only if

$$\begin{vmatrix} \sigma_2 & \sigma_3 & \cdots & \cdots & \cdots & \sigma_{m+1} \\ \sigma_3 & \sigma_4 & \cdots & \cdots & \cdots & \sigma_{m+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{m+1} & \sigma_{m+2} & \cdots & \cdots & \cdots & \sigma_{2m} \end{vmatrix} = 0 \quad (N = 2m+1)$$

and

$$\begin{vmatrix} \sigma_3 & \sigma_4 & \cdots & \cdots & \cdots & \sigma_{m+2} \\ \sigma_4 & \sigma_5 & \cdots & \cdots & \cdots & \sigma_{m+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{m+2} & \sigma_{m+3} & \cdots & \cdots & \cdots & \sigma_{2m+1}, \end{vmatrix} = 0 \quad (N = 2m+2),$$

where σ_j are the coefficients in the formal expansion of the function

$$f(t) = \sqrt{\left(t + \frac{1}{a^2}\right)\left(t + \frac{1}{b^2}\right)\left(t + 1 - \frac{c^2}{a^2} - \frac{d^2}{b^2}\right) + t\left(\frac{c^2}{a^4} + \frac{d^2}{b^4}\right) + \frac{c^2}{a^4 b^2} + \frac{d^2}{a^2 b^4}}$$

into a power series $f(t) = \sum_{j=0}^{\infty} \sigma_j t^j$.

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A Simple Proof of Sion's Minimax Theorem

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The following theorem due to Sion [**3**] is fundamental in convex analysis and in the theory of games.

Theorem. *Let X and Y be nonvoid convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is upper semicontinuous and quasi-concave in the first variable and lower semicontinuous and quasi-convex in the second variable. Then*

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

The assumptions on f mean that for every real α the upper sets $U_\alpha(y) := \{x \in X : f(x, y) \geq \alpha\}$ for y in Y and the lower sets $L_\alpha(x) := \{y \in Y : f(x, y) \leq \alpha\}$ for x in X are closed and convex. It is well known and easy to see that the minima and maxima in the theorem exist under our topological assumptions.

Sion proved his theorem with the aid of Helly's theorem and the "KKM-Theorem" of Knaster, Kuratowski, and Mazurkiewicz. The special case where X and Y are finite-dimensional simplexes and f is a continuous function was proved by von Neumann [**4**] by reducing the problem to the 1-dimensional case.

We present a proof that is close in spirit to von Neumann's original proof. It uses only the 1-dimensional KKM-theorem (i.e., every interval in $\mathbb{R}$ is connected) and the 1-dimensional Helly theorem (i.e., any family of pairwise intersecting compact intervals in $\mathbb{R}$ has nonvoid intersection).

The following lemma was inspired by Groemer's paper [**1**] and by [**2**].

Lemma. *In the situation of the theorem, let K_1 and K_2 be nonvoid convex and compact subsets of X with $X = K_1 \cup K_2$. Then*

$$\xi := \max_{i\in\{1,2\}} \min_{y\in Y} \max_{x\in K_i} f(x, y) = \beta := \min_{y\in Y} \max_{x\in X} f(x, y).$$

Proof. Choose z_i in Y with $\max_{x\in K_i} f(x, z_i) \le \xi$ for $i = 1, 2$. Then we have $\xi \ge \max_{x\in X} \min_{y\in I} f(x, y)$, where $I = [z_1, z_2]$ is the convex hull of $\{z_1, z_2\}$. Suppose that $\min_{y\in I} \max_{x\in X} f(x, y) > \delta > \xi$ for some real δ. Then we get $\bigcap_{x\in X} L_\delta(x) \cap I = \emptyset$ and $\bigcap_{y\in I} U_\delta(y) = \emptyset$. By the 1-dimensional Helly theorem, there exist x_1 and x_2 in X with

$$L_\delta(x_1) \cap L_\delta(x_2) \cap I = \emptyset \tag{1}$$

and there exist y_1 and y_2 in I with

$$U_\delta(y_1) \cap U_\delta(y_2) \cap [x_1, x_2] = \emptyset,$$

which implies that

$$L_\delta(x_i) \cap \{y_1, y_2\} \ne \emptyset \qquad (i = 1, 2). \tag{2}$$

Since the sets $L_\delta(x_i)$ are closed and $[y_1, y_2]$ is connected, relations (1) and (2) ensure the existence of y_0 such that

$$y_0 \in [y_1, y_2] \setminus \big(L_\delta(x_1) \cup L_\delta(x_2)\big). \tag{3}$$

A similar argument shows that there exists x_0 such that

$$x_0 \in [x_1, x_2] \setminus \big(U_\delta(y_1) \cup U_\delta(y_2)\big). \tag{4}$$

By (3) we have $\{x_1, x_2\} \subset U_\eta(y_0)$ for some $\eta > \delta$. Since $x_0 \in [x_1, x_2] \subset U_\eta(y_0)$ we infer that $f(x_0, y_0) > \delta$. Similarly, (4) implies that $f(x_0, y_0) < \delta$, a contradiction. Therefore we have $\xi \ge \min_{y\in I} \max_{x\in X} f(x, y) \ge \beta \ (\ge \xi)$. ■

Proof of the theorem. We fix a finite subset A of X and set

$$\beta = \min_{y\in Y} \max_{x\in[A]} f(x, y),$$

where $[A]$ denotes the convex hull of A. Let $\mathcal{K}$ denote the system of all nonvoid closed convex subsets K of $[A]$ with $\min_{y\in Y} \max_{x\in K} f(x, y) = \beta$. Let $\mathcal{C}$ be a chain in $\mathcal{K}$. Then the fact that $U_\beta(y) \cap C \ne \emptyset$ for all C in $\mathcal{C}$ and y in Y implies that $U_\beta(y) \cap \bigcap \mathcal{C} \ne \emptyset$ for each y in Y (i.e., $\bigcap \mathcal{C} \in \mathcal{K}$). Therefore, $\mathcal{K}$ is inductively ordered by inclusion.

By Zorn's Lemma, $\mathcal{K}$ has a minimal element K_0. Suppose that K_0 has proper closed convex subsets K_1 and K_2 with $K_0 = K_1 \cup K_2$. Then our lemma tells us that either $K_1 \in \mathcal{K}$ or $K_2 \in \mathcal{K}$, contradicting the minimality of K_0. Hence, K_0 must be a singleton $\{x_0\}$. This implies that

$$\alpha := \max_{x\in X}\min_{y\in Y} f(x,y) \geq \min_{y\in Y} f(x_0,y) = \min_{y\in Y}\max_{x\in[A]} f(x,y) \geq \min_{y\in Y}\max_{x\in A} f(x,y),$$

whence $\bigcap_{x\in A} L_\alpha(x) \neq \emptyset$ for each finite subset A of X. Since Y is compact and all the sets $L_\alpha(x)$ are closed, there exists a y_0 in $\bigcap_{x\in X} L_\alpha(x)$. Accordingly,

$$\alpha \geq \max_{x\in X} f(x,y_0) \geq \min_{y\in Y}\max_{x\in X} f(x,y) \ (\geq \alpha). \qquad \blacksquare$$

Remark. If one does not want to use Zorn's Lemma, one can also proceed in the foregoing proof as in [**1**] or [**2**]: if K in $\mathcal{K}$ is not a singleton, then it can be split by some hyperplane into two convex compact subsets $K = K_1 \cup L_1$, and by our lemma we may assume that K_1 belongs to $\mathcal{K}$. Continuing this process, one can arrange the construction of sets K_n in $\mathcal{K}$ ($n = 1, 2, 3, \ldots$) such that their diameters decrease to 0, hence, $\bigcap_{n\in\mathbb{N}} K_n$ is a singleton $\{x_0\}$. Now $U_\beta(y) \cap K_n \neq \emptyset$ for each n, so x_0 lies in $U_\beta(y)$ for each y in Y (i.e., $\{x_0\}$ is a member of $\mathcal{K}$).

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A Constructive Approach to Singular Value Decomposition and Symmetric Schur Factorization

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1. INTRODUCTION. Fundamental to understanding linear transformations in $\mathbb{R}^n$ is the fact that the image of a sphere under a matrix mapping $\mathbf{A}$ is an ellipsoid. This geometrical point is seldom emphasized in sophomore-level linear algebra courses because its demonstration depends on the "Singular Value Decomposition," the proof of which, in most texts, requires the somewhat intimidating "Symmetric Schur Factorization." So the opportunity to make a very appealing geometrical point is lost. We believe that this is both unfortunate and unnecessary.