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1.

(a) $f(x_i|\beta_0,\beta_1,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y_i-\beta_0-\beta_1 x_i)^2}{2\sigma^2}}$

$$L = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y_i-\beta_0-\beta_1 x_i)^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \prod_{i=1}^{N} e^{\frac{-(y_i-\beta_0-\beta_1 x_i)^2}{2\sigma^2}}$$

$$\Rightarrow L_n = N\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(y_i-\beta_0-\beta_1 x_i)^2$$

$$\frac{\partial L_n}{\partial \beta_0} = \frac{1}{\sigma^2}\sum_{i=1}^{N}(y_i-\beta_0-\beta_1 x_i) = 0 \Rightarrow \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

$$\frac{\partial L_n}{\partial \beta_1} = \frac{1}{\sigma^2}\sum_{i=1}^{N}(y_i-\beta_0-\beta_1 x_i)x_i = 0 \Rightarrow \hat{\beta}_1 = \frac{\sum_1^N x_i y_i}{\sum_1^N x_i^2}$$

) same as OLS estimator #

(b) Same procedure as (a), we can derive $L_n = -N\ln\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$

$$= -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$\frac{\partial L_n}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\mu) = 0 \Rightarrow \hat{\mu} = \bar{x}$$ $\Rightarrow$ same as OLS

$$\frac{\partial L_n}{\partial \sigma^2} = -\frac{N}{2\sigma^2} - \frac{1}{2\sigma^4}\sum(x_i-\mu)^2 = 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{N}\sum(x_i-\hat{\mu})^2$$ $\Rightarrow$ different from OLS, which considering DOF #

2. $\Phi(x_i'\theta) = \int_{-\infty}^{x_i'\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv$, $\frac{d\Phi(u)}{du} = \phi(u)$

$$\nabla L_n(\theta) = \sum_1^n \frac{y_i-\Phi}{\Phi(1-\Phi)}\phi \cdot x_i$$

$$H(\theta_0) = E(\nabla^2 \ln L(\theta_0)) = E\left[\nabla\left(\frac{y_i-\Phi}{\Phi(1-\Phi)}\phi\right)x_i\right] = -E(\phi x_i x_i')$$

$$B(\theta_0) = Var\left(\frac{y_i-\Phi}{\Phi(1-\Phi)}\phi x_i\right) = E\left(\left(\frac{y_i-\Phi}{\Phi(1-\Phi)}\right)^2 \phi^2 x_i x_i'\right) = E\left(\frac{\phi^3}{\Phi^2(1-\Phi)^2} x_i x_i'\right)$$

conditional variance

$$= E(\phi x_i x_i')$$

3.

$$F(x_i';\beta) = G(x_i'\beta) = \frac{1}{1+e^{(-\beta_0-\beta_1 x_i)}}$$

$$\frac{\partial G}{\partial \beta_0} = \ln(1+e^{(\ )})\cdot e^{(\ )}\cdot(-1)$$

$$\frac{\partial G}{\partial \beta_1} = \ln(1+e^{(\ )})\cdot e^{(\ )}\cdot(-x_i)$$

$$L_n(\theta) = \sum_{i=1}^{12}\left[y_i \ln G + (1-y_i)\ln(1-G)\right]$$

$$\frac{\partial L_n}{\partial \beta_0} = \sum\left[y_i \frac{\frac{\partial G}{\partial \beta_0}}{G} - (1-y_i)\frac{\frac{\partial G}{\partial \beta_0}}{1-G}\right]\cdot \hat{\beta}_0 = 0$$

$$\Rightarrow 1\cdot\frac{-\ln(1+e^{-(\beta_0+\beta_1)})e^{-(\beta_0+\beta_1)}}{\frac{1}{1+e^{-(\beta_0+\beta_1)}}}\cdot 3 + \frac{\ln(1+e^{-(\beta_0+\beta_1)})\cdot e^{-(\beta_0+\beta_1)}}{\frac{e^{(\beta_0+\beta_1)}}{1+e^{-(\beta_0+\beta_1)}}}\cdot 3 - \frac{\ln(1+e^{-\beta_0})e^{-\beta_0}}{\frac{1}{1+e^{-\beta_0}}}\cdot 4 + \frac{\ln(1+e^{-\beta_0})e^{-\beta_0}}{\frac{e^{\beta_0}}{1+e^{-\beta_0}}}\cdot 2 = 0$$

$$\frac{\partial L_n}{\partial \beta_1} = \ldots$$

$$= 1\cdot\frac{-\ln(1+e^{-(\beta_0+\beta_1)})e^{-(\beta_0+\beta_1)}}{\frac{1}{1+e^{-(\beta_0+\beta_1)}}}\cdot 3 + \frac{\ln(1+e^{-(\beta_0+\beta_1)})\cdot e^{-(\beta_0+\beta_1)}}{\frac{e^{(\beta_0+\beta_1)}}{1+e^{-(\beta_0+\beta_1)}}}\cdot 3 = 0$$

Based on the above 2 equation, $\hat{\beta}_0$ & $\hat{\beta}_1$ can be solved.