Applications of the Laplace Transform

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Abstract

In this article, we review some non-trivial applications of the Laplace transform in signal analysis and in solving linear ordinary differential equations with constant coefficients.

1 Time-domain convolution

Let's begin by evaluating the convolution of a continuous-time signal with itself. Let

$$x(t) = \begin{cases} 1 & -2a \leq t \leq 2a \\ 0 & \text{else} \end{cases} , \tag{1}$$

and let $y(t) = x(t) * x(t)$.

Convolution in the time-domain is equivalent to multiplication in the Laplace domain. So, we can find $X(s) = \mathcal{L}(x(t))$ and obtain the convolution as $x(t) * x(t) = \mathcal{L}^{-1}(X(s) \cdot X(s))$. First, to find $X(s)$, we note that $x(t)$ can be expressed in terms of unit step functions. Since $x(t)$ is 1 between $-2a$ and $2a$, it can be written as

$$x(t) = u(t + a) - u(t - a) \ . \tag{2}$$

Recall that a shift in the time-domain causes an exponential function to appear in the Laplace transform. Specifically, knowing that $\mathcal{L}(u(t)) = 1/s$, we also know that $\mathcal{L}(u(t - t_0)) = e^{st_0}/s$. Using this property, we have

$$X(s) = \mathcal{L}\{u(t + a) - u(t - a)\} = \frac{e^{as}}{s} - \frac{e^{-as}}{s} \ . \tag{3}$$

Now, multiply in the Laplace domain.

$$X(s) \cdot X(s) = \frac{e^{2as}}{s^2} - \frac{2}{s^2} + \frac{e^{-2as}}{s^2} \ . \tag{4}$$

To find the time-domain convolution, take the inverse Laplace transform of the above product. As mentioned, we can use the fact that an exponential function in the Laplace domain corresponds to a shift in the time-domain. Specifically, knowing that $\mathcal{L}^{-1}(1/s^2) = tu(t)$, we also know that $\mathcal{L}^{-1}(e^{as}/s^2) = (t+a)u(t+a)$. Using this property, we have

$$y(t) = \mathcal{L}^{-1}\{X(s) \cdot X(s)\} = (t+2a)u(t+2a) - 2tu(t) + (t-2a)u(t-2a)\ . \tag{5}$$

So, $y(t) = 0$ for $t \leq 2a$. The other cases of t are not so obvious, so let's consider them one at a time.

The signal $y(t)$ seems to change shape as t crosses $-2a$, 0, and $2a$, as we can see when rewriting $y(t)$ for each interval:

$$\begin{aligned} -2a \leq t \leq 0 \quad &\Rightarrow \quad y(t) = (t+2a) - 0 + 0 \\ 0 \leq t \leq 2a \quad &\Rightarrow \quad y(t) = (t+2a) - 2t + 0 \\ 2a \leq t \quad &\Rightarrow \quad y(t) = (t+2a) - 2t + (2a-t)\ . \end{aligned}$$

Notice that the final line results in $y(t) = 0$. From these expressions, we can write $y(t)$ a bit more neatly by combining terms.

$$y(t) = \begin{cases} 2a+t & -2a \leq t \leq 0 \\ 2a-t & 0 \leq t \leq 2a \\ 0 & else \end{cases} \tag{6}$$

We could clean the result even further by expressing it in terms of absolute values.

$$y(t) = \begin{cases} 2a-|t| & |t| \leq 2a \\ 0 & else \end{cases} \tag{7}$$

We've somewhat neglected the distinction between $<$ and $\leq$, but our neglect has little consequence in practice. Our intuition about the signal remains the same, regardless of the strictness of inequalities.

1.1 Convolving with time-derivatives

An interesting and useful phenomenon arises when taking the Laplace transform of a time-derivative. Given any $x(t)$, we may not have an idea of how to easily evaluate

$$x(t) * \frac{d}{dt}\delta(t)\ , \tag{8}$$

as it involves the derivative of an impulse. (If a derivative roughly indicates a slope, then what's the slope of a vertical line?) To avoid this issue, we use the fact that $\mathcal{L}(dx(t)/dt) = s \cdot \mathcal{L}(x(t))$. So, to evaluate the above convolution we can first take multiplication in the Laplace domain as

$$\begin{aligned}
\mathcal{L}\{x(t)\} \cdot \mathcal{L}\left\{\frac{d}{dt}\delta(t)\right\} &= X(s) \cdot (s \cdot \mathcal{L}\{\delta(t)\}) \\
&= X(s) \cdot (s \cdot 1) \\
&= sX(s) \\
&= \mathcal{L}\left\{\frac{d}{dt}x(t)\right\} \ ,
\end{aligned} \tag{9}$$

and taking the inverse Laplace transform of the above product, we see that Eq. **??** is simply $dx(t)/dt$. By using convenient properties of the Laplace transform, we've effectively dodged the bullet of differentiating an impulse!