MATH 579 Project: An Integrating Maxwell Equation Solver

Justin Cardona

May 6, 2024

1 Introduction

Electromagnetic scattering problems are highly relevant to many applications in science, engineering and medicine. It is often the case that these problems, although they can be described by Maxwell's equations, can not be solved for analytically in any obvious way. As a result, it is important to have fast and accurate numerical solvers for these complex problems. This document discusses an integral method solver for Maxwell's equations called Gila. In order to do this, a few preliminary steps are required. First, in Section 2 the statement of the electromagnetic physics problem is presented. Sections 3 and 4 present different ways of formulating this problem using different physical quantities and shows that they are well posed[1]. Finally, in Section 5, the operating principles of Gila are presented using the given context.

2 Electromagnetic Scattering

Consider some $\mathcal{D} \subset \mathbb{R}^3$. To denote the electric and magnetic fields in this region of space for any particular time, take $E, B : \mathcal{D} \times [0, \infty) \to \mathbb{R}^3$. These fields are related to the electric current and charge densities $J : \mathcal{D} \times [0, \infty) \to \mathbb{R}^3$ and $\rho : \mathcal{D} \times [0, \infty) \to \mathbb{R}$. However, interesting scattering problems involve a medium through which the fields must travel. These materials will be described using time-constant permittivity $\epsilon : \mathcal{D} \to \mathbb{R}^{3\times 3}$. It then becomes natural to use the auxiliary electric field $D = \epsilon E$ and magnetic field $H = \frac{1}{\mu_0} B - M$ (these are called the constitutive relations). Here $M : \mathcal{D} \times [0, \infty) \to \mathbb{R}^3$ is the magnetization of the material. With these quantities it is now possible to write governing equations of electromagnetism.

Definition 1. *Maxwell's Equations:* $\forall A \in \mathcal{D}$ that are 2D smooth bounded, and orientable manifolds; $\forall V \in \mathcal{D}$ that are volumes with a piecewise smooth boundary:

$$\frac{\partial}{\partial t}\int_A B\cdot da + \int_{\partial A} E\cdot dl = 0 \quad (1a) \qquad \int_{\partial V} D\cdot da = \int_V \rho dV \quad (2a)$$

$$\frac{\partial}{\partial t}\int_A D\cdot da + \int_{\partial A} H\cdot dl = -\int_S J\cdot da \quad (1b) \qquad \int_{\partial V} B\cdot da = 0 \quad (2b)$$

The above is the integral formulation of Maxwell's equations, there is an equivalent differential formulation that can be written as follows. $\forall(x,t)\in\mathcal{D}\times[0,\infty)$:

$$\frac{\partial B}{\partial t} + \nabla\times E = 0 \quad (3a) \qquad \nabla\cdot D = \rho \quad (4a)$$

$$\frac{\partial D}{\partial t} - \nabla\times H = -J \quad (3b) \qquad \nabla\cdot B = 0 \quad (4b)$$

Although integral methods are the topic here, it will actually be the differential versions that come in handy. Gila is a frequency domain solver so the Fourier transform of the equations is considered instead. Using the convention,

$$f(\omega) := \frac{1}{\sqrt{2\pi}}\int f(t)e^{-i\omega t}dt$$

equations 3a and 3b may be written in Fourier space as

$$-i\omega B + \nabla\times E = 0 \tag{5a}$$

$$i\omega D + \nabla\times H = J \tag{5b}$$

It only really matters that these two equations are solved, given that equations 4a and 4b are satisfied at any time. This is because equations 3a and 3b preserve the divergence of the fields over time evolution; this can be seen by taking the divergence of these equations and using that $\frac{\partial\rho}{\partial t} + \nabla\cdot J = 0$. The scattering problem asks what is the total field produced, given an incident field and a dielectric profile. Due to the linearity of Maxwell's equations, one may decompose solution fields $E = E^i + E^s$ and $H = H^i + H^s$, where the i and s superscripts denote the incident and scattered fields respectively. Effectively, the scattering problems asks for E and H given E^i, H^i, and ϵ (we don't consider the permeability μ since for most materials it does not differ appreciably from free space). However, as will be seen in Section 3 there might be more effective ways to ask this question.

3 Well-posedness

This section discusses the well-posedness of Maxwell's equations and alternate formulations of the scattering problem. While it may seem intuitive that Maxwell's equations are well posed (electric fields do exist in reality), it is important to show that they have unique solutions for the purposes of mathematical numerics. Another important consideration is that one does not necessarily have to solve for the the E and H in their numerical scheme. One may instead choose D and B or perhaps J and M. The reasons for why one might choose a particular formulation will be better described in Section 5. For now it will be shown that:

- One formulation has a solution $\iff$ the other formulations have a (unique) solution.
- All formulation have solutions (that are equivalent).

These formulations will be developed explicitly before moving on. Many of the details of this section are adapted the most relevant parts of an existing paper[1] with some modifications to more exactly fit Gila's context. For convenience let $\chi : \mathcal{D} \to \mathbb{R}^{3\times 3}$ so that $\chi = \frac{1}{\epsilon_0}\epsilon - I$. The scattered fields may be written in Fourier space as,

$$\begin{aligned}
\nabla \times H^s &= \omega\epsilon_0 E^s + \omega\epsilon_0\chi E := \omega\epsilon_0 E^s + J^c \\
\nabla \times E^s &= \omega\mu_0 H^s
\end{aligned}$$

Assuming that ϵ is point-wise invertible, the contrast current density may be written as

$$J^c = \omega\epsilon_0\chi\epsilon^{-1}D \tag{6}$$

This can be used to write the fields in terms of a vector potential A[2] such that

$$E^s = \frac{1}{\omega\epsilon_0}\left(\nabla\nabla\cdot A - \omega^2\epsilon_0\mu_0 A\right) \tag{7a}$$

$$H^s = \nabla \times A \tag{7b}$$

$A : \mathcal{D} \to \mathbb{R}^3$ is defined for continuously differentiable J^c such that:

$$A(r) = \int_{\mathcal{D}} G(r - r')J^c(r')dV' \tag{8}$$

where the Green's function G is given as:

$$G(r) = \frac{e^{-\omega|r|/\epsilon_0}}{4\pi|r|} \tag{9}$$

Decomposing the fields into the incident and scattered ones, one may write

$$\omega\epsilon_0 E^i = \omega\epsilon_0 E + \frac{\omega^2}{c^2} A - \nabla\nabla\cdot A \tag{10}$$

where c is the speed of light, and using $J^c = \omega\epsilon_0\chi E$ to compute A. This is first formulation to obtain E. An alternative one uses D:

$$\omega\epsilon_0 E^i = \omega\epsilon_0\epsilon^{-1} D + \frac{\omega^2}{c^2} A - \nabla\nabla\cdot A \tag{11}$$

using $J^c = \omega\epsilon_0\chi\epsilon^{-1}D$ to compute A. Finally, the equation for J^c is just

$$\omega\epsilon_0\chi E^i = J^c + \chi\left(\frac{\omega^2}{c^2} A - \nabla\nabla\cdot A\right) \tag{12}$$

In order to make expressions more compact, define

$$NJ^c = \frac{\omega^2}{c^2} A(r) - \nabla\nabla\cdot A(r) \tag{13}$$

Equations 10, 11, and 12 may now be written

$$(I + N\chi)E = E^i \tag{14a}$$

$$(I + N\chi)\epsilon^{-1}D = E^i \tag{14b}$$

$$\frac{1}{\omega\epsilon_0}(I + \chi N)J^c = \chi E^i \tag{14c}$$

Suppose that equation 14a has been solved to give E, since E satisfies 14a it is clear that $J^c = \omega\epsilon_0\chi E$ satisfies 14c (denote this relation 14a→14c). Since χ is not invertible, the opposite direction is not as trivial. The solution lies in equation 7a, which is exactly the formula to compute the scattered electric field given the current. Adding this to the incident field gives the the total field so 14a↔14c. In order to obtain an equivalence with 14b, note that $D = \epsilon E$ so 14a→14b. Finally, equation 6 gives how to get a unique J^c given D and 14b→14c. The result of this is that if one of the formulations has a solution then they all do.

In order to show that such a solution is unique, begin by noting that the following system has only the trivial solution[3]:

$$\begin{cases} \nabla\times H^s = \omega\epsilon E^s \\ \nabla\times E^s = \omega\mu_0 H^s \\ \text{radiation conditions} \ \ \& \ \ E, H \in L^2_{\text{loc}}(\mathbb{R}^3)^3 \end{cases} \tag{15}$$

where $L^2_{\text{loc}}(\mathbb{R})^3$ denotes the 3D vectors functions that are locally square integrable. Now assume that J_1^c and J_2^c are solutions to 14c with the same incident fields. Call $J_3^c = J_2^c - J_2^c$, it must be a solution to the homogeneous counterpart of 14c.

$$J_{1,2}^c = \omega\epsilon_0\chi(E^i + E_{1,2}^s) \implies J_3^c = \omega\epsilon_0\chi(E_1^s - E_s^2) = \omega\epsilon_0\chi E_3^s$$

Therefore E_3^s, H_3^s satisfy 15 so $E_3^s = 0$ and so does J_3^c. This means that 14c (and by extension 14a and 14b) have a unique solution.

4 Decomposing the Operators

It has been shown that if one of the formulations admit a solution, they all do and they are unique. It must then be shown that the exists a solution in general.

Theorem 1. Let H_1, H_2 be Hilbert spaces. Let $C : H_1 \to H_2$ be a compact operator and let $T : H_1 \to H_2$ be an invertible bounded operator with T^{-1} bounded as well. $T + C$ is one-to-one $\implies$ $(T + C)^{-1}$ exists and is bounded.

The goal is to apply this $I + \chi N$ so it must be shown that $I + \chi N$ is one-to-one. It has already been shown that if a solution exists, its unique. All that remains is to split $I + \chi N = T + C$ such that T, C are as in Theorem 1.

4.1 Conditions on Permittivity

$\forall f \in L^2(G)^3$ on a space G, ϵ must satisfy (for $0 < \alpha, \beta < \infty$) coerciveness and boundedness respectively:

$$\alpha \int_G f \cdot f^* dV \leq \left| \int_G f^* \cdot (\epsilon f) dV \right| \leq \beta \int_G f \cdot f^* dV \tag{16}$$

This implies that ϵ^{-1} must exist almost everywhere and the elements of ϵ are modulus bounded from above by

$$\epsilon_{\max} = \sup_{k,l,r\in\bar{G}} |\epsilon_{k,l}(r, \omega)| < \infty$$

Splitting ϵ into real and imaginary parts:

$$\epsilon = \epsilon' + \epsilon'' \tag{17}$$

finite energy and causality principles gives (with $\xi \in \mathbb{C}^3, a', a''$ independent of r and ω):

$$0 \leq \xi \cdot (\epsilon(r,\omega)'\xi) \leq a'\xi^* \cdot \xi$$
$$0 \leq \xi \cdot (\epsilon(r,\omega)''\xi) \leq a''\xi^* \cdot \xi$$

This holds almost everywhere on G so $\epsilon', \epsilon'' \succcurlyeq 0$.

4.2 Helmholtz Decomposition

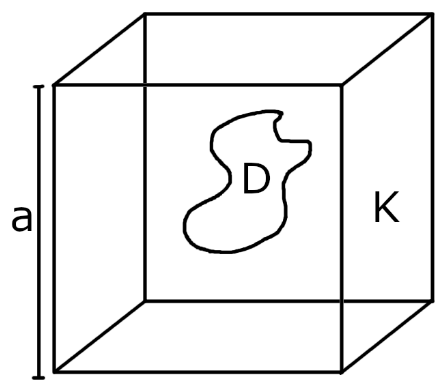

Figure 1: Extend the domain of $\mathcal{D}$ to a cube $\mathcal{K}$ that has the same origin, faces perpendicular to the axes. The distance between ∂K (with normal unit vector n) and $\mathcal{D}$ is strictly positive. Furthermore, extend $J^c(r \in K \setminus \bar{\mathcal{D}})$.

For any $f \in C^1(\mathcal{K}^3) \bigcup C(\bar{\mathcal{K}}^3)$, the Helmholtz decomposition theorem states $\exists p \in C^2(\mathcal{K}), w \in C^1(\mathcal{K}^3) \cap C(\bar{\mathcal{K}}^3)$ such that:

$$f = \nabla p + \nabla \times w \tag{19}$$

$$n \times w|_{\partial\mathcal{K}} = 0 \tag{20}$$

Using the identity

$$\nabla \cdot (g \times h) = h \cdot (g \times h) - g \cdot \nabla \times h \tag{21}$$

one can obtain

$$\int_{\mathcal{K}} \nabla \cdot (w^* \times \nabla p) dV = \int_{\partial\mathcal{K}} \nabla p \cdot \nabla \times w^* dA. \tag{22}$$

From Gauss's theorem and 22, the desired form of this expression is found:

$$\int_{\partial\mathcal{K}} \nabla p \cdot \nabla \times w^* dA = \int_{\partial\mathcal{K}} \nabla p \cdot (n \times w^*) dA \tag{23}$$

This reveals that the following satisfies the inhomogeneous Helmholtz equation[4].

$$s(r) = \int_{\mathcal{K}} G(r - r')p(r')dV' \tag{24}$$

$$\nabla^2 s(r) - \frac{\omega^2}{c_0} s(r) = -p(r) \tag{25}$$

For a continuously differentiable scalar function $q(r)$, using integration by parts and using the translational invariance of G,

$$\int_{\mathcal{K}} G(r - r')\partial_{i'} q(r')dV' = \partial_{i'} \int_{\mathcal{K}} G(r - r')q(r')dV' + \sum_{\pm} \pm \int_{\partial K|_{i=\pm a}} G(r - r')q(r')dA'. \tag{26}$$

Here $i \in \{x, y, z\}$, $i' \in \{x', y', z'\}$. Applying this to the Helmholtz equation with boundary condition 20:

$$\int_{\mathcal{K}} G(r - r')\nabla' \times w(r')dV = \nabla \times \int_{\mathcal{K}} G(r - r')w(r')dV' \tag{27a}$$

$$\int_{\mathcal{K}} G(r - r')\nabla' p(r')dV = \nabla \int_{\mathcal{K}} G(r - r')p(r')dV' + \int_{\partial\mathcal{K}} G(r - r')p(r')n(r')dA' \tag{27b}$$

From 24, one can use the vector potential representation of the fields to write,

$$\omega\epsilon_0 E^s = \left(\nabla\nabla \cdot -\frac{\omega^2}{c_0^2}\right) \int_{\mathcal{K}} G(r - r')J^c(r')dV'. \tag{28}$$

Finally, this will be be expanded as follows:

$$\begin{aligned}\left(\nabla\nabla \cdot -\frac{\omega^2}{c_0^2}\right) \int_{\mathcal{K}} G(r - r')J^c(r')dV' = &\left(\nabla\nabla \cdot -\frac{\omega^2}{c_0^2}\right) \int_{\partial\mathcal{K}} G(r - r')p(r')n(r')dA' - \nabla p \\ &-\frac{\omega^2}{c_0^2} \int_{\mathcal{K}} G(r - r')\nabla \times w dV' \end{aligned} \tag{29}$$

4.3 Finite Energy Conditions

Provided that $\chi E^i \in L^2(\mathcal{K}^3)$, $J^c \in L^2(\mathcal{K})^3$, since continuously differentiable functions are dense in $L^2(\mathcal{K})$, 19 and 20 can be extended to vector fields[4] in $L^2(\mathcal{K})^3$. Therefore, it is natural to define the projection operators as self maps on $L^2(\mathcal{K})^3$:

$$Pf = \nabla p \tag{30a}$$

$$Qf = (I - P)f = \nabla \times w \tag{30b}$$

Denote $p \in H^1(\mathcal{K})$, $w \in H^1_{t0}(\mathcal{K})^3$ with:

$$H^1(\mathcal{K}) = \{g|\ g \in L^2(\mathcal{K}, \nabla g \in L^2(\mathcal{K})^3)\} \tag{31a}$$
$$H^1_{t0}(\mathcal{K})^3 = \{g|\ g \in H^1(\mathcal{K})^3, n \times g\_\partial\mathcal{K} = 0\} \tag{31b}$$

The p and w are unique solutions if one imposes the conditions for each regular closed surface $S \in \bar{\mathcal{K}}$:

$$\int_{\mathcal{K}} p dV = 0 \tag{32}$$
$$\int_S w \cdot n dA = 0 \tag{33}$$

Once again generalizing to $f \in L^2(\mathcal{K})^3$, define two more operators $K : L^2(\mathcal{K})^3 \to H^2(\mathcal{K})^3$ and $L : L^2(\mathcal{K})^3 \to C^\infty(\mathcal{K})^3$ such that:

$$Kf = \frac{\omega^2}{c_0^2} \int_{\mathcal{K}} G(r - r')f(r')dV' \tag{34a}$$
$$Lf = \left(\frac{\omega^2}{c_0^2} - \nabla\nabla\cdot\right) \int_{\partial\mathcal{K}} p(r')G(r - r')n(r')dA' \tag{34b}$$

This allows for a useful decomposition of the currents formulation of the scattering problem.

$$\chi E^i = \frac{1}{\omega\epsilon_0}(I + \chi P + \chi KQ + \chi L)J^c \tag{35}$$

It is with this that one should take $T = I + \chi P$ and $C = \chi KQ + \chi L$ as in Theorem 1. It will be shown here that C has the required properties, the same will be done for T in the next subsection. The Green's function in K is square integrable on $\mathcal{K}$ so K is a Hilbert-Schmidt operator. K is compact on $L^2(\mathcal{K})^3 \to L^2(\mathcal{K})^3$. This means χKQ is compact since χ and Q and bounded. This just leaves χL, note that χ vanishes outside $\bar{\mathcal{D}}$. For this reason one could define $\phi \in C_0^\infty(\mathcal{K})$ such that:

$$\begin{cases} \phi(r) = 1 & \text{for } \inf_{x\in\partial\mathcal{K}} |x - r| \geq 2d/3 \\ 0 \leq \phi(r) \leq 1 & \text{for } d/3 < \inf_{x\in\partial\mathcal{K}} |x - r| < 2d/3 \\ \phi(r) = 0 & \text{for } \inf_{x\in\partial\mathcal{K}} |x - r| \leq d/3 \end{cases} \tag{36}$$

where d is the minimum distance between $\mathcal{K}$ and $\mathcal{D}$. Since the contrast function vanishes outside the dielectric object, $\chi = \chi\phi$. The point of this detour is to show that ϕL (and thus χL) is compact. First note that it is bounded since

$$\sup_{r'\in\partial\mathcal{K},\ r\in\text{supp}(\phi)} \left|\partial^2 G(r - r')\right| < \infty \tag{37}$$

Note that G is a C^∞ function away from its singularity. Since $C_0^\infty(\mathcal{K})^3$ is a subspace of $H^1(\mathcal{K})^3$, by the compact embedding theorem for Sobolev spaces[5] ϕL is compact.

4.4 The Operator $I + \chi P$

T is bounded because ϵ is bounded, but there is no such simple argument for its inverse. For T^{-1}, a different decomposition is considered instead. Denote $\mathcal{R}(P)$ as the range of P. Since P and Q are mutually orthogonal projections, $\mathcal{R}(P) \cup \mathcal{R}(Q) = L^2(\mathcal{K})^3$. Therefore, T will be written as a matrix with type

$$T : \begin{pmatrix} \mathcal{R}(P) \\ \mathcal{R}(Q) \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{R}(P) \\ \mathcal{R}(Q) \end{pmatrix} \tag{38}$$

with the definition

$$T = \begin{bmatrix} P(I+\chi)P & 0 \\ Q\chi P & Q \end{bmatrix} \tag{39}$$

P and Q are the identities on $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ so this can be inverted as

$$T^{-1} = \begin{bmatrix} PAP & 0 \\ -Q\chi PAP & Q \end{bmatrix} \tag{40}$$

where $PAP = (P(I+\chi)P)^{-1}$ on $\mathcal{R}(P)$. Therefore T^{-1} exists $\iff$ PAP exists, and PAP does always exist since $I + \chi$ is coercive (and so is $P(I+\chi)P$). In addition, since $\mathcal{R}(P)$ is a Hilbert space with the inner product on $L^2(\mathcal{K})^3$. The Lax-Milgram theorem guarantees the existence and boundedeness of PAP in this case[6]. The result is that T and C satisfy the requirements outlines in Theorem 1 and the problems are well-posed.

5 The Gila Solver

Gila is a frequency domain Maxwell's equation solver that the uses the JM formulation and integral methods written in Julia. The end goal of the development of this tool is to use it for solving scattering problems in materials that are highly inhomogeneous and anisotropic. As seen in Section 3 each formulation is equivalent, but what makes the currents so useful is that there is no demand on continuity of the integration kernel. The result is that one can use discontinuous basis functions, which give better numerical performance in inhomogeneous and anisotropic materials[7]. Here, the analytic expressions for what the Gila is doing is presented in subsection 5.1, then the numerical implementation details are presented in subsection 5.2. Finally a toy example of Gila is presented to demonstrate its capabilities.

5.1 Mathematical Principles

Recall the Maxwell's equations under investigation:

$$\mu_0 \frac{\partial}{\partial t}(H + M) + \nabla \times E = 0$$
$$\epsilon \frac{\partial}{\partial t} E - \nabla \times H = -J$$

Fourier transforming in time, one may isolate for the currents in terms of the fields,

$$\begin{pmatrix} J \\ -M \end{pmatrix} = \begin{pmatrix} i\epsilon\omega & \nabla\times \\ \frac{i}{\mu_0 \omega}\nabla\times & 1 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}. \tag{41}$$

To write the matrix as Hermitian, choose to adopt a change of coordinates that allows. Taking $j = J$ and $m = i\mu_0 \omega M$, let $Z = \sqrt{\frac{\mu_0}{\epsilon}}$ and $k_0 = \omega\sqrt{\mu_0 \epsilon}$.

$$\frac{i}{k_0}\begin{pmatrix} j \\ -m \end{pmatrix} = -\begin{pmatrix} Z^{-1} & -\frac{i}{k_0}\nabla\times \\ \frac{i}{k_0}\nabla\times & Z \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \tag{42}$$

For the sake of brevity, the vector containing the current j and m will be denoted p, that of the fields E and H will be denoted f and the Maxwell operator relating them as M. That is to say Equation 42 can be written as $\frac{i}{k_0}p = -Mf$. Additionally, M_0 will be used to refer to the Maxwell operator in free space. However, in general one wishes to look at how light scatter off objects. In this formalism, they will be characterized by permittivity and permeability response functions $X : \mathcal{D} \to \mathbb{C}^{2\times 2}$ is the following sense[8]:

$$\frac{i}{k_0}\begin{pmatrix} j^s \\ -m^s \end{pmatrix} = -\begin{pmatrix} z^{-1}X_{je} & zX_{jh} \\ z^{-1}X_{me} & zX_{mh} \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \tag{43}$$

Here the superscript s is used to denote scattered currents and $z = \sqrt{\mu_0/\epsilon_0}$. This may be written more compactly as $\frac{i}{k_0}p^s = Xf$. Using this and decomposing the fields into scattered and initial ones, the Maxwell equations may be written as:

$$(M_0 - X)f = \frac{i}{k_0}p^i \tag{44}$$

Taking G_0 to the Green's function of M_0, the above means that $f^s = G_0 X f$ and therefore

$$
\begin{aligned}
(I - G_0 X) f &= f_i \\
(I - X G_0) p^s &= -ik_0 X f_0 + X G_0 p^i \\
(I - X G_0) p &= p_i + p_0
\end{aligned}
\tag{45}
$$

where $f_i = f_0 + \frac{i}{k_0} G_0 p^i$ with f_0 representing a possible incoming radiative field—free solution entering through the boundary of the computational domain and $p_0 = -ik_0 X f_0$. Equation 45 is essentially what is to be solved. As such, one of the central use cases of Gila is implementing iterative solution methods to evaluate $W = (I - X G_0)^{-1}$ for specific input vectors.

In order to shed some light on how this is done, an appropriate thing to do is actually Fourier transform in the spatial components (transform variable k). Additionally the frequency is extended to the complex plane $\zeta = \omega + i\delta$ (for reasons that will soon become clear). The vacuum Maxwell operator in this setting will be written as:

$$
\breve{M}_0 = -\begin{bmatrix} \zeta Z^{-1} & \times_k \\ -\times_k & \zeta Z \end{bmatrix} \tag{46}
$$

where $\times_k$ is the cross product operator with vector k:

$$
\times_k = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_z & 0 \end{bmatrix}
$$

The corresponding Green's function is then,

$$
\breve{G}_0 = \begin{bmatrix} \zeta Z^{-1} & -\times_k \\ \times_k & \zeta Z \end{bmatrix} \begin{bmatrix} \breve{G}_0^0 & 0 \\ 0 & \breve{G}_0^0 \end{bmatrix} \tag{47}
$$

where $\breve{G}_0^0 = [(k^2 - \zeta^2) I - k \otimes k]^{-1}$. This can also be written $\breve{G}_0^0 = \frac{1}{(k^2-\zeta^2)}(I - \zeta^{-2} k \otimes k)$ and in order to do the inverse Fourier transform, Jordan's lemma is used. Specifically, one can write $G_0^0 = g_0 I + \frac{1}{k_0^2}(\nabla\cdot) g_0 (\nabla\cdot)^\dagger$. g_0 is defined as follows: take $s = \frac{r - \tilde{r}}{\lambda}$ to be the wavelength scaled separation between two points r and $\tilde{r}$

so that

$$
\begin{aligned}
g_0(s) &= \frac{1}{2\pi}\int_0^\infty dk k^2 \int_0^\pi d\theta i \sin\theta \frac{e^{2\pi iks\cos\theta}}{(k^2-\zeta^2)} \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^\infty dk \frac{ke^{2\pi iks}}{s(k^2-\zeta^2)} = i\frac{e^{2\pi i\zeta s}}{4\pi s}
\end{aligned}
\tag{48}
$$

In the last equality, Jordan's lemma is applied. Since $\breve{G}_0^0 = [(k^2-\zeta^2)I - k\otimes k]^{-1}$, Schur complementation can be used to instead write:

$$
\begin{aligned}
(\frac{1}{k_0^2}(\nabla\times)(\nabla\times) - \zeta^2 I)\breve{G}_0^0 &= I \\
\zeta^2 G_0^0 &= \frac{1}{k_0^2}(\nabla\times)(\nabla\times)\left(g_0 I + \frac{1}{k_0^2}(\nabla\cdot)g_0(\nabla\cdot)^\dagger\right) - I \\
&= \frac{1}{\zeta^2}\left(\frac{1}{k_0^2}(\nabla\times)(\nabla\times)g_0 I - I\right)
\end{aligned}
\tag{49}
$$

For complex frequencies then, Equation 45 can be written as:

$$
\left[(I+zX) - zX\begin{bmatrix} (\frac{1}{k_0}\nabla\times)^2 & i\frac{z}{\zeta^2}(\frac{1}{k_0}\nabla\times)^3 \\ -i\frac{z}{\zeta^2}(\frac{1}{k_0}\nabla\times)^3 & (\frac{1}{k_0}\nabla\times)^2 \end{bmatrix} g_0 I\right] p = p_i + p_0 \tag{50}
$$

5.2 Implementation

Gila solves Equation 49 by discretizing the space $\mathcal{K}$, partitioning it into identical cuboids, which are divided into smaller cells of the same shape. The integrals can be done over volumes, but the presence of the $(\nabla\times)(\nabla\times)$ operator in 50 allows this to instead be expressed in terms of surface integrals, which eliminate the error associated with the discretization of the differential operator. All the numerical error of this method comes from the integration and space ultimately being discrete (as well as the iteration used to solve 50).

$$
q_{m,j}^{b\dagger}\left[\frac{1}{k_0^2}(\nabla\times)(\nabla\times)g_0 I\right] q_{l,i}^a = \sum_{h,k}(n_h\times j)\cdot(n_k\times i)\oint_h\oint_k g_0(r_m^b - r_l + (v_m^b - v_l^a)) \tag{51}
$$

where a and b are volume labels, m and l are cube labels, and i and j are direction labels. The qs are the (constant) basis functions on the cells. The rs and vs represent the centers of the cubes and the positions to the integration surfaces with respect to the centers. This equation can effectively be though of as a stiffness matrix equation. For the integration, Gauss quadrature is used for cells that share a face and h-adaptive cubature if they dont. In the codebase repository, the iterative solve is not yet implemented, but for the example in this document BICSTABL is used. In general, the matrix is dense so these solves are lengthy.

However, it will be quickly noted that the problem has some symmetric block Toeplitz structure, and there are attempts underway to leverage this for speed increases. As an example, consider the following computation of the currents produced by an electric line current. The code for producing these plots can be seen in Appendix A.

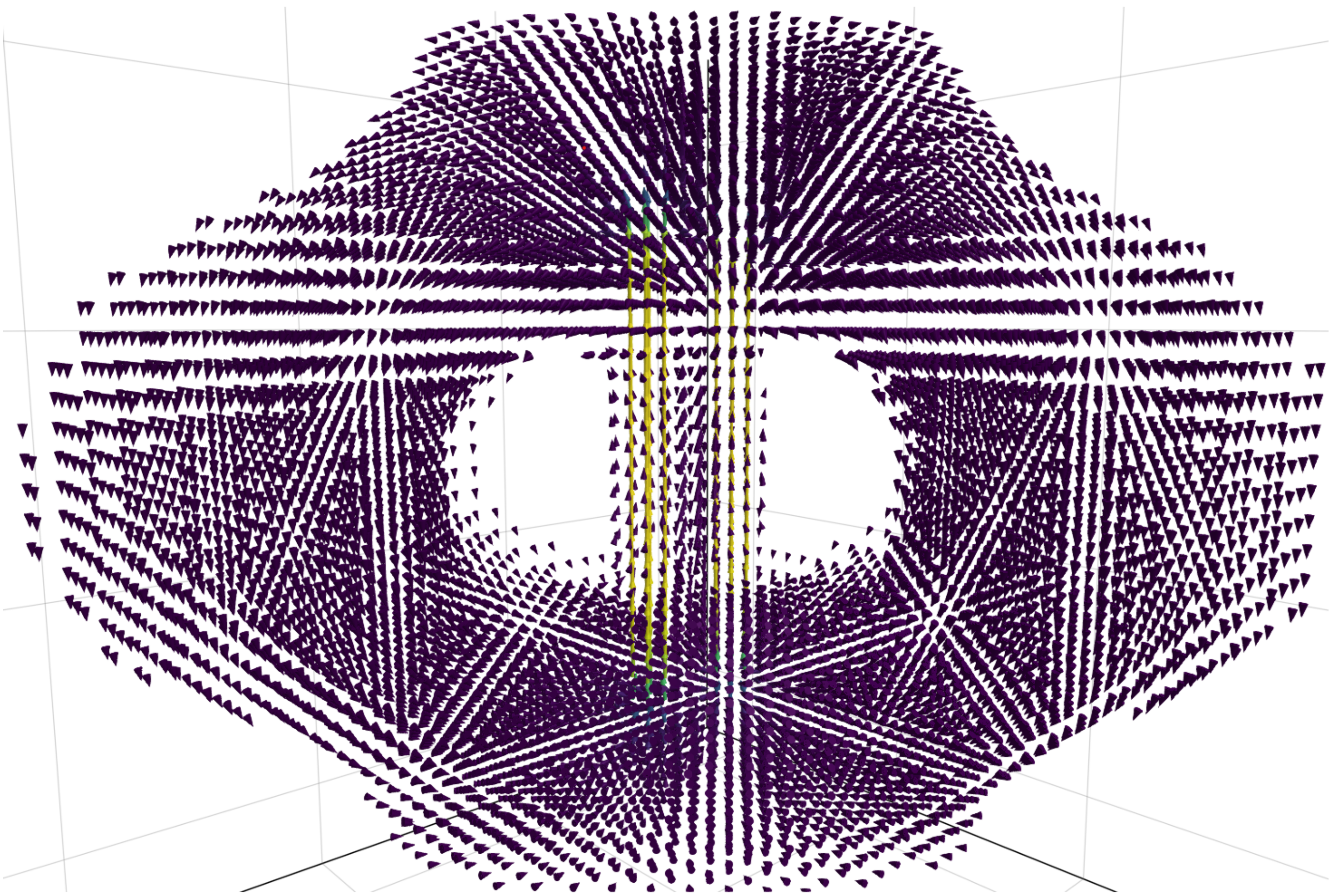

Figure 2: Four close and parallel lines of current and the total currents they cause, as calculated by Gila and BICSTABL. The qualitative behaviour where the flow forms circulant path connecting the ends of the line current are present.

A Code for Example graph

```
include("GilaOperators.jl")

using Base.Threads
using FFTW
using GLMakie
using GeometryBasics
```

```
using IterativeSolvers
using LinearAlgebra
using LinearAlgebra.BLAS
using Random
using Serialization
using Statistics
using ..GilaOperators

Random.seed!(0);

num_threads = nthreads()
BLAS.set_num_threads(num_threads)
FFTW.set_num_threads(num_threads)

num_cells = 32
cells = [num_cells, num_cells, num_cells] # Cells in volume
scale = (1//100, 1//100, 1//100) # Size of cells in units of wavelength
coord = (0//1, 0//1, 0//1) # Center position of volume

"""
Solves t = (1 - XG)^{-1}i
"""
function solve(ls::LippmanSchwinger, i::AbstractArray{ComplexF64, 4}; solver=bicgstabl
    )
  out = solver(ls, reshape(deepcopy(i), prod(size(i))))
  return reshape(out, size(i))
end

# You can choose your chi as a function of space here
medium = fill(5.0 + 0im, num_cells, num_cells, num_cells, 1)
ls = LippmanSchwinger(cells, scale, coord, medium)

## VISUALIZATION
source = zeros(ComplexF64, num_cells, num_cells, num_cells, 3)

# Sources
A = 10.0
for i in 5:25
  source[5, 7, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]
  source[6, 7, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]
```

```
  source[5, 8, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]
  source[6, 8, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]

  source[10, 7, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]
  source[11, 7, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]
  source[10, 8, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]
  source[11, 8, i, :] = A * [0.0+0.0im, 0.0+0.0im, 1.0+0.0im]
end

points = [Point3f((x-1)*scale[1] + coord[1], (y-1)*scale[2] + coord[2], (z-1)*scale[3]
     + coord[3]) for x in 1:num_cells for y in 1:num_cells for z in 1:num_cells]
field = [Vec3f(real.(source[x, y, z, :])...) for x in 1:num_cells for y in 1:num_cells
     for z in 1:num_cells]

out = solve(ls, source)
field_out = [Vec3f(real.(out[x, y, z, :])...) for x in 1:num_cells for y in 1:
    num_cells for z in 1:num_cells]

which_field = field_out

view_scale = 1000
grid_points = [Point3f(view_scale*((x-1)*scale[1] + coord[1]), view_scale*((y-1)*scale
    [2] + coord[2]), view_scale*((z-1)*scale[3] + coord[3])) for x in 1:num_cells for
    y in 1:num_cells for z in 1:num_cells]
cube = Rect3f((0.0, 0.0, 0.0), scale .* view_scale)

color = norm.(which_field)
color[color .< 1e-1] .= NaN

scene = arrows(grid_points, which_field; color=color, arrowsize=(view_scale * 0.3 *
    minimum(scale), view_scale * 0.3 * minimum(scale), view_scale  * 0.5 * minimum(
    scale)))
display(scene)
readline()
```

Listing 1: The code used for plotting a line current.

```
module GilaOperators

using LinearAlgebra
using Serialization
```

```julia
using Statistics
using GilaElectromagnetics

export GilaOperator, LippmanSchwinger, LippmanSchwingerNoDiagonal, YGi

abstract type GilaOperator end

function LinearAlgebra.mul!(y::AbstractVector{ComplexF64}, op::GilaOperator, x::
    AbstractVector{ComplexF64})
  y .= op * x
  return y
end

mutable struct LippmanSchwinger <: GilaOperator
  self_mem::GlaOprMem
  medium::AbstractArray{ComplexF64, 4}
end

internal_size(op::LippmanSchwinger) = (op.self_mem.trgVol.cel..., 3)

function Base.:*(op::LippmanSchwinger, x::AbstractVector{ComplexF64})
  x_copy = deepcopy(x)
  acted_vec = egoOpr!(op.self_mem, reshape(deepcopy(x), internal_size(op)))
  acted_vec .*= op.medium
  acted_vec .= reshape(x_copy, size(acted_vec)) .- acted_vec
  return reshape(acted_vec, size(x))
end

function Base.:*(op::LippmanSchwinger, x::AbstractArray{ComplexF64, 4})
  x_copy = deepcopy(x)
  acted_vec = egoOpr!(op.self_mem, x)
  acted_vec .*= op.medium
  acted_vec .= x_copy .- acted_vec
  return acted_vec
  # return reshape(acted_vec, prod(size(x)))
end

Base.eltype(_::LippmanSchwinger) = ComplexF64
Base.size(op::LippmanSchwinger) = (prod(op.self_mem.trgVol.cel)*3, prod(op.self_mem.
    trgVol.cel)*3)
```

```
Base.size(op::LippmanSchwinger, _::Int) = prod(op.self_mem.trgVol.cel)*3

function LippmanSchwinger(cells::AbstractVector{Int}, scale::NTuple{3, Rational{Int}}, coord::NTuple{3, Rational{Int}}, medium::AbstractArray{ComplexF64, 4})
  options = GlaKerOpt(false)
  self_volume = GlaVol(cells, scale, coord)
  filename = "preload/$(cells[1])x$(cells[2])x$(cells[3])_$(float(scale[1]))x$(float(scale[2]))x$(float(scale[3]))@$(float(coord[1])),$(float(coord[2])),$(float(coord[3])).fur"
  if isfile(filename)
    fourier = deserialize(filename)
    self_mem = GlaOprMem(options, self_volume, egoFur=fourier, setType=ComplexF64)
  else
    self_mem = GlaOprMem(options, self_volume, setType=ComplexF64)
    serialize(filename, self_mem.egoFur)
  end
  return LippmanSchwinger(self_mem, medium)
end
end
```

Listing 2: Helper code to interface between Gila and Visualization code (Listing 1).

References

[1] MC Van Beurden and SJL Van Eijndhoven. “Well-posedness of domain integral equations for a dielectric object in homogeneous background”. In: *Journal of Engineering Mathematics* 62 (2008), pp. 289–302.

[2] A.T. De Hoop. *Handbook of radiation and scattering of waves: Acoustic waves in fluids, elastic waves in solids, electromagnetic waves: with corrections*. Academic Press, 1995.

[3] Douglas Samuel Jones. *The theory of electromagnetism*. Elsevier, 2013.

[4] Richard Courant and David Hilbert. *Methods of mathematical physics: partial differential equations*. John Wiley & Sons, 2008.

[5] Robert A Adams and John JF Fournier. *Sobolev spaces*. Elsevier, 2003.

[6] Rainer Kress, Vladimir Maz’ya, and Vladimir Kozlov. *Linear integral equations*. Vol. 82. Springer, 1989.

[7] Johannes Markkanen, Pasi Yla-Oijala, and Ari Sihvola. “Discretization of volume integral equation formulations for extremely anisotropic materials”. In: *IEEE Transactions on Antennas and Propagation* 60.11 (2012), pp. 5195–5202.

[8] Lukas Novotny and Bert Hecht. *Principles of nano-optics*. Cambridge university press, 2012.