Problem Set 3: Solution

Part One: Hand-Written Exercise

1. (a) Let

$$\hat{\sigma}_r^2 = \frac{1}{n-(k+1-q)} \sum_{1=1}^{n} \hat{e}_{i,r}^2 = \frac{SSR_r}{n-(k+1-q)}$$
$$\hat{\sigma}_{ur}^2 = \frac{1}{n-(k+1)} \sum_{1=1}^{n} \hat{e}_{i,ur}^2 = \frac{SSR_{ur}}{n-(k+1)},$$

where $\hat{e}_{i,r}^2$ and $\hat{e}_{i,ur}^2$ are the residuals from restricted and unrestricted models respectively. The fact that

$$\frac{(n-k-1+q)\hat{\sigma}_r^2}{\sigma^2} - \frac{(n-k-1)\hat{\sigma}_{ur}^2}{\sigma^2} \sim \chi^2(q)$$

implies

$$\frac{SSR_r - SSR_{ur}}{\sigma^2} \sim \chi^2(q).$$

Moreover, we know that

$$\frac{(n-k-1)\hat{\sigma}_{ur}^2}{\sigma^2} = \frac{SSR_{ur}}{\sigma^2} \sim \chi^2(n-k-1).$$

Finally, since $(\mathrm{SSR}_r - \mathrm{SSR}_{ur})$ and SSR_{ur} are independent of each other, we have the following result:

$$\begin{aligned}\frac{(\mathrm{SSR}_r - \mathrm{SSR}_{ur})/q}{\mathrm{SSR}_{ur}/(n-k-1)} &= \frac{SSR_r - SSR_{ur}}{\sigma^2 q} \Big/ \frac{SSR_{ur}}{\sigma^2(n-k-1)} \\ &\sim \frac{\chi^2(q)/q}{\chi^2(n-k-1)/(n-k-1)} \sim F(q, n-k-1). \quad \blacksquare\end{aligned}$$

(b)

$$\begin{aligned}F &= \frac{(\mathrm{SSR}_r - \mathrm{SSR}_{ur})/q}{\mathrm{SSR}_{ur}/(n-k-1)} = \frac{(\frac{\mathrm{SSR}_r}{\mathrm{SST}} - \frac{\mathrm{SSR}_{ur}}{\mathrm{SST}})/q}{\frac{\mathrm{SSR}_{ur}}{\mathrm{SST}}/(n-k-1)} \\ &= \frac{(1-R_r^2-1+R_{ur}^2)/q}{(1-R_{ur}^2)/(n-k-1)} = \frac{(R_{ur}^2-R_r^2)/q}{(1-R_{ur}^2)/(n-k-1)}. \quad \blacksquare\end{aligned}$$

2. (a)

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 & -6 & 0 \\ 0 & -2 & 1 & 0 & 4 & 0 \\ 1 & 3 & 0 & 1 & 0 & 5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ -1 \\ 7 \end{bmatrix}$$

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(b) Under the null hypothesis $\mathbf{R}\boldsymbol{\beta} = \mathbf{c}$, the test statistic F and its distribution is given by:

$$F = \frac{(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{c})' \left[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\right]^{-1} (\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{c})}{6\hat{\sigma}^2} \sim F(6, 144).$$

3. (a) $y_i = \alpha_0 + \alpha_1 D_{i1} + \alpha_2 D_{i2} + \beta_0 x_i + \beta_1(x_i D_{i1}) + \beta_2(x_i D_{i2}) + u_i$

(b) i. When $D_{i1} = 1, D_{i2} = 0$: $y_i = (\alpha_0 + \alpha_1) + (\beta_0 + \beta_1)x_i + u_i$

ii. When $D_{i1} = 0, D_{i2} = 1$: $y_i = (\alpha_0 + \alpha_2) + (\beta_0 + \beta_2)x_i + u_i$

iii. When $D_{i1} = 0, D_{i2} = 0$: $y_i = \alpha_0 + \beta_0 x_i + u_i$

4. (a) An 1% increase in "expendA" will lead to an $0.01\beta_1$ unit increase for "voteA".

(b) $H_0: \ \beta_1 + \beta_2 = 0$.

(c) Let $\mathbf{R} = (0, 1, 1, 0)$, and $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)'$. Then under the null hypothesis, our test statistic t and its distribution is then given by:

$$t = \frac{\mathbf{R}\hat{\boldsymbol{\beta}}}{\sqrt{\mathbf{R}\widehat{\mathrm{Var}}(\hat{\boldsymbol{\beta}})\mathbf{R}'}} = \frac{\mathbf{R}\hat{\boldsymbol{\beta}}}{\hat{\sigma}\sqrt{\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'}} \sim t(n-4)$$

(d) $H_0: \ \beta_1 = \beta_2 = 0$.

(e) Let $\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)'$. Then under the null hypothesis, our test statistic F and its distribution is then given by:

$$F = \frac{(\mathbf{R}\hat{\boldsymbol{\beta}})' \left[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\right]^{-1} (\mathbf{R}\hat{\boldsymbol{\beta}})}{2\hat{\sigma}^2} \sim F(2, n-4).$$