Spring 2020

Quantitative Analysis
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Problem Set 6

Due: 4/13

Part One: Hand-Written Exercise

1. Please verify the fact that

$$\left(\mathbf{A}-\mathbf{x}_i\mathbf{x}_i'\right)\left(\mathbf{A}^{-1}+\frac{\mathbf{A}^{-1}\mathbf{x}_i\mathbf{x}_i'\mathbf{A}^{-1}}{1-\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i}\right)=\mathbf{I}$$

for any invertible matrix $\mathbf{A}$ that has the same dimension as $\mathbf{x}_i\mathbf{x}_i'$.

$\left(\mathbf{A}-\mathbf{x_i}\mathbf{x_i}'\right)\left(\mathbf{A}^{-1}+\frac{\mathbf{A}^{-1}\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}}{1-\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i}}\right)$

$$\begin{aligned}
&=\mathbf{A}\mathbf{A}^{-1}+\frac{\mathbf{A}\mathbf{A}^{-1}\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}}{1-\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i}}-\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}-\frac{\mathbf{x_i}(\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i})\mathbf{x_i}'\mathbf{A}^{-1}}{1-\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i}}\\
&=\mathbf{I}+\frac{\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}-(1-\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i})\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}-(\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i})\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}}{1-\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i}}\\
&=\mathbf{I}+\frac{\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}-\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}+\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i}\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}-\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i}\mathbf{x_i}\mathbf{x_i}'\mathbf{A}^{-1}}{1-\mathbf{x_i}'\mathbf{A}^{-1}\mathbf{x_i}}\\
&=\mathbf{I}
\end{aligned}$$

2. Suppose that we obtain a bootstrap sample from a set of N observations.

   (a) For $i=1,...,N$ and $j=1,...,N$, what is the probability that the ith bootstrap observation is *not* the jth observation form the original samples? Dose your answer depends on i or j? Justify your answer.
   For this situation, the i^{th} bootstrap observation can take anyone except the j^{th} of the N original observations. Thus, the probability is $\frac{N-1}{N}$, which doesn't depend on i and j.

   (b) What is the probability that the jth observation form the original samples is *not* in the N bootstrap samples? Justify your answer.
   The j^{th} observation from original sample is not selected for N times. So the probability is $(\frac{N-1}{N})^N$

   (c) Continue with part (b), calculate the probability for $N=5$ and $N=5000$.
   When $N=5$, $(\frac{5-1}{5})^5=0.32768$;
   When $N=5000$, $(\frac{5000-1}{5000})^{5000}=0.36784$.

(d) Continue with part (b), calculate the probability when $N \to \infty$.
$\lim_{N\to\infty}(\frac{N-1}{N})^N = e^{-1} = 0.36788$.