Problem Set 1: Solution

Part One: Hand-Written Exercise

1. The least-squares(LS) criterion function is

$$\mathrm{Q}_n(\beta_0, \beta_1) := \textstyle\sum_{i=1}^n u_i^2 = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

The first order conditions(FOCs) are

$$\frac{\partial Q_n(\beta_0, \beta_1)}{\partial \beta_0} = -2\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0 \tag{1}$$

$$\frac{\partial Q_n(\beta_0, \beta_1)}{\partial \beta_1} = -2\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)x_i = 0 \tag{2}$$

By (1)(2),

$$n\beta_0 + \sum_{i=1}^n x_i \beta_1 = \sum_{i=1}^n y_i \tag{3}$$

$$\sum_{i=1}^n x_i \beta_0 + \sum_{i=1}^n x_i^2 \beta_1 = \sum_{i=1}^n x_i y_i \tag{4}$$

By (3)(4), we can get

$$\hat\beta_1 = \frac{\sum_{i=1}^n (y_i - \bar y)(x_i - \bar x)}{\sum_{i=1}^n (x_i - \bar x)^2}$$

$$\hat\beta_0 = \bar y - \hat\beta_1 \bar x$$

2. We already know that for model A:

$$\hat\beta_0 = \bar y - \hat\beta_1 \bar x$$

$$\hat\beta_1 = \frac{\sum (x_i - \bar x)(y_i - \bar y)}{\sum (x_i - \bar x)^2}$$

Now for model B, from the F.O.C. of $\sum \left(y_i - \alpha_0 - \alpha_1(x_i - \bar x)\right)^2$ we have:

$$\begin{cases} -2\sum \left(y_i - \hat\alpha_0 - \hat\alpha_1(x_i - \bar x)\right) = 0 \\ -2\sum \left(y_i - \hat\alpha_0 - \hat\alpha_1(x_i - \bar x)\right)(x_i - \bar x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sum y_i = n\hat\alpha_0 + \hat\alpha_1 \sum (x_i - \bar x) \\ \sum y_i(x_i - \bar x) = \hat\alpha_0 \sum (x_i - \bar x) + \hat\alpha_1 \sum (x_i - \bar x)^2 \end{cases}$$

$\hat{\alpha}_0$ and $\hat{\alpha}_1$ is therefore given by:

$$\hat{\alpha}_0 = \bar{y}$$
$$\hat{\alpha}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

(a) $\hat{\alpha}_0$ and $\hat{\beta}_0$ are not identical, and their variance is given by:

$$\mathrm{Var}(\hat{\alpha}_0) = \mathrm{Var}(\bar{y}) = \frac{\sigma^2}{n}$$
$$\mathrm{Var}(\hat{\beta}_0) = \frac{\sigma^2}{n} \cdot \frac{\sum x_i^2}{\sum(x_i - \bar{x})^2}.$$

Since for any sample of data, $\sum x_i^2 \geq \sum(x_i - \bar{x})^2$ (please verify), hence $\mathrm{Var}(\hat{\beta}_0) \geq \mathrm{Var}(\hat{\alpha}_0)$.

(b) $\hat{\alpha}_1$ and $\hat{\beta}_1$ are identical.

3. (a)

$$\tilde{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2}$$
$$\mathbb{E}(\tilde{\beta}_1) = \frac{\sum x_i \mathbb{E}(y_i)}{\sum x_i^2} = \beta_0 \cdot \frac{\sum x_i}{\sum x_i^2} + \beta_1.$$

(b) If $\beta_0 \neq 0$, then $\tilde{\beta}_1$ is biased iff $\sum x_i \neq 0$.

(c)

$$\mathrm{Var}(\tilde{\beta}_1) = \mathrm{Var}\left(\frac{\sum x_i y_i}{\sum x_i^2}\right) = \mathrm{Var}\left(\frac{\sum x_i u_i}{\sum x_i^2}\right) = \frac{\sigma^2}{\sum x_i^2}$$

(d)

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum(x_i - \bar{x})^2}.$$

As $\sum x_i^2 \geq \sum(x_i - \bar{x})^2$ for any sample of data, $\mathrm{Var}(\hat{\beta}_1) \geq \mathrm{Var}(\tilde{\beta}_1)$ in general.

(e) No, since $\tilde{\beta}_1$ is NOT unbiased in general. In the case where $\tilde{\beta}_1$ is unbiased, then we have $\bar{x} = 0$, causing $\mathrm{Var}(\hat{\beta}_1) = \mathrm{Var}(\tilde{\beta}_1)$.