Problem Set 4: Solution

Part One: Hand-Written Exercise

1. (a) False. Recall that $\hat{\sigma}^2_{OLS} = \frac{1}{n-k-1}\sum \hat{e}_i^2$ is an unbiased estimator of σ^2. Hence, for $s \in \mathbb{R}$, $\hat{\sigma}^2(s) = \frac{1}{n-s}\sum \hat{e}_i^2$ is a biased estimator as long as $s \neq k+1$. However, $\forall s \in \mathbb{R}$, $\hat{\sigma}^2(s)$ is consistent for σ^2.

   (b) False. Consider a case where we have the data $x_i,\ i = 1, ..., n$, and the true population mean $\mu_x = 0$. Our estimator is designed as:

$$\hat{\mu}_n = \begin{cases} -1 & \text{with } \ p = 1/2 \\ 1 & \text{with } \ p = 1/2 \end{cases}.$$

   This estimator, although completely disregards the data x_i, is still unbiased. It is, however, not consistent ($\lim\limits_{n\to\infty} \hat{\mu}_n \neq 0$).

   (c) False. CLT can work only when the population mean and variance exist. For example, CLT doesn't work if the original distribution is t(1) or t(2).

   (d) False. Take the OLS estimator $\hat{\beta}$ for example, a $\hat{\beta}$ that is weakly consistent for β_0 does not automatically imply that $\text{Var}(x_i u_i) = \mathbf{V}$ exist. Therefore, $\mathbf{V}^{-\frac{1}{2}} \frac{1}{\sqrt{n}} \sum x_i u_i$ may not exist asymptotically, let along following a normal distribution.

   (e) True. Since if $\hat{\beta}$ is weakly consistent for $\alpha_0 \neq \beta_0$, or if $\hat{\beta}$ diverge, then $\lim\limits_{n\to\infty} \sqrt{n}(\hat{\beta} - \beta_o) = \pm\infty$.

2. We know that

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\
&= \frac{1}{n}\sum_{i=1}^{n}(b_0 + b_1 x_i + u_i) - \hat{\beta}_1 \bar{x} \\
&= b_0 + b_1 \bar{x} + \frac{1}{n}\sum_{i=1}^{n} u_i - \hat{\beta}_1 \bar{x}
\end{aligned}$$

   Since

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} u_i \xrightarrow{p} E(u_i) = 0 \quad \text{(WLLN and assumption(ii))} \\
&\bar{x} \xrightarrow{p} \mu_x \quad \text{(WLLN)} \\
&\hat{\beta}_1 \xrightarrow{p} b_1
\end{aligned}$$

Therefore, $\hat{\beta}_0 \xrightarrow{p} b_0 + b_1\mu_x + 0 - b_1\mu_x = b_0$

3. (a) By the given assumptions, we have

$$\begin{aligned}
\hat{\gamma}_1 &= \frac{\sum w_i y_i}{\sum w_i^2} = \frac{\sum (x_i + \epsilon_i)(\beta_1 x_i + u_i)}{\sum (x_i^2 + 2x_i\epsilon_i + \epsilon_i^2)} \\
&= \frac{\beta_1 \frac{1}{n}\sum x_i^2 + \frac{1}{n}\sum x_i u_i + \beta_1 \frac{1}{n}\sum x_i \epsilon_i + \frac{1}{n}\sum \epsilon_i u_i}{\frac{1}{n}\sum x_i^2 + \frac{2}{n}\sum x_i\epsilon_i + \frac{1}{n}\sum \epsilon_i^2} \\
&\xrightarrow{p} \beta_1 \cdot \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\epsilon^2}
\end{aligned}$$

(b) $\hat{\gamma}_1$ is consistent for β_1 only when $\sigma_\epsilon^2 = 0$, that is, there is actually no measurement error. Otherwise $\hat{\gamma}_1$ is not consistent. Moreover, when $\sigma_\epsilon^2 \neq 0$, then $\hat{\gamma}_1$ underestimates β_1 by $\beta_1 \cdot \sigma_\epsilon^2/(\sigma_x^2 + \sigma_\epsilon^2)$ as $n \to \infty$.

(c) As $\sigma_x^2 \to \infty$, $\hat{\gamma}_1 \xrightarrow{p} \beta_1$. Since $\sigma_x^2 \to \infty$ means that the variation of our independent variable x_i becomes infinitely large, in this case, the measurement error dose not matter anymore.

(d) As $\sigma_\epsilon^2 \to \infty$, $\hat{\gamma}_1 \xrightarrow{p} 0$. Since $\sigma_\epsilon^2 \to \infty$ means that the observable variable $w_i = x_i + \epsilon_i$ is extremely noisy. The information in x_i is completely dominated by the noise in ϵ_i. Therefore,

$$\hat{\gamma}_1 \xrightarrow{p} \frac{\text{Cov}(w_i y_i)}{\text{Var}(w_i)} \to 0.$$

4. (a)

$$\begin{aligned}
\tilde{\alpha}_1 &= \frac{\sum (x_{i1} - \bar{x}_1) y_i}{\sum (x_{i1} - \bar{x}_1)^2} = \frac{\sum (x_{i1} - \bar{x}_1)(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + u_i)}{\sum (x_{i1} - \bar{x}_1)^2} \\
&= \beta_1 + \beta_2 \frac{\sum (x_{i1} - \bar{x}_1) x_{2i}}{\sum (x_{i1} - \bar{x}_1)^2} + \frac{\sum (x_{i1} - \bar{x}_1) u_i}{\sum (x_{i1} - \bar{x}_1)^2}.
\end{aligned}$$

By the given assumptions, we have:

$$\begin{aligned}
&\frac{1}{n}\sum (x_{i1} - \bar{x}_1) u_i \xrightarrow{p} \mathbb{E}(x_1 u) - \mu_{x1}\mathbb{E}(u) = 0 \\
&\frac{1}{n}\sum (x_{i1} - \bar{x}_1)^2 \xrightarrow{p} \sigma_{x_1}^2 \\
&\frac{1}{n}\sum (x_{i1} - \bar{x}_1) x_{i2} = \frac{1}{n}\sum (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2) \xrightarrow{p} \sigma_{x_1 x_2}.
\end{aligned}$$

Therefore, we have:

$$\tilde{\alpha}_1 \xrightarrow{p} \beta_1 + \beta_2 \frac{\sigma_{x_1 x_2}}{\sigma_{x_1}^2}.$$

$\tilde{\alpha}_1$ is consistent for β_1 only when $\sigma_{x_1 x_2} = 0$, otherwise $\tilde{\alpha}_1$ is not consistent.

(b) Since $\sigma_{x_1x_2} > 0$, then if $\beta_2 > 0$, $\tilde{\alpha}_1$ overestimates β_1 by $\beta_2 \frac{\sigma_{x_1x_2}}{\sigma_{x_1}^2}$ as $n \to \infty$. On the other hand, if $\beta_2 < 0$, then $\tilde{\alpha}_1$ underestimates β_1 by $-\beta_2 \frac{\sigma_{x_1x_2}}{\sigma_{x_1}^2}$ as $n \to \infty$.

(c) $\sqrt{n}(\tilde{\alpha}_1 - \beta_1)$ does not follow any distributions. Since $\sigma_{x_1x_2} > 0$, thus

$$\tilde{\alpha} - \beta_1 \xrightarrow{p} \beta_2 \frac{\sigma_{x_1x_2}}{\sigma_{x_1}^2} \neq 0.$$

So $\sqrt{n}(\tilde{\alpha}_1 - \beta_1)$ clearly diverge as $n \to \infty$.

5.

$$\mathbf{R}\tilde{\mathbf{D}}\mathbf{R}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1(k+1)} \\ d_{21} & d_{22} & \cdots & d_{2(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{(k+1)1} & d_{(k+1)2} & \cdots & d_{(k+1)(k+1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} d_{21} & d_{22} & \cdots & d_{2(k+1)} \\ d_{31} & d_{32} & \cdots & d_{3(k+1)} \\ d_{41} & d_{42} & \cdots & d_{4(k+1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} d_{22} & d_{23} & d_{24} \\ d_{32} & d_{33} & d_{34} \\ d_{42} & d_{43} & d_{44} \end{bmatrix}$$