The Fourier transform of a discrete-time averaging filter

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\begin{aligned}
H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \\
&= \frac{1}{M+1} \left(\sum_{k=0}^{M} e^{-j\omega k} \right) \\
&= \frac{1}{M+1} \left(\sum_{k'=-M/2}^{M/2} e^{-j\omega (k'+M/2)} \right) \\
&= \frac{e^{-j\omega M/2}}{M+1} \left(\sum_{k'=-M/2}^{M/2} e^{-j\omega k'} \right) \\
&= \frac{e^{-j\omega M/2}}{M+1} \left(\frac{(e^{-j\omega})^{-M/2} - (e^{-j\omega})^{M/2+1}}{1 - e^{-j\omega}} \right) \\
&= \frac{e^{-j\omega M/2}}{M+1} \left(\frac{e^{j\omega M/2} - e^{-j\omega (M/2+1)}}{1 - e^{-j\omega}} \right) \\
&= \frac{e^{-j\omega M/2}}{M+1} \left(\frac{e^{j\omega M/2} - e^{-j\omega (M/2+1)}}{1 - e^{-j\omega}} \right) \left(\frac{e^{j\omega/2}}{e^{j\omega/2}} \right) \\
&= \frac{e^{-j\omega M/2}}{M+1} \left(\frac{e^{j\omega (M+1)/2} - e^{-j\omega (M+1)/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right) \\
&= \frac{e^{-j\omega M/2}}{M+1} \left(\frac{e^{j\omega (M+1)/2} - e^{-j\omega (M+1)/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right) \left(\frac{j2}{j2} \right) \\
&= \frac{e^{-j\omega M/2}}{M+1} \left(\frac{\sin(\omega (M+1)/2)}{\sin(\omega/2)} \right).
\end{aligned}
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