sets. A **function** from A to B is nonempty relation $f \subseteq A \times B$ that satisfies the following two conditions:

1. *Existence*: For all a in A, there exists a b in B such that $(a, b) \in f$.
2. *Uniqueness*: If $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

That is, given any element a in A, there is one and only one element b in B such that $(a, b) \in f$. Set A is called the domain of f and is denoted by dom f. Set B is referred to as the codomain of f. We may write $f : A \to B$ to indicate f has domain A and codomain B. The range of f, denoted rng f, is the set of all second elements of members of f. That is,

$$\{b \in B : \exists a \in A \ni (a, b) \in f\}.$$

Definition 2 (Surjective)**.** A function $f : A \to B$ is called **surjective** (or is said to map A **onto** B) if $B = \text{rng } f$. A surjective function is also referred to as a **surjection**

Definition 3 (Injective)**.** A function $f : A \to B$ is called **injective** (or **one-to-one**) if, for all a and a' in A, $f(a) = f(a')$. An injective function is also referred to as an **injection**.

Definition 4 (Bijective)**.** A function $f : A \to B$ is called **bijective** or a **bijection** if it is both surjective and injective.

Definition 5 (Composition)**.** If f and g are functions with $f : A \to B$ and $g : B \to C$, then for any $a \in A$, $f(a) \in B$. But B is the domain of g, so g can be applied to $f(a)$. This yields $g(f(a))$, an element of C. Thus we have established a correspondance between a in A and $g(f(a))$ in C. This correspondance is called the composition function of f and g and is denoted by $g \circ f$ (read "g of f"). It defines a function $g \circ f : A \to C$ given by

$$(g \circ f)(a) = g(f(a)) \text{ for all } a \in A.$$

Definition 6 (Inverse)**.** Let $f : A \to B$ be bijective. The inverse function of f is the function f^{-1} given by

$$f^{-1} = \{(y, x) \in B \times A : (x, y) \in f\}.$$

2.2 Cardinality

Definition 7 (Equinumerous)**.** Two sets S and T are called **equinumerous**, written $S \sim T$, if there exists a bijective function from S onto T.

Definition 8 (Finite or Infinite)**.** A set S is said to be **finite** if $S = \varnothing$ or if there exists $n \in \mathbb{N}$ and a bijection $f : \{1, 2, ..., n\} \to S$. If a set is not finite, it is said to be **infinite**.

Definition 9 (Cardinal number)**.** The **cardinal number** of I_n is n, and if $S \sim I_n$, then we say that S **has *n* elements**. The cardinal number of $\varnothing$ is taken to be 0. If a cardinal number is not finite, it is called **transfinite**.

Theorem 1. Let S be a countable set and let $T \subseteq S$. Then T is countable.

Definition 10 (Power Set)**.** Given any set S, let $\mathscr{P}(S)$ denote the collection of all the subsects of S. The set $\mathscr{P}(S)$ is called the **power set** of S.

Theorem 2. For any set S, we have $|S| < |\mathscr{P}(S)|$.

2.3 The Real Numbers

Axiom. (Well-ordering property of $\mathbb{N}$) If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \leq k$ for all $k \in S$.

Theorem 3. (Mathematical Induction) A technique of mathematical proof. Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$, provided that

1. $P(1)$ is true
2. Whenever $P(k)$ is true, for some number k, then $P(k+1)$ is true.

Proof. (By contradiction) Given statements $P(n)$, $n \in \mathbb{N}$. Show if we have properties 1) and 2), then $P(n)$ holds for all n. Suppose $P(n)$ false for some n. Let $F = \{n \in \mathbb{N} : P(n) \text{ false}\}$. F is non-empty by assumption, so by well-ordering principle it has a least element, say $n_0 \neq 1$. Consider $n_0 - 1$, a natural number, so that $P(n_0 - 1)$ is true, since otherwise n_0 wasn't the smallest element of F. ■

Remark (*Slight Generalization*)**.** For proving $p(n)$ for all $n \geq n_0$, then prove:

1. $p(n_0)$ is true.
2. if $p(k)$ is true for k, then $p(k+1)$ is true with $k \geq n_0$

2.4 Ordered Fields

Axiom (Axioms of an Ordered Field)**.** We begin by assuming the existence of a set R, called the set of real numbers, and two operations "+" and "·", called addition and multiplication, such that the following properties apply :—

A1. For all $x, y \in \mathbb{R}$, $x + y \in \mathbb{R}$ and if $x = w$ and $y = z$, then $x + y = w + z$.

A2. For all $x, y \in \mathbb{R}$, $x + y = y + x$.

A3. For all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.

A4. There is a unique real number 0 such that $x + 0 = x$, for all $x \in \mathbb{R}$.

A5. For each $x \in \mathbb{R}$ there is a unique real number $-x$ such that $x + (-x) = 0$.

M1. For all $x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$ and if $x = w$ and $y = z$, then $x \cdot y = w \cdot z$.

M2. For all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.

M3. For all $x, y, z \in \mathbb{R}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

M4. There is a unique real number 1 such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

M5. For each $x, y \in \mathbb{R}$ with $x \neq 0$, there is a unique real number $1/x$ such that $x \cdot (1/x) = 1$. We also write x^{-1} or $\frac{1}{x}$ in place of $1/x$.

DL. For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Remark. These first 11 axioms are called the field axioms because they describe a system know as a field in the study of abstract algebra. Axioms A2 and M2 are called the ***commutative laws*** and axioms A3 and M3 are the ***associative laws***. Axiom DL is the ***distributive law*** that shows how addition and multiplication relate to each other. Because of A1 and M1, we can think of addition and multiplication as functions that map $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. When writing multiplication we often omit the raised dot and write xy instead of $x \cdot y$.

In addition to the field axioms, the real numbers also satisfy four order axioms.

O1. For all $x, y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, or $x < y$ holds (***trichotomy law***).

O2. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

O3. For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.

O4. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $xz < yz$.

Definition 11 (Ordering of Rational Functions)**.** If p/q and f/g are rational functions, then we say that

$$\begin{aligned} \frac{p}{q} > \frac{f}{g} &\iff \frac{p}{q} - \frac{f}{g} > 0 \\ &\iff \frac{pg - fq}{qg} > 0 \end{aligned}$$

Definition 12 (Absolute Value)**.** If $x \in \mathbb{R}$, then the absolute value of x, denoted by $|x|$, is defined by:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then

(a) $|x| \geq 0$,

(b) $|x| \leq a \iff -a \leq x \leq a$,

(c) $|xy| = |x| \cdot |y|$,

(d) $|x + y| \leq |x| + |y|$.

Remark. Part (d) is referred to as the ***triangle inequality***, and has other forms. For example, letting $x = a - c$ and $y = c - b$, we obtain

$$|a - b| \leq |a - c| + |c - b| .$$

Theorem 4. Let $x, y \in \mathbb{R}$ such that $x \leq y + \varepsilon$ for every $\varepsilon > 0$. Then $x \leq y$.

2.5 Completeness Axiom

Definition 13 (Irrational)**.** Let p be a prime number. Then $\sqrt{p}$ is not a rational number.

Definition 14 (Bounds)**.** Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s$ for all $s \in S$, then m is called an ***upper bound*** of S, and we say that S is bounded above. If $m \leq s$ for all $s \in S$, then m is a ***lower bound*** of S and S is bounded below. The set S is said to be ***bounded*** if it is bounded above and bounded below.

Definition 15 (Maximum and Minimum)**.** If an upper bound m of S is a member of S, then m is called the **maximum** (or largest element) of S, and we write

$$m = \max S.$$

Similarly, if a lower bound of S is a member of S, then it is called the **minimum** (or least element) of S, denoted by

$$m = \min S.$$

Definition 16 (Supremum and Infimum)**.** Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the least upper bound of S is called its ***supremum*** and is denoted by $\sup S$. Thus $m = \sup S$ iff

1. $m \geq s$, for all $s \in S$, and
2. if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

If S is bounded below, then the greatest lower bound of S is called its **infimum** and is denoted by $\inf S$, so $m = \inf S$.

Definition 17 (Completeness Axiom)**.** Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem 5 (Archimedean Property of $\mathbb{R}$)**.** The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Theorem 6 (Alternative Archimedean Properties)**.** Each of the following is equivalent to the Archimedean Property.

1. For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$.
2. For each $x > 0$ and for each $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$
3. For each $x > 0$, there exists an $n \in \mathbb{N}$ such that $0 < 1/n < x$.

Theorem 7 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** If x and y are real numbers with $x < y$, then there exists a rational number r such that $x < r < y$.

Theorem 8. If x and y are real numbers with $x < y$, then there exists an irrational number w such that $x < w < y$.

2.6 Examples

- Example: For which n is $n! > n^n$?

  Expect $n! > 2^n$ if $n \geq 4$. (i.e. for all $n \geq 4$).

  Base case: True if $n_0 = 4$ since $24 > 16$. Suppose $k! > 2^k$.

  Inductive step: Show: $(k+1)! > 2^{k+1}$. Easy:

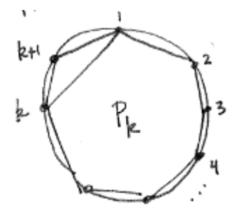
  $$k! > 2^k \Rightarrow (k+1)! > 2^k(k+1) > 2^{k+1}$$

  since $k + 1 > 2$ if $k \geq 4$.

- Example: For $n \geq 3$, if we connect n points on circle w with straight line segments, the interior angles of the resulting polygon add up to $(n-2) \cdot 180$.

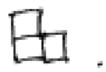
  Base case: $n = 3$. Angles of triangle add up to $180°$.

  Inductive Step: Suppose true for k. Prove true for k+1. By hypothesis, P_k has interior angles $(k-2) \cdot 180\,\mathrm{deg}$. Triangle P_{k+1} has interior angles defined to be the sum of the p_k angles and the triangle with vertices $k, k+1, 1$. That is $(k-2) \cdot 180\,\mathrm{deg} + 180\,\mathrm{deg} = ((k+1)-2) \cdot 180\,\mathrm{deg}$. ✓.

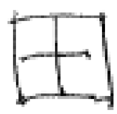


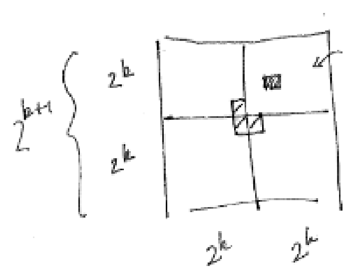
Imagine, any number of edges k, where the first edge is named 1. One could simply add another $k+1$ edge to the list.

- Example: Prove that any $2^n \times 2^n$ grid of squares with any one square removed can be tiled with L-shaped tiles.

 $\Leftarrow$ *L-shaped tiles*

Removing any box results in the remaining boxes being an L-shaped tile.

 $\Leftarrow$ *box*



A large block can be covered by L-shapes by hypothesis. What about other 3 blocks?

Inductive hypothesis doesn't apply to grid $2^k \times 2^k$ without removing a square. Meaning without removing one atomic box.

Solution: consider removing a single L instead, taking away a box from three quadrants. It is an equivalent procedure.

3 Midterm three

3.1 Boundaries

Definition 18 (Interior Point and Boundary Point)**.** Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an ***interior point*** of S if there exists a neighborhood N of x such that $N \subseteq S$. If for every neighborhood N of x, $N \cap S \neq \varnothing$ and $N \cap (\mathbb{R} \setminus S) \neq \varnothing$, then x is called a ***boundary point*** of S. The set of all interior points of S is denoted by int S, and the set of all boundary points of S is denoted by bd S.

Definition 19 (Neighborhood)**.** Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A ***neightborhood*** of x (or an ε***-neightborhood*** of x) is a set of the form $N(x;\varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\}$.

Remark. The professor uses the notation:

$$N_\varepsilon(x) = \{y \in \mathbb{R} : |x - y| < \varepsilon\},$$

which is probably nicer.

Definition 20 (Deleted Neighborhood)**.** Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A ***deleted neighborhood*** of x is a set of the form $N^*(x;\varepsilon) = \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\}$. Clearly, $N^*(x;\varepsilon) = N(x;\varepsilon) \setminus \{x\}$

Remark. The professor uses the notation:

$$N^*_\varepsilon(x) = \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\},$$

which is probably nicer.

Definition 21 (Open and Closed Sets)**.** Let $S \subseteq \mathbb{R}$. If bd $S \subseteq S$, then S is said to be ***closed***. If bd $S \subseteq \mathbb{R} \setminus S$, then S is said to be ***open***.

Theorem 9.

- A set S is open $\iff$ $S = \text{int } S$. Equivalently, S is open $\iff$ every point in S is an interior point of S.
- A set S is closed $\iff$ its complement $\mathbb{R} \setminus S$ is open.
- The union of any collection of open sets is an open set.
- The intersection of any finite collection of open sets is an open set.

3.2 Topology of the Real Numbers

Every bounded sequence has a convergent subsequence.

- If $\{s_n\}$ bounded, then
    1. for every ε, $\exists N \in \mathbb{N} \ni s_n < m + \varepsilon$ when $n \geq N$. (Else there are infinitely many $s_n \geq m + \varepsilon$, so there can't be a lim sup.)
    2. for every $\varepsilon > 0$, $\forall i \in \mathbb{N}, \exists k > i$ with $s_k > m - \varepsilon$. (There are infinitely many $s_k \in (m - \varepsilon, m + \varepsilon)$, else $m - \varepsilon$ is upper bound for all limits of subsequences.)

Definition 22 (Accumulation Points)**.** Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an **accumulation point** of S if every deleted neighborhood of x contains a point of S. That is, for every $\varepsilon > 0$, $N^*(x,\varepsilon) \cup S \neq \varnothing$. The set of all accumulation points of S is denoted by S'. If $x \in S$ and $x \notin S'$, then x is called an **isolated point** of S.

Definition 23 (Closure)**.** Let $S \subseteq \mathbb{R}$. Then the closure of S, denoted cl S, is defined by

$$\text{cl } S = S \cup S',$$

where S' is the set of all accumulation points of S.

Also,

$$\text{cl } S = S \cup \text{bd } S.$$

3.3 Compact Sets

Definition 24 (Compact, Open Cover, and Subcover)**.** A set S is said to be ***compact*** if whenever it is contained in the union of a family $\mathscr{F}$ of open sets, it is contained in the union of some finite number of the sets in $\mathscr{F}$. If $\mathscr{F}$ is a family of open sets whose union contains S, then $\mathscr{F}$ is called an ***open cover*** of S. If $\mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{G}$ is also an open cover of S, then $\mathscr{G}$ is called a ***subcover*** of S.

Corollary. S is compact $\overset{Heine-Borel}{\Longleftrightarrow}$ S is closed and bounded $\iff$ every infinite subset of S has an accumulation point in S.

S is a nonempty closed bounded subset of $\mathbb{R} \Rightarrow S$ has a maximum and a minimum.

Definition 25 (Heine–Borel)**.** A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.

Definition 26 (Bolzano–Weierstrass)**.** If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.

3.4 Sequences

Definition 27 (Sequence)**.** A sequence S is a function whose domain is the set $\mathbb{N}$ of natural numbers. Denoted by its value of n at s_n instead of $S(n)$ or by listing its values $(s_1, s_2, s_3, ...)$. s_n is the n^{th} term of the sequence.

Definition 28 (Convergence, Divergence, Limit)**.** A sequence (s_n) is said to ***converge*** to the real number s provided that

for every $\varepsilon > 0$ there exists a natural number N such that for all $n \in \mathbb{N}$, $n \geq N$ implies that $|s_n - s| < \varepsilon$.

If (s_n) converges to s, then s is called the ***limit*** of the sequence (s_n), and we write $\lim\limits_{n\to\infty} s_n = s$, $\lim s_n = s$, or $s_n \to s$. If a sequence does not converge to a real number, it is said to ***diverge***.

Definition 29 (Subsequence)**.** Let $(s_n)_{n=1}^{\infty}$ be a sequence and let $(n_k)_{k=1}^{\infty}$ be any sequence of natural numbers such that $n_1 < n_2 < ....$ The sequence $(s_{n_k})_{k=1}^{\infty}$ is called a ***subsequence*** of $(s_n)_{n=1}^{\infty}$.

Definition 30 (Limit Superior and Limit Inferior)**.** Let (s_n) be a bounded sequence. A ***subsequential limit*** of (s_n) is any real number that is the limit of some subsequence of (s_n). If S is the set of all subsequential limits of (s_n), then we define the ***limit superior*** (or ***upper limit***) of (s_n) to be

$$\lim\sup s_n = \sup S.$$

Similarly, we define the ***limit inferior*** (or ***lower limit***) of (s_n) to be

$$\lim\inf s_n = \inf S.$$

Definition 31 (Bounded Sequence)**.** A sequence (s_n) is said to be ***bounded*** if the range $\{s_n : n \in \mathbb{N}\}$ is a bounded set, that is, if there exists an $M \geq 0$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$

Every convergent sequence is bounded.
If a sequence converges, its limit is unique.
Every bounded sequence has a convergent subsequence.

3.5 Limit Theorem

Definition 32 (Limit Theorems)**.**

1. $\lim(s_n + t_n) = s + t$
2. $\lim(ks_n) = ks$ and $\lim(k + s_n) = k + s$, for any $k \in \mathbb{R}$
3. $\lim(s_n t_n) = st$
4. $\lim(s_n/t_n) = s/t$, provided that $t_n \neq 0$ for all n and $t \neq 0$

Definition 33 (Lesser Convergence)**.** Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$, and $\lim t_n = t$. If $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $s \leq t$.

Corollary. If (t_n) converges to t and $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$.

Definition 34 (Ratio Convergence)**.** Suppose that (s_n) is a sequence of positive terms and that the sequence of rations (s_{n+1}/s_n) converges to L. If $L < 1$, then $\lim s_n = 0$

Definition 35 (Divergence)**.** A sequence (s_n) is said to ***diverge to*** $+\infty$, and we write $\lim s_n = +\infty$ provided that

for every $M \in \mathbb{R}$ there exists a natural number N such that $n \geq N$ implies that $s_n > M$.

A sequence (s_n) is said to ***diverge to*** $-\infty$, and we write $\lim s_n = +\infty$ provided that

for every $M \in \mathbb{R}$ there exists a natural number N such that $n \geq N$ implies that $s_n < M$.

Definition 36 (Greater Divergence)**.** Suppose that (s_n) and (t_n) are sequences such that $s_n \leq t_n$ for all $n \in \mathbb{N}$.

1. If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.
2. If $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

Definition 37 (Inverse of Divergence)**.** Let (s_n) be a sequence of positive numbers. Then $\lim s_n = +\infty \iff \lim(1/s_n) = 0$.

3.6 Monotone Sequences and Cauchy Sequences

Definition 38 (Monotone Sequences)**.** A sequence (s_n) of real numbers is ***increasing*** if $s_n \leq s_{s_n+1}$ for all $n \in \mathbb{N}$ and is ***decreasing*** if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. A sequence is ***monotone*** if it is increasing or decreasing.

Definition 39 (Monotone Convergence Theorem)**.** A monotone sequence is convergent $\iff$ it is bounded.

Definition 40 (Cauchy Sequence)**.** If, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then $|s_n - s_m| < \varepsilon$.

Every convergent sequence is ***Cauchy***.

If (s_n) is a ***Cauchy*** sequence, then (s_n) converges.

Proof. Given any $\varepsilon > 0$, choose N such that $|s_n - s| < \frac{\epsilon}{2}$ if $n \geq N$ (which is possible to do since $s_n \to s$). Then $|s_n - s_m| = |s_n - s + s - s_m|$ because adding and subtracting by the limit is the same as doing nothing, and, by the triangle inequality, $|s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. ■

4 Final

4.1 Limits of Functions

Definition 41 (Limit of f at c)**.** Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that a real number L is a limit of f at c, if

for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L)| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$.

Theorem 10. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L \iff$ for each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.

Theorem 11. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L \iff$ for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequences $(f(s_n))$ converges to L.

Theorem 12. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then the following are equivalent:

1. f does not have a limit at c.
2. There exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c, but $(f(s_n))$ is not convergent in $\mathbb{R}$.

4.2 Continuity

Definition 42 (Continuous)**.** Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that f is continuous at c, if

$$\text{for each } \varepsilon > 0 \text{ there exists a } \delta > 0 \text{ such that } |f(x) - L)| < \varepsilon \text{ whenever } x \in D \text{ and } |x - c| < \delta.$$

If f is continuous at each point of a subset S of D, then f is said to be **continuous on** S. If f is continuous on its domain D, then f is said to be a **continuous function**.

Theorem 13. Let $f : D \to \mathbb{R}$ and let $c \in D$. Then the following three conditions are equivalent:

1. f is continuous at c.
2. If (x_n) is any sequence in D such that (x_n) converges to c, then $\lim_{n\to\infty} f(x) = f(c)$.
3. For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$.
4. If c is an accumulation point of D, then f has a limit at c and $\lim_{x\to c} f(x) = f(c)$.

Theorem 14. Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at $c \iff$ there exists a sequence (x_n) in D such that (x_n) converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.

Theorem 15. Let $f : D \to \mathbb{R}$ and let $c \in D$. Suppose that f and g are continuous at c. Then

1. $f + g$ and fg are continuous at c, and
2. f/g is continuous at c if $g(c) \neq 0$

Theorem 16. Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at a point $c \in D$ and g is continuous at $f(c)$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at c.

Theorem 17. A function $f : D \to \mathbb{R}$ is continuous on $D \iff$ for every open set $G \subseteq \mathbb{R}$ there exists an open set $H \subseteq \mathbb{R}$ such that $H \cap D = f^{-1}(G)$.

Corollary. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous $\iff$ $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$.

4.3 Properties of Continuous Functions

Definition 43 (Bounded)**.** A function $f : D \to \mathbb{R}$ is said to be bounded if its range $f(D)$ is a bounded subset of $\mathbb{R}$. That is, f is bounded if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$

Theorem 18. Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Corollary. Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then f assumes minimum and maximum values on D. That is, there exist points x_1 and x_2 in D such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in D$.

Lemma. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(c) = 0$.

Definition 44 (Intermediate Value Theorem)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f has the intermediate value property on $[a, b]$. That is, if k is any value between $f(a)$ and $f(b)$ [*i.e.*, $f(a) < k < f(b)$ or $f(b) < k < f(a)$], then there exists $c \in (a, b)$ such that $f(c) = k$.

Theorem 19. Let I be a compact interval and suppose that $f : I \to \mathbb{R}$ is a continuous function. Then the set $f(I)$ is a compact interval.

4.4 Uniform Continuity

Definition 45 (Uniform Continuity)**.** Let $f : D \to \mathbb{R}$. We say that f is **uniformly continuous** on D if

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in D$.

A function is continuous at a point, but uniform continuity is a property of a that applies to a function *on a set*. We never speak of a function being uniformly continuous at a point.

4.5 Differentiation

Definition 46 (Derivative)**.** Let f be a real-valued function defined on an interval I containing the point c. (We allow the possibility that c is an endpoint of I.) We say that f is differentiable at c (or has a derivative at c) if the limit

$$lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. We denote the derivative of f at c by $f'(c)$ so that

$$f'(c) = lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

whenever the limit exists and is finite. If the function f is differentiable at each point of the set $S \subseteq I$, then f is said to be differentiable on S, and the function $f' : S \to \mathbb{R}$ is called the derivative of f on S.

Theorem 20. Let I be an interval containing the point c and suppose that $f : I \to \mathbb{R}$. Then f is differentiable at $c \iff$ for every sequence (x_n) in I that converges to c with $x_n \neq c$ for all n, the sequence

$$\left(\frac{f(x_n) - f(c)}{x_n - c}\right)$$

converges. Furthermore, if f is differentiable at c, then the sequence of quotients above will converge to $f'(c)$.

Theorem 21. If $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$, then f is continuous at c.

Theorem 22. Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then

1. If $k \in \mathbb{R}$, then the function kf is differentiable at c and
$$(kf)'(c) = k \cdot f'(c).$$
2. The function $f + g$ is differentiable at c and
$$(f + g)'(c) = f'(c) + g'(c)$$
3. (Product Rule) The function fg is differentiable at c and
$$(fg)'(c) = f(c)g'(c) + g(c)f'(c)$$
4. (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c and
$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$$

Theorem 23. (Chain Rule) Let I and J be intervals in $\mathbb{R}$, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where $f(I) \subseteq J$, and let $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

4.6 The Mean Value Theorem

Theorem 24. If f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

Theorem 25. (Rolle's Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and such that $f(a) = f(b)$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Theorem 26. (Mean Value Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 27. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Corollary. Let f and g be continuous an $[a, b]$ and differentiable on (a, b). Suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant C such that $f = g + C$ on $[a, b]$.

Theorem 28. Let f be differentiable on an interval I. Then

1. if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I, and
2. if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

Theorem 29. (Intermediate Value Theorem for Derivatives) Let f be differentiable on $[a, b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = k$.

Theorem 30. (Inverse Function Theorem) Let f be differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$. Then f is injective, f^{-1} is differentiable on $f(I)$, and

$$(f^{-1})'(y) = \frac{1}{f'(x)},$$

where $y = f(x)$

4.7 Taylor's Theorem

Theorem 31. (Taylor's Theorem) Let f and its first n derivatives be continuous on $[a, b]$ and differentiable on (a, b), and let $x_0 \in [a, b]$. Then for each $x \in [a, b]$ with $x \neq x_0$ there exists a point c between x and x_0 such that

$$\begin{aligned} f(x) = f(x_0) + f'(x)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + ... \\ + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \end{aligned}$$

4.8 Integration

Definition 47 (Partition)**.** Let $[a, b]$ be an interval in $\mathbb{R}$. A **partition** P of $[a, b]$ is a finite set of points $\{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < ... < x_n = b.$$

If P and Q are two partitions of $[a, b]$ with $P \subseteq Q$, then Q is called a **refinement** of P.

Definition 48 (Upper and Lower Sum)**.** Suppose that f is a bounded function defined on $[a, b]$ and that $P = \{x_0, ..., x_n\}$ is a partition of $[a, b]$. For each $i = 1, ..., n$ we let

$$M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

When only one function is under consideration, we may abbreviate thes to M_i and m_i, respectively. Letting $\Delta x_i = x_i - x_{i-1}(i = 1, ..., n)$, we define the **upper sum** of f with respect to P to be

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

and the **lower sum** of f with respect to P to be

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Since we are assuming that f is a bounded function on $[a, b]$, there exist numbers m and M such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Thus for any partition P of [a,b] we have

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a).$$

This implies that the uuper and lower sums for f form a bounded set, and it guarantees the existence of the following upper and lower integrals of f.

Remark (History)**.** (Sometimes $U(f, P)$ and $L(f, P)$ are called the upper and lower Darboux sums in honor of Gaston Darboux (1842–1917), who first developed this approach to the Riemann integral.)

Definition 49 (Upper and Lower Integral)**.** Let f be a bounded function defined on $[a, b]$. Then

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

is called the **upper integral** of f on $[a, b]$. Similarly,

$$L(f) = \{L(f, P) : P \text{ is a partition of } [a, b]\}$$

is called the **lower integrals** of f on $[a, b]$, and we denote their common value by $\int_a^b f$ or by $\int_a^b f(x)dx$. That is, if L(f) = U(f), then

$$\int_a^b f = \int_a^b f(x)dx = L(f) = U(f)$$

is the **Riemann integral** of f on $[a, b]$.

When convenient, refer to the function f as being **integrable** on $[a, b]$ and call $\int_a^b f$ the **integral** of f on $[a, b]$.

Theorem 32. Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and Q is a refinement of P, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Theorem 33. Let f be a bounded function on $[a, b]$. Then $L(f) \leq U(f)$.

Theorem 34. Let f be a bounded function on $[a, b]$. Then f is integrable $\iff$ for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Theorem 35. (The Fundamental Theorem of Calculus I) Let f be integrable on $[a, b]$. For each $x \in [a, b]$, let

$$F(x) = \int_a^x f(t)dt.$$

Then F is uniformly continuous on $[a, b]$. Furthermore, if f is continuous on $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

4.9 Infinite Series