Notes on Manifolds

Justin H. Le
Department of Electrical & Computer Engineering
University of Nevada, Las Vegas
lejustin.lv@gmail.com

August 3, 2016

1 Multilinear maps

A tensor $\mathcal{T}$ of order r can be expressed as the *tensor product* of r vectors:

$$\mathcal{T} = u_1 \otimes u_2 \otimes \ldots \otimes u_r \tag{1}$$

We herein fix $r = 3$ whenever it eases exposition. Recall that a vector $u \in U$ can be expressed as the combination of the basis vectors of U. Transform these basis vectors with a matrix A, and if the resulting vector u' is equivalent to uA, then the components of u are said to be *covariant*. If $u' = A^{-1}u$, i.e., the vector changes inversely with the change of basis, then the components of u are *contravariant*. By *Einstein notation*, we index the covariant components of a tensor in subscript and the contravariant components in superscript.

Just as the components of a vector u can be indexed by an integer i (as in u_i), tensor components can be indexed as $\mathcal{T}_{ijk}$. Additionally, as we can view a matrix to be a linear map $M : U \to V$ from one finite-dimensional vector space to another, we can consider a tensor to be *multilinear* map $\mathcal{T} : V^{*r} \times V^s \to \mathbb{R}$, where V^s denotes the s-th-order Cartesian product of vector space V with itself and likewise for its algebraic dual space V^*. In this sense, a tensor maps an ordered sequence of vectors to one of its (scalar) components. Just as a linear map satisfies $M(a_1u_1 + a_2u_2) = a_1M(u_1) + a_2M(u_2)$, we call an r-th-order tensor multilinear if it satisfies

$$\mathcal{T}(u_1, \ldots, a_1v_1 + a_2v_2, \ldots, u_r) = a_1\mathcal{T}(u_1, \ldots, v_1, \ldots, u_r) + a_2\mathcal{T}(u_1, \ldots, v_2, \ldots, u_r), \tag{2}$$

for scalars a_1 and a_2.

Let $\{\mathcal{T}_t\}$ be the set of all tensors. Endow $\{\mathcal{T}_t\}$ with the binary operator $+$ that maps two tensors, $\mathcal{T}_1$ and $\mathcal{T}_2$, to a tensor whose ijk-th component is the scalar addition of the ijk-th components of $\mathcal{T}_1$ and $\mathcal{T}_2$. Let scalar multiplication be defined on $\{\mathcal{T}_t\}$ such that the multiplication of $\mathcal{T} \in \{\mathcal{T}_t\}$ by a scalar a results in a multiplication of its components by a. Further endow the set with the tensor product operator as defined above. Then with these operators, $\{\mathcal{T}_t\}$ forms an algebra over the field $\{a_t\}$, the set of all a for which the above properties hold.

2 Smooth manifolds

Let X be any set and T be a collection of subsets of X. Call the members of T *open sets.* T forms a topology on X if its members satisfy:

- T contains X and the empty set $\varnothing$
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open

Then (X, T) is a *topological space.* For simplicity, we often omit T and refer to a topological space by its underlying set X.

An open set containing $x \in X$ is a *neighborhood* of x. T is *Hausdorff* if there exist disjoint sets $A, B \in T$ such that $a \in A$ and $b \in B$, $\forall a, b \in X$. That is, on a Hausdorff topological space, any two points lie in disjoint neighborhoods.

Proposition 2.1. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Let H and $G \subseteq H$ be topological spaces. (We omit their topologies for simplicity.) Suppose G is not Hausdorff. Then for some $a, b \in G$, there exist $A, B \subseteq G$ such that $a \in A$, $b \in B$, and $A \cap B \neq \varnothing$. Yet, $a, b \in H$ and $A, B \subseteq H$, so the same can be said for H. Thus, G not Hausdorff implies H not Hausdorff. $\square$

Recall that a map $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $c \in \mathbb{R}$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$, $\forall x \in \mathbb{R}$. Intuitively, this definition of continuity seems to highlight a relationship between "neighborhoods" of size ϵ and δ. We can analogously understand continuity of maps between topological spaces.

Let X, Y be two topological spaces and $A \in X, B \in Y$ be open sets. A map $f : X \to Y$ is continuous if $\forall B \in Y$ and $\forall x \in X$, there exists a neighborhood A of x such that $f(A) \in B$. Equivalently, we can say that the inverse image of an open set in Y is open.

If both f and its inverse are continous, then f is a *homeomorphism.* Then there exists a continuous map $g : Y \to X$ such that $f \circ g = g \circ f = 1$. Then X and Y are *homeomorphic* or *topologically equivalent.*

Recall that a map $f : \mathbb{R} \to \mathbb{R}$ is smooth if $\partial^k f \ \partial x^k$ exists and is continuous $\forall x \in \mathbb{R}$ and $k = 1, 2, 3, \ldots$. Equivalently, we say that f is $C^\infty(\mathbb{R})$. We can similarly define smoothness for maps between open sets.

Let $X, Y \subseteq \mathbb{R}^n$ be open sets. The map $f : X \to Y$ is smooth if every component of the Jacobian matrix

$$Df(x) := \left[\frac{\partial^i f}{\partial x^j}\right] \tag{3}$$

exists and is continuous $\forall i, j = 1, 2, 3, \ldots$. Here, we note that, for a single component of f (i.e., a map $f^i : \mathbb{R}^n \to \mathbb{R}$),

$$\frac{\partial f^i}{\partial x} = \frac{\partial^n f^i}{\partial x^1 \ldots \partial x^n}. \tag{4}$$

Let $f : X \to Y$ be a homeomorphism, where $X, Y \subseteq \mathbb{R}$ are open sets. If both f and its inverse are smooth, then f is a *diffeomorphism.*

A topological space X is *locally Euclidean* if $\forall x_i \in X$ with neighborhood U_i, there exists a homeomorphism ϕ_i from U_i to a subset of $\mathbb{R}^n$. We refer to the set $\{U_i\}$ as the coordinate neighborhoods of X, $\{\phi_i\}$ as the coordinate maps, and each pair (U_i, ϕ_i) as a chart. An *atlas* on X is a set of charts whose neighborhoods form a countable covering of X. An atlas $\mathcal{A}$ is maximal if any other atlas containing $\mathcal{A}$ is equal to $\mathcal{A}$, i.e., no atlas is larger.

A *topological manifold M* (or simply a *manifold*) is a topological space that is:

- second countable (its basis is countable)
- Hausdorff (pairs of points belong to some disjoint neighborhoods)
- locally Euclidean ($\exists \phi : U \to (V \subseteq \mathbb{R}^n)$ as above)

Let (U_i, ϕ_i) and (U_j, ϕ_j) be two charts with overlapping U_i and U_j, i.e., $U_i \cap U_j \neq \varnothing$. We call $\phi_j \circ \phi_i^{-1}$ a *transition function* of the atlas formed by these charts. We say that ϕ_i and ϕ_j are *compatible.*

A space with a maximal atlas is a manifold. Furthermore, if a space has any atlas, it is also a manifold due to the following proposition.

Proposition 2.2. *On a locally Euclidean space, every atlas is a subset of a unique maximal atlas.*

A *smooth manifold* is a topological manifold whose transition functions $\phi_j \circ \phi_i^{-1}$ are diffeomorphisms. In this case, we say that ϕ_i and ϕ_j are C^∞-compatible, and we refer to the atlas as a C^∞ atlas or a smooth structure. As before, a topological space is a smooth manifold if it is second countable, Hausdorff, and has a maximal smooth structure (i.e., has any C^∞ atlas).

References

[1] James Raymond Munkres. *Topology.* Prentice Hall, 2000.

[2] Peter Petersen. "Manifold Theory".

[3] Paul Renteln. *Manifolds, Tensors, and Forms.* Cambridge University Press, 2014.

[4] Loring W Tu. *An Introduction to Manifolds.* Springer, 2000.