Assignment 3

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1.a. The form of the differential equation seen in class becomes modified to include a magnetic force:

$$\dot{p} = -\frac{1}{\tau}p - |e|E - \frac{|e|}{m}p\times B$$

$$= -\frac{1}{\tau}p - |e|\left(E_x(\hat{x}\pm i\hat{y}) + \frac{1}{m}(p_x\hat{y} - p_y\hat{x})|B|\right)$$

Since $E = |E|e^{i(kx-\omega t)}$ (& therefore $j = |j|e^{i(kx-\omega t)}$), we can use $j = -|e|nv_d = \frac{-|e|n}{m}p$ to write:

$$\frac{dj}{dt} = \frac{1}{\tau}j + \frac{|e|^2n}{m}E_x(\hat{x}\pm i\hat{y}) - \frac{|e||B|}{m}(j_x\hat{y} - j_x\hat{x})$$

$$(i\omega\tau - 1)j = \frac{|e|^2n\tau}{m}E_x(\hat{x}\pm i\hat{y}) - \frac{|e||B|\tau}{m}(j_x\hat{y} - j_y\hat{x})$$

$$\Longrightarrow \begin{pmatrix} i\omega\tau - 1 & -\frac{|e||B|\tau}{m} \\ \frac{|e||B|\tau}{m} & i\omega\tau - 1 \end{pmatrix} j = \frac{|e|^2n\tau E_x}{m}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

$$\Longrightarrow j = \frac{\sigma_0 E_x}{(i\omega\tau - 1)^2 + \left(\frac{|e||B|\tau}{m}\right)^2}\begin{pmatrix} 1 - i\left(\omega \pm \frac{|e||B|\tau}{m}\right) \\ \pm i\left(1 - i\left(\omega \pm \frac{|e||B|}{m}\tau\right)\right) \end{pmatrix}$$

$$= \frac{\sigma_0}{(i\omega\tau - 1)^2 + \omega_c^2\tau^2}\begin{pmatrix} 1 - i(\omega\pm\omega_c) \\ \pm i(1 - i(\omega\pm\omega_c)) \end{pmatrix} E_0$$

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\sigma_0 E_x}{1 - i(\omega\mp\omega_c)\tau}\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \quad \square$$

b. $\nabla\times E + \frac{\partial B}{\partial t} = 0$, taking the curl:

$$\nabla(\underbrace{\nabla\cdot E}_{0}) - \nabla^2 E + \mu_0\frac{\partial}{\partial t}(\nabla\times H) = 0$$

$$-\nabla^2 E + \mu_0\frac{\partial}{\partial t}\left(j + \frac{\partial D}{\partial t}\right) = 0$$

$$\nabla^2 E - \mu_0\frac{\partial j}{\partial t} + \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} = 0$$

$$\Longrightarrow -k^2E + i\omega\mu_0 j + \omega^2\mu_0\varepsilon_0 E = 0 \quad \text{(in Fourier space)}$$

$$k^2 - \omega^2\mu_0\varepsilon_0 = \frac{i\omega\mu_0\sigma_0}{1 - i(\omega\mp\omega_c)\tau}$$

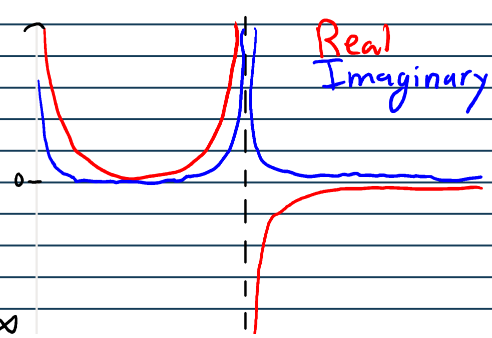
$$k^2c^2 = \frac{i\omega\sigma_0}{\varepsilon_0(1 - i(\omega\mp\omega_c)\tau)} + \omega^2$$

$$\Longrightarrow \varepsilon(\omega) = 1 + \frac{i\sigma_0}{\varepsilon_0\omega(1 - i(\omega\mp\omega_c)\tau)}$$

$$= 1 - \frac{\omega_p^2}{\omega(i/\tau + \omega\mp\omega_c)} \quad \square$$

c. $$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\frac{1}{\omega - \omega_c + i/\tau}$$

$$= 1 - \frac{\omega_p^2}{\omega}\frac{\omega - \omega_c - i/\tau}{(\omega - \omega_c)^2 + 1/\tau^2}$$



$$\mathrm{Re}(\varepsilon) = 1 - \frac{\omega_p^2}{\omega}\frac{\omega - \omega_c}{(\omega-\omega_c)^2 + 1/\tau^2} = 1 - \frac{\omega_p^2\tau^2}{\omega}\frac{\omega - \omega_c}{(\omega-\omega_c)^2\tau^2 + 1}$$

$$\mathrm{Im}(\varepsilon) = \frac{\omega_p^2}{\omega\tau}\frac{1}{(\omega-\omega_c)^2 + 1/\tau^2} = \frac{\tau\omega_p^2}{\omega(1 + (\omega-\omega_c)^2\tau^2)}$$

$$\lim \mathrm{Re}(\varepsilon) = \lim \mathrm{Im}(\varepsilon) = \lim_{-} \mathrm{Re}(\varepsilon) = -\lim_{+} \mathrm{Re}(\varepsilon) = \lim \mathrm{Im}(\varepsilon) \to \infty$$

$\mathrm{Im}(\varepsilon) = \frac{\omega_p^2}{\omega\tau}\frac{1}{(\omega-\omega_c)^2 + 1/\tau^2} = \frac{\tau\omega_p^2}{\omega(1+(\omega-\omega_c)^2\tau^2)}$

$\lim_{\omega\to 0} \mathrm{Re}(\varepsilon) = \lim_{\omega\to 0} \mathrm{Im}(\varepsilon) = \lim_{\omega\to\omega_c^-} \mathrm{Re}(\varepsilon) = -\lim_{\omega\to\omega_c^+} \mathrm{Re}(\varepsilon) = \lim_{\omega\to\omega_c} \mathrm{Im}(\varepsilon) \to \infty$

$\lim_{\omega\to\infty} \mathrm{Re}(\varepsilon) = \lim_{\omega\to\infty} \mathrm{Im}(\varepsilon) = 0$

Therefore, for $\omega > \omega_p$, $\omega < \omega_c$ ε exists for arbitrary k.

d. For $\omega \ll \omega_c$ $\varepsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega_c\omega} \approx \frac{\omega_p^2}{\omega_c\omega} \implies k^2c^2 = \frac{\omega_p^2}{\omega_c}\omega \implies \omega = \omega_c\left(\frac{k^2c^2}{\omega_p^2}\right)$ □.

2\. a. In periodic boundary condition with period L, $k_i = \frac{2\pi}{L}n_i$ where $i \in \{1,2\}$ & $n_i \in \mathbb{Z}$. Therefore each electron occupies an area $\frac{1}{2}\frac{(2\pi)^2}{V_c} = \frac{2\pi^2}{L^2}$.

The ½ is because electrons have spin ½ so 2 electrons can have this k state. Since N electrons would be occupying πk_f^2 area in k-space, $k_f^2 = 2\pi\frac{N}{L^2}$

Therefore $\boxed{k_f^2 = 2\pi n}$.

b. πr_s^2 is the area per electron $\left(\frac{L^2}{N}\right)$ so using the result from a, $r_s^2 = \frac{1}{\pi n} \implies \boxed{r_s^2 = \frac{2}{k_f^2}}$.

c. Integrating over quantum numbers:

$\int dn_x dn_y = \int d^2n = \frac{L^2}{(2\pi)^2}\int d^2k = \frac{L^2}{2\pi}\int dk\,k$, $\varepsilon = \frac{\hbar^2k^2}{2m} \Rightarrow d\varepsilon = \frac{\hbar^2 k}{m}dk$

$= \frac{L^2}{2\pi}\int \frac{m}{\hbar^2}d\varepsilon \implies g(\varepsilon) = \frac{mL^2}{2\pi\hbar^2}$, but $D(\varepsilon)$ also takes spin degeneracy & area into account so $D(\varepsilon) = \frac{2}{L^2}g(\varepsilon)$.

$\implies \boxed{D(\varepsilon) = \frac{m}{\pi\hbar^2}}$

Therefore the density of states is therefore a constant for $\varepsilon > 0$. For $\varepsilon < 0$ there are no states, so of course the density of states is also zero. □

d. $n = \int_{-\infty}^{\infty} d\varepsilon\, D(\varepsilon) f(\varepsilon)$

$= \int_0^{\varepsilon_F} d\varepsilon\, D(\varepsilon) + \frac{\pi^2}{6}(k_BT)^2 D'(\varepsilon_F) + \dots$

The higher order terms are all proportional to derivatives of $D(\varepsilon)$ but these vanish since $D(\varepsilon)$ is a constant. Therefore $\boxed{\varepsilon_F = \frac{\pi\hbar^2 n}{m}}$

$\mu = \frac{\partial U}{\partial N} = \frac{\partial}{\partial N}\int_0^{\varepsilon_F} \varepsilon g_s g(\varepsilon)\, d\varepsilon$ ($g_s = 2$ for spin ½)

$= \frac{\partial}{\partial N}\left(\frac{mL^2}{\pi\hbar^2}\varepsilon_F^2\right)$

$= \frac{\partial}{\partial N}(\varepsilon_F N)$ $\left(\varepsilon_F = \frac{\pi\hbar^2 N}{mL^2}\right)$

$= \varepsilon_F$

e. $n = \int_{-\infty}^{\infty} d\varepsilon\, D(\varepsilon) f(\varepsilon)$

$= \frac{m}{\pi\hbar^2}\int_0^{\infty} \frac{1}{1+e^{\beta(\varepsilon-\mu)}}d\varepsilon$ Let $u = \beta(\varepsilon-\mu)$, $du = \beta d\varepsilon$

$= \frac{m}{\pi\hbar^2\beta}\left(\beta\mu + \ln(1+e^{-\beta\mu})\right)$

$\implies \varepsilon_F = \mu + k_BT\ln(1+e^{-\mu/k_BT})$ □

f. It differs by $k_BT\ln(1+e^{-\mu/k_BT})$, but since $\frac{\mu}{k_BT} \ll 1$ for room temperature metals the ln terms essentially vanishes. Mathematically, $D(\varepsilon)$ is not smooth at $\varepsilon = 0$ so the series expansion cannot be applied.