Input-to-state stability of soft-reset systems with nonlinear data

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Abstract

Input-to-state stability (ISS) is considered for a nonlinear "soft-reset" system with inputs. The latter is a system that approximates a hard-reset system, which is modeled as a hybrid system with inputs. In contrast, a soft-reset system is modeled as a differential inclusion with inputs. Lyapunov conditions on the hard-reset system are given that guarantee ISS for the soft-reset system. In turn, it is shown when global asymptotic stability for the origin of the zero-input reset system guarantees ISS for nonzero inputs. Examples are given to demonstrate the theory.

Keywords: Reset Control, Input-to-State Stability, Differential Inclusions, Strong Convexity, Homogeneous Systems

1 Introduction

This paper is dedicated to Eduardo Sontag, on the occasion of his 70th birthday. In the nonlinear control systems community, Eduardo is perhaps best known for introducing the input-to-state stability (ISS) property in his seminal 1989 paper [1]. This property became a bedrock for analyzing nonlinear dynamical systems with inputs, including to establish closed-loop stability results based on the ISS nonlinear small-gain theorem [2]. In this paper, we investigate the ISS property for a class of reset control systems.

Though most nonlinear control algorithms are based on either continuous-time systems or discrete-time systems, reset control systems employ hard state resets that lead to a mixture of continuous-time and discrete-time dynamics. Reset systems reset their states or a subset of their states to zero based on a determined condition. Typically, in a closed-loop control system, resets occur in the controller state rather than the plant state.

Reset systems have various applications in control systems. A reset integrator, also referred to as a Clegg integrator (CI) [3], is one of the earliest examples of a reset system. This integrator circuit has a describing function similar to the frequency response of a linear integrator but with a different phase lag. Later, in the 1970's, Horowitz and co-authors attempted to provide a systematic approach for designing reset controllers in order to add flexibility in linear controller designs and eliminate the limitations of Clegg's integrator [4],[5]. In the 1990's, Hollot presented performance and stability analysis for general linear reset control systems [6],[7],[8]. Later, Beker and co-authors achieved controller design specifications that demonstrated some advantages of reset controllers over linear controllers in [9], [10], and [11]. In the last two decades, researchers have begun to approach reset systems from a hybrid dynamical systems point of view [12]. Due to to the nature of continuous/discrete interplay, reset systems can be modeled within the framework of hybrid dynamical systems in the sense of [13], [14], and [15]. Looking at reset controllers from the hybrid systems point of view has led to a more rigorous analysis on stability and robustness of reset systems; see, for example, [16] and [17]. In many practical instances it has been shown that embedding reset-like behavior in an otherwise continuous controller results in desirable performance; see, for example, [18] and [19]. However, due to the difficulty in providing rigorous analytical tools that can handle instantaneous changes of system solutions and a lack of performance analysis in nonlinear settings, reset systems have not been applied extensively.

Recently, attempts have been made to broaden the applicability of reset systems by introducing an alternative "soft-reset" implementation, which does not require the framework of hybrid systems but instead is modeled by a differential inclusion. The idea was introduced in [20] for linear reset systems without inputs. Follow-up work related to numerical verification of stability for these soft-reset systems is given in [21]. Soft resets were studied from a passivity point of view for nonlinear reset systems with inputs in [22].

In this work, we study the soft-reset implementation of hard-reset systems having nonlinear data, establishing conditions for ISS of the soft-reset system. We begin in Section 3 by exploiting strong convexity of a Lyapunov function for the hard-reset system, as done in [20]. However, in contrast with [20], which studies global asymptotic stability given linear data, we study ISS given nonlinear data that satisfy certain local sector growth conditions. Then, in Section 4, under a different sector growth condition, we infer ISS of the soft-reset system using global asymptotic stability of the origin for the zero-input system. We do this using two approaches, one involving strong Lyapunov conditions and the other involving homogeneity of the system data. In Section 5, by assuming a strongly convex weak Lyapunov function for the zero-input hard-reset system, we establish global exponential stability of the origin of the zero-input soft-reset system, using homogeneity of both the data and the Lyapunov function. We then use this result to claim that the conditions for ISS in Section 4 are satisfied for the soft-reset system with nonzero inputs. Lastly, some numerical examples on asymptotic stability and ISS of a soft-reset system in a closed loop are provided.

2 Notation

The set of (nonnegative) real numbers is denoted by ($\mathbb{R}_{\geq 0}$) $\mathbb{R}$. The set of (nonnegative) integers is denoted by ($\mathbb{Z}_{\geq 0}$) $\mathbb{Z}$. For any two vectors $u, v \in \mathbb{R}^n$, we use $\langle u, v\rangle := u^T v$. For $x \in \mathbb{R}^n$, we use $|x| := \sqrt{\langle x, x\rangle}$. We denote by $\mathcal{G}$ the set of functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ that are continuous, nondecreasing, and zero at zero. The subset of strictly increasing functions in $\mathcal{G}$ is denoted by $\mathcal{K}$. The subset of unbounded functions in $\mathcal{K}$ is denoted by $\mathcal{K}_\infty$. Moreover, $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if $\beta(\cdot, s)$ belongs to class $\mathcal{K}$ for each $s \geq 0$, and for each fixed $r \geq 0$, the mapping $\beta(r, \cdot)$ is decreasing to zero. A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be *sector bounded near the origin* if there exist $\delta > 0$ and $L > 0$ such that $|f| \leq L|z|$ for all $z \in \mathbb{R}^n$ satisfying $|z| \leq \delta$ and all $f \in F(z)$. It is said to be *quadratically bounded near the origin* if there exist $\delta > 0$ and $L > 0$ such that $|f| \leq L|z|^2$ for all $z \in \mathbb{R}^n$ satisfying $|z| \leq \delta$ and all $f \in F(z)$. It is said to be homogeneous of degree $k \in \mathbb{Z}_{\geq 0}$ if $F(\lambda x) = \lambda^k F(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. We use C^1 for any function that is continuously differentiable. A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called *strongly convex* if there exists a $\mu > 0$ such that, for all $x, y \in \mathbb{R}^n$, we have

$$V(y) \geq V(x) + \langle \nabla V(x), y - x\rangle + \mu|x - y|^2. \tag{1}$$

Given $x \in \mathbb{R}^n$ and a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, the distance of x to $\mathcal{A}$ is denoted $|x|_{\mathcal{A}}$ and is defined by $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}}|x - y|$. The origin of the system $\dot{x} \in F(x)$ is said to be *(Lyapunov) stable* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|x(0)| \leq \delta$ implies $|x(t)| \leq \varepsilon$ for all $t \geq 0$. It is said to be *globally attractive* if every solution x satisfies $\lim_{t\to\infty}|x(t)| = 0$. It is said to be *globally asymptotically stable (GAS)* if it is both stable and globally attractive. It is said to be *globally exponentially stable (GES)* if there exist positive constants c_0 and c_1 such that every solution x satisfies $|x(t)| \leq c_0|x(0)| \exp(-c_1 t)$ for all $t \geq 0$. The system $\dot{x} \in F(x, d)$ is said to be *input-to-state stable (ISS)* if, for each locally essentially bounded input d, maximal solutions are defined on $[0, \infty)$ and there exist a class $\mathcal{KL}$ function β and a class $\mathcal{G}$ function γ such that every solution x satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|d\|_\infty) \qquad \forall t \geq 0. \tag{2}$$

3 ISS for nonlinear soft-reset systems

A hard-reset system with input is a hybrid dynamical system in the modeling framework of [23] with state $x \in \mathbb{R}^n$ and an external disturbance $d \in \mathbb{R}^m$ given as:

$$x \in C := \{x \in \mathbb{R}^n : \varphi(x) \leq 0\}, \qquad \dot{x} \in \widehat{F}(x,d), \tag{3a}$$
$$x \in D := \{x \in \mathbb{R}^n : \varphi(x) \geq 0\}, \qquad x^+ = g(x). \tag{3b}$$

The set C indicates where the continuous change of the state is allowed, and the set D indicates where the instantaneous change of the state is allowed. Continuous change is governed by the input-driven differential inclusion in (3a) while instantaneous change is governed by the difference equation in (3b). The mappings that define these entities have the following properties:

Assumption 1

1. $\widehat{F} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semi-continuous (that is, its graph is closed) and locally bounded with nonempty, convex values.
2. $g : D \to \mathbb{R}^n$ is continuous.

In this paper, we are interested in the ISS property for reset, or reset-inspired, systems with inputs. If we were to consider ISS for the hard-reset system (3), a potential Lyapunov condition would be the existence of a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that admits a continuous, positive definite function $\widehat{W} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a function $\gamma \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \forall x \in \mathbb{R}^n, \tag{4}$$

and

$$x \in C,\ |x| \geq \gamma(|d|),\ \widehat{f} \in \widehat{F}(x,d) \quad \Longrightarrow \quad \left\langle \nabla V(x), \widehat{f} \right\rangle \leq -\widehat{W}(x), \tag{5a}$$
$$x \in D \quad \Longrightarrow \quad V(g(x)) \leq V(x). \tag{5b}$$

However, there is a problem with this condition. In particular, (5b) does not rule out the possibility of solutions that exclusively jump without decreasing the Lyapunov function V. To avoid having to worry about such solutions, it is of interest to recast the hard-reset system (3) via its "soft-reset" implemention, first introduced in [20] for linear reset systems without inputs. This implementation corresponds to the differential inclusion

$$\dot{x} \in \widehat{F}(x,d) + \kappa(x) \Big(\mathrm{SGN}\left(\varphi(x)\right) + 1 \Big) \left(g(x) - x\right) =: F(x,d), \tag{6}$$

where $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$ and the set-valued mapping SGN is defined as

$$\mathrm{SGN(s)} := \begin{cases} \frac{s}{|s|} & s \neq 0 \\ [-1, 1] & s = 0. \end{cases} \tag{7}$$

Lemma 1 If Assumption 1 holds and $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$ is continuous then the set-valued mapping F defined in (6) is outer semicontinous and locally bounded with nonempty, convex values.

Proof (Sketch) See [24, Prop. 2.23(a), Prop. 5.51(a),(b)]. □

As we will show below, the Lyapunov conditions (4)-(5) guarantee ISS of (6) at least for $\kappa(\cdot)$ sufficiently large if the following additional conditions hold:

Assumption 2

1. The function V in (4)-(5) is strongly convex.
2. There exists a $M = M^T$ such that $\varphi(x) = x^T M x$ for all $x \in \mathbb{R}^n$.
3. $\widehat{F} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ sector bounded near the origin, $g : D \to \mathbb{R}^n$ is sector bounded near the origin, and $\widehat{W}$ is quadratically bounded near the origin.
4. Jumps starting in the jump set $x \in D$ land in the flow set i.e., $g(x) \in C$ for all $x \in D$.
5. The function γ in (5) belongs to $\mathcal{K}_\infty$ and satisfies

$$\limsup_{s \to 0^+} \frac{s}{\gamma(s)} < \infty. \tag{8}$$

6. The condition (5a) holds with C replaced by the inflated set $C_\varepsilon := \left\{x \in \mathbb{R}^n : \varphi(x) \leq \varepsilon x^T x\right\}$ for some $\varepsilon > 0$.

The condition (8) can always be satisfied by, for example, adding a linear term to γ near the origin. By assuming Item 4 of Assumption 2, we allow for solutions of the hard-reset system (3) to flow without immediately jumping after a prior jump, but this does not remove the purely discrete-time solutions to (3) that do not decrease the function V. The strong convexity of V in Item 1 of Assumption 2 enables establishing ISS of the soft-reset system (6), as in the following theorem:

Theorem 1 *The conditions (4)-(5), augmented with the conditions in Assumptions 1 and 2, guarantee that there exists a continuous function $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$ with sufficiently large values and a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $|x| \geq \gamma(|d|)$ implies $\langle \nabla V(x), f\rangle \leq -W(x)$ for all $f \in F(x, d)$.*

Proof Since the function g is assumed to be continuous and sector bounded near the origin, there exists a continuous function $\sigma_0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive definite and sector bounded near the origin satisfying

$$|M(g(x) + x)| \leq \sigma_0(x), \quad \forall x \in D. \tag{9}$$

Using the inequality from (9) and the Cauchy-Schwarz inequality, $M = M^T$, and Item 4 of Assumption 2, giving $g(x)^T M g(x) \leq 0$ when $x^T M x \geq 0$, it follows that

$$\begin{aligned} & x \neq 0,\ x^T M x \geq 0 \quad \Longrightarrow \\ & \qquad |g(x) - x)| \geq \frac{-(g(x) - x)^T M(g(x) + x)}{\sigma_0(x)} = \frac{x^T M x - g(x)^T M g(x)}{\sigma_0(z)} \\ & \qquad \geq \frac{x^T M x}{\sigma_0(x)}. \end{aligned} \tag{10}$$

By the strong convexity of V, as in (1), from Item 1 of Assumption 1, we can write

$$V(g(x)) \geq V(x) + \langle \nabla V(x), g(x) - x \rangle + \mu |x - g(x)|^2. \tag{11}$$

From (11) and (5b), we have:

$$x \in D \quad \Longrightarrow \quad \langle \nabla V(x), x - g(x) \rangle \geq \mu |x - g(x)|^2. \tag{12}$$

It then follows from the definition of the set D in (3b) and the definition of SGN below (6) that (12) can be rewritten as

$$\begin{aligned} & s \in \mathrm{SGN}(x^T M x) \quad \Longrightarrow \\ & \qquad \langle \nabla V(x), (s+1)(g(x) - x) \rangle \leq -(s+1)\mu |x - g(x)|^2. \end{aligned} \tag{13}$$

Combining (13), (10), and Assumption 1 gives

$$\begin{aligned} & x \neq 0,\ s \in \mathrm{SGN}(x^T M x) \quad \Longrightarrow \\ & \qquad \langle \nabla V(x), (s+1)(g(x) - x) \rangle \leq -2\mu \max\{0, x^T M x\} \frac{x^T M x}{\sigma_0(x)^2}. \end{aligned} \tag{14}$$

Due to Item 5 of Assumption 1, there exist $\varepsilon, m > 0$ such that $s/\gamma(s) \leq m$ for all $s \in (0, \varepsilon]$. We will show that there exists a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is quadratically bounded near the origin such that, for all $x \in \mathbb{R}^n$ and $|d| \leq \varepsilon$, we have

$$\widehat{f} \in \widehat{F}(x, d),\ |x| \geq \gamma(|d|) \geq \frac{|d|}{m} \quad \Longrightarrow \quad \left\langle \nabla V(x), \widehat{f} \right\rangle + \widehat{W}(x) \leq \sigma(x). \tag{15}$$

Due to Item 6 of Assumption 2, $\sigma(x)$ may take any nonnegative values for $x \in C_\varepsilon$. If $\varepsilon > \overline{\sigma}(M)$, the latter denoting the maximum singular value of M, then $C_\varepsilon = \mathbb{R}^n$. Since V is a C^1 function, ∇V is sector bounded near the origin, and, due to Assumption 2, $\widehat{W}$ is quadratically bounded near the origin. Hence, using the Cauchy-Schwarz inequality on the left-hand side of the inequality (15), and the local sector boundedness of $\widehat{F}$ and ∇V, as well as W being quadratically bounded near the origin, we have that there exist positive constants $L_{\nabla V}$, L_f, and L_W such that, for values of $x \in \mathbb{R}^n$ near the origin,

$$\begin{aligned} & \widehat{f} \in \widehat{F}(x, d),\ |x| \geq \gamma(|d|) \geq \frac{|d|}{m} \quad \Longrightarrow \\ & \qquad |\nabla V(x)||\widehat{f}| + \widehat{W}(x) \leq L_{\nabla V}|x| L_F\left(|x| + |d|\right) + L_W |x|^2 \\ & \qquad \qquad \leq L_{\nabla V} L_f |x|^2 + m L_{\nabla V} L_f |x|^2 + L_W |x|^2 \\ & \qquad \qquad \leq \left(L_{\nabla V} L_f (1 + m) + L_W\right) |x|^2. \end{aligned}$$

Let us define $\sigma_1(x) := \big(L_{\nabla V} L_f(1+m) + L_W\big)|x|^2$. We have now shown that (15) holds for small values of x and d. For large values of x and d, due to the fact that $\widehat{F}$, ∇V, and $\widehat{W}$ are each locally bounded, there exist $h \in \mathcal{G}$ and a constant $c > 0$ such that

$$\begin{aligned}\widehat{f} \in \widehat{F}(x,d),\ |x| \geq \gamma(|d|) \quad &\Longrightarrow \\ &\Big\langle \nabla V(x), \widehat{f} \Big\rangle + \widehat{W}(x) \leq h(|x| + |d| + c) \\ &\qquad\qquad\qquad \leq h(|x| + \gamma^{-1}(|x|) + c) =: \sigma_2(x). \end{aligned} \tag{16}$$

Let $\xi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth function satisfying $\xi(r) = 0$ for $r \leq \gamma(0.5\varepsilon)$ and satisfying $\xi(r) = 1$ for $r \geq \gamma(\varepsilon)$. Then, letting $\sigma(x) := \sigma_1(x) + \xi(|x|)\sigma_2(x)$, and noting that (15) holds for $\gamma^{-1}(|x|) \leq \varepsilon$, we have from (16) that

$$\widehat{f} \in \widehat{F}(x,d),\ |x| \geq \gamma(|d|) \quad \Longrightarrow \quad \Big\langle \nabla V(x), \widehat{f} \Big\rangle + \widehat{W}(x) \leq \sigma(x). \tag{17}$$

It follows from (17) that

$$\begin{aligned} x \neq 0,\ \widehat{f} \in \widehat{F}(x,d),\ |x| \geq \gamma(|d|),\ x^T M x \geq \varepsilon |x|^2 \quad &\Longrightarrow \\ \Big\langle \nabla V(x), \widehat{f} \Big\rangle + \widehat{W}(x) &\leq \sigma(x) \\ &\leq \frac{\sigma_0(x)^2 \sigma(x)}{\varepsilon^2 |x|^4} \max\{0, x^T M x\} \frac{x^T M x}{\sigma_0(x)^2}. \end{aligned} \tag{18}$$

Due to σ and σ_0 being quadratically bounded and sector bounded near the origin, respectively, we can establish

$$\limsup_{x \to 0, x \in D\setminus\{0\}} \frac{\sigma_0(x)^2 \sigma(x)}{\varepsilon^2 |x|^4} < \infty.$$

Pick the continuous function $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that

$$\frac{1}{2\mu}\left(\kappa_0(x) + \frac{\sigma_0(x)^2 \sigma(x)}{\varepsilon^2 |x|^4}\right) \leq \kappa(x), \qquad \forall x \in D\setminus\{0\}, \tag{19}$$

where $\kappa_0 : \mathbb{R}^n \to \mathbb{R}_{>0}$ is continuous. Here, we can combine our so far derived bounds from (14), (18) with (19) and write

$$\begin{aligned} x \neq 0,\ \widehat{f} \in \widehat{F}(x,d),\ |x| \geq \gamma(|d|),\ s \in \mathrm{SGN}(x^T M x) \quad &\Longrightarrow \\ \Big\langle \nabla V(x), \widehat{f} - \kappa(x)(s+1)(x - g(x)) \Big\rangle & \\ \leq -\widehat{W}(x) - \kappa_0(x) \max\{0, x^T M x\} \frac{x^T M x}{\sigma_0(x)^2}&. \end{aligned} \tag{20}$$

Let us define

$$W(x) := \widehat{W}(x) + \kappa_0(x) \max\{0, x^T M x\} \frac{x^T M x}{\sigma_0(x)^2}.$$

Due to Item 3 of Assumption 1, the positivity and continuity of $\kappa_0(\cdot)$, and the fact that σ_0 is positive definite and sector bounded near the origin, we have that $W(\cdot)$ is continuous and positive definite. From (20), we have

$$\begin{aligned} \widehat{f} \in \widehat{F}(x,d),\ |x| \geq \gamma(|d|),\ s \in \mathrm{SGN}(x^T M x) \quad &\Longrightarrow \\ \Big\langle \nabla V(x), \widehat{f} - \kappa(x)(s+1)(x - g(x)) \Big\rangle &\leq -W(x) \end{aligned}$$

It follows that the soft-reset system (6) is ISS for large enough κ (see [25, Theorem 1]), particularly for κ satisfying (19). □

Remark 1 Note that the gain function γ is preserved in the input-to-state stability property from the hard-reset system (3) to the soft-reset system (6).

4 From Global Asymptotic Stability to ISS

In this section, we demonstrate two methods for inferring ISS from a globally asymptotically stable (GAS) soft-reset system (6) together with the following sector growth condition:

Assumption 3 There exists a positive constant L such that

$$|\widehat{f}|_{\widehat{F}(x,0)} \leq L|d|, \qquad \forall x \in \mathbb{R}^n,\ d \in \mathbb{R}^m,\ \widehat{f} \in \widehat{F}(x,d).$$

The first method takes advantage of the strong Lyapunov conditions for GAS of the origin for soft-reset system (6) with zero disturbance to achieve ISS. The second method establishes ISS by exploiting a homogeneity assumption on the data of (6) and a GAS assumption for the origin of (6) with zero disturbance. The latter can be establish under weakened Lyapunov conditions with a strong convexity assumption, as in Section 5.

4.1 ISS from Strong Lyapunov Conditions

ISS of (6) is established under the following assumptions involving strong Lyapunov conditions on the system having zero disturbance.

Assumption 4 There exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

1. the below inequality holds

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \forall x \in \mathbb{R}^n,\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty \tag{21}$$

2. there exist positive constants c and $\hat{c}$ such that, recalling the definition of F in (6),

$$\langle \nabla V(x), f_0 \rangle \leq -c|x|^2, \qquad \forall x \in \mathbb{R}^n,\ f_0 \in F(x,0) \tag{22a}$$

$$|\nabla V(x)| \leq \hat{c}|x|, \qquad \forall x \in \mathbb{R}^n, \tag{22b}$$

Theorem 2 *If Assumptions 3 and 4 hold then the system (6) is ISS.*

Proof Let $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, and $f \in F(x,d)$. Let $\widehat{f} \in \widehat{F}(x,d)$ and $h \in \kappa(x)\big(\mathrm{SGN}\,(\varphi(x)) + 1\big)(g(x) - x)$ be such that $f = \widehat{f} + h$. Let $\widehat{f}_0^* \in \widehat{F}(x,0)$ be such that $|\widehat{f}|_{\widehat{F}(x,0)} = |\widehat{f} - \widehat{f}_0^*|$. Note that $\widehat{f}_0^* + h \in F(x,0)$. Then, using (22), we have

$$\langle \nabla V(x), f \rangle = \left\langle \nabla V(x), \widehat{f} + h \right\rangle \tag{23}$$

$$
\begin{aligned}
&= \left\langle \nabla V(x), \widehat{f}_0^* + h \right\rangle + \left\langle \nabla V(x), \widehat{f} - \widehat{f}_0^* \right\rangle \\
&\leq -c|x|^2 + |\nabla V(x)| \, |\widehat{f} - \widehat{f}_0^*| \\
&= -c|x|^2 + |\nabla V(x)| \, |\widehat{f}|_{\widehat{F}(x,0)} \\
&\leq -c|x|^2 + \hat{c}L|x||d|.
\end{aligned}
$$

To dominate the term $\hat{c}L|x||d|$ for large $|x|$, we introduce $0 < \theta < 1$ in the previous inequality as follows:

$$
\begin{aligned}
\frac{\partial V}{\partial x} f &\leq -c(1-\theta)|x|^2 - c\theta|x|^2 + \hat{c}L|x||d| \\
&\leq -c(1-\theta)|x|^2, \qquad \forall |x| \geq \frac{\hat{c}L|d|}{c\theta}.
\end{aligned}
$$

Hence, the conditions for input-to-state stability from [25, Theorem 1] are satisfied with $\rho(r) = (\hat{c}L/c\theta)r$, and we conclude that system (6) is input-to-state stable with $\gamma(r) := \alpha_1^{-1} \circ \alpha_2 \circ \rho(r)$. □

4.2 ISS from Homogeneity Conditions and GAS

In this section, we emulate the results from the previous section and obtain ISS by using GAS of the origin and homogeneity for the soft-reset system (6) with $d \equiv 0$.

Assumption 5

1. The origin of the soft-reset system (6) is GAS when $d \equiv 0$.
2. The function g and the mapping $x \mapsto \widehat{F}(x, 0)$ are homogeneous of degree 1, and the function κ is homogeneous of degree 0.

Theorem 3 *If Assumptions 1 and 5 hold and there exists a $M = M^T$ such that $\varphi(x) = x^T M x$ for all $x \in \mathbb{R}^n$ then the system (6) is ISS.*

Proof Due to Item 2 of Assumption 5 and the assumption that $\varphi(x) = x^T M x$, we have that F is homogeneous of degree 1. Indeed,

$$
\begin{aligned}
F(\lambda x, 0) &= \widehat{F}(\lambda x, 0) + \kappa(\lambda x) \left(\mathrm{SGN}\left(\varphi(\lambda x)\right) + 1 \right) (g(\lambda x) - \lambda x), \\
&= \lambda \widehat{F}(x, 0) + \kappa(x) \left(\mathrm{SGN}\left(\varphi(x)\right) + 1 \right) \lambda (g(x) - x), \\
&= \lambda \left[\widehat{F}(x, 0) + \kappa(x) \left(\mathrm{SGN}\left(\varphi(x)\right) + 1 \right) (g(x) - x) \right], \\
&= \lambda F(x, 0).
\end{aligned} \tag{24}
$$

Next, due to Item 1 of Assumption 1 and Lemma 1, $x \mapsto F(x, 0)$ is outer semi-continuous and locally bounded. Therefore $F(x, 0)$ is compact for each $x \in \mathbb{R}^n$; see [24, Theorem 5.19]. It follows from Item 1 of Assumption 5 and [26, Theorem 1.2] that there exists a C^∞ function V and a positive definite, continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that (21) holds and, for all $x \in \mathbb{R}^n$ and $f_0 \in F(x, 0)$, $\langle \nabla V(x), f_0 \rangle \leq$

$-W(x)$. Then, from [27, Prop. 8], there exists a C^1 function $\overline{V}:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ that is homogeneous of degree 2 and a positive definite, continuous function $\overline{W}:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ such that

$$\overline{\alpha}_1(|x|)\leq \overline{V}(x)\leq \overline{\alpha}_2(|x|),\qquad \forall x\in\mathbb{R}^n,\ \overline{\alpha}_1,\overline{\alpha}_2\in\mathcal{K}_\infty, \tag{25a}$$

$$\big\langle \nabla\overline{V}(x), f_0\big\rangle \leq -\overline{W}(x),\qquad \forall x\in\mathbb{R}^n,\ f_0\in F(x,0). \tag{25b}$$

Note that $\overline{V}$ can be constructed from V using [27, Eq. 36] and [27, Eq. 37]. We will show that Assumption 4 holds with V replaced by $\overline{V}$ and thereby invoke Theorem 2 to conclude that (6) is ISS.

Due to the fact that $\overline{V}$ is homogeneous of degree 2, we have that, for all $x\in\mathbb{R}^n$ such that $x\neq 0$, with $w:=x/|x|$,

$$\overline{V}(x)=\overline{V}(|x|w)=|x|^2\overline{V}(w)\geq |x|^2\min_{v\colon |v|=1}\overline{V}(v). \tag{26}$$

Letting $a_1:=\min_{v\colon |v|=1}\overline{V}(v)$, we have from (25a) that $a_1>0$. It follows from (26) that

$$\overline{V}(x)\geq a_1|x|^2,\quad \forall x\in\mathbb{R}^n. \tag{27}$$

Due to the fact that $\overline{V}$ is homogeneous of degree 2, the Euler homogeneous function theorem ensures that $\nabla\overline{V}$ is homogeneous of degree 1. Then, due to the homogeneity of degree 1 of $\nabla\overline{V}$ and $x\mapsto F(x,0)$, along with the homogeneity of degree 2 of $\overline{V}$, it can be shown that, for all $x\in\mathbb{R}^n$ and $f_0\in F(x,0)$,

$$c_1:=\sup_{v\colon |v|=1}\frac{\langle\nabla\overline{V}(v),f_0\rangle}{\overline{V}(v)},$$

$$\big\langle\nabla\overline{V}(x),f_0\big\rangle\leq c_1\overline{V}(x). \tag{28}$$

From (25), we have that c_1 must be negative. Hence, with $\lambda:=-c_1>0$, we have

$$\big\langle\nabla\overline{V}(x),f_0\big\rangle\leq -\lambda\overline{V}(x),\quad \forall x\in\mathbb{R}^n,\ f_0\in F(x,0). \tag{29}$$

Combining (29) with (27), we have that $\overline{V}$ satisfies (22a) with $c:=a_1\lambda$. Due to the fact that $\nabla\overline{V}$ is homogeneous of degree 1, $\overline{V}$ satisfies (22b) with $\hat{c}:=\max_{v\colon |v|=1}\big|\nabla\overline{V}(v)\big|$. We conclude that Assumption 4 holds with V replaced by $\overline{V}$, and it follows from Theorem 2 that (6) is ISS. □

5 GAS for homogeneous soft-reset systems

The goal of this section is to give weak Lyapunov conditions on the hard-reset system with $d\equiv 0$ that guarantee the assumptions of the previous sections 4.1 and 4.2 on the soft-reset system with $d\equiv 0$. We consider the continuous-time implementation (6) of the hybrid, reset control system (3) with $d\equiv 0$ and show that the origin of the soft implementation is globally exponentially stable (GES) if the following assumption holds.

Assumption 6

1. The function g and the mapping $x\mapsto\widehat{F}(x,0)$ are homogeneous of degree 1, and the function κ is homogeneous of degree 0.

2. Let $M = M^T$. There exist $\varepsilon > 0$ and a C^1 and strongly convex, homogeneous of degree 2, positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that, with the definition $C_\varepsilon := \{x \in \mathbb{R}^n : x^T M x \leq \varepsilon x^T x\}$, the following inequalities hold:

$$\left\langle \nabla V(x), \widehat{f}_0 \right\rangle \leq 0, \qquad \forall x \in C_\varepsilon,\ \widehat{f}_0 \in \widehat{F}(x,0), \tag{30a}$$

$$V(g(x)) \leq V(x), \qquad \forall x \in D. \tag{30b}$$

3. Jumps starting in the jump set $x \in D$ land in the flow set i.e., $g(x) \in C$ for all $x \in D$.
4. There is no solution to (3a) in C with an unbounded time domain that keeps the function V equal to a nonzero constant.

According to Assumption 6, we have stability of the origin for system (3) with $d \equiv 0$. However, these assumptions are not strong enough to establish GES of the origin. It is possible for hard-reset systems to have discrete-time solutions which do not converge to zero. This can be addressed by considering the corresponding continuous-time soft-reset implementation (6) of the hard-reset system (3), for which GES of the origin can be obtained with $d \equiv 0$ as in the following result.

Theorem 4 *Under Assumptions 1 and Assumption 6, the origin of (6) with $d \equiv 0$ is globally exponentially stable for $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$ taking sufficiently large values.*

Proof For the origin of (6) with $d \equiv 0$, consider the Lyapunov candidate V, which is positive definite and radially unbounded due to Item 2 of Assumption 6. First, we bound the inner product $\langle \nabla V(x), \big(\mathrm{SGN}(x^T M x) + 1\big)(g(x) - x)\rangle$. By using (3*b*), strong convexity of V as in (1), and (30b), we have

$$x^T M x \geq 0 \quad \Longrightarrow \quad \langle \nabla V(x), (g(x) - x)\rangle \leq -\mu |g(x) - x|^2. \tag{31}$$

We can rewrite (31) as follows:

$$\begin{aligned} s \in \mathrm{SGN}(x^T M x) \quad &\Longrightarrow \\ &\langle \nabla V(x), (s+1)(g(x) - x)\rangle \leq -(s+1)\mu |g(x) - x|^2. \end{aligned} \tag{32}$$

Let $\delta_0 > 0$ be such that, for all $x \in \mathbb{R}^n$, we have $|M(g(x) + x)|_2 \leq \delta_0 |x|_2$, and then, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} x \neq 0,\ x^T M x \geq 0 \quad \Longrightarrow \quad |g(x) - x| &\geq \frac{-(g(x) - x)^T M (g(x) + x)}{\delta_0 |x|} \\ &= \frac{x^T M x - g(x)^T M g(x)}{\delta_0 |x|} \\ &\geq \frac{x^T M x}{\delta_0 |x|}. \end{aligned} \tag{33}$$

Combining (32) and (33) results in

$$\begin{aligned}
&x \neq 0,\ s \in \mathrm{SGN}(x^T Mx) \quad \Longrightarrow \\
&\qquad \langle \nabla V(x), (s+1)(g(x)-x) \rangle \leq -2\mu \max\{0, x^T Mx\} \frac{x^T Mx}{\delta_0^2 |x|^2}.
\end{aligned} \tag{34}$$

The next step is to bound $\left\langle \nabla V(x), \widehat{f}_0 \right\rangle$ for $\widehat{f}_0 \in \widehat{F}(x,0)$ and for all $x \in \mathbb{R}^n$. Due to Assumption 6, this quantity is negative when the inequality $x^T Mx \leq \varepsilon |x|_2^2$ holds. Using homogeneity of degree 2 for V and thus, homogeneity of degree 1 of ∇V due to Euler's homogeneous function theorem, along with homogeneity of degree 1 of $x \mapsto \widehat{F}(x,0)$, it follows for $x^T Mx \geq \varepsilon |x|_2^2$ that there exists a $\Gamma > 0$ such that, for all $\widehat{f}_0 \in \widehat{F}(x,0)$,

$$\begin{aligned}
x^T Mx \geq \varepsilon |x|_2^2 > 0 \quad \Longrightarrow \quad \left\langle \nabla V(x), \widehat{f}_0 \right\rangle &\leq \Gamma |x|^2 \\
&\leq \frac{\Gamma}{\varepsilon} x^T Mx \\
&\leq \frac{\Gamma \delta_0^2}{\varepsilon^2} \max\{0, x^T Mx\} \frac{x^T Mx}{\delta_0^2 |x|^2}.
\end{aligned} \tag{35}$$

It follows from (34), (35), and Assumption 6 that, for each constant $v > 0$, there exists κ taking sufficiently large values such that, for $\widehat{f}_0 \in \widehat{F}(x,0)$,

$$\begin{aligned}
&s \in \mathrm{SGN}(x^T Mx) \quad \Longrightarrow \\
&\left\langle \nabla V(x), \widehat{f}_0 - \kappa(x)(s+1)(x-g(x)) \right\rangle \leq -v \max\{0, x^T Mx\} \frac{x^T Mx}{|x|^2} \leq 0.
\end{aligned}$$

Due to the invariance principle for differential inclusion [28, Theorem 11] and properties of F, the origin is globally asymptotically stable if and only if there is no $c > 0$ and solution x such that $V(x(t)) = c$ for all $t \geq 0$. Let us assume that $V(x(t))$ is equal to a nonzero constant. Let $\widehat{f}_0(t) \in \widehat{F}(x(t),0)$ and $s(t) \in \mathrm{SGN}\left(x(t)^T Mx(t)\right)$ satisfy, for almost all $t \geq 0$,

$$\dot{x}(t) = \widehat{f}_0(t) - \kappa(x(t))(s(t)+1)(x(t) - g(x(t))). \tag{36}$$

Then we have

$$\left\langle \nabla V(x(t)), \widehat{f}_0(t) - \kappa(x(t))(s(t)+1)(x(t)-g(x(t))) \right\rangle = 0.$$

According to (35), such solutions require $x^T(t)Mx(t) \leq 0$ for all $t \geq 0$. As a result, it follows from (30a), (32), and the positivity of κ that, for almost all $t \geq 0$,

$$\begin{aligned}
&\left\langle \nabla V(x(t)), \widehat{f}_0(t) \right\rangle = 0, \\
&\langle \nabla V(x(t)), -\kappa(x(t))(s(t)+1)(g(x(t)-x(t)) \rangle = 0.
\end{aligned}$$

With (32) and the positivity of $\kappa(\cdot)$ and μ, it follows that for almost all $t \geq 0$

$$(s(t)+1)|g(x(t)) - x(t)| = 0.$$

It follows from (36) and the definition of $\widehat{f}(t)$ that $x(\cdot)$ is also a solution of (3a). Since we have assumed in Item 4 of Assumption 6 that solutions of (3a) do not keep V equal to a nonzero constant, it follows that no solution keeps V equal to a nonzero constant and thus we have estalished GAS of the origin of $\dot{x} \in F(x,0)$. That is, GAS of the origin of (6) with $d \equiv 0$ can be concluded. With this conclusion, along with

Assumption 1, it can be shown using [26, Theorem 1.2] and [27, Prop. 8], as done in the proof of Theorem 3, that there exists a C^1 function $\overline{V}$ that is homogeneous of degree 2 and a constant $\lambda > 0$ such that (29) holds. Furthermore, due to the fact that $\overline{V}$ is homogeneous of degree 2, it can be shown using steps similar to those in the proof of Theorem 3 that, with $a_1 \coloneqq \min_{v:\, |v|=1} \overline{V}(v)$ and $a_2 \coloneqq \max_{v:\, |v|=1} \overline{V}(v)$, we have $a_1|x|^2 \le \overline{V}(x) \le a_2|x|^2$ for all $x \in \mathbb{R}^n$. It follows that the origin of (6) with $d \equiv 0$ is GES. $\square$

The following corollary establishes that the soft-reset system (6) is ISS by combining Theorems 3 and 4.

Corollary 1 If Assumptions 3 and 6 hold then the system (6) is ISS for $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$ taking sufficiently large values.

Proof Given that Assumption 6 holds, Theorem 4 ensures that Item 1 of Assumption 5 holds for $\kappa(\cdot)$ taking sufficiently large values. The remaining items of Assumption 5 follow immediately from the assumptions stated here. Hence, Theorem 3 can be invoked to conclude ISS of (6). $\square$

In the next section, we present a numerical example.

6 Examples

6.1 GAS example

Consider the soft-reset system (6) with state x comprising a plant state $x_p \in \mathbb{R}$ and a compensator state $x_c \in \mathbb{R}$, i.e., $x \coloneqq (x_p, x_c)^T \in \mathbb{R}^2$, with the system data given by

$$\widehat{F}(x,0) = \begin{bmatrix} x_c \\ -K\,\mathrm{SGN}(x_c)|x| - x_p \end{bmatrix}, \tag{38a}$$

$$g(x) = \begin{bmatrix} x_p \\ 0 \end{bmatrix}, \qquad \varphi(x) = x^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x, \tag{38b}$$

for some $K \in \mathbb{R}_{>0}$. We choose $\kappa(\cdot)$ in (6) to be constant, namely with $\kappa(x) = \kappa \in \mathbb{R}_{\ge 0}$ for all $x \in \mathbb{R}^2$. Consequently, $\kappa(\cdot)$ is homogeneous of degree 0, and, noting that $x \mapsto \widehat{F}(x,0)$ and g in (38) are homogeneous of degree 1, we have that Item 1 of Assumption 6 holds. For all $x \in \mathbb{R}^2$, consider $V(x) \coloneqq x^T x$, which is a C^1, strongly convex, and positive definite function that is homogeneous of degree 2 and satisfies

$$\left\langle \nabla V(x), \widehat{f}_0 \right\rangle = -2K|x|x_c \le 0, \qquad \forall x \in \mathbb{R}^2,\ \widehat{f}_0 \in \widehat{F}(x,0), \tag{39a}$$

$$V(g(x)) = x_p^T x_p \le V(x), \qquad \forall x \in \mathbb{R}^2. \tag{39b}$$

Hence, V satisfies Item 2 of Assumption 6 with $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $C_\varepsilon = \mathbb{R}^n$. For all $x \in \mathbb{R}^2$, $g(x)^T M g(x) = 0$, and thus Item 3 of Assumption 6 holds. Supposing that a solution x of system (3a) keeps $V(x(t))$ at a nonzero constant for all $t \geq 0$, it follows from (39a) that $x_c \equiv 0$ and, therefore, that $\dot{x}_c \equiv 0$. It then follows from (38a) that solutions x of system (3a) satisfy $\dot{x}_p = x_c \equiv 0$ and $\dot{x}_c = -x_p \equiv 0$. It follows that $x \equiv 0$ and $V(x) = x^T x \equiv 0$, contradicting the premise that $V(x)$ is a nonzero constant. We have thus shown by contradiction that Item 4 of Assumption 6 holds. It is now verified that Assumption 6 holds, and Theorem 4 may be applied to conclude GES of the origin of (6) with $d \equiv 0$ and with the system data given by (38).

Setting $K = 0.5$, Figures 1 and 2 respectively show the evolution of the states x_p and x_c, highlighting the effect of setting κ equal to various constant values. Figure 3 shows the evolution of $V(x) = x^T x$.

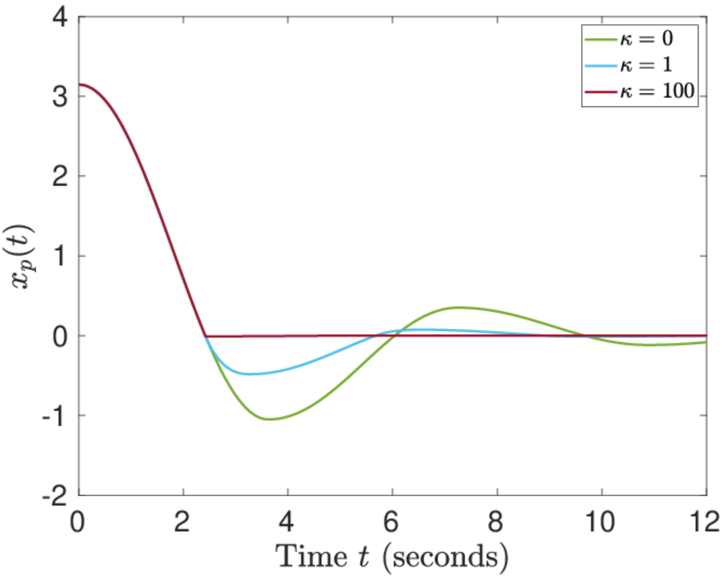

Fig. 1 The value of $x_p(t)$ as a function of t using nonlinear homogeneous system data and initializing the system state at $(x_p^\circ, 0)^T$ with x_p° being randomly selected.

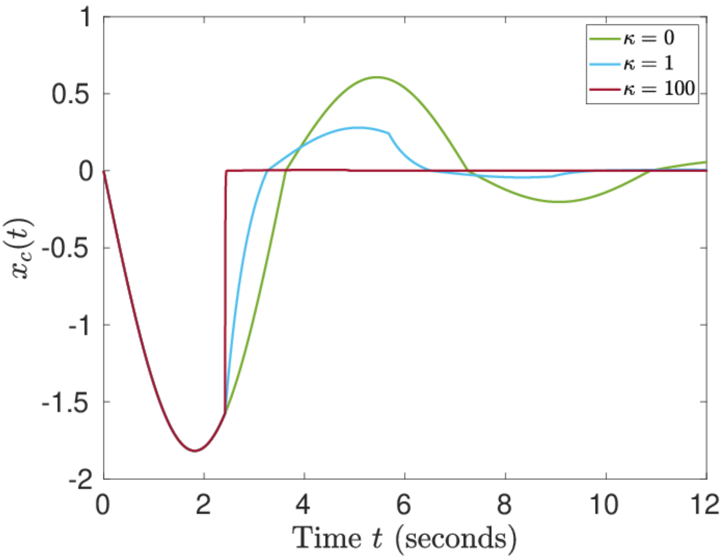

Fig. 2 The value of $x_c(t)$ as a function of t using nonlinear homogeneous system data and initializing the system state at $(x_p^\circ, 0)^T$ with x_p° being randomly selected.

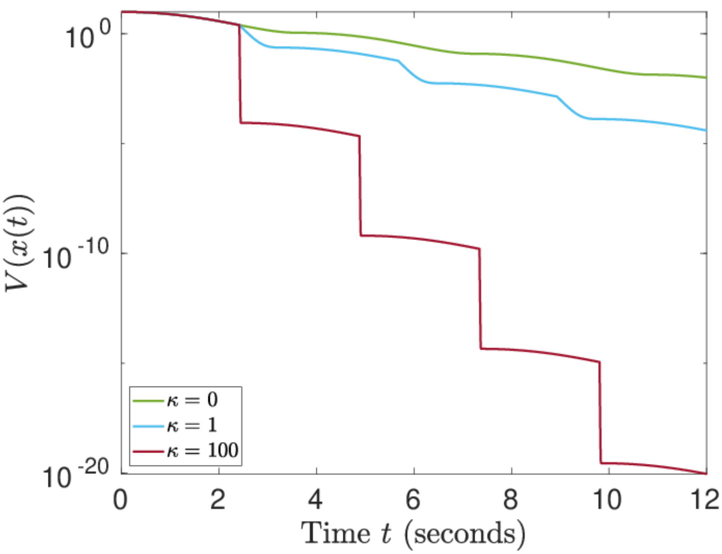

Fig. 3 The value of $V(x(t))$ as a function of t using nonlinear homogeneous system data and initializing the system state at $(x_p^\circ, 0)^T$ with x_p° being randomly selected.

6.2 ISS example

Figures 4-6 show the same example from Section 6.1 but with

$$\widehat{F}(x,d) = \begin{bmatrix} x_c + d \\ -K\,\mathrm{SGN}(x_c)|x| - x_p \end{bmatrix} \tag{40}$$

and a disturbance of $d(t) = 0.1\sin(t)$ for all $t \geq 0$.

Setting $d \equiv 0$ makes (40) equivalent to (38a), and thus, Assumption 6 holds for the same reasons given in Section 6.1. With F given by (6), (38b), and (40), for any $x \in \mathbb{R}^2$, $d \in \mathbb{R}$, and $\widehat{f} \in \widehat{F}(x,d)$, letting $\widehat{f}_0^*$ be such that $|\widehat{f}|_{\widehat{F}(x,0)} = |\widehat{f} - \widehat{f}_0^*|$, we have that $\widehat{f} - \widehat{f}_0^* = d$, and thus, Assumption 3 holds with $L = 1$. We conclude that the conditions of Corollary 1 hold for the soft-reset system (6) with system data given by (38b) and (40), ensuring that, for sufficiently large $\kappa(\cdot)$, (6) is ISS with this system data. Figures 4-6 show results for the various constant values of κ considered in Section 6.1.

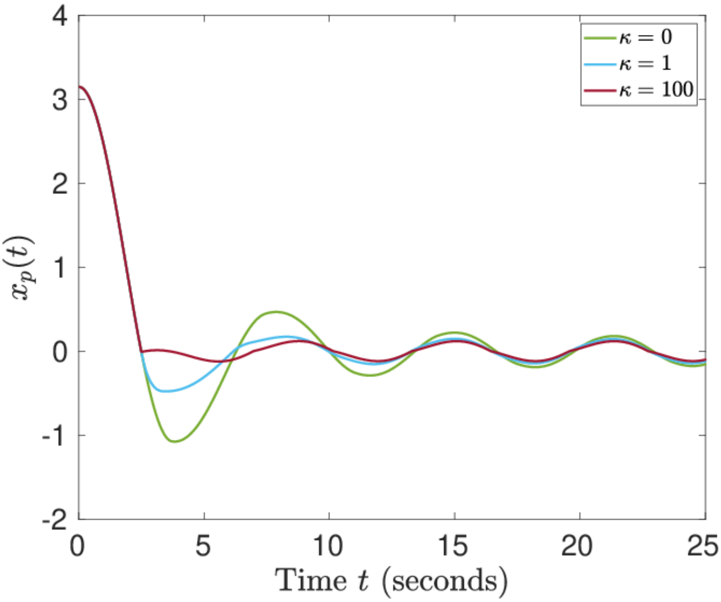

Fig. 4 The value of $x_p(t)$ as a function of t using nonlinear homogeneous system data, a sinusoidal input disturbance in the plant, and initializing the system state at $(x_p^\circ, 0)^T$ with x_p° being randomly selected.

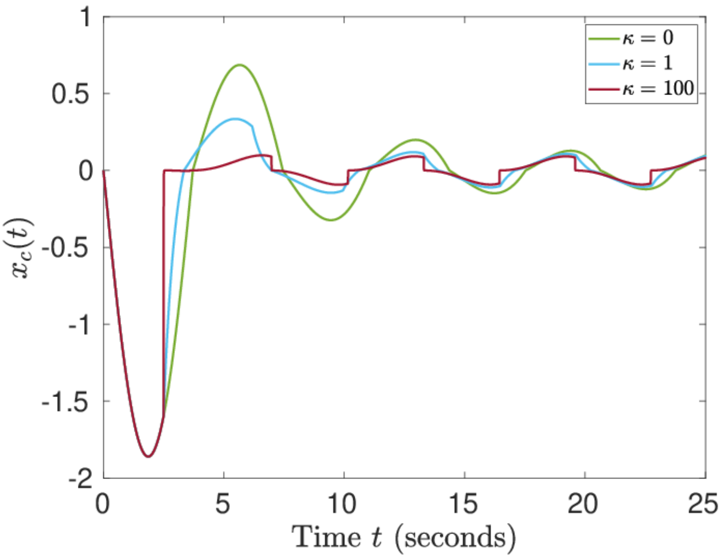

Fig. 5 The value of $x_c(t)$ as a function of t using nonlinear homogeneous system data, a sinusoidal input disturbance in the plant, and initializing the system state at $(x_p^\circ, 0)^T$ with x_p° being randomly selected.

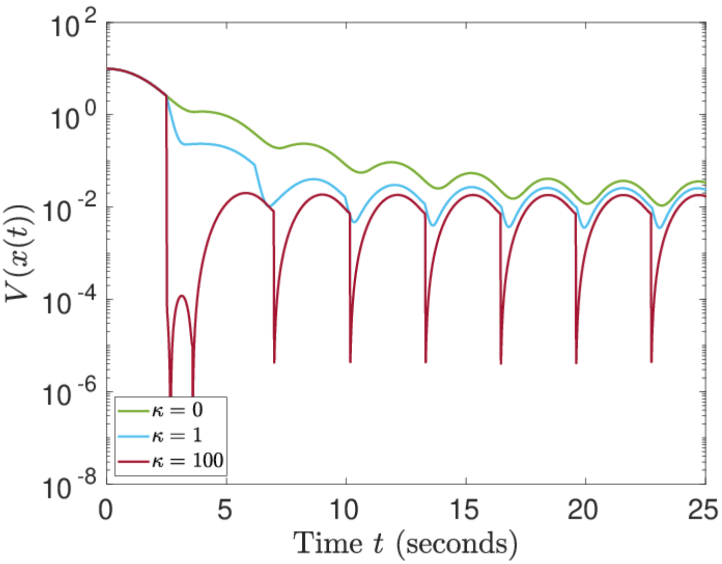

Fig. 6 The value of $V(x(t))$ as a function of t using nonlinear homogeneous system data, a sinusoidal input disturbance in the plant, and initializing the system state at $(x_p^\circ, 0)^T$ with x_p° being randomly selected.

7 Conclusion

Conditions were provided for input-to-state stability of nonlinear soft-reset systems with inputs. Assuming strong convexity of a Lyapunov function for the hard-reset system, sufficient conditions are provided for input-to-state stability of the corresponding soft-reset system. Moreover, two methods are described for obtaining input-to-state stability for the soft-reset system via asymptotic stability for the origin of the zero-input soft-reset system. Lastly, exponential stability is established for a soft-reset system using strong convexity of a weak Lyapunov function for the hard-reset system, along with homogeneity of the data and the Lyapunov function. Numerical examples are given, illustrating global asymptotic stability and input-to-state stability of a soft-reset control system in a closed loop.

Acknowledgments. Research supported by the Air Force Office of Scientific Research under grant AFOSR FA9550-21-1-0452.

References

[1] Sontag, E.D.: Smooth stabilization implies coprime factorization. IEEE Transactions on Automatic Control **34**, 435–443 (1989)

[2] Jiang, Z.P., Teel, A.R., Praly, L.: Small-gain theorem for iss systems and applications. Math. Control Signal Systems **7**, 95–120 (1994)

[3] Clegg, J.C.: A nonlinear integrator for servomechanisms. Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry **77**(1), 41–42 (1958)

[4] Horowitz, I., Rosenbaum, P.: Non-linear design for cost of feedback reduction in systems with large parameter uncertainty. International Journal of Control **21**(6), 977–1001 (1975)

[5] Krishnan, K., Horowitz, I.: Synthesis of a non-linear feedback system with significant plant-ignorance for prescribed system tolerances. International Journal of Control **19**(4), 689–706 (1974)

[6] Hu, H., Zheng, Y., Chait, Y., Hollot, C.: On the zero-input stability of control systems with clegg integrators. In: Proceedings of the 1997 American Control Conference, vol. 1, pp. 408–410 (1997)

[7] Hollot, C., Zheng, Y., Chait, Y.: Stability analysis for control systems with reset integrators. In: Proceedings of the 36th IEEE Conference on Decision and Control, vol. 2, pp. 1717–1719 (1997)

[8] Chait, Y., Hollot, C.: On Horowitz's contributions to reset control. International Journal of Robust and Nonlinear Control **12**(4), 335–355

(2002)

[9] Beker, O., Hollot, C., Chait, Y., Han, H.: Fundamental properties of reset control systems. Automatica **40**(6), 905–915 (2004)

[10] Beker, O., Hollot, C.V., Chait, Y.: Plant with integrator: an example of reset control overcoming limitations of linear feedback. IEEE Transactions on Automatic Control **46**(11), 1797–1799 (2001)

[11] Beker, O., Hollot, C., Chait, Y.: Stability of a MIMO reset control system under constant inputs. In: Proceedings of the 38th IEEE Conference on Decision and Control, vol. 3, pp. 2780–2781 (1999)

[12] Haddad, W.M., Chellabsina, V., Kablar, N.: Active control of combustion instabilities via hybrid resetting controllers. In: Proceedings of the 2000 American Control Conference. ACC, vol. 4, pp. 2378–2382 (2000)

[13] Goebel, R., Hespanha, J., Teel, A.R., Cai, C., Sanfelice, R.: Hybrid systems: generalized solutions and robust stability. IFAC Proceedings Volumes **37**(13), 1–12 (2004)

[14] Goebel, R., Teel, A.R.: Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. Automatica **42**(4), 573–587 (2006)

[15] Goebel, R., Sanfelice, R.G., Teel, A.R.: Hybrid dynamical systems. IEEE Control Systems Magazine **29**(2), 28–93 (2009)

[16] Nešić, D., Zaccarian, L., Teel, A.R.: Stability properties of reset systems. Automatica **44**(8), 2019–2026 (2008)

[17] Nesic, D., Teel, A.R., Zaccarian, L.: Stability and performance of SISO control systems with first-order reset elements. IEEE Transactions on Automatic Control **56**(11), 2567–2582 (2011)

[18] Zheng, J., Guo, Y., Fu, M., Wang, Y., Xie, L.: Improved reset control design for a PZT positioning stage. In: 2007 IEEE International Conference on Control Applications, pp. 1272–1277 (2007)

[19] Fernández, A., Barreiro, A., Banos, A., Carrasco, J.: Reset control for passive teleoperation. In: 34th Annual Conference of IEEE Industrial Electronics, pp. 2935–2940 (2008)

[20] Teel, A.R.: Continuous-time implementation of reset control systems. In: Trends in Nonlinear and Adaptive Control – A Tribute to Laurent Praly for His 65th Birthday, Springer, pp. 27–42 (2021)

[21] Bertollo, R., Teel, A.R., Zaccarian, L.: Continuous-time reset control with

max-of-quadratics Lyapunov certificates. Submitted.

[22] Le, J.H., Teel, A.R.: Passive soft-reset controllers for nonlinear systems. In: 2021 60th IEEE Conference on Decision and Control (CDC), pp. 5320–5325 (2021)

[23] Goebel, R., Sanfelice, R.G., Teel, A.R.: Hybrid Dynamical Systems: Modeling, Stability, and Robustness. Princeton University Press, New Jersey (2012)

[24] Rockafellar, R.T., Wets, R.J.B.: Variational Analysis vol. 317. Springer, Heidelberg (1998)

[25] Dellarossa, M., Tanwani, A., Zaccarian, L.: Non-pathological iss-Lyapunov functions for interconnected differential inclusions. IEEE Transactions on Automatic Control (2021)

[26] Clarke, F.H., Ledyaev, Y.S., Stern, R.J.: Asymptotic stability and smooth Lyapunov functions. Journal of differential Equations **149**(1), 69–114 (1998)

[27] Nakamura, H., Yamashita, Y., Nishitani, H.: Smooth Lyapunov functions for homogeneous differential inclusions. In: Proceedings of the 41st SICE Annual Conference. SICE 2002., vol. 3, pp. 1974–1979 (2002)

[28] Ryan, E.: A universal adaptive stabilizer for a class of nonlinear systems. Systems & Control Letters **16**(3), 209–218 (1991)