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21MA101
ENGINEERING MATHEMATICS – I

Department: MATHEMATICS
Batch/Year:2021 – 2022/ I
Created by: Department of Mathematics
Date: 09.10.2021

Table of Contents

Course Objectives

S. No.	Course Objectives
The syllabus is designed to:	
1	Explain the concepts of matrix algebra.
2	Make the students understand the idea of curvature, evolutes and envelopes.
3	Impart the knowledge of functions of several variables.
4	Introduce the concepts of Gamma and Beta integral.
5	Develop an understanding on the basics of multiple integrals.

Prerequisites

Subject Code: **21MA101**
Subject Name: ENGINEERING MATHEMATICS – I
Prerequisites
To learn engineering mathematics one has to be strong in mathematics including the basic concepts of algebra, trigonometry, geometry and precalculus.
Definition of differentiation and integration from the first principle and how to use some properties and rules to find the derivatives and integration of more complicated functions.

Syllabus

21MA101	ENGINEERING MATHEMATICS – I	L T P C 3 2 0 4

UNIT I MATRICES **9+6**

Eigenvalues and Eigenvectors of a real matrix – Characteristic equation – Properties of eigenvalues and eigenvectors – Statement and applications of Cayley-Hamilton Theorem – Diagonalization of matrices by orthogonal transformation – Reduction of a quadratic form to canonical form by orthogonal transformation – Nature of quadratic forms.

UNIT II APPLICATIONS OF DIFFERENTIAL CALCULUS **9+6**

Curvature in Cartesian and Polar Co-ordinates – Centre and radius of curvature – Circle of curvature – Evolutes – Envelopes (excluding Evolute as envelope of normals).

UNIT III FUNCTIONS OF SEVERAL VARIABLES **9+6**

Limits – Continuity – Partial derivatives (excluding Euler's theorem) – Total derivative – Differentiation of implicit functions – Jacobian and properties – Taylor's series for functions of two variables – Maxima and minima of functions of two variables – Lagrange's method of undetermined multipliers.

UNIT IV GAMMA, BETA INTEGRALS AND APPLICATIONS **9+6**

Gamma and Beta Integrals – Properties – Relation between Gamma and Beta functions, Evaluation of integrals using Gamma and Beta functions.

UNIT V MULTIPLE INTEGRALS **9+6**

Double integrals – Change of order of integration – Double integrals in polar coordinates – Area enclosed by plane curves – Triple integrals – Volume of solids.

TOTAL: 75 PERIODS

Course Outcomes

CO's	Course Outcomes	Highest Cognitive Level
After the successful completion of the course, the student will be able to:		
C101.1	Diagonalize a matrix by orthogonal transformation.	K2
C101.2	Determine the Evolute and Envelope of curves.	K3
C101.3	Examine the maxima and minima of function of several variables.	K3
C101.4	Apply Gamma and Beta integrals to evaluate improper integrals.	K3
C101.5	Evaluate the area and volume by using multiple integrals.	K3

CO-PO/CO-PSO Mapping

CO's	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1	2	1	1	1	1	1	-	-	-	-	-	1
CO2	3	2	1	-	-	1	-	-	-	-	-	1
CO3	2	1			1	1	1	-	-	-	-	1
CO4	2	2	1	1	1	-	-	-	-	-	-	1
CO5	3	2	1	1	1	1	-	-	-	-	-	1

CO's	PSO1	PSO2	PSO3
CO1	-	-	-
CO2	-	-	-
CO3	-	-	-
CO4	-	-	-
CO5	-	-	-
CO6	-	-	-

Lecture Plan

S. No	Topics to be covered	No. of periods	Proposed Date	Actual Date	Pertain -ing CO	Taxonomy Level	Mode of Delivery
1.	Introduction-Matrices	1			CO1	K1	PPT, Chalk & Talk
2.	Eigenvalues and Eigenvectors of a Real Matrix	2			CO1	K1	Chalk and Talk
3.	Properties of Eigenvalues and Eigenvectors	1			CO1	K2	PPT, Chalk & Talk
4.	Statement and Applications of Cayley-Hamilton Theorem	2			CO1	K2	PPT, Chalk & Talk
5.	Diagonalization of Matrices by Orthogonal Transformation	2			CO1	K2	PPT, Chalk & Talk
6.	Reduction of a Quadratic Form to Canonical Formby Orthogonal Transformation	2			CO1	K2	PPT, Chalk & Talk
7.	Nature of Quadratic Forms	1			CO1	K1	PPT, Chalk & Talk

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Activity Based Learning

S. No	Topic	Pertain-ing CO's	Highest Cognitive Level	Activity
1	Eigenvalues and Eigenvectors (Group)	CO1	K2	1. Students work in small groups to calculate the eigenvalues and eigenvectors for a designated 2×2 or 3×3 matrix. 2. Students then discuss the results of their calculations.
2	Eigenvalues and Eigenvectors (Individual)	CO1	K2	Test Yourself!

1. **Eigenvalues and eigenvectors each group will be assigned one of the following matrices.**

This is a small group activity for groups of 3-4. The students will be given one of 10 matrices. The students are then instructed to find the eigenvectors and eigenvalues for this matrix and record their calculations on board. In the class discussion that follows students report their finding and compare and contrast the properties of the eigenvalues and eigenvectors they find. Two topics that should specifically discussed are the case of repeated eigenvalues (degeneracy) and complex eigenvectors, e.g., in the case of some pure rotations, special properties of the eigenvectors and eigenvalues of hermitian matrices, common eigenvectors of commuting operators.

Activity Based Learning

Students' Task

Estimated Time: 15 min for student task, 30 min for class discussion.

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad A_6 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad A_7 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \quad A_8 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad A_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For your matrix: 1. Find the eigenvalues.

2. Find the (unnormalized) eigenvectors.

3. Normalize your eigen state.

4. Describe what this transformation does.

When you are finished, write your solutions on the board. If you finish early, try another matrix with a different structure, i.e. real vs. complex entries, diagonal vs. non-diagonal, 2 × 2 vs. 3 × 3, with vs. without explicit dimensions.

2. To Remember the concepts: Test Yourself (Individual):

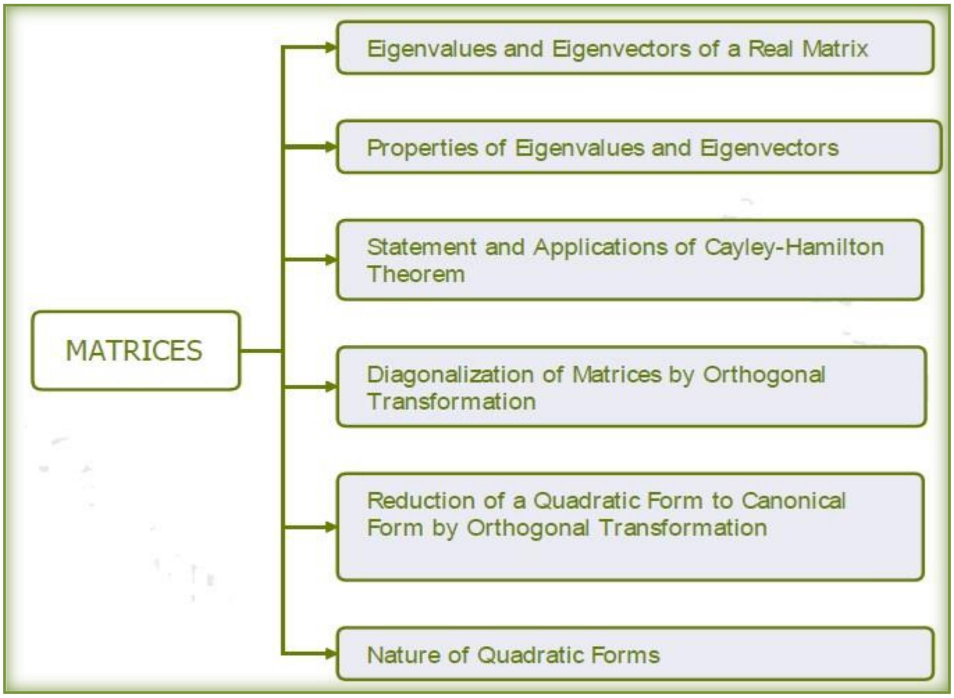
https://edurev.in/course/quiz/attempt/-1_Eigenvalues-And-Eigenvectors-MCQ-Test-1/49b0110f-d8a9-4263-a5e6-8963dffe185f

UNIT I

MATRICES

1. INTRODUCTION

The application of the mathematics lies in the formulation of the physical problem into a mathematical model. Most of such models are related to one concept known as matrices which are useful in other branches of mathematics applied to all branches of engineering and technology. A very few branches worth mentioning are mechanical engineering, electrical engineering, electronics engineering, instrumentation and control engineering, information technology and especially in the field of information technology, the matrices plays a significant role in the branches like data structure, data compression, data mining, image processing, network theory parallel computing, grid computing and so on.



In this chapter, the basic terminologies related to matrices are given. The main focus is on eigenvalues and eigenvectors. All the properties related to eigenvalues and eigenvectors has been listed out and proved with numerous examples. Using eigenvalues and eigenvectors, the process of the diagonalisation of matrices have been vigorously discussed using the orthogonal reduction process. The inversion of matrices, which are useful in solving simultaneous algebraic equations, are given in detail using Cayley-Hamilton theorem.

1.1 Eigenvalues and Eigenvectors of a Real Matrix

Let $A=\left[a_{ij}\right]$ be a square matrix of order n. Suppose there exists a non-zero (at least one entry is non zero) column vector X and a scalar λ such that $AX=\lambda X$. Then λ is called the Eigen value of the matrix A and X is called an Eigen vector corresponding to the Eigenvalue λ.

Note:

In particular A is a transformation taking X to λX.

$$AX=\lambda X \Rightarrow AX-\lambda IX=0$$
$$\Rightarrow (A-\lambda I)X=0\,(zero\,matrix) \qquad \rightarrow (1)$$

If this homogenous linear system $(A-\lambda I)X=0$ has a non-zero solution X then $A-\lambda I$ must be a singular matrix. Suppose if $A-\lambda I$ is nonsingular then its regular inverse exists.

Multiplying both sides of (1) by $\left(A-\lambda I\right)^{-1}$ we get

$$\left(A-\lambda I\right)^{-1}\left(A-\lambda I\right)X=0$$
$$\Rightarrow X=0$$

Thus we get the only solution $X=0$.Hence $A-\lambda I$ must be singular matrix and its determinant value is zero. That is $\left|A-\lambda I\right|=0$.

Characteristic Matrix:

The matrix $A-\lambda I$ is called characteristic matrix of the given matrix A which is obtained by subtracting λ from diagonal elements of A.

Characteristic Polynomial:

The determinant $\left|A-\lambda I\right|$ when expanded will give a polynomial of degree n in λ which is called characteristic polynomial of matrix A.

Characteristic Equation:

The equation $|A-\lambda I|=0$ is called characteristic equation.

Characteristic Roots (or) Eigen Values (or) Latent Roots:

The roots $\lambda_1, \lambda_2, \lambda_3, \ldots..\lambda_n$ of the characteristic equation are called characteristic roots or Eigen values or Latent roots.

Characteristic vectors (or) Eigenvectors:

Corresponding to each characteristic root λ there corresponds non-zero vector X which satisfies the equation $(A-\lambda I)X=0$. The non-zero vectors X are called characteristic vectors or Eigenvectors.

Spectrum of a Matrix:

The set of all Eigen values of A is called the spectrum of A.

Method for finding Characteristic equation

Case (i)

If A is a square matrix of order 2 then its characteristic equation can be simply written as $\lambda^2 - S_1\lambda + S_2 = 0$, where

$S_1 =$ Sum of the main diagonal elements.

$S_2 = |A|$

Case (ii)

If A is a square matrix of order 3 then its characteristic equation can be simply written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 =$ Sum of the main diagonal elements.

$S_2 =$ Sum of the minors of the main diagonal elements.

$S_3 = |A|$

Example 1: Write down the characteristic polynomial of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Solution: The characteristic polynomial of A is given by

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - 4$$
$$= 1 + \lambda^2 - 2\lambda - 4$$
$$= \lambda^2 - 2\lambda - 3$$

Example 2: Find the Eigen values of $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

The Characteristic equation of A is given by $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 =$ Sum of the main diagonal elements.

=1+1+1 =3

$S_2 =$ Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$
$$= 1 + 0 + 1$$
$$= 2$$

$$S_3 = |A| = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 0$$

Therefore, the characteristic equation is $\lambda^3 - (3)\lambda^2 + (2)\lambda - (0) = 0$

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

Solving the above equation we get the Eigen values as 0, 1, 2.

Example 3: Find the Eigen values and Eigen vectors of $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

The Characteristic equation of A is given by $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 =$ Sum of the main diagonal elements.

=3+2+5

=10

$S_2 =$ Sum of the minors of the main diagonal elements.

$$\begin{vmatrix} 2 & 6 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 10 + 15 + 6$$

$$= 31$$

$$S_3 = |A| = \begin{vmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{vmatrix} = 30$$

Therefore, the characteristic equation is $\lambda^3 - (10)\lambda^2 + (31)\lambda - (30) = 0$

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

Solving the above equation we get the Eigen values as $\lambda =$ 2, 3 and 5.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue λ.

Then $(A-\lambda I)X = 0 \quad \Rightarrow \begin{pmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (1)$

Case (i)

When $\lambda = 2$

Equation (1) gives the system of homogenous equation as

$$(1) \Rightarrow \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 + x_2 + 4x_3 = 0$$
$$6x_3 = 0$$
$$3x_3 = 0$$
$$\Rightarrow x_3 = 0 \ \& \ x_1 + x_2 = 0$$

$$x_1 = -x_2$$
$$\frac{x_1}{-1} = \frac{x_2}{1}$$

Therefore the Eigen vector corresponding to $\lambda = 2$ is given by $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Case (ii)

When $\lambda = 3$

Equation (1) gives the system of homogenous equation as

$$(1) \Rightarrow \begin{pmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow x_2 + 4x_3 &= 0 \\ -x_2 + 6x_3 &= 0 \\ +2x_3 &= 0 \end{aligned}$$

$$\Rightarrow x_3 = 0 \Rightarrow x_2 = 0$$

x_1 is arbitrary, say $x_1 = 1$

Therefore the Eigen vector corresponding to $\lambda = 3$ is given by $X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Case (iii) When $\lambda = 5$

Equation (1) gives the system of homogenous equation as

$$(1) \Rightarrow \begin{pmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow 2x_1 + x_2 + 4x_3 &= 0 \\ -3x_2 + 6x_3 &= 0 \end{aligned}$$

Solving the two equations by cross multiplication rule, we get

$$\frac{x_1}{6+12} = \frac{x_2}{0+12} = \frac{x_3}{6-0}$$

$$\frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6}$$

Therefore the Eigen vector corresponding to $\lambda = 5$ is given by $X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Example 4: Find the Eigen values and Eigen vectors of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

The Characteristic equation of A is given by $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 =$ Sum of the main diagonal elements.

=-2+1=0

=-1

$S_2 =$Sum of the minors of the main diagonal elements.

$$\begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix}$$

$$= -6 - 3 - 12$$

$$= -21$$

$S_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$ = 45

Therefore, the characteristic equation is $\lambda^3 - (-1)\lambda^2 + (-21)\lambda - (45) = 0$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

Solving the above equation we get the Eigen values as $\lambda =$ -3,- 3 and 5.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the Eigen vector corresponding to the Eigen value λ.

Then $(A-\lambda I)X = 0 \quad \Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (1)$

Case (i) When $\lambda = 5$

Equation (1) gives the system of homogenous equation as

$$(1) \Rightarrow \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - 4x_2 - 6x_3 = 0$$
$$-x_1 - 2x_2 - 5x_3 = 0$$

Solving the first two equations by the rule of cross multiplication, we have

$$\frac{x_1}{-12-12} = \frac{x_2}{-42-6} = \frac{x_3}{28-4}$$
$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$
$$\frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Therefore the Eigen vector corresponding to $\lambda = 5$ is given by $X_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Case (ii)

When $\lambda = -3$

Equation (1) gives the system of homogenous equation as

$$(1) \Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}\Rightarrow x_1 + 2x_2 - 3x_3 &= 0\\ 2x_1 + 4x_2 - 6x_3 &= 0\\ -x_1 - 2x_2 + 3x_3 &= 0\end{aligned}$$

The system has the similar equations. Choose anyone of the equation to find the eigen vector. Setting $x_3 = 0$ in the first equation we have,

$$x_1 + 2x_2 = 0$$
$$2x_2 = -x_1$$
$$\frac{x_1}{-2} = \frac{x_2}{1}$$

Therefore the Eigen vector corresponding to $\lambda = -3$ is given by $X_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

If $x_2 = 0$ in the first equation we have,

$$x_1 - 3x_3 = 0$$
$$3x_3 = x_1$$
$$\frac{x_1}{3} = \frac{x_2}{1}$$

Therefore the Eigen vector corresponding to $\lambda = -3$ is given by $X_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Example 5: Find the Eigen values and Eigen vectors of $\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

The Characteristic equation of A is given by $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

S_1 = Sum of the main diagonal elements

=2+1-1

=2

$S_2 =$ Sum of the minors of the main diagonal elements.

$$\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$$
$$= -4 - 4 + 4$$
$$= -4$$

$$S_3 = |A| = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = -8$$

Therefore, the characteristic equation is $\lambda^3 - (2)\lambda^2 + (-4)\lambda - (-8) = 0$

$$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

Solving the above equation we get the Eigen values as $\lambda =$ 2,2 and -2.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the Eigen vector corresponding to the Eigen value λ.

Then $(A - \lambda I)X = 0 \quad \Rightarrow \begin{pmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (1)$

Case (i)

When $\lambda = -2$

Equation (1) gives the system of homogenous equation as

$$(1) \Rightarrow \begin{pmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$$
$$x_1 + 3x_2 + x_3 = 0$$
$$x_1 + 3x_2 + x_3 = 0$$

Solving the first two equations by the rule of cross multiplication, we have

$$\frac{x_1}{-2-6} = \frac{x_2}{2-4} = \frac{x_3}{12+2}$$
$$\frac{x_1}{-8} = \frac{x_2}{-2} = \frac{x_3}{14}$$
$$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7}$$

Therefore the Eigen vector corresponding to $\lambda = -2$ is given by $X_1 = \begin{pmatrix} -4 \\ -1 \\ 7 \end{pmatrix}$

Case (ii)

When $\lambda = 2$

Equation (1) gives the system of homogenous equation as

$$(1) \Rightarrow \begin{pmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -2x_2 + 2x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 3x_2 - 3x_3 = 0$$

Solving the first two equations by cross multiplication rule

$$\frac{x_1}{-2+2} = \frac{x_2}{2-0} = \frac{x_3}{0+2}$$
$$\frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2}$$
$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore the Eigen vector corresponding to $\lambda = 2$ is given by $X_2 = X_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Exercise:

1. Find the Eigen values and Eigen vectors of $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$
2. Find the Eigen values and Eigen vectors of $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
3. Find the Eigen values and Eigen vectors of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

Answers:

1. $\lambda = -2, 3, 6$ & Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.
2. $\lambda = 2, -1, -1$ & Eigen vectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
3. $\lambda = 14, 0, 0$ & Eigen vectors are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

Matrix Calculator: Try this!

- **https://www.mathsisfun.com/algebra/matrix-calculator.html**

- Use any online eigenvector calculator available and try to find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 6 \\ 3 & 6 & 9 \end{pmatrix}$, compare with the problems we have done and give your comments.

- **https://www.mathopolis.com/questions/q.html?id=17820&t=mif&qs=17820_17821_17804_17805_17806_17807_17814_17818_17819_17808_17809_17810_17811_17812_17813_17815_17816_17817_17822_17823&site=1&ref=2f616c67656272612f656967656e76616c75652e68746d6c&title=456967656e766563746f7220616e6420456967656e76616c7565#**

To Learn More:

- **https://www.mathsisfun.com/algebra/eigenvalue.html**

- **http://math.mit.edu/~gs/linearalgebra/ila0601.pdf**

1.2 Properties of eigenvalues and eigenvectors

Property 1: A square matrix A and its transpose A^T have the same eigenvalues.

Proof: Let $A=\left[a_{ij}\right]; i, j=1,2,3,\ldots,n.$

The characteristic polynomial of A is

$$|A-\lambda I|=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} \qquad (1)$$

The characteristic polynomial of A^T is

$$\left|A^T-\lambda I\right|=\begin{vmatrix} a_{11}-\lambda & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn}-\lambda \end{vmatrix} \qquad (2)$$

Determinant (2) can be obtained by changing rows into columns of determinant (1).

Therefore $|A-\lambda I|=\left|A^T-\lambda I\right|$

$\therefore$ The characteristic equations of A and A^T are identical.

$\therefore$ The eigenvalues of A and A^T are the same.

Property 2: The sum of the eigenvalues of a matrix A is equal to the sum of the principal diagonal elements of A.

Property 3: The product of the eigenvalues of a matrix A is equal to $|A|$.

Property 4: The eigenvalues of real symmetric matrix is real.

Proof: Let λ_r be an eigenvalue of a real symmetric matrix A and X_r the corresponding eigenvector.

Then, by definition $AX_r=\lambda_r X_r$ (1)

Multiplying both sides of (1) by $\bar{X}_r^{\,T}$ where $\bar{X}_r^{\,T}$ means the transpose of the conjugate of X_r.

$$\begin{aligned}\bar{X}_r^{\,T} A X_r &= \bar{X}_r^{\,T} \lambda_r X_r \\ &= \lambda_r \bar{X}_r^{\,T} X_r\end{aligned} \qquad (2)$$

Taking conjugate of (2)

$$\overline{\bar{X}_r^{\,T} A X_r} = \overline{\lambda_r \bar{X}_r^{\,T} X_r}$$
$$X_r^{\,T} A \overline{X_r} = \overline{\lambda_r} X_r^{\,T} \overline{X_r} \qquad (3)$$

Taking transpose of (3)

$$\left(X_r^{\,T} A \overline{X_r}\right)^T = \left(\overline{\lambda_r} X_r^{\,T} \overline{X_r}\right)^T$$
$$\overline{X_r}^{\,T} A^T X_r = \overline{\lambda_r}\, \overline{X_r}^{\,T} X_r$$
$$\overline{X_r}^{\,T} A X_r = \overline{\lambda_r}\, \overline{X_r}^{\,T} X_r \qquad \because A = A^T \qquad (4)$$

From (2) and (4) we have

$$\lambda_r \bar{X}_r^{\,T} X_r = \bar{\lambda}_r \bar{X}_r^{\,T} X_r$$
$$\left(\lambda_r - \bar{\lambda}_r\right) \bar{X}_r^{\,T} X_r = 0$$
$$\left(\lambda_r - \bar{\lambda}_r\right) = 0$$
$$\lambda_r = \bar{\lambda}_r$$

This shows that eigenvalues of real symmetric matrix is real.

Property 5: If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the eigenvalues of a matrix A, then

(i) $k\lambda_1, k\lambda_2, k\lambda_3, \dots, k\lambda_n$ are the eigenvalues of the matrix kA, where k is a non-zero scalar.

(ii) $\lambda_1^{\,p}, \lambda_2^{\,p}, \lambda_3^{\,p}, \dots, \lambda_n^{\,p}$ are the eigenvalues of the matrix A^p, where p is a positive integer.

(iii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots, \frac{1}{\lambda_n}$ are the eigenvalues of the inverse matrix A^{-1}, provided A is non-singular.

(iv) $\lambda_1 - k, \lambda_2 - k, \lambda_3 - k, \dots, \lambda_n - k$ are the eigenvalues of $A - kI$.

Proof:

(i) Let λ_r be an eigenvalue of A and X_r the corresponding eigenvector.

Then, by definition

$$AX_r = \lambda_r X_r \qquad (1)$$

Multiplying both sides of (1) by k,

$$kAX_r = k\lambda_r X_r \qquad (2)$$

From (2), $k\lambda_r$ is an eigenvalue of kA and the corresponding eigenvector X_r.

(ii) Let λ_r be an eigenvalue of A and X_r the corresponding eigenvector.

Then, by definition

$$AX_r = \lambda_r X_r \qquad (1)$$

Multiplying both sides of (1) by A,

$$A \times AX_r = A \times \lambda_r X_r$$

$$A^2 X_r = \lambda_r \left(AX_r \right)$$

$$A^2 X_r = \lambda_r \left(\lambda_r X_r \right)$$

$$A^2 X_r = \lambda_r^{\ 2} X_r$$

This shows that $\lambda_r^{\ 2}$ is the eigenvalue of A^2 and the corresponding eigenvector X_r.

Similarly $A^3 X_r = \lambda_r^{\ 3} X_r$ and so on.

In general, $\qquad A^p X_r = \lambda_r^{\ p} X_r$.

(iii) Premultiplying both sides of (1) by A^{-1},

$$A^{-1} AX_r = A^{-1} \lambda_r X_r$$

$$IX_r = \lambda_r A^{-1} X_r$$

$$A^{-1} X_r = \frac{1}{\lambda_r} X_r$$

This shows that $\frac{1}{\lambda_r}$ is the eigenvalue of A^{-1} and the corresponding eigenvector X_r.

(iv) Let λ_r be an eigenvalue of A and X_r the corresponding eigenvector.

Then, by definition

$$AX_r = \lambda_r X_r \qquad (1)$$

$$AX_r - kX_r = \lambda_r X_r - kX_r$$

$$\left(A - kI \right) X_r = \left(\lambda_r - k \right) X_r$$

This shows that $\lambda_r - k$ is the eigenvalue of $A - kI$ and the corresponding eigenvector X_r

Property 6: The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Property 7: The eigenvalues of a triangular matrix are the diagonal elements of the matrix.

Property 8: Two similar matrices have the same eigenvalues.

Example 1: Find the sum and product of the eigenvalues of the matrix

$$A=\begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}.$$

Solution: Sum of the eigenvalues = Sum of the diagonal elements

$= 1+0-3$

$= -2$.

Product of the eigenvalues $=|A|$

$= -1$

Example 2: If sum of two eigenvalues and trace of a 3×3 matrix A are equal. Find the value of $|A|$.

Solution: Sum of the eigenvalues = Trace of the matrix

= Sum of the diagonal elements of the matrix.

Given:

Sum of two eigenvalues = Trace of a 3×3 matrix A

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix A, then

$$\lambda_1+\lambda_2 = a_{11}+a_{22}+a_{33}$$

$$\Rightarrow \lambda_3 = 0.$$

Product of the eigenvalues = $|A|$

i.e., $\lambda_1\lambda_2\lambda_3 = |A|$

$\Rightarrow |A| = 0$.

Example 3: Find the eigenvalues of the matrix $A=\begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$ and also find the sum of the eigenvalues of A^{-1}.

Solution: Eigenvalues of the matrix $A = 3, 4, 5$.

Eigenvalues of the matrix $A^{-1} = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

Sum of the eigenvalues of $A^{-1} = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

$$= \frac{47}{60}.$$

Example 4: Find the eigenvalues of $A, A^3, 5A, A-3I, A^{-1}$ for the matrix $A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

Solution: Eigenvalues of $A = 2,3,4$.

Eigenvalues of $A^3 = 2^3, 3^3, 4^3 = 8, 27, 64$.

Eigenvalues of $5A = 5\times 2, 5\times 3, 5\times 4 = 10, 15, 20$.

Eigenvalues of $A-3I = 2-3, 3-3, 4-3 = -1, 0, 1$.

Eigenvalues of $A^{-1} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

Example 5: Two eigenvalues of a matrix $A=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to one each. Find the third eigenvalue.

Solution: Let $\lambda_1 = \lambda_2 = 1$ and λ_3 be the third eigenvalue.

Sum of the eigenvalues = Sum of the diagonal elements.

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$

$$1 + 1 + \lambda_3 = 2 + 3 + 2$$

$$\Rightarrow \lambda_3 = 5.$$

Therefore the third eigenvalue $= 5$.

Example 6: Find the sum of the squares of the eigenvalues of $A=\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

Solution: The eigenvalues of $A = 3, 2, 5$.

The eigenvalues of $A^2 = 3^2, 2^2, 5^2 = 9, 4, 25$.

The sum of squares of the eigenvalues of $A = 9 + 4 + 25 = 38$.

Example 7: If $2,-1,-3$ are the eigenvalues of the matrix A, then find the eigenvalues of A^2-2I.

Solution: Eigenvalues of $A^2 = 4,1,3$

The eigenvalues of $A^2-2I = 2,-1,1$.

Example 8: The product of two eigenvalues of a matrix $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16.Find the third eigenvalue.

Solution: Product of the eigenvalues = $|A|$

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$16\lambda_3 = 32$$

$$\Rightarrow \lambda_3 = 2.$$

Therefore the third eigenvalue is 2.

Example 9: If the eigenvalues of the matrix A of order 3×3 are $2,3,1$, then find the eigenvalues of $Adj(A)$.

Solution: We know that $A^{-1} = \frac{1}{|A|}Adj(A)$

$$Adj(A) = |A|A^{-1}.$$

$$|A| = 2\times3\times1 = 6.$$

Eigenvalues of $A^{-1} = \frac{1}{2},\frac{1}{3},\frac{1}{1}$.

Eigenvalues of $Adj(A) = |A|A^{-1} = 6\times\frac{1}{2}, 6\times\frac{1}{3}, 6\times1$

$$= 3,2,6.$$

Example 10: If 2, 3 are the eigenvalues of $A=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ k & 0 & 2 \end{bmatrix}$. Find k and third eigenvalue.

Solution: Sum of the eigenvalues = Sum of the diagonal element.

$$\lambda_1+\lambda_2+\lambda_3 = a_{11}+a_{22}+a_{33}$$

$$2+3+\lambda_3 = 2+2+2 = 6$$

$$\Rightarrow \lambda_3 = 1.$$

Product of the eigenvalues = $|A|$

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$6 = 8-2k$$

$$\Rightarrow k = 1.$$

Example 11: If λ is the eigenvalue of A, then prove that $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

Solution: We know that $AX = \lambda X$ (1)

$$(1)\times A^{-1} \Rightarrow A^{-1}AX = A^{-1}\lambda X$$

$$IX = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

Therefore $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

Exercise:

1. Find the eigenvalues of $3A+2I$ where $A = \begin{bmatrix} 5 & 4 \\ 0 & 2 \end{bmatrix}$.

2. Find the sum and product of the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

3. Two eigenvalues of $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}$ are equal and they are double the third.

   Find the eigenvalues of $\frac{A^{-1}}{2}$ and A^2.

4. Find the eigenvalues of A^3 and A^{-1} if $A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

5. The product of two eigenvalues of $A=\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is 3.Find the third eigenvalue.

6. If λ_1 and λ_2 are the eigenvalues of $A=\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$ form the matrix whose eigenvalues are λ_1^2 and λ_2^2.

7. Find the constants a and b of the matrix $A=\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ whose eigenvalues are 1, 6.

Answers:

1. $17,8$ 2. $6,1$ 3. $\frac{1}{2},\frac{1}{4},\frac{1}{4}$ and $1,4,4$ 4. $8,27,64$ and $\frac{1}{2},\frac{1}{3},\frac{1}{4}$,

5. 9 6. $A^2=\begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$ 7. If b=5 then a=2. If b=2 then a=5

References:

1.https://youtu.be/i8FukKfMKCI

1.3 Statement and Applications of Cayley - Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation."

Applications: To find (i) Inverse of a given square matrix

(ii) Higher integral powers of the given matrix.

Example 1: Verify Cayley - Hamilton theorem for the matrix $A=\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ and hence find A^4.

Solution: The characteristic equation of A is $|A-\lambda I|=0$

$$(i.e.,)\ \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$$S_1 = 6,\ S_2 = 8\ \&\ S_3 = 3.$$

The characteristic equation of A is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0.$

By Cayley Hamilton theorem $A^3 - 6A^2 + 8A - 3I = 0. \quad \rightarrow (1)$

Now, $A^2 = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$ & $A^3 = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$

Verification:

$$LHS = A^3 - 6A^2 + 8A - 3I$$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 6\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 8\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$= RHS.$$

Thus Cayley Hamilton theorem is verified.

To find A^4:

Premultiplying (1) by A,

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$\Rightarrow A^4 = 6A^3 - 8A^2 + 3A$$

$$A^4 = 6\begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 8\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 3\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}.$$

Example 2: Verify that the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ satisfies its own characteristic equation and hence find A^{-1}.

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$(i.e.,)\ \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$$S_1 = 3,\ S_2 = 1\ \&\ S_3 = 3.$$

The characteristic equation of A is $\lambda^3 - 3\lambda^2 + \lambda - 3 = 0$.

By Cayley Hamilton theorem $A^3 - 3A^2 + A - 3I = 0. \quad \rightarrow (1)$

Now $A^2 = \begin{pmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{pmatrix}$ & $A^3 = \begin{pmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{pmatrix}$

Verification:

$$LHS = A^3 - 3A^2 + A - 3I$$

$$= \begin{pmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{pmatrix} - 3\begin{pmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$= RHS.$$

Thus Cayley Hamilton theorem is verified.

To find A^4:

Premultiplying (1) by A^{-1},

$$A^2 - 3A + I - 3A^{-1} = 0$$

$$\Rightarrow 3A^{-1} = A^2 - 3A + I$$

$$3A^{-1} = \begin{pmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{pmatrix}.$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{pmatrix}.$$

Example 3: Use Cayley Hamilton theorem to find the value of the matrix given by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$, if the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$(i.e.,)\ \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$$S_1 = 5,\ S_2 = 7\ \&\ S_3 = 3.$$

The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

By Cayley Hamilton theorem $A^3 - 5A^2 + 7A - 3I = 0. \quad \rightarrow (1)$

Now the given polynomial $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$,

$$= A^5\left(A^3-5A^2+7A-3I\right)+A\left(A^3-5A^2+8A-2I\right)+I$$

$$= 0 + A\left(A^3-5A^2+7A-3I\right)+A^2+A+I$$

$$= A^2+A+I. \qquad \rightarrow (2)$$

Now $A^2 = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$

Substituting A^2 in (2), the given polynomial

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$$

$$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}.$$

Example 4: Find the eigen values of A and hence find A^n (n is a positive integer), given that $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and also find A^3.

Solution: The characteristic equation of A is $|A-\lambda I| = 0$

$$(i.e.,)\ \lambda^2 - S_1\lambda + S_2 = 0.$$

$$S_1 = 4,\ S_2 = -5.$$

The characteristic equation of A is $\lambda^2-4\lambda-5=0.$

By Cayley Hamilton theorem $A^2-4A-5I=0. \qquad \rightarrow (1)$

The eigen value of A are $\lambda = -1, 5.$

When λ^n is divided by $\lambda^2-4\lambda-5,$ let the quotient be $Q(\lambda)$ and the remainder be $a\lambda+b.$

Then $\qquad \lambda^n = \left(\lambda^2-4\lambda-5\right)Q(\lambda)+(a\lambda+b) \qquad \rightarrow (2)$

Put $\lambda=-1$ in (2), $\qquad -a+b=(-1)^n \qquad \rightarrow (3)$

Put $\lambda=5$ in (2), $\qquad 5a+b=(5)^n \qquad \rightarrow (4)$

Solving (3) and (4) we get,

$$a = \frac{5^n-(-1)^n}{6} \quad \& \quad b = \frac{5^n+5(-1)^n}{6}$$

Replacing λ by the matrix A in (2) we have

$$A^n = \left(A^2 - 4A - 5I\right)Q(A) + aA + bI$$

$$= (0)Q(A) + aA + bI \qquad \text{By (1)}$$

$$A^n = \left\{\frac{5^n - (-1)^n}{6}\right\}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \left\{\frac{5^n + 5(-1)^n}{6}\right\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To find A^3

$$A^3 = \left\{\frac{125+1}{6}\right\}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \left\{\frac{125-5}{6}\right\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 42 \\ 84 & 63 \end{pmatrix} + \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} 41 & 42 \\ 84 & 83 \end{pmatrix}.$$

Exercise:

1. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ and hence find A^{-1}.

2. Verify that the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ satisfies its own characteristic equation and hence find A^4.

3. Use Cayley Hamilton theorem to find the value of the matrix given by $A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A - 36I$, if the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$.

Answers:

1. $A^{-1} = \dfrac{1}{35}\begin{pmatrix} -4 & 11 & -5 \\ -1 & -5 & 25 \\ 6 & 1 & -10 \end{pmatrix}$

2. $A^4 = \begin{pmatrix} 209 & 208 & 208 \\ 208 & 209 & 208 \\ 208 & 208 & 209 \end{pmatrix}$

3. $A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A - 36I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

To Know more:

https://www.slideshare.net/AdilAslam4/cayleyhamilton-theorem-with-examples

https://yutsumura.com/tag/cayley-hamilton-theorem

1.4 Diagonalization of Matrices by Orthogonal Transformation

The orthogonal transformation or orthogonal reduction can be applicable only for real symmetric matrices. If A is a real symmetric matrix then the eigenvectors are linearly independent and pairwise orthogonal.

Normalize each eigenvectors by dividing each element by the square root of sum of the square of its elements and the normalized modal matrix N can be formed by these normalized eigenvectors.

Moreover the resulting matrix is orthogonal. By the property of orthogonal matrices,

$$N^{-1} = N^T$$

Hence $N^{-1}AN = D$ becomes $N^T AN = D$ where D is a diagonal matrix whose diagonal elements are the eigenvalues of A. Transforming A to D by means of the transform $N^T AN = D$ known as the orthogonal transformation or orthogonal reduction.

Working Rule for Diagonalization (Orthogonal Transformation)

Step 1: Find the characteristic equation.

Step 2: Solve the characteristic equation. To get eigenvalues or characteristic roots.

Step 3: Find the eigenvectors.

Step 4: Check eigenvectors are orthogonal to each other. That is,

$$X_1^T X_2 = 0; \;\; X_2^T X_3 = 0; \;\; X_3^T X_1 = 0.$$

Step 5: Form normalized modal Matrix N.

For example $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then normalized vector of X is $\begin{bmatrix} \dfrac{a}{\sqrt{a^2+b^2+c^2}} \\ \dfrac{b}{\sqrt{a^2+b^2+c^2}} \\ \dfrac{c}{\sqrt{a^2+b^2+c^2}} \end{bmatrix}$,

Step 6: Find N^T

Step 7: Calculate AN

Step 8: Calculate $D = N^T AN.'$

Example 1: Diagonalize the matrix $A=\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Solution: Given that $A=\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

The characteristic equation is $|A-\lambda I|=0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0,$ where

$$
\begin{aligned}
S_1 &= \text{sum of the main diagonal elements} \\
&= 8+7+3=18 \\
S_2 &= \text{Sum of the minors of main diagonal elements} \\
&= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} \\
&= (21-16)+(21-4)+(56-36) \\
&= 5+20+20=45
\end{aligned}
$$

$$
\begin{aligned}
S_3 &= \text{Determinant of A} \\
&= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} \\
&= 8(21-16)+6(-18+8)+2(24-14) \\
&= 40-60+20=0
\end{aligned}
$$

The characteristic equation is

$$
\begin{aligned}
&\lambda^3 - 18\lambda^2 + 45\lambda = 0 \\
&\lambda(\lambda-15)(\lambda-3)=0 \qquad \Rightarrow \lambda = 0,3,15 \\
&\text{The eigenvalues are } \lambda = 0,3,15
\end{aligned}
$$

The Eigen vector are given by $(A-\lambda I)X=0$

$$
\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad (1)
$$

For $\lambda = 0$, (1) becomes

$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 7x_2 - 4x_3 = 0$$
$$x_1 - 4x_2 + 3x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36}$$
$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 0$ is $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

For $\lambda = 3$, (1) becomes

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 0x_3 = 0$$

Solving the last two equations we have

$$\frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8}$$
$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$
$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 3$ is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

For $\lambda = 15,$ (1) becomes

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 8x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 - 12x_3 = 0$$

Solving the last two equations we have

$$\frac{x_1}{24+16} = \frac{-x_2}{28+12} = \frac{x_3}{56-36}$$
$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$
$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 15$ is $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Hence the Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

And the normalized vectors are

$$X_1^N = \begin{pmatrix} \frac{1}{\sqrt{1^2+2^2+2^2}} \\ \frac{2}{\sqrt{1^2+2^2+2^2}} \\ \frac{2}{\sqrt{1^2+2^2+2^2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad X_2^N = \begin{pmatrix} \frac{2}{\sqrt{2^2+1^2+(-2)^2}} \\ \frac{1}{\sqrt{2^2+1^2+(-2)^2}} \\ \frac{-2}{\sqrt{2^2+1^2+(-2)^2}} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$X_3^N = \begin{pmatrix} \dfrac{2}{\sqrt{2^2+(-2)^2+1^2}} \\ \dfrac{-2}{\sqrt{2^2+(-2)^2+1^2}} \\ \dfrac{1}{\sqrt{2^2+(-2)^2+1^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{2}{3} \\ \dfrac{-2}{3} \\ \dfrac{1}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}, \quad N^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T AN = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \times \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \times \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

2. Diagonalize the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

Solution:

Given that $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

The characteristic equation is $|A-\lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where

$S_1 =$ sum of the main diagonal elements

$= 2+1+1 = 4$

$S_2 =$ Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= (1-4)+(2-1)+(2-1)$

$= -3+1+1 = -1$

$S_3 =$ Determinant of A

$$= \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$$

$= 2(1-4) - 1(1-2) - 1(-2+1)$

$= -6+1+1 = -4$

The characteristic equation is

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$\lambda = 4, 1, -1$

The eigenvalues are $\lambda = 4, 1, -1$

The Eigen vector are given by $(A - \lambda I)X = 0$

$$A = \begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad (1)$$

For $\lambda = 4$, (1) becomes

$$-2x_1 + x_2 - x_3 = 0$$
$$x_1 - 3x_2 - 2x_3 = 0$$
$$-x_1 - 2x_2 - 3x_3 = 0$$

Solving the equations we have

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 4$ is

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda = 1,\quad (1)$ becomes

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 + 0x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + 0x_3 &= 0 \end{aligned}$$

Solving the equations we have

$$\frac{x_1}{-2-0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 1$ is

$$X_2 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda = -1,\quad (1)$ becomes

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$$

Solving the equations we have

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = -1$ is

$$X_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

Hence the Eigen vectors are

$$X_1=\begin{pmatrix}1\\1\\-1\end{pmatrix},\ X_2=\begin{pmatrix}-2\\1\\-1\end{pmatrix},\ X_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

And the normalized vectors are

$$X_1^{\ N}=\begin{pmatrix}\dfrac{1}{\sqrt{1^2+1^2+(-1)^2}}\\ \dfrac{1}{\sqrt{1^2+1^2+(-1)^2}}\\ \dfrac{-1}{\sqrt{1^2+1^2+(-1)^2}}\end{pmatrix}=\begin{pmatrix}\dfrac{1}{\sqrt{3}}\\ \dfrac{1}{\sqrt{3}}\\ \dfrac{-1}{\sqrt{3}}\end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}\quad X_2^{\ N}=\begin{pmatrix}\dfrac{-2}{\sqrt{(-2)^2+1^2+(-1)^2}}\\ \dfrac{1}{\sqrt{(-2)^2+1^2+(-1)^2}}\\ \dfrac{-1}{\sqrt{(-2)^2+1^2+(-1)^2}}\end{pmatrix}=\begin{pmatrix}\dfrac{-2}{\sqrt{6}}\\ \dfrac{1}{\sqrt{6}}\\ \dfrac{-1}{\sqrt{6}}\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\-1\end{pmatrix}$$

$$X_3^{\ N}=\begin{pmatrix}\dfrac{0}{\sqrt{0^2+(1)^2+1^2}}\\ \dfrac{1}{\sqrt{0^2+(1)^2+1^2}}\\ \dfrac{1}{\sqrt{0^2+(1)^2+1^2}}\end{pmatrix}=\begin{pmatrix}\dfrac{0}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\\ \dfrac{1}{\sqrt{2}}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

The normalized modal matrix is

$$N=\begin{pmatrix}\dfrac{1}{\sqrt{3}} & \dfrac{-2}{\sqrt{6}} & \dfrac{0}{\sqrt{2}}\\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{2}}\\ \dfrac{-1}{\sqrt{3}} & \dfrac{-1}{\sqrt{6}} & \dfrac{1}{\sqrt{2}}\end{pmatrix},\ N^T=\begin{pmatrix}\dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{-1}{\sqrt{3}}\\ \dfrac{-2}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} & \dfrac{-1}{\sqrt{6}}\\ 0 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}}\end{pmatrix}$$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & \frac{0}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3. Diagonalize the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

Solution:

Given that $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The characteristic equation is $|A - \lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where

$S_1 =$ sum of the main diagonal elements

$= 3 + 3 + 3 = 9$

$S_2 =$ Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$= (9-1) + (9-1) + (9-1)$

$= 8 + 8 + 8 = 24$

$S_3 = \text{Determinant of A}$

$$= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(9-1) - 1(3+1) + 1(-1-3)$$

$$= 24 - 4 - 4 = 16$$

The characteristic equation is

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$$

$$\lambda = 1, 4, 4$$

The eigenvalues are $\lambda = 1, 4, 4$

The Eigen vector are given by $(A - \lambda I)X = 0$

$$A = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad (1)$$

For $\lambda = 1$, (1) becomes

$$2x_1 + x_2 + x_3 = 0$$

$$x_1 + 2x_2 - x_3 = 0$$

$$x_1 - x_2 + 2x_3 = 0$$

Solving the equations we have

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 1$ is

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 4$, (1) becomes

$$-x_1 + x_2 + x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$

Solving the equations we have

Put $x_1 = 0$ in the second equation we get,

$$0 - x_2 - x_3 = 0$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 4$ is

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Again $\lambda = 4,$

Now assume $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$$X_1^T \ X_3 = 0 \Rightarrow -l + m + n = 0$$
$$X_2^T \ X_3 = 0 \Rightarrow -m + n = 0$$
$$\therefore \frac{l}{2} = \frac{m}{1} = \frac{n}{1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 4$ is

$$X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Hence the Eigen vectors are

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

And the normalized vectors are

$$X_1^N = \begin{pmatrix} \dfrac{-1}{\sqrt{(-1)^2+1^2+(1)^2}} \\ \dfrac{1}{\sqrt{(-1)^2+1^2+(1)^2}} \\ \dfrac{1}{\sqrt{(-1)^2+1^2+(1)^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{-1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad X_2^N = \begin{pmatrix} \dfrac{0}{\sqrt{(0)^2+1^2+(-1)^2}} \\ \dfrac{1}{\sqrt{(0)^2+1^2+(-1)^2}} \\ \dfrac{-1}{\sqrt{(0)^2+1^2+(-1)^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{0}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$X_3^N = \begin{pmatrix} \dfrac{2}{\sqrt{2^2+(1)^2+1^2}} \\ \dfrac{1}{\sqrt{2^2+(1)^2+1^2}} \\ \dfrac{1}{\sqrt{2^2+(1)^2+1^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{2}{\sqrt{6}} \\ \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \dfrac{-1}{\sqrt{3}} & \dfrac{0}{\sqrt{2}} & \dfrac{2}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \end{pmatrix}, N^T = \begin{pmatrix} \dfrac{-1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ 0 & \dfrac{1}{\sqrt{2}} & \dfrac{-1}{\sqrt{2}} \\ \dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} \end{pmatrix}$$

$$N^T AN = \begin{pmatrix} \dfrac{-1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ 0 & \dfrac{1}{\sqrt{2}} & \dfrac{-1}{\sqrt{2}} \\ \dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} \dfrac{-1}{\sqrt{3}} & \dfrac{0}{\sqrt{2}} & \dfrac{2}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

4. Diagonalize the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Solution:

Given that $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

The characteristic equation is $|A - \lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where

$S_1 =$ sum of the main diagonal elements

$= 6 + 3 + 3 = 12$

$S_2 =$ Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$= (9-1) + (18-4) + (18-4)$

$= 8 + 14 + 14 = 36$

$S_3 =$ Determinant of A

$$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 6(9-1) + 2(-6+2) + 2(2-6)$

$= 48 - 8 - 8 = 32$

The characteristic equation is

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 8, 2, 2$

The eigenvalues are $\lambda = 8, 2, 2$

The Eigen vector are given by $(A - \lambda I)X = 0$

$$A=\begin{pmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=0 \qquad (1)$$

For $\lambda=8$, (1) becomes

$$-2x_1-2x_2+2x_3=0$$
$$-2x_1-5x_2-1x_3=0$$
$$2x_1-1x_2-5x_3=0$$

Solving the equations we have

$$\frac{x_1}{2+10}=\frac{x_2}{-4-2}=\frac{x_3}{10-4}$$
$$\frac{x_1}{12}=\frac{x_2}{-6}=\frac{x_3}{6}$$
$$\frac{x_1}{2}=\frac{x_2}{-1}=\frac{x_3}{1}$$

The Eigen vector corresponding to the eigenvalue $\lambda=8$ is

$$X_1=\begin{pmatrix}2\\ -1\\ 1\end{pmatrix}$$

For $\lambda=2$, (1) becomes

$$4x_1-2x_2+2x_3=0$$
$$-2x_1+x_2-x_3=0$$
$$2x_1-x_2+x_3=0$$

Solving the equations we have

Put $x_1=0$ in the second equation we get,

$$0+x_2-x_3=0$$
$$\frac{x_2}{1}=\frac{x_3}{1}$$

The Eigen vector corresponding to the eigenvalue $\lambda=2$ is

$$X_2=\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$$

Again $\lambda = 2,$

Now assume $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$$X_1^T \ X_3 = 0 \Rightarrow 2l - m + n = 0$$
$$X_2^T \ X_3 = 0 \Rightarrow -m + n = 0$$
$$\therefore \frac{l}{1} = \frac{m}{1} = \frac{n}{-1}$$

The Eigen vector corresponding to the eigenvalue $\lambda = 2$ is

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Hence the Eigen vectors are

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

And the normalized vectors are

$$X_1{}^N = \begin{pmatrix} \frac{2}{\sqrt{(2)^2 + (-1)^2 + (1)^2}} \\ \frac{-1}{\sqrt{(2)^2 + (-1)^2 + (1)^2}} \\ \frac{1}{\sqrt{(2)^2 + (-1)^2 + (1)^2}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad X_2{}^N = \begin{pmatrix} \frac{0}{\sqrt{(0)^2 + 1^2 + (1)^2}} \\ \frac{1}{\sqrt{(0)^2 + 1^2 + (1)^2}} \\ \frac{1}{\sqrt{(0)^2 + 1^2 + (1)^2}} \end{pmatrix} = \begin{pmatrix} \frac{0}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$X_3{}^N = \begin{pmatrix} \frac{1}{\sqrt{1^2 + (1)^2 + (-1)^2}} \\ \frac{1}{\sqrt{1^2 + (1)^2 + (-1)^2}} \\ \frac{-1}{\sqrt{1^2 + (1)^2 + (-1)^2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

The normalized modal matrix is

$$N=\begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}}\end{pmatrix},\quad N^T=\begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\end{pmatrix}$$

$$N^TAN=\begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\end{pmatrix}\times\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\times\begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}}\end{pmatrix}$$

$$\therefore D=\begin{pmatrix}8 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$$

5. Diagonalize the matrix

$$A=\begin{pmatrix}10 & -2 & -5\\ -2 & 2 & 3\\ -5 & 3 & 5\end{pmatrix}$$

Solution:

Given that $A=\begin{pmatrix}10 & -2 & -5\\ -2 & 2 & 3\\ -5 & 3 & 5\end{pmatrix}$

The characteristic equation is $|A-\lambda I|=0 \Rightarrow \lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

Where

$S_1 =$ sum of the main diagonal elements

$= 10+2+5 = 17$

$S_2 =$ Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix}$$

$= (10-9)+(50-25)+(20-4)$

$= 1+25+16 = 42$

$S_3 =$ Determinant of A

$$= \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{vmatrix}$$

$= 10(10-9)+2(-10+15)-5(-6-10)$

$= 10+10-20 = 0$

The characteristic equation is

$\lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\lambda(\lambda-14)(\lambda-3) = 0$

$\lambda = 0, 3, 14$

The eigenvalues are $\lambda = 0, 3, 14$

The Eigen vector are given by $(A-\lambda I)X = 0$

$$A = \begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad (1)$$

For $\lambda = 0$, (1) becomes

$$10x_1 - 2x_2 - 5x_3 = 0$$
$$-2x_1 + 2x_2 + 3x_3 = 0$$
$$-5x_1 + 3x_2 + 5x_3 = 0$$

Solving the equations we have

$$\frac{x_1}{-6+10}=\frac{x_2}{10-30}=\frac{x_3}{20-4}$$

$$\frac{x_1}{4}=\frac{x_2}{-20}=\frac{x_3}{16}$$

$$\frac{x_1}{1}=\frac{x_2}{-5}=\frac{x_3}{4}$$

The Eigen vector corresponding to the eigenvalue $\lambda=0$ is

$$X_1=\begin{pmatrix}1\\-5\\4\end{pmatrix}$$

For $\lambda=3,\quad (1)$ becomes

$$\begin{aligned}7x_1-2x_2-5x_3&=0\\-2x_1-x_2+3x_3&=0\\-5x_1+3x_2+2x_3&=0\end{aligned}$$

Solving the equations we have

$$\frac{x_1}{-6-5}=\frac{x_2}{10-21}=\frac{x_3}{-7-4}$$

$$\frac{x_1}{-11}=\frac{x_2}{-11}=\frac{x_3}{-11}$$

$$\frac{x_1}{-1}=\frac{x_2}{-1}=\frac{x_3}{-1}$$

The Eigen vector corresponding to the eigenvalue $\lambda=3$ is

$$X_2=\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$

For $\lambda=14,\quad (1)$ becomes

$$\begin{aligned}-4x_1-2x_2-5x_3&=0\\-2x_1-12x_2+3x_3&=0\\-5x_1+3x_2-9x_3&=0\end{aligned}$$

Solving theequations we have

$$\frac{x_1}{-6-60}=\frac{-x_2}{10+12}=\frac{x_3}{48-4}$$

$$\frac{x_1}{-66}=\frac{x_2}{22}=\frac{x_3}{44}$$

$$\frac{x_1}{-3}=\frac{x_2}{1}=\frac{x_3}{2}$$

The Eigen vector corresponding to the eigenvalue $\lambda=14$ is

$$X_3=\begin{pmatrix}-3\\1\\2\end{pmatrix}$$

Hence the Eigen vectors are

$$X_1=\begin{pmatrix}1\\-5\\4\end{pmatrix}\quad X_2=\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\quad X_3=\begin{pmatrix}-3\\1\\2\end{pmatrix}$$

And the normalized vectors are

$$X_1{}^N=\begin{pmatrix}\frac{1}{\sqrt{1^2+(-5)^2+4^2}}\\ \frac{-5}{\sqrt{1^2+(-5)^2+4^2}}\\ \frac{4}{\sqrt{1^2+(-5)^2+4^2}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{42}}\\ \frac{-5}{\sqrt{42}}\\ \frac{4}{\sqrt{42}}\end{pmatrix}=\frac{1}{\sqrt{42}}\begin{pmatrix}1\\-5\\4\end{pmatrix}$$

$$X_2{}^N=\begin{pmatrix}\frac{-1}{\sqrt{(-1)^2+(-1)^2+(-1)^2}}\\ \frac{-1}{\sqrt{(-1)^2+(-1)^2+(-1)^2}}\\ \frac{-1}{\sqrt{(-1)^2+(-1)^2+(-1)^2}}\end{pmatrix}=\begin{pmatrix}\frac{-1}{\sqrt{3}}\\ \frac{-1}{\sqrt{3}}\\ \frac{-1}{\sqrt{3}}\end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$

$$X_3^N = \begin{pmatrix} \frac{-3}{\sqrt{(-3)^2+(1)^2+2^2}} \\ \frac{1}{\sqrt{(-3)^2+(1)^2+2^2}} \\ \frac{2}{\sqrt{(-3)^2+(1)^2+2^2}} \end{pmatrix} = \begin{pmatrix} \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}, \quad N^T = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \times \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

Exercise Problems:

1. Diagonalize the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

2. Diagonalize the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

3. Diagonalize the matrix $A=\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

4. Diagonalize the matrix $A=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

5. Diagonalize the matrix $A=\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Answers:

1. $D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ 2. $D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ 3. $D=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

4. $D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ 5. $D=\begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

1.5 Reduction of a Quadratic Form to Canonical Form By Orthogonal Transformation

Quadratic form

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

For example:

$(i)\, x^2+4xy+4y^2$ is a 2 variable quadratic form.

$(ii)\, ax^2+by^2+cz^2+2hxy+2gyz+2fzx$ is a 3 variable quadratic form.

$(iii)\, x_1^2+x_2^2+3x_3^2+x_4^2$ is a 4 variable quadratic form.

General Quadratic form

The general quadratic form in n variables $x_1, x_2, ..., x_n$ is $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_{j,}$ where a_{ij} are real numbers such that $a_{ij}=a_{ji}$ for all $i, j=1,2,3,..,n$. Usually the quadratic form is denoted by Q and so $Q=\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$.

1. Matrix form of Q

If $X=\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$, where $a_{ij}=a_{ji}$, then A is a symmetric matrix and the quadratic form $Q=\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_{ij}$ can be written as $Q=X^TAX$. A is called the matrix of the quadratic form.

For example the quadratic form $x^2+4xy+4y^2$ can be written in the matrix form $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Here $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $A=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Note: Since the quadratic form $Q=X^TAX$, it is obvious that the characteristics or properties of Q depend on the characteristics of A.

2. Canonical form of Q

A quadratic form Q which contains only the square terms of the variables is said to be in canonical form.

For example, $x^2+y^2, x^2-y^2, x^2+y^2-4z^2$ and $x_1^2+x_2^2+2x_3^2+x_4^2$ are in canonical forms because they contain only square terms.

3. Reduction of canonical form Q by orthogonal transformation

Let $Q = X^T AX$ be a quadratic form in n variables and $A=[a_{ij}]$ be the symmetric matrix of order n of the quadratic form. We will reduce A to diagonal form by an orthogonal transformation $X = NY$, where N is the normalized modal matrix of A .Then $N^T AN = D$, where D is the diagonal matrix containing the eigenvalues of A.

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, then

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_1 \end{bmatrix}$$

$$\therefore Q = X^T AX = (NY)^T A(NY)$$

$$= Y^T (N^T AN)Y$$

$$Q = Y^T DY$$

If $Y = \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{bmatrix}$, then $Q=\begin{bmatrix} y_1 & y_2 & ... & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1 y_1 & \lambda_2 y_2 & ... & \lambda_n y_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$$

This is the required quadratic form.

Note: In the canonical form the coefficients are the eigenvalues of A.Since A is a symmetric matrix, the eigenvalues of A are all real. So, the eigenvalues may be positive, negative or zero. Hence the terms of the canonical form may positive, negative or zero. By using the canonical form or the eigenvalues we can characterize the quadratic form.

Example 1: Write down the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$.

Solution: The quadratic form is $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$.

It has 3 variables x_1, x_2, x_3.

So, the matrix of the quadratic form is a 3×3 symmetric matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here

$$a_{11} = \text{coefficient of } x_1^2 = 2 \qquad a_{12} = a_{21} = \frac{1}{2}(\text{coefficient of } x_1x_2) = \frac{1}{2}(2) = 1$$

$$a_{22} = \text{coefficient of } x_2^2 = -2 \qquad a_{13} = a_{31} = \frac{1}{2}(\text{coefficient of } x_1x_3) = \frac{1}{2}(-6) = -3$$

$$a_{33} = \text{coefficient of } x_3^2 = 4 \qquad a_{23} = a_{32} = \frac{1}{2}(\text{coefficient of } x_2x_3) = \frac{1}{2}(6) = 3$$

$$\therefore \quad A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

Example 2: Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$, which is a 3×3 symmetric matrix.

So the quadratic form has 3 variables x_1, x_2, x_3.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then the quadratic form is

$$Q = X^T AX = [x_1 x_2 x_3]\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Q = [2x_1 + 4x_2 + 5x_3 4x_1 + 3x_2 + x_3 5x_1 + x_2 + x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Q = (2x_1 + 4x_2 + 5x_3)x_1 + (4x_1 + 3x_2 + x_3)x_2 + (5x_1 + x_2 + x_3)x_3$$

$$Q = 2x_1^2 + 4x_1x_2 + 5x_3x_1 + 4x_1x_2 + 3x_2^2 + x_3x_2 + 5x_1x_3 + x_2x_3 + x_3^2$$

$$Q = 2x_1^2 + 3x_2^2 + x_3^2 + 8x_1x_2 + 10x_1x_3 + 2x_2x_3$$

Example 3: Reduce the quadratic form $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$ to canonical form through an orthogonal transformation.

Solution: Given quadratic form is $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$

The matrix of the quadratic form is $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

The characteristic equation of A is $\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Where $S_1 =$ sum of the diagonal elements of $A = 1 + 5 + 1 = 7$

$S_2 =$ sum of minors of diagonals of A

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} 4 + (-8) + 4 = 0$$

$$S_3 = |A| = 1(4) - (-2) + 3(-14) = 4 + 2 - 42 = -36$$

$\therefore$ The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$

By trial λ=-2 is a root.

Other roots are given by

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 6)(\lambda - 3) = 0$

$\lambda = 6, 3$

$\therefore$ The eigen values are $\lambda = -2, 6, 3$

To find eigen vectors:

Let $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be an eigen vector corresponding to eigen value λ.

$$(A-\lambda I)X=0\Rightarrow\begin{pmatrix}1-\lambda & 1 & 3\\ 1 & 5-\lambda & 1\\ 3 & 1 & 1-\lambda\end{pmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Then

$$\left.\begin{aligned}&(1-\lambda)x_1+x_2+3x_3=0\\ \Rightarrow\ &x_1+(5-\lambda)x_2+x_3=0\\ &3x_1+x_2+(1-\lambda)x_3=0\end{aligned}\right\}\qquad\text{(I)}$$

Case (i) If $\lambda=-2$, then the equation (I) becomes

$$3x_1+x_2+3x_3=0$$
$$x_1+7x_2+x_3=0$$
$$3x_1+x_2+3x_3=0$$

From first two equations, we get

$$\frac{x_1}{1-21}=\frac{x_2}{3-3}=\frac{x_3}{21-1}$$
$$\Rightarrow\frac{x_1}{-20}=\frac{x_2}{0}=\frac{x_3}{20}$$
$$\Rightarrow\frac{x_1}{1}=\frac{x_2}{0}=\frac{x_3}{-1}$$

Choosing $x_1=-1, x_2=0, x_3=-1$we get an eigen vector $X_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$

Case (ii) If $\lambda=3$, then the equation (I) becomes

$$-2x_1+x_2+3x_3=0$$
$$x_1+2x_2+x_3=0$$
$$3x_1+x_2-2x_3=0$$

From first two equations, we get

$$\frac{x_1}{1-6}=\frac{x_2}{3+2}=\frac{x_3}{-4-1}$$
$$\Rightarrow\frac{x_1}{-5}=\frac{x_2}{5}=\frac{x_3}{-5}$$
$$\Rightarrow\frac{x_1}{1}=\frac{x_2}{-1}=\frac{x_3}{1}$$

Choosing $x_1 = 1, x_2 = -1, x_3 = 1$ we get an eigen vector $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case (iii) If $\lambda = 6$, then the equation (I) becomes

$$-5x_1 + x_2 + 3x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0$$

From first two equations, we get

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$
$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$
$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

Choosing $x_1 = 1, x_2 = 2, x_3 = 1$ we get an eigen vector $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Since the eigenvalues are different, the eigenvectors are mutually orthogonal.

Normalised eigen vectors are $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

The normalised modal matrix $N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

Then

$$N^T AN = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$N^T AN = D$$

The transformation $X = NY$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ reduces the quadratic form into

$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -2y_1^2 + 3y_2^2 + 6y_3^2$ which is a canonical form.

Example 4: Reduce the given quadratic form Q to its canonical form using orthogonal transformation $Q = x^2 + 3y^2 + 3z^2 - 2yz$.

Solution: Given quadratic form is $Q = x^2 + 3y^2 + 3z^2 - 2yz$.

The matrix of the quadratic form is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

Where $S_1 =$ sum of the diagonal elements of $A = 1+3+3 = 7$

$S_2 =$ sum of minors of elements of diagonals of A

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 9 - 1 + 3 - 0 + (3 - 0) = 14$$

$$S_3 = |A| = 1(9-1) = 8$$

$\therefore$ The characteristic equation is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

Since the sum of the coefficients is 0, $\lambda = 1$ is a root.

Other roots are given by

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 2)(\lambda - 4) = 0$$

$$\lambda = 2, 4$$

$\therefore$ The eigen values are $\lambda = 1, 2, 4$

To find eigen vectors:

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to eigen value λ of A.

$$(A - \lambda I)X = 0 \Rightarrow \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then

$$\left.\begin{aligned} (1-\lambda)x_1 + 0x_2 + 0x_3 = 0 \\ \Rightarrow 0x_1 + (3-\lambda)x_2 + (-1)x_3 = 0 \\ 0x_1 + (-1)x_2 + (3-\lambda)x_3 = 0 \end{aligned}\right\} \qquad \text{(I)}$$

Case (i) If $\lambda = 1$, then the equation (I) becomes

$$0x_1 + 0x_2 + x_3 = 0 \Rightarrow x_3 = 0$$

$$0x_1 + 2x_2 - x_3 = 0 \Rightarrow 2x_2 = x_3$$

$$0x_1 - x_2 + 2x_3 = 0 \Rightarrow 2x_2 = 2x_3$$

$$x_3 = 0 \Rightarrow x_2 = 0$$

x_1 can be any real value. We shall take $x_1 = 1$

$\therefore$ an eigenvector is $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case (ii) If $\lambda = 2$, , then the equation (I) becomes

$$-x_1 = 0 \Rightarrow x_1 = 0$$

$$0x_1 + x_2 - x_3 = 0 \Rightarrow x_2 = x_3$$

$$0x_1 - x_2 + x_3 = 0 \Rightarrow x_2 = x_3$$

Take $x_2 = 1 \therefore x_3 = 1$

$\therefore$ an eigenvector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case (iii) If $\lambda = 4$, then the equation (I) becomes

$$-3x_1 + 0x_2 + 0x_3 = 0 \Rightarrow x_1 = 0$$
$$0x_1 - x_2 - x_3 = 0 \Rightarrow x_2 = -x_3$$
$$0x_1 - x_2 - 5x_3 = 0 \Rightarrow x_2 = -x_3$$

Take $x_3 = 1 \therefore x_2 = 1$

$\therefore$ an eigenvector $X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Thus the eigenvalues are $\lambda = 1, 2, 4$ and the corresponding eigenvectors are

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Normalised eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

So, the normalised modal matrix $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & -\frac{4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}$$

Then

$$\therefore N^T AN = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The orthogonal transformation $X = NY$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ reduces the quadratic form into

$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 2y_2^2 + 4y_3^2$ which is a canonical form.

Example 5: Reduce the quadratic $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ to the canonical form by an orthogonal transformation. Find one set of values of x, y, z (not all zero) which will make the quadratic form zero.

Solution: Given quadratic form is $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$.

The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

Where $S_1 = \text{sum of the diagonal elements of } A = 8+7+3=18$

$S_2 = \text{sum of minors of diagonals of A}$

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = (21-16)+(24-4)+(56-36)=45$$

$S_3 = |A| = 8(21-16)+6(-18+18)+2(24-14)=0$

$\therefore$ The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-15)=0$$

$$\Rightarrow \lambda = 0, 3, 15$$

To find eigenvectors:

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to eigenvalue λ of A.

Then

$$(A-\lambda I)X = 0 \Rightarrow \begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (8-\lambda)x_1 - 6x_2 + 2x_3 &= 0 \\ \Rightarrow -6x_1 + (7-\lambda)x_2 + (-4)x_3 &= 0 \\ 2x_1 + (-4)x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \qquad (\mathrm{I})$$

Case (i) If $\lambda = 0$, then the equation (I) becomes

$$8x_1 - 6x_2 + 2x_3 = 0 \Rightarrow 4x_1 - 3x_2 + x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

From the first and third equations we get

$$\frac{x_1}{-9+4} = \frac{x_2}{2-12} = \frac{x_3}{-16+6}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{-10} = \frac{x_3}{-10}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Choosing $x_1 = 1, x_2 = 2, x_3 = 2$, we get an eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Similarly, we can find for $\lambda = 3\ X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and for $\lambda = 15, X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ The normalised modal matrix

$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\therefore N^T A N = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

The orthogonal transformation $X = NY$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ reduces the quadratic form

into $Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0y_1^2 + 3y_2^2 + 15y_3^2$

The transformation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\therefore \quad x = \frac{1}{3}(y_1 + 2y_2 + 2y_3)$$

$$y = \frac{1}{3}(2y_1 + y_2 - 2y_3)$$

$$z = \frac{1}{3}(2y_1 + y_2 - 2y_3)$$

Quadratic form =0

$$\Rightarrow 0y_1^2 + 3y_2^2 + 15y_3^2 = 0$$

$$\Rightarrow 3y_2^2 + 15y_3^2 = 0$$

$$\Rightarrow y_2 = 0 \text{ and } y_3 = 0$$

and y_1 can take any value, we shall choose y_1 =3

$$x = 1, y = 2, z = 2$$

Hence, this set of values will make the quadratic form=0.

Example 6: Reduce $6x^2+3y^2+3z^2-4xy-2yz+4xz$ into a canonical form by an orthogonal reduction and find the rank, signature, index and the nature of quadratic form.

Solution: Given the quadratic form is $6x^2+3y^2+3z^2-4xy-2yz+4xz$

The matrix of the quadratic form of $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow \lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where S_1 = sum of the diagonal elements of $A=6+3+3=12$

S_2 = sum of minors of elements of diagonals of A

$$=\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}+\begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}+\begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$=8+14+14=36$$

$$S_3=|A|=6(9-1)+2(-6+2)+2(2-6)=48-8-8=32$$

$\therefore$ The characteristic equation is $\lambda^3-12\lambda^2+36\lambda-32=0$

By trial, λ=2 is a root.

Other roots are given by

$$\lambda^2-10\lambda+16=0$$

$$(\lambda-2)(\lambda-8)=0$$

$$\lambda=2,8$$

$\therefore$ The eigen values are $\lambda=2,2,8$

To find eigen vectors:

Let $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to eigen value λ of A.

Then

$$(A-\lambda I)X=0 \Rightarrow \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (6-\lambda)x_1-2x_2+2x_3=0 \\ \Rightarrow -2x_1+(3-\lambda)x_2+(-1)x_3=0 \\ 2x_1+(-1)x_2+(3-\lambda)x_3=0 \end{aligned}\right\} \qquad \text{(I)}$$

Case (i) If $\lambda=8$, then the equation (I) becomes

$$\begin{aligned} &-2x_1-2x_2+2x_3=0 \Rightarrow x_1+x_2-x_3=0 \\ &-2x_1-5x_2-x_3=0 \Rightarrow 2x_1+5x_2+x_3=0 \\ &2x_1-x_2+5x_3=0 \end{aligned}$$

From the first two equations we get

$$\frac{x_1}{1+5}=\frac{x_2}{-2-1}=\frac{x_3}{5-2}$$

$$\Rightarrow \frac{x_1}{6}=\frac{x_2}{-3}=\frac{x_3}{3}$$

$$\Rightarrow \frac{x_1}{2}=\frac{x_2}{-1}=\frac{x_3}{1}$$

Choosing $x_1=2, x_2=-1, x_3=1$ we get an eigen vector is $X_1=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (ii) If $\lambda=2$, , then the equation (I) becomes

$$\begin{aligned} &4x_1-2x_2+2x_3=0 \Rightarrow 2x_1-x_2+x_3=0 \\ &-2x_1+x_2-x_3=0 \Rightarrow 2x_1-x_2+x_3=0 \\ &2x_1-x_2+x_3=0 \end{aligned}$$

So we get only one equation $2x_1-x_2+x_3=0$ (1)

Choosing $x_3=0$, we get $2x_1-x_2=0 \Rightarrow x_2=2x_1$

Choosing $x_1=1$, we get $x_2=2$

$\therefore$ an eigenvector $X_2=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$\text{Let } X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore a + 2b = 0 \Rightarrow a = -2b$$

Also X_3 satisfies (1)

$$\therefore 2a - b + c = 0$$
$$\Rightarrow -4b - b + c = 0$$
$$\Rightarrow c = 5b$$

Choosing $b = 1$ we get $c = 5$ and $a = -2$ $X_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

Thus the eigen values are 8,2,2 and the corresponding eigen vectors are

Clearly X_3 is to X_1 and X_2.

But X_1 and X_2 are also orthogonal.

The Normalised eigen vectors are $\begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix}$

So, the normalised modal matrix $N = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$

Then $N^T AN = D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

The orthogonal transformation $X = NY$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ reduces the quadratic form into $Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 4y_3^2$ which is a canonical form.

$\therefore$ Rank of the quadratic form=3,

Index=3,

Signature=3

The Quadratic form is positive definite since all the eigenvalues are positive.

Exercise:

1. Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form by orthogonal reduction.
2. Reduce the quadratic form $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2$ to canonical form by orthogonal reduction.
3. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to canonical form through anorthogonal transformation and hence show that it is positive semi-definite.

Answers:

1. $-2y_1^2 + 6y_2^2 + 6y_3^2$ 2. $-2y_1^2 + 3y_2^2 + 6y_3^2$ 3. $3y_2^2 + 15y_3^2$

To Know more:

- https://math.stackexchange.com/questions/581925/quadratic-form-in-canonical-form
- https://unacademy.com/lesson/reduction-of-quadratic-form-to-canonical-form-through-orthogonal-transformation/0ZRV2TAH
- http://egyankosh.ac.in/bitstream/123456789/17828/1/Unit-14.pdf

1.6 Nature of the quadratic form

Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, x_3, ..., x_n$

Rank of the quadratic form

Number of square terms in canonical form.

i.e., The rank of A is r, then the canonical form of Q consists only $'r'$ square terms.

Index of the quadratic form

The number of positive square terms in the canonical form

Therefore, the number of positive square terms is index = s

Signature of the quadratic form

The difference of number of positive and negative square terms

i.e., $p = s - (r - s) = 2s - r$

Types of Quadratic Forms

Quadratic forms can be classified according to the nature of the eigenvalues of the matrix of the quadratic form:

1. If all λ_i are positive, the form is said to be positive definite.
2. If all λ_i are negative, the form is said to be negative definite.
3. If all λ_i are non-negative (positive or zero), the form is said to be positive semidefinite.
4. If all λ_i are non-positive (zero or negative), the form is said to be negative semidefinite.
5. If λ_i represent a mixture of positive, zero, and negative values, the form is said to be indefinite

Direct Method

A general quadratic form of 3 variables is a real-valued function $a_{11}x^2 + a_{22}y^2 + a_{22}z^2 + a_{12}xy + a_{13}xz + a_{23}yz.$

Here

$$a_{11} = \text{coefficient of } x^2 \qquad a_{12} = a_{21} = \frac{1}{2}(\text{coefficient of xy})$$

$$a_{22} = \text{coefficient of } y^2 \qquad a_{13} = a_{31} = \frac{1}{2}(\text{coefficient of xz})$$

$$a_{33} = \text{coefficient of } z^2 \qquad a_{23} = a_{32} = \frac{1}{2}(\text{coefficient of yz})$$

As we see a quadratic form is determined by the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$D_1 = |a_{11}|$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

SI. No	Nature	Conditions
1	Positive definite	$D_i > 0,\ i = 1, 2, 3 \ldots, n$ (or) All eigenvalues are positive
2	Negative definite	$(-1)^i D_i > 0$ for all n $i.e., D_1, D_3, D_5, \ldots\ldots$ are all negative $D_2, D_4, D_6, \ldots\ldots$ are all positive (or) All eigenvalues are negative
3	Positive semi definite	$D_i \geq 0$ and at least one value is zero (or) All the eigenvalues are positive and at least one value is zero
4	Negative semi definite	$(-1)^i D_i \geq 0$ and at least one value is zero (or) All the eigenvalues are negative and at least one value is zero
5	Indefinite	All other cases

Example 1: Prove that the quadratic form $x_1^3+2x_2^2+3x_3^2+2x_1x_2+2x_2x_3-2x_1x_3$ is indefinite.

Solution: $A=\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 3\end{bmatrix}$

$$D_1=1$$

$$D_2=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1$$

$$D_3=\begin{vmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 3\end{vmatrix}=-2$$

$$D_1=+ve$$
$$D_2=+ve$$
$$D_3=-ve$$

$\therefore$ The nature is indefinite

Example 2: Discuss the nature of the quadratic form $2x_1x_2+2x_2x_3-2x_1x_3$ without reducing to the canonical form

Solution: $A=\begin{bmatrix}0 & 1 & -1\\ 1 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}$

$$D_1=0$$

$$D_2=\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}=-1$$

$$D_3=\begin{vmatrix}0 & 1 & -1\\ 1 & 0 & 1\\ -1 & 1 & 0\end{vmatrix}=-2$$

$$D_1=0$$
$$D_2=-ve$$
$$D_3=-ve$$

$\therefore$ The nature is indefinite

Example 3: Find the index, signature and the nature of the quadratic form $x_1^2+2x_2^2-3x_3^2$

Solution: $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -3\end{bmatrix}$

$D_1=1$

$D_2=\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}=2$

$D_3=\begin{vmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -3\end{vmatrix}=-6$

$D_1=+ve$
$D_2=+ve$
$D_3=-ve$

$\therefore$ The nature is indefinite

Index $=2$

Signature $=2-1=1$

Example 4. Determine the nature of the quadratic form $f(x_1,x_2,x_3)=x_1^2+2x_2^2$

Solution:

$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$

$D_1=1$

$D_2=\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}=2$

$D_3=\begin{vmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{vmatrix}=0$

Since D_i's are both positive and zero , the nature is semi positive definite.

Exercise

1. Find the index, signature and nature of the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 - 4x_1x_2 + 6x_2x_3 - 10x_1x_3$
2. Find the nature of the quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$
3. Find the nature of the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$

Answers:

1. Index - 2, Signature - 2, Rank - 2. Nature – Positive semi definite
2. Nature – Positive definite
3. Nature – Positive definite

Additional Problems:

Exercise 1.1: Eigenvalues and Eigenvectors of a Real Matrix

Find the Eigen values and Eigen vectors of the following Problems

1. $\begin{pmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

5. $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

Answers:

1. $\lambda = -3, 3, 9$ & Eigen vectors are $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

2. $\lambda = 0, -1, 2$ & Eigen vectors are $\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$.

3. $\lambda = 1, 3, -4$ & Eigen vectors are $\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$.

4. $\lambda = 5, 1, 1$ & Eigen vectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

5. $\lambda = 2, 3, 6$ & Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Exercise 1.2: Properties of Eigenvalues and Eigenvectors

1. Let A be a 3x3 Matrix with eigenvalues 1, -1, 0. Then the determinant of I + A^{100} is
2. A is a singular matrix of order 3, 2 and 3 are the eigenvalues. Find the third eigenvalue.
3. If the sum of the eigenvalues and trace of a 3x3 matrix A are equal. Then the value of $|A| = ?$
4. If the sum of the eigenvalues of the matrix of the quadratic form equal to zero, then what will be the nature of the quadratic form
5. If $A = \begin{pmatrix} a & b & c \\ x & y & z \\ u & v & w \end{pmatrix}$ is an orthogonal matrix, what is its inverse?

Answers:

1. 4 2. 0 3. 0 4. Indefinite 5. $A = \begin{pmatrix} a & x & u \\ b & y & v \\ c & z & w \end{pmatrix}$

Exercise 1.3: Statement and Applications of Cayley-Hamilton Theorem

1. Verify Cayley- Hamilton theorem for $A = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$ and hence find A^{-1}.
2. Show that the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$ satisfies its own characteristic equation.

   Hence find the inverse of the matrix A.
3. Verify Cayley- Hamilton theorem for $A = \begin{pmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{pmatrix}$ and hence find A^{-1}.

4. Verify Cayley- Hamilton theorem for $A=\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ and hence find A^{-1} & A^4.

5. Verify that the matrix $A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$ satisfies its own characteristic equation. Hence find A^4.

6. Verify that the matrix $A=\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ satisfies its own characteristic equation. Hence find A^4.

7. Verify Cayley-Hamilton theorem for $A=\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and hence find A^{-1}. Also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A.

8. Use Cayley-Hamilton theorem to find the value of the matrix $A^8-5A^7+7A^6-3A^5+8A^4-5A^3+8A^2-2A+I$ if the matrix $A=\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

9. Find A^n, using Cayley-Hamilton theorem, when $A=\begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$, Hence find A^4.

10. Find A^n, using Cayley-Hamilton theorem, when $A=\begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}$, Hence find A^3.

Answers:

1. $A^{-1}=\frac{1}{3}\begin{pmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{pmatrix}$.

2. $A^{-1}=\frac{-1}{11}\begin{pmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{pmatrix}$.

3. $A^{-1}=\begin{pmatrix} 8 & 0 & -3 \\ -43 & 1 & 17 \\ 3 & 0 & -1 \end{pmatrix}$.

4. $A^{-1}=\frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$, $A^4=\begin{pmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{pmatrix}$.

5. $A^4=\begin{pmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{pmatrix}$.

6. $A^4=\begin{pmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{pmatrix}$.

7. $A^{-1}=\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$ & $A^5-4A^4-7A^3+11A^2-A-10I=A+5I$.

8. $127A^2-223A+106I=\begin{pmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{pmatrix}$.

9. $A^n=\left(\frac{6^n-2^n}{4}\right)\begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}+\left(\frac{3\times 2^n-6^n}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^4=\begin{pmatrix} 976 & 960 \\ 320 & 336 \end{pmatrix}$.

10. $A^n=\left(\frac{9^n-4^n}{5}\right)\begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}+\left(\frac{9\times 4^n-4\times 9^n}{5}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^3=\begin{pmatrix} 463 & 399 \\ 266 & 330 \end{pmatrix}$.

Exercise 1.4: Diagonalization of Matrices by Orthogonal Transformation

1. $A=\begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$

2. $A=\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

3. $A=\begin{pmatrix}2 & 2 & 1\\1 & 3 & 1\\1 & 2 & 2\end{pmatrix}$

4. $A=\begin{pmatrix}2 & 1 & -1\\1 & 1 & -2\\-1 & -2 & 1\end{pmatrix}$

5. $A=\begin{pmatrix}2 & 0 & 4\\0 & 6 & 0\\4 & 0 & 2\end{pmatrix}$

Answers:

1. $D=\begin{pmatrix}-4 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{pmatrix}$ 2. $D=\begin{pmatrix}3 & 0 & 0\\0 & 6 & 0\\0 & 0 & 9\end{pmatrix}$ 3. $D=\begin{pmatrix}5 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

4. $D=\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 4\end{pmatrix}$ 5. $D=\begin{pmatrix}-2 & 0 & 0\\0 & 6 & 0\\0 & 0 & 6\end{pmatrix}$

Exercise 1.5: Reduction of a Quadratic Form to Canonical Form by Orthogonal Transformation

1. Reduce the quadratic form $10x_1^2+2x_2^2+5x_3^2-4x_1x_2+6x_2x_3-10x_3x_1$ to canonical form by orthogonal reduction.
2. Reduce the quadratic form $2x_1^2+5x_2^2+3x_3^2+4x_1x_2$ to canonical form by orthogonal reduction.
3. Reduce the quadratic form $3x_1^2-3x_2^2-5x_3^2-2x_1x_2-6x_2x_3-6x_3x_1$ to canonical form through an orthogonal transformation.
4. Reduce the quadratic form $2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3+2x_3x_1$ to canonical form through an orthogonal transformation.
5. Reduce the quadratic form $2x_1^2+3x_2^2+2x_3^2+2x_3x_1$ to canonical form through an orthogonal transformation.

Answers:

1. $3y_2^2 + 14y_3^2$ 2. $y_1^2 + 3y_2^2 + 6y_3^2$ 3. $4y_1^2 - y_2^2 - 8y_3^2$
4. $4y_1^2 + y_2^2 + y_3^2$ 5. $y_1^2 + 3y_2^2 + 3y_3^2$

Exercise 1.6: Nature of Quadratic Forms

Find the nature of the quadratic form for the following

1. $2x^2 + 3y^2 + 2z^2 + 2xy$
2. $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$
3. $8x^2 + 7y^2 + 3z^2 - 12xy + 4yz - 8zx$
4. $u^2 + v^2 + w^2$ of four variables
5. $-4u^2 + v^2 + 3w^2$

Answers:

1. Nature – Positive definite
2. Nature- Indefinite
3. Nature – Positive semi definite
4. Nature – Positive semi definite
5. Nature- Indefinite

Practice Quiz

Multiple Choice Questions

1.	How many Eigen values does a 2 X2 matrix have? (a) 0 (b) 1 (c) 2 (d) infinite
2.	The value of x if one of the Eigen values of the matrix $A=\begin{pmatrix}2 & x\\ 1 & 3\end{pmatrix}$ is zero (a) 5 (b) 6 (c) 0 (d) none of the above.
3.	The characteristic polynomial of the matrix $\begin{pmatrix}3 & 2 & 9\\ 7 & 5 & 13\\ 6 & 17 & 19\end{pmatrix}$ is given by (a) $\lambda^3-27\lambda^2-122\lambda-313$ (b) $\lambda^3+27\lambda^2-122\lambda-313$ (c) $\lambda^3-27\lambda^2-122\lambda+133$ (d) $\lambda^3-27\lambda^2+212\lambda+125$
4.	The sum and product of the eigenvalues of $A=\begin{bmatrix}1 & 2 & 5\\ 2 & 2 & 2\\ 1 & 2 & 7\end{bmatrix}$ are (a) $10,-4$ (b) $10,4$ (c) $-4,10$ (d) $4,10$
5.	If $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 0 & -2\\ 1 & 2 & 3\end{bmatrix}$, then the sum of the eigenvalues of $2A$ are (a) 4 (b) 16 (c) 8 (d) 6
6.	The matrix $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$ is singular. One of the eigenvalue is 3, the other eigenvalues are (a) $0,15$ (b) $0,0$ (c) $0,3$ (d) $7,8$

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Practice Quiz

7.	For a given matrix A of order $3, \lvert A\rvert = 32$ and two eigenvalues are 8 and 2. Then the sum of the eigenvalues is (a) 24 (b) 12 (c) 32 (d) 10
8.	If two of the eigenvalues of a matrix whose determinant is 4 are -1 and 2. Then the third eigenvalue is (a) 2 (b) 4 (c) -4 (d) -2
9.	Find the symmetric matrix A, whose eigenvalues are 1 and 3 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$
10.	For which eigenvectors the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ can be diagonalized? a) $\lambda = 0,1$ b) $\lambda = 1,1$ c) $\lambda = 1,0$
11.	What is the condition for a matrix A to be orthogonal? Justify. a) $B^T B = 1$ b) $B^T B = 0$ c) $B^T + B = 0$
12.	If A is an orthogonal matrix then a) |A| = 0 b) |A| = 1 c) |A| $\pm$ 1
13.	The matrix of the quadratic form $Q(x, y) = 3x^2 + 2y^2 - 4xy$. $(a)\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} (b)\begin{bmatrix} -2 & 3 \\ 3 & 1 \end{bmatrix} (c)\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} (d)\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$
14.	The matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$. $(a)\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 6 \\ 3 & 6 & 2 \end{bmatrix} (b)\begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix} (c)\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix} (d)\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$

Practice Quiz

15.	The matrix of the quadratic form $2x_1^{\,2}+5x_2^{\,2}+4x_1x_2+4x_1x_2+2x_3x_1$. $(a)\begin{bmatrix}2&2&1\\2&5&0\\1&0&0\end{bmatrix}(b)\begin{bmatrix}-2&-2&1\\-2&0&-1\\1&-1&0\end{bmatrix}(c)\begin{bmatrix}-2&2&1\\2&-5&0\\1&0&0\end{bmatrix}(d)\begin{bmatrix}-2&2&1\\2&-5&1\\1&1&0\end{bmatrix}$
16.	Find the nature of the quadratic form $x^2-2y^2+z^2$ (a) Positive Definite (b)Negative Definite (c) Indefinite (d)Positive semi definite
17.	Find the index of the quadratic form $x^2-2y^2+z^2$ (a) 1 (b) 2 (c) 0 (d) 3
18.	Find the Signature of the quadratic form $2x^2+3y^2+2z^2$ a) 1 (b) 2 (c) 0 (d) 3

Answers:

1 (c) 2 (b) 3 (a) 4 (a) 5 (c) 6 (a) 7 (b) 8 (d)
9 (a) 10 (b) 11 (a) 12 (c) 13 (d) 14 (b) 15 (a) 16 (c)
17 (a) 18 (d)

Assignments

Assignment -1

Q. No	Questions	K level	CO
1.	Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$.	K1	CO1
2.	Verify that the matrix $A=\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ satisfies its characteristic equation and hence find A^4	K1	CO1
3.	Verify Cayley – Hamilton Theorem and hence find A^{-1} if $A=\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$.	K1	CO1
4.	Reduce the quadratic form $8x^2+7y^2+3z^2-12xy-8zy+4xz$ to the canonical form through an orthogonal transformation and hence show that it is Positive Semi definite.	K2	CO1
5.	Given that $\mathrm{A}=\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$, find the value of $A^6-5A^5+8A^4-2A^3-9A^2+31A-36I$ using Cayley – Hamilton theorem.	K3	CO1
6.	The eigenvectors of a 3 x 3 real symmetric matrix A corresponding to the eigenvalues 2, 3, 6 are $[1\ \ 0\ \ -1]^{T}, [1\ \ 1\ \ 1]^{T}, [-1\ \ 2\ \ -1]^{T}$ respectively. Find the matrix A.	K3	CO1
7.	Find all values of parameters p which the matrix $A=\begin{bmatrix} 2 & 33 & -1 \\ 0 & p-1 & 3 \\ 0 & 0 & p+1 \end{bmatrix}$ has eigenvalues equal to 1 and 2 and 3.	K3	CO1

Assignments

Assignment -2

Q. No	Questions	K level	CO
1.	Compute the Eigen values and Eigen vectors of the following system $10x_1 + 2x_2 + x_3 = \lambda x_1$ $2x_1 + 10x_2 + x_3 = \lambda x_2$ $2x_1 + x_2 + 10x_3 = \lambda x_3$	K1	CO1
2.	Find the Eigen values and Eigen vectors of $\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$	K1	CO1
3.	Verify that the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ satisfies its own characteristic equation and hence find A^4.	K1	CO1
4.	Verify that the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ satisfies its own characteristic equation and hence find A^{-1}.	K2	CO1
5.	Use Cayley Hamilton theorem to find the value of the matrix given by $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$, if the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.	K3	CO1

Assignments

6.	Reduce the quadratic form $x_1^2+2x_2^2+x_3^2-2x_1x_2+2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive definite. Also give a non-zero set of values (x_1,x_2,x_3) which makes the quadratic form zero.	K3	CO1
7.	Find out the type of conic represented by $17x^2-30xy+17y^2=128$ after reducing the quadratic form $17x^2-30xy+17y^2$ to canonical form by an orthogonal transformation.	K3	CO1

Answers:

1. $\lambda=8,9,13$ & Eigen vectors are $\begin{pmatrix}3\\-2\\-2\end{pmatrix},\begin{pmatrix}1\\1\\-3\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}$.

2. $\lambda=-1,-1,-1$ & Eigen vectors are $\begin{pmatrix}-5\\0\\7\end{pmatrix},\begin{pmatrix}6\\7\\0\end{pmatrix},\begin{pmatrix}0\\5\\6\end{pmatrix}$.

3. $A^4=\begin{pmatrix}-86&85&-85\\85&-86&85\\-85&85&-86\end{pmatrix}$

4. $A^{-1}=\frac{1}{9}\begin{pmatrix}0&3&3\\3&2&-7\\3&-1&-1\end{pmatrix}$

5. $A^8-5A^7+7A^6-3A^5+8A^4-5A^3+8A^2-2A+I=\begin{pmatrix}295&285&285\\0&10&0\\285&285&295\end{pmatrix}$

6. $y_2^2+3y_3^2, x_1=1, x_2=1, x_3=-1$

7. $2y_1^2+32y_2^2, \frac{y_1^2}{64}+\frac{y_2^2}{4}$ which represents an ellipse

Part A

Q. No.	Questions	K Level	CO
1.	If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a square matrix of order n, then show that $\lambda_1^3, \lambda_2^3, \lambda_3^3, \ldots, \lambda_n^3$ are the eigenvalues of A^3. **Solution:** Let X_r be the eigenvector corresponding to the eigenvalue λ_r. Then $AX_r = \lambda_r X_r$ ------(1) Pre multiplying (1) by A and using (1), $A^2 X_r = A(\lambda_r X_r) = \lambda_r (AX_r) = \lambda_r \lambda_r X_r = \lambda_r^2 X_r$ Similarly, $A^3 X_r = \lambda_r^3 X_r$ ------(2) From (2) λ^3 is the eigenvalue of A^3.	K1	CO1
2.	If λ is the eigenvalues of the matrix A then show that A^{-1} has the eigenvalue $\frac{1}{\lambda}$. **Solution:** Let λ be the eigenvalue of A. Then AX = λX --------------(1) Pre multiplying both sides of (1) by A^{-1}, we get $A^{-1}(AX) = A^{-1}(\lambda X) \quad \Rightarrow (A^{-1}A)X = (A^{-1}\lambda)X$ $\mathrm{IX} = \lambda(\mathrm{A}^{-1}\mathrm{X}) \quad \Rightarrow \mathrm{X} = \lambda(\mathrm{A}^{-1}\mathrm{X})$ $\Rightarrow A^{-1}X = \frac{1}{\lambda}X$ $\Rightarrow \frac{1}{\lambda}X = A^{-1}X$ --------------(2) Comparing (1) and (2), we have $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.	K1	CO1
3.	Find the sum and product of all the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$. **Solution:** We know that sum of the eigenvalues = sum of diagonal elements $= 1+2+3 = 6$ The product of the eigenvalues = Determinant of the matrix	K1	CO1

Part A

	$=\begin{vmatrix}1&1&1\\1&2&2\\1&2&3\end{vmatrix}=1.$		
4.	Two eigenvalues of the matrix $A=\begin{bmatrix}2&2&1\\1&3&1\\1&2&2\end{bmatrix}$ are equal to 1 each. Find the eigenvalues of A^{-1}. **Solution:** Let the third eigenvalue be λ. The other two eigen values are 1,1, We know that sum of the eigenvalues = Trace of A $1+1+\lambda=2+3+2 \quad \Rightarrow 2+\lambda=7 \quad \Rightarrow \lambda=5.$ The eigenvalues of A are 1, 1, 5. The eigenvalues of A^{-1} are $1,1,\frac{1}{5}$.	K1	CO1
5.	Find the sum of squares of eigenvalues of the matrix A= $\begin{pmatrix}1&7&5\\0&2&9\\0&0&5\end{pmatrix}$. **Solution:** The given matrix A is an upper triangular matrix. Hence its eigenvalues are given by 1,2,5. Sum of squares of eigenvalues = 1+4+25=30.	K1	CO1
6.	If the eigenvalues of the matrix A of order 3 x 3 are 2, 3, 1, then find the eigenvalues of adjoint of A. **Solution:** We know that $A^{-1}=\frac{1}{\|A\|}adjA \quad \Rightarrow adjA=\|A\|A^{-1}$ Eigen values of A^{-1} are $\frac{1}{2},\frac{1}{3}\&1$ and $\|A\|$ = Product of the eigen values = 6 Eigen values of adjoint of A are $6\times\frac{1}{2},6\times\frac{1}{3},6\times1$ (i.e) 3, 2, 6.	K1	CO1
7.	The product of two eigenvalues of the matrix $A=\begin{pmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{pmatrix}$ is 16. Find the third eigenvalue. **Solution:** Given, $\lambda_1.\lambda_2=16.$ WKT $\lambda_1\lambda_2\lambda_3=\|A\|$	K1	CO1

Part A

	$$16\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1)+2(-6+2)+2(2-6) \quad = 32$$ $\therefore \lambda_3 = 2.$ Hence the third eigenvalue is 2.		
8.	If one of the eigenvalues of $\begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{pmatrix}$ is -9. Find the other two. **Solution:** Let λ_1 , λ_2, λ_3 be the three eigenvalues. Given $\lambda_1 = -9$. We know that sum of the eigenvalues = sum of diagonal elements. -9+λ_2+λ_3 = 7-8-8 λ_2+λ_3= 0 $\lambda_2 = -\lambda_3$. Also The product of the eigenvalues = Determinant of the matrix =441. (ie) (-9)($-\lambda_3$)(λ_3) = 441. Therefore $\lambda_3 = 7$ & $\lambda_2 = -7$.	K1	CO1
9.	If α and β are the eigenvalues of $A = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$ form the matrix whose eigenvalues are α^3 and β^3 . **Solution:** If α and β are the eigenvalues of A, α^3 and β^3 are the eigenvalues of $A^{3.}$ $A^3 = \begin{pmatrix} 38 & -50 \\ -50 & 138 \end{pmatrix}$.	K1	CO1
10.	Find the matrix whose eigenvalues are the eigenvalues of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ reduced by 4. **Solution:** We know that the matrix A – KI has the eigenvalues $\lambda_1 - K, \lambda_2 - K, ..., \lambda_n - K$.	K1	CO1

Part A

	Here $A=\begin{pmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{pmatrix}$ and K = 4. So the matrix is Find the matrix whose eigenvalues are the eigenvalues of $A=\begin{pmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{pmatrix}$ reduced by 4. **Solution:** We know that the matrix A – KI has the eigenvalues $\lambda_1-K,\lambda_2-K,........,\lambda_n-K.$ Here $A=\begin{pmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{pmatrix}$ and K = 4. So the matrix is $\begin{pmatrix}2&-2&2\\-2&-1&-1\\2&-1&-1\end{pmatrix}.$		
11.	Find the eigenvalues of A^{-1}, if $A=\begin{pmatrix}3&5&3\\0&4&6\\0&0&1\end{pmatrix}$. **Solution:** Since A is the upper triangular matrix, the eigenvalues of A are 3,4,1. The eigenvalues of A^{-1} are $\frac{1}{3},\frac{1}{4},1.$	K1	CO1
12.	Find the constants a and b such that matrix $\begin{pmatrix}a&4\\1&b\end{pmatrix}$ has 3 and -2 as its eigenvalues. **Solution:** Given $A=\begin{pmatrix}a&4\\1&b\end{pmatrix}$ $\lambda_1+\lambda_2=a+b=1 \quad --------(1)$ $\lambda_1\lambda_2=ab-4=-6$ $\Rightarrow ab=-2 \quad --------(2)$ From (1) and (2), we get $a=2,\ b=-1$ and $a=-1,\ b=2.$	K1	CO1

Part A

13.	If the sum of two eigenvalues and trace of (3×3) matrix A are equal, find $\lvert A\rvert$. **Solution:** Trace of the Matrix (A) = $\lambda_1+\lambda_2+\lambda_3=\lambda_1+\lambda_2 \Rightarrow \lambda_3=0 \Rightarrow \lvert A\rvert=\lambda_1\lambda_2\lambda_3=0$	K1	CO1
14.	Two eigenvalues of $A=\begin{pmatrix}4&6&6\\1&3&2\\-1&-5&-2\end{pmatrix}$ are equal and they are double the third. Find the eigenvalues of A^2. **Solution:** Eigen values of A are $2\lambda, 2\lambda, \lambda$. $\Rightarrow 5\lambda=4+3-2=5$ $\Rightarrow \lambda=1$. Eigen values of A^2 are 4, 4, 1.	K1	CO1
15.	The matrix $A=\begin{pmatrix}1&2&3\\2&0&-2\\1&2&3\end{pmatrix}$ is singular. One of the eigenvalues is 2. Find the other two eigenvalues. **Solution:** $\lambda_1=2,\ \lvert A\rvert=0=\lambda_1\lambda_2\lambda_3 \Rightarrow \lambda_2=0$ $\lambda_1+\lambda_2+\lambda_3=4 \quad \Rightarrow \lambda_3=2.$	K1	CO1
16.	If $\begin{pmatrix}4\\1\end{pmatrix}$ is an eigenvector of the matrix $A=\begin{pmatrix}5&4\\1&2\end{pmatrix}$. Find the corresponding eigen value. **Solution:** If λ is an eigenvalue of A, then $(A-\lambda I)\,X=0$ $\Rightarrow \begin{pmatrix}5-\lambda&4\\1&2-\lambda\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\therefore\ \lambda=6$	K1	CO1
17.	If $A=\begin{pmatrix}3&1&4\\0&2&6\\0&0&5\end{pmatrix}$, find the eigenvalues of $A^2-5A+3I$.	K1	CO1

	Solution: A has eigenvalues 3, 2, 5. Eigen values of $A^2-5A+3I$ are 3^2-15+3, 2^2-10+3, 5^2-25+3. (i.e) -3,-3,3		
18.	If the sum of eigenvalues of the matrix of a quadratic form is equal to zero, then what will be the nature of the quadratic form. **Solution:** If the sum of eigenvalues of the matrix of a quadratic form is zero then the eigenvalues of the matrix should consist of both positive and negative values. (i.e) the canonical form should consist of both negative and positive terms. Therefore, the quadratic form is indefinite.	K1	CO1
19.	State Cayley – Hamilton theorem and its uses. **Solution: "** Every square matrix satisfies its own characteristic equation." It is used to find (i) Inverse of a given square matrix (ii) Higher integral powers of the given matrix.	K1	CO1
20.	Given $A=\begin{pmatrix}1 & 2\\4 & 3\end{pmatrix}$, find A^{-1} using Cayley – Hamilton theorem. **Solution:** The characteristic equation is $\lvert A-\lambda I\rvert=0$ $(1-\lambda)(3-\lambda)-8=0 \quad \Rightarrow \lambda^2-4\lambda-5=0$. By Cayley Hamilton Theorem, $A^2-4A-5I=0$ Post multiplying by A^{-1}, we get $A^2A^{-1}-4AA^{-1}-5IA^{-1}=0$ $A-4I-5A^{-1}=0 \qquad \Rightarrow 5A^{-1}=A-4I$ $5A^{-1}=\begin{pmatrix}1 & 2\\4 & 3\end{pmatrix}-\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}=\begin{pmatrix}-3 & 2\\4 & -1\end{pmatrix} \Rightarrow A^{-1}=\frac{1}{5}\begin{pmatrix}-3 & 2\\4 & -1\end{pmatrix}$.	K1	CO1
21.	Check whether $A^3-20A+8I=0$, where A= $\begin{pmatrix}1 & 1 & 3\\1 & 3 & -3\\-2 & -4 & -4\end{pmatrix}$ **Solution:** The characteristic equation is $\lvert A-\lambda I\rvert=0$	K1	CO1

Part A

	$\lambda^3 - 20\lambda + 8 = 0$ By Cayley Hamilton Theorem, we get $A^3 - 20A + 8 = 0$		
22.	Write down the matrix of the quadratic form $x^2 + y^2 + z^2 + xy + yz + zx.$ **Solution:** The required matrix is $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$	K1	CO1
23.	Prove that the quadratic form $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$ is indefinite. **Solution:** The matrix A of the quadratic form is $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{pmatrix}$ $D_1 = \lvert 1 \rvert = 1 > 0$, $D_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5 < 0$ $D_3 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 0.$ $\therefore$ The quadratic form is indefinite.	K1	CO1
24.	Determine the nature of the following quadratic form $x_1^{\ 2} + 2x_2^{\ 2}.$ **Solution:** The matrix A of the quadratic form is $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$ $D_1 = \lvert 1 \rvert = 1 > 0,\quad D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 > 0$ The given quadratic form is positive definite.	K1	CO1

Part A

25.	Give the nature of a quadratic form whose matrix is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. **Solution:** The nature is negative definite. Since all the eigen values are negative.	K1	CO1

Part B

Q. No.	Questions	K Level	CO
1.	Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$. **Ans:** The Characteristic equation is $\lambda^3-2\lambda^2-4\lambda+8=0$ The eigenvalues are –2, 2, 2. The eigenvectors are $X_1=\begin{pmatrix} -4 \\ -1 \\ 7 \end{pmatrix}$ & $X_2=X_3=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	K2	CO1
2.	Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ **Ans:** The Characteristic equation is $(\lambda-1)^2(\lambda-5)=0$. The eigenvalues are 5, 1, 1. The eigenvectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.	K2	CO1
3.	Verify that the matrix $A=\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ satisfies its characteristic equation and hence find A^4. **Ans:** The Characteristic equation is $\lambda^3-6\lambda^2+8\lambda-3=0$ and $A^4=\begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$.	K2	CO1
4.	Given that $A=\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$, find the value of $A^6-5A^5+8A^4-2A^3-9A^2+31A-36I$ using Cayley – Hamilton theorem. **Ans:** Characteristic eqn. is $\lambda^3-3\lambda^2-\lambda+9=0$,	K2	CO1

Part B

5.	Verify Cayley – Hamilton Theorem and hence find A^{-1} if $A=\begin{pmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{pmatrix}$. **Ans:** The Characteristic equation is $\lambda^3-5\lambda^2+9\lambda-1=0$ & $A^{-1}=\begin{pmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{pmatrix}$.	K1	CO1
6.	Find the characteristic equation of $A=\begin{pmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{pmatrix}$ and hence express the matrix A^5 in terms $A^2, A \& I$. **Ans:** The characteristic equation is $\lambda^3-5\lambda^2+7\lambda-3\ =0$ & $A^5=58A^2-111A+54I$.	K1	CO1
7.	Diagonalise $A=\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}$ by an orthogonal transformation. **Ans:** The characteristic equation is $\lambda^3-12\lambda^2+36\lambda-32=0$. The eigen values are 8, 2, 2 The eigen vectors are $\begin{pmatrix}2\\ -1\\ 1\end{pmatrix},\begin{pmatrix}0\\ 1\\ 1\end{pmatrix},\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}$ $\& D=\begin{pmatrix}8 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$	K2	CO1
8.	Diagonalise the matrix $A=\begin{pmatrix}2 & 1 & -1\\ 1 & 1 & -2\\ -1 & -2 & 1\end{pmatrix}$ by means of an orthogonal transformation. **Ans:** The characteristic equation is $\lambda^3-4\lambda^2-\lambda+4=0$. The eigen values are –1,1,4. The eigen vectors are $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}\begin{pmatrix}2\\ -1\\ 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}\& D=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{pmatrix}$	K2	CO1

Part B

9.	The eigenvectors of a 3 x 3 real symmetric matrix A corresponding to the eigenvalues 2, 3, 6 are $[1 \ \ 0 \ \ -1]^T, [1 \ \ 1 \ \ 1]^T, [-1 \ \ 2 \ \ -1]^T$ respectively. Find the matrix A. **Ans:** $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$.	K1	CO1
10.	Reduce $2x_1x_2 + 2x_3x_2 + 2x_3x_1$ to canonical form by an orthogonal reduction and hence find its nature. **Ans:** The matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The characteristic equation is $\lambda^3 - 3\lambda - 2 = 0$. The eigenvalues are 2, -1, -1. The eigenvectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$. The canonical form is $2y_1^2 - y_2^2 - y_3^2$. Nature of the quadratic form is Indefinite.	K1	CO1
11.	Reduce the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy - 8zy + 4xz$ to the canonical form through an orthogonal transformation and hence show that it is Positive Semi definite. **Ans:** The matrix is $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$. The eigenvalues are 0, 3, 15. The eigenvectors are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. The canonical form is $3y_2^2 + 15y_3^2$.	K2	CO1

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- Online Course: NPTEL Online Videos, Courses-IIT Video Lectures
  Course Name: Lecture Series on Mathematics - II
  Course Instructor: Prof. Sunita Gakkhar, Prof. H. G. Sharma, Prof. Tanuja Srivastava, Department of Mathematics, IIT Roorkee.
  Mod-2 Lec-13 - Mod-2 Lec-17
  http://www.nptelvideos.in/2012/11/mathematics-ii.html

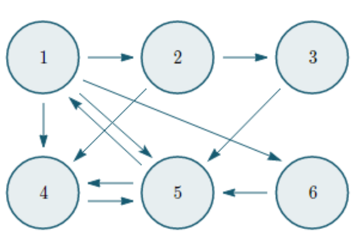
Real Time Applications

1. Google's PageRank

Google's extraordinary success as a search engine was due to their clever use of eigenvalues and eigenvectors. From the time it was introduced in 1998, Google's methods for delivering the most relevant result for our search queries has evolved in many ways, and PageRank is not really a factor any more in the way it was at the beginning.

Google's home page in 1998

But for this discussion, let's go back to the original idea of PageRank.Let's assume the Web contains 6 pages only. The author of Page 1 thinks pages 2, 4, 5, and 6 have good content, and links to them. The author of Page 2 only likes pages 3 and 4 so only links from her page to them. The links between these and the other pages in this simple web are summarised in this diagram.



A simple Internet web containing 6 pages

Google engineers assumed each of these pages is related in some way to the other pages, since there is at least one link to and from each page in the web.

Their task was to find the "most important" page for a particular search query, as indicated by the writers of all 6 pages. For example, if everyone linked to Page 1, and it was the only one that had 5 incoming links, then it would be easy - Page 1 would be returned at the top of the search result.

However, we can see some pages in our web are not regarded as very important. For example, Page 3 has only one incoming link. Should its outgoing link (to Page 5) be worth the same as Page 1's outgoing link to Page 5?

The beauty of PageRank was that it regarded pages with many incoming links (especially from other popular pages) as more important than those from mediocre pages, and it gave more weighting to the outgoing links of important pages.

Google's Use of Eigenvalues and Eigenvectors

For the 6-page web illustrated above, we can form a "link matrix" representing the relative importance of the links in and out of each page.

Considering Page 1, it has 4 outgoing links (to pages 2, 4, 5, and 6). So in the first column of our "links matrix", we place value $\frac{1}{4}$ in each of rows 2, 4, 5 and 6, since each link is worth $\frac{1}{4}$ of all the outgoing links. The rest of the rows in column 1 have value 0, since Page 1 doesn't link to any of them.Meanwhile, Page 2 has only two outgoing links, to pages 3 and 4. So in the second column we place value $\frac{1}{2}$ in rows 3 and 4, and 0 in the rest. We continue the same process for the rest of the 6 pages.

$$A = \begin{vmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 1 & 1 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Next, to find the eigenvalues.

We have

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & -\lambda & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\lambda & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & -\lambda & 0 & 0 \\ \frac{1}{4} & 0 & 1 & 1 & -\lambda & 1 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda^6 - \frac{5\lambda^4}{8} - \frac{\lambda^3}{4} - \frac{\lambda^2}{8}$$

This expression is zero for $\lambda = -0.72031$, $-0.13985 \pm 0.39240j$, 0, 1. (I expanded the determinant and then solved it for zero using Wolfram Alpha.)

We can only use non-negative, real values of λ (since they are the only ones that will make sense in this context), so we conclude $\lambda=1$. (In fact, for such PageRank problems we always take $\lambda=1$.)

We could set up the six equations for this situation, substitute and choose a "convenient" starting value, but for vectors of this size, it's more logical to use a computer algebra system. Using Wolfram Alpha, we find the corresponding eigenvector is:

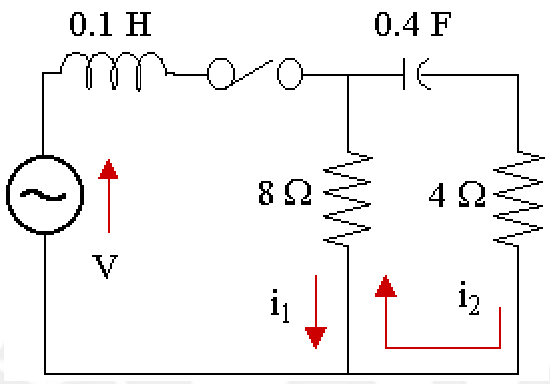
$$\mathbf{v}_1 = [4\ 1\ 0.5\ 5.5\ 8\ 1]^{\mathrm{T}}$$

As Page 5 has the highest PageRank (of 8 in the above vector), we conclude it is the most "important", and it will appear at the top of the search results.

We often **normalize** this vector so the sum of its elements is 1. (We just add up the amounts and divide each amount by that total, in this case 20.) This is OK because we can choose any "convenient" starting value and we want the relative weights to add to 1. I've called this normalized vector **P** for "PageRank".

$$P = [0.2\ 0.05\ 0.025\ 0.275\ 0.4\ 0.05]^T$$

2. Electronics: RLC Circuits



An electrical circuit consists of 2 loops, one with a 0.1 H inductor and the second with a 0.4 F capacitor and a 4 Ω resistor, and sharing an 8 Ω resistor, as shown in the diagram. The power supply is 12 V. (We'll learn how to solve such circuits using systems of differential equations in a later chapter, beginning at Series RLC Circuit.)

Let's see how to solve such a circuit (that means finding the currents in the two loops) **using matrices and their eigenvectors and eigenvalues.** We are making use of Kirchhoff's voltage law and the definitions regarding voltage and current in the differential equations chapter linked to above.

NOTE: There is no attempt here to give full explanations of where things are coming from. It's just to illustrate the way such circuits can be solved using eigenvalues and eigenvectors.

For the left loop: $0.1\frac{di_1}{dt} + 8(i_1 - i_2) = 120.1\frac{di_1}{dt} + 8(i_1 - i_2) = 12$

Multiplying by 10 and rearranging gives: $\frac{di_1}{dt} = -80i_1 + 80i_2 + 120 \ldots (1)$

For the right loop: $4i_2 + 2.5\int i_2 dt + 8(i_2 - i_1) = 12$

Real Time Applications

Differentiating gives: $4\frac{di_2}{dt} + 2.5i_2 + 8\frac{di_2}{dt} - 8\frac{di_1}{dt} = 12$

Rearranging gives: $12\frac{di_2}{dt} = 8\frac{di_1}{dt} - 2.5i_2 + 12$

Substituting (1) gives:

$$12\frac{di_2}{dt} = 8\ \ -80i_1 + 80i_2 + 120\ \ - 2.5i_2 + 12$$
$$= -640i_1 + 637.5i_2 + 972$$

Dividing through by 12 and rearranging gives:

$$\frac{dt}{di_2} = -53.333i_1 + 53.125i_2 + 81\ \ldots\ 2$$

$$\frac{dK}{dt} = AK + v,\ where\ \ K = \begin{bmatrix} i_1 \\ 1_2 \end{bmatrix},\ \ A = \begin{bmatrix} -80 & 80 \\ -53.333 & 53.125 \end{bmatrix},\ v_1 = \begin{bmatrix} 120 \\ 81 \end{bmatrix}$$

The characteristic equation for matrix A is

$$\lambda_1 = -26.2409,\ v_1 = \begin{bmatrix} 1.4881 \\ 1 \end{bmatrix},\ \lambda_2 = -0.6341,\ v_2 = \begin{bmatrix} 1.008 \\ 1 \end{bmatrix},$$

which yields the eigenvalue-eigenvector pairs

$$K = c_1\begin{bmatrix} 1.4881 \\ 1 \end{bmatrix} e^{-1.4881t} + c_2\begin{bmatrix} 1.008 \\ 1 \end{bmatrix} e^{-1.008t}$$

To Know More:

https://www.intmath.com/matrices-determinants/8-applications-eigenvalues-eigenvectors.php#rep

Contents Beyond Syllabus

Data Science Machine Learning

This course is part of the Mathematics for Machine Learning Specialization Mathematics for Machine Learning: Linear Algebra Offered By Imperial College London

- **https://www.coursera.org/learn/linear-algebra-machine-learning#about**

SKILLS YOU WILL GAIN:

- In this course on Linear Algebra we look at what linear algebra is and how it relates to vectors and matrices. Then we look through what vectors and matrices are and how to work with them, including the knotty problem of ***eigenvalues and eigenvectors,*** and how to use these to solve problems. Finally we look at how to use these to do fun things with datasets - like how to rotate images of faces and how to extract eigenvectors to look at how the Pagerank algorithm works.

- At the end of this course you will have an intuitive understanding of ***vectors and matrices*** that will help you bridge the gap into linear algebra problems, and how to apply these concepts to machine learning.

To Explore More:

- **https://www.youtube.com/watch?v=R13Cwgmpuxc&feature=youtu.be**

- **Google's PageRank: https://www.intmath.com/matrices-determinants/8-applications-eigenvalues-eigenvectors.php**

This example is the web has 6 pages, whereas Google (and Bing and other search engines) needs to cope with billions of pages. This requires a lot of computing power, and clever mathematics to optimize processes.

Additional Resources

- https://web.ma.utexas.edu/users/olenab/Fall-2011-341/341solshwk9.pdf
- https://www.math.tamu.edu/~stecher/LinearAlgebraPdfFiles/chapterFive.pdf
- https://www.analyzemath.com/linear-algebra/matrices/eigenvalues-and-eigenvectors-questions-with-solutions.html

Mini Project

To create matrices and performing basic matrix calculations in MATLAB.

1. Creating Matrices
2. Adding and Subtracting Matrices
3. Vector Products and Transpose
4. Multiplying Matrices
5. Identity Matrix
6. Matrix Inverse

Reference:

https://in.mathworks.com/help/matlab/math/matrices-in-the-matlab-environment.html

Prescribed Text Books & Reference Books

ENGINEERINGMATHEMATICS–I	**21MA101**
S. No.	**TEXT BOOKS**
1	Erwin Kreyszig, "Advanced Engineering Mathematics", John Wiley and Sons, 10th Edition, New Delhi, 2016.
2	B.S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, New Delhi, 43rd Edition, 2014.
3	T. Veerarajan, "Engineering Mathematics", Tata McGraw Hill, 2nd Edition, New Delhi, 2011.
REFERENCES :	
1	M. K. Venkataraman, "Engineering Mathematics, Volume I", 4thEdition, The National Publication Company, Chennai, 2003.
2	Sivaramakrishna Dass, C. Vijayakumari, "Engineering Mathematics", Pearson Education India, 4thEdition 2019.
3	H. K. Dass, and Er. Rajnish Verma, "Higher Engineering Mathematics", S. Chand Private Limited, 3rd Edition 2014.
4	B.V. Ramana, "Higher Engineering Mathematics", Tata McGraw Hill Publishing Company,6th Edition, New Delhi, 2008.
5	S.S. Sastry, "Engineering Mathematics", Vol.I &II, PHI Learning

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Thank you

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