r08631036 熊紘鉾

1. $Q_n(\beta_0, \beta_1) = \sum_1^n (y_i - \beta_0 - \beta_1 x_i)^2$ — ①

$$\frac{\partial Q_n}{\partial \beta_0} = -2\sum_1^n (y_i - \beta_0 - \beta_1 x_i) = 0 \Rightarrow \Sigma y_i = \Sigma(\beta_0 + \beta_1 x_i) = n\beta_0 + \Sigma \beta_1 x_i$$

$$\Rightarrow \bar{y} = \beta_0 + \beta_1 \bar{x}$$

$$\Rightarrow \beta_0 = \bar{y} - \beta_1 \bar{x} \longrightarrow ①$$

$$\Rightarrow Q_n = \Sigma(y_i - \bar{y} + \beta_1 \bar{x} - \beta_1 x_i)^2 = \sum_1^n [(y_i - \bar{y}) - \beta_1(x_i - \bar{x})]^2$$

$$\Rightarrow \frac{\partial Q_n}{\partial \beta_1} = -2\sum_1^n [(y_i - \bar{y}) - \beta_1(x_i - \bar{x})](x_i - \bar{x}) = 0$$

$$\Rightarrow \beta_1 = \frac{\sum_1^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_1^n (x_i - \bar{x})^2}, \quad \beta_0 = \bar{y} - \beta_1 \bar{x} \quad \#$$

2. Model A: $y_i = \beta_0 + \beta_1 x_i + u_i$, use the result from 1.

$$\Rightarrow \beta_1 = \frac{\Sigma(y_i - \bar{y})(x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2}, \quad \beta_0 = \bar{y} - \beta_1 \bar{x}$$

Model B: $y_i = \alpha_0 + \alpha_1(x_i - \bar{x}) + v_i$, use the same idea from 1.

$$\Rightarrow \alpha_1 = \frac{\sum_1^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_1^n (x_i - \bar{x})^2}, \quad \alpha_0 = \bar{y}$$

(a) No, they are not identical. The variance of $\beta_0$ is larger.

(b) Yes, they are identical. Therefore, their variance are the same.

3.

(a) $\tilde{\beta}_1 = \frac{\sum_1^n x_i y_i}{\sum_1^n x_i^2} \Rightarrow E(\tilde{\beta}_1) = E\left(\frac{\Sigma x_i y_i}{\Sigma x_i^2}\right) = E\left(\frac{\Sigma x_i(\beta_0 + \beta_1 x_i + u_i)}{\Sigma x_i^2}\right) = \beta_1 + \beta_0 \frac{\Sigma x_i}{\Sigma x_i^2} + \cancel{\frac{\Sigma x_i u_i}{\Sigma x_i^2}}^{\,0}$ ※

(b) No

(c) Considering the matrix representative, $\tilde{\beta}_1 = Cy$ be an estimator with $\beta_0 = 0$ where

$C = (X'X)^{-1}X' + D$

$Var(\tilde{\beta}) = Var(Cy)$

$=$ ↓

$Var(Cy)$

$= C \cdot Var(y) \cdot C'$

$= \sigma^2 CC'$

$= \sigma^2 [(X'X)^{-1}X' + D][X(X'X)^{-1} + D']$

$= \sigma^2 ((X'X)^{-1}\cancel{X'X(X'X)^{-1}} + (X'X)^{-1}X'D' + DX(X'X)^{-1} + DD')$

$= \sigma^2 ((X'X)^{-1} + X^{-1}D' + DX'^{-1} + DD')$

$= \sigma^2 (X'X)^{-1} + \sigma^2 DD' = Var(\hat{\beta}) + \sigma^2 DD'$ #     From wiki/Gauss-Markov

(d) No, it's $Var(\tilde{\beta}_1) \geq Var(\hat{\beta}_1)$ in general since D is positive semi-definite.

(e) No violation. As explained above that $\sigma^2 DD' \geq 0$, $Var(\hat{\beta}_1)$ has the smallest variance.