# Problems in Riemann integration

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## 1 Introduction

**Proposition 1.1.** *Let $x \in [0,1]$ and define*

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{n} & x = \frac{m}{n} \end{cases}, \tag{1}$$

*where $m$ and $n$ are integers with no factors in common (except 1). Then $f$ is Riemann integrable.*

*Proof.* Define a partition of $[0,1]$ such that $0 = x_0 < x_1 < \ldots < x_k = 1$ and

$$\begin{aligned} x_1 - x_0 &< \frac{\epsilon}{k} \\ x_2 - x_1 &< \frac{\epsilon}{k} \\ &\vdots \\ x_k - x_{k-1} &< \frac{\epsilon}{k}\ . \end{aligned} \tag{2}$$

Let $P$ be the set of all the above intervals for a fixed $k$. As $k$ increases, the intervals grow smaller, $\epsilon$ decreases, and we say that "the partition $P$ grows finer".

To satisfy Riemann integrability, we'd like to show that $\inf_P U_P(f) = \sup_P L_P(f)$, where

$$L_P(f) = \sum_{i=0}^{k} m_i(x_k - x_{k-1}) \tag{3}$$

$$U_P(f) = \sum_{i=0}^{k} M_i(x_k - x_{k-1}) \tag{4}$$

$$m_i = \inf_{[x_i,\, x_{i-1})} f(x) \tag{5}$$

$$M_i = \sup_{[x_i,\, x_{i-1})} f(x) \tag{6}$$

Since an irrational number lies in every interval $[x_i,\ x_{i-1})$, the infimum of $f$ on each interval is 0. So, $m_i = 0$ for all $i$ and $L_P(f) = 0$. Then $\sup_P L_P(f) = 0$. Now, we only need to show that $\inf_P U_P(f) = 0$.

Since a rational number lies in every interval, the supremum of $f$ on each interval is the supremum of $1/n$, which is 1. So, $M_i = 1$ for all $i$. Then, by the partition that we defined in 2, we have

$$U_P(f) = \sum_{i=0}^{k} M_i(x_k - x_{k-1}) = 1 \cdot (k \cdot \frac{\epsilon}{k}) = \epsilon\ . \tag{7}$$

As the partition becomes finer, the intervals grow smaller and $\epsilon$ tends to zero. So, we've shown that $U_P(f)$ tends to 0. That is, $\inf_P U_P(f) = 0$. Then $\inf_P U_P(f) = \sup_P L_P(f) = 0$. ☐

**Proposition 1.2.** *Let $x \in [0,1]$ and define*

$$f(x) = \begin{cases} 1 & x \neq \frac{1}{2} \\ 2 & x = \frac{1}{2} \end{cases}\ . \tag{8}$$

*Then $f$ is Riemann integrable.*

*Proof.* Define a partition of $[0,1]$ such that $0 = x_0 < x_1 < \ldots < x_k = 1$ and

$$\begin{aligned} x_1 - x_0 &< \epsilon \\ x_2 - x_1 &< \epsilon \\ &\vdots \\ x_k - x_{k-1} &< \epsilon\ . \end{aligned} \tag{9}$$

Let the partition $P$ denote the set of all the above intervals for a fixed $k$.

The infimum of each interval is 1, regardless of whether the interval contains $x = \frac{1}{2}$. Knowing this, let's determine whether there exists a partition $P$ such that $U_P(f) - L_P(f) < \epsilon$ for any $\epsilon$, as in Lemma A.1.

For any $P$, there exists $\ell$ such that $\frac{1}{2} \in [x_{\ell-1}, x_\ell)$. Then on this interval, $M_\ell = 2$. Computing the upper sum,

$$\begin{aligned} U_P(f) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &= M_\ell(x_\ell - x_{\ell-1}) + \sum_{i=1,\ i\neq\ell}^{n} M_i(x_i - x_{i-1}) \\ &< 2\epsilon + 1 \cdot (n-1)\epsilon \\ &= \epsilon(n+1)\ , \end{aligned} \tag{10}$$

where $(n-1)$ arises from the fact that there are $(n-1)$ terms in the sum remaining after excluding the interval that contains $\frac{1}{2}$. (Only one interval may contain $\frac{1}{2}$, by the definition of a partition.)

From the above expression, we can show that there exists $n$ (and therefore a $P$) such that Lemma A.1 is satisfied. To satisfy the lemma, we must have

$$U_P(f) - L_P(f) < \epsilon(n+1) - 1 \leq \epsilon \tag{11}$$

So, given any $\epsilon$, there exists $n \leq \frac{1}{\epsilon}$ that satisfies the above inequality. Then, $f$ is Riemann integrable. □

**Proposition 1.3.** *Suppose that $f$ is Riemann integrable on $[a, b]$ and $f(x) \geq 0$ for all $x$. Then*

*(A) $\int_a^b f(x)dx \geq 0$ and*

*(B) if $\int_a^b f(x)dx = 0$ and $f$ is continuous, then $f(x) = 0$ for all $x \in [a, b]$.*

*Proof.* Define a partition of $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_k = b$. Let $x_k^* \in [x_k, x_{k-1}]$. Then, for all $n$,

$$\begin{aligned} S_n(f) &= \sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}) \\ &\geq \sum_{k=1}^{n} 0 \cdot (x_k - x_{k-1}) \\ &= 0 \; . \end{aligned} \tag{12}$$

Since $S_n(f) \rightarrow$ as $n \rightarrow$, the above result implies that $\int_a^b f(x)dx \geq 0$.

To prove $(B)$, suppose that there exists $x_0 \in [a, b]$ such that $f(x_0) \neq 0$. Without loss of generality, assume that $f(x_0) > 0$. Also, suppose that $f$ is continuous on its domain, so that

$$\begin{aligned} \forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - x_0| \leq \delta \quad &\Rightarrow \quad |f(x) - f(x_0)| \leq \epsilon \\ &\Rightarrow \quad f(x_0) - \epsilon \leq f(x) \; . \end{aligned} \tag{13}$$

Then, we can show that the integral of $f$ over its domain must be greater than zero.

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^{x_0-\delta} f(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)dx + \int_{x_0+\delta}^{b} f(x)dx \\ &= 0 + \int_{x_0-\delta}^{x_0+\delta} f(x)dx + 0 \\ &> \int_{x_0-\delta}^{x_0+\delta} f(x_0)dx \\ &> 2\delta f(x_0) \\ &> 0 \; , \end{aligned} \tag{14}$$

where the final line follows from the fact that both $\delta$ and $f(x_0)$ are greater than zero. So, the integral is non-zero, and $(B)$ is proven by contraposition. □

## 2 Supplement

**Proposition 2.1.** *Let $f$ be a continuous function on $[a, b]$ and*

$$F(x) = \int_a^x f(t)dt \; . \tag{15}$$

*Then $F$ is Lipschitz continuous on $[a, b]$.*

*Proof.* First, note that $f$ is bounded, since it's continuous on a closed and bounded interval. That is, $\exists M$ such that $f(x) \leq M$ for all $x$.

Let $x_1 \in [a, b]$ and $x_2 \in [a, b]$. Without loss of generality, let $x_1 < x_2$. Then,

$$\begin{aligned} |F(x_2) - F(x_1)| &= \left| \int_a^{x_2} f(x)dx - \int_a^{x_1} f(x)dx \right| \\ &\overset{(a)}{=} \left| \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx - \int_a^{x_1} f(x)dx \right| \\ &= \left| \int_{x_1}^{x_2} f(x)dx \right| \\ &\overset{(b)}{<} \int_{x_1}^{x_2} |f(x)|dx \\ &\leq \int_{x_1}^{x_2} M dx \\ &= M(x_2 - x_1) \; . \end{aligned} \tag{16}$$

Line $(a)$ follows from Lemma A.3. Line $(b)$ follows from Lemma A.2. By the assumption that $x_1 < x_2$, we see that $x_2 - x_1$ is always positive. So, the final line implies that $|F(x_2) - F(x_1)| < M|x_2 - x_1|$, which proves the Lipschitzianity of $F$. $\square$

**Proposition 2.2.** *Let $f$ be a continuous function on $[a, b]$ and*

$$\int_{x_1}^{x_2} f(x)dx = 0 \; , \tag{17}$$

*for every $x_1$ and $x_2$ in $[a, b]$. Then $f(x) = 0$ for all $x$.*

*Proof.* Suppose that there exists $x^* \in [a, b]$ such that $f(x^*) \neq 0$. Without loss of generality, let $f(x^*) > 0$. Since $f$ is continuous on its domain, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x^*| \leq \delta$ implies $f(x) - f(x^*) \leq \epsilon$. Then, $f(x^*) - \epsilon \leq f(x)$.

Now, suppose that $x_1 = x^* - \delta$ and $x_2 = x^* + \delta$. Then,

$$\begin{aligned} \int_{x_1}^{x_2} f(x)dx &\geq \int_{x_1}^{x_2} f(x^*)dx \\ &= \int_{x^*-\delta}^{x^*+\delta} f(x^*)dx \\ &= (x^* + \delta - (x^* - \delta))f(x^*) \\ &= 2\delta f(x^*) \\ &> 0 \end{aligned} \tag{18}$$

Then there exists $x_1$ and $x_2$ such that the integral is non-zero, and the proposition is proven by contraposition. $\square$

**Proposition 2.3.** *Let $f$ be a continuous function on $[a,b]$. Then, there exists $x_0 \in [a,b]$ such that*

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)dx \ . \tag{19}$$

*Proof.* Since $f$ is continuous on a closed and bounded interval, it achieves its infimum and supremum on that interval. So, let $c$ and $d$ be in $[a,b]$ such that $c = \inf_{[a,b]} f(x)$ and $d = \sup_{[a,b]} f(x)$. Then,

$$\begin{aligned}
\int_a^b f(c)dx &\leq \int_a^b f(x)dx && \leq \int_a^b f(d)dx \\
(b-a)f(c) &\leq \int_a^b f(x)dx && \leq (b-a)f(d) \\
f(c) &\leq \frac{1}{b-a}\int_a^b f(x)dx && \leq f(d)
\end{aligned}$$

Then, by the Intermediate Value Theorem, there exists $x_0 \in [c,d]$ such that $f(x_0) = \frac{1}{b-a}\int_a^b f(x)dx$. Since $[c,d] \subseteq [a,b]$, it follows that $x_0$ is also in $[a,b]$. $\square$

# A Some useful results

**Lemma A.1.** *Let $f$ be a bounded function on $[a, b]$. Suppose that for each $\epsilon > 0$ there is a partition $P$ such that*

$$U_P(f) - L_P(f) \leq \epsilon \; . \tag{20}$$

*Then $f$ is Riemann integrable.*

**Lemma A.2.** *Let $f$ be a continuous function on $[a, b]$. Then,*

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \tag{21}$$

**Lemma A.3.** *Let $f$ be a continuous function on $[a, b]$ and $a \leq c \leq b$. Then,*

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \tag{22}$$