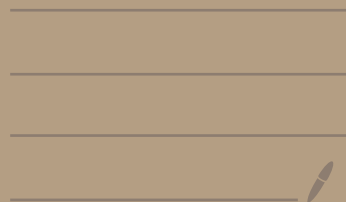


# 线性判别分析推导

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$$X = (x_1 \ x_2 \ \dots \ x_N)^T = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_N^T \end{pmatrix}_{N \times p}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}_{N \times 1}$$

$$\{ (x_i, y_i) \}_{i=1}^N, \quad x_i \in \mathbb{R}^p, \quad y_i \in \{+1, -1\}$$

$$X_{c_1} = \{x_i \mid y_i = +1\}, \quad X_{c_2} = \{x_i \mid y_i = -1\}$$

$$|X_{c_1}| = N_1, \quad |X_{c_2}| = N_2, \quad N_1 + N_2 = N$$

$$z_i = w^T x_i \quad \|w\| = 1$$

$$\bar{z} = \frac{1}{N} \sum_{i=1}^N w^T x_i$$

$$S_Z = \frac{1}{N} \sum_{i=1}^N (z_i - \bar{z})(z_i - \bar{z})^T$$

$$= \frac{1}{N} \sum_{i=1}^N (w^T x_i - \bar{z})(w^T x_i - \bar{z})^T$$

$$C_1: \bar{z}_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} w^T x_i$$

$$s_1 = \frac{1}{N} \sum_{i=1}^N (w^T x_i - \bar{z}) (w^T x_i - \bar{z})^T =$$

$$C_2: \bar{Z}_2 = \frac{1}{N_2} \sum_{i=1}^{N_2} w^T x_i$$

$$S_2 = \frac{1}{N_2} \sum_{i=1}^N (\omega^T x_i - \bar{z}_2) (\omega^T x_i - \bar{z}_2)^T$$

类间:  $(\bar{z}_1 - \bar{z}_2)^2$

类内:  $S_1 + S_2$

目标函数:  $J(w) = \frac{(\bar{z}_1 - \bar{z}_2)^2}{S_1 + S_2}$

$$\hat{w} = \arg \max_w J(w)$$

$$\begin{aligned}(\bar{Z}_1 - \bar{Z}_2)^2 &= \left[ \frac{1}{N_1} \bar{Z} W^T X_1 - \frac{1}{N_2} \bar{Z} W^T X_1 \right]^2 \\&= \left\{ W^T \left[ \frac{1}{N_1} \bar{Z} X_1 - \frac{1}{N_2} \bar{Z} X_1 \right] \right\}^2 \\&= \left[ W^T (\bar{X}_1 - \bar{X}_2) \right]^2 = W^T (\bar{X}_1 - \bar{X}_2) (X_1 - X_2)^T W\end{aligned}$$

$$\begin{aligned} S_1 + S_2 &= \frac{1}{N_1} \sum (w^T x_i - \frac{1}{N_1} \sum w^T x_i) (w^T x_i - \frac{1}{N_1} \sum w^T x_i) + \dots \\ &= \frac{1}{N_1} \sum w^T (x_i - \bar{x}_1) (x_i - \bar{x}_1)^T w + \dots \\ &= w^T \left[ \frac{1}{N_1} \sum_{i=1}^{N_1} (x_i - \bar{x}_1) (x_i - \bar{x}_1)^T \right] \cdot w + \dots \\ &= w^T \cdot S_1 \cdot w + w^T \cdot S_2 \cdot w \end{aligned}$$

$$= w^T (S_1 + S_2) \cdot w$$

$$J(w) = \frac{w^T (\bar{x}_1 - \bar{x}_2) (\bar{x}_1 - \bar{x}_2)^T w}{w^T (S_1 + S_2) w}$$

$$= \frac{w^T \cdot S_b \cdot w}{w^T \cdot S_w \cdot w}$$

$S_b$ : between-class 类间方差

$S_w$ : within-class 类内方差

$$\frac{\partial J(w)}{\partial w} = 2 S_b \cdot w (w^T S_w \cdot w)^{-1}$$

$$- w^T S_b \cdot w (w^T S_w \cdot w)^{-2} \cdot 2 S_w \cdot w$$

$$= 0$$

$$\Rightarrow S_b \cdot w \cdot (w^T S_w \cdot w)^{-1} - w^T S_b \cdot w \cdot (w^T S_w \cdot w)^{-2} \cdot S_w \cdot w$$

$$\Rightarrow S_b \cdot w \cdot (w^T S_w \cdot w) = w^T \cdot S_b \cdot w \cdot S_w \cdot w$$

$$\Rightarrow \underbrace{w^T \cdot S_b \cdot w}_{\text{标量}} \cdot S_w \cdot w = S_b \cdot w \cdot \underbrace{w^T S_w \cdot w}_{\text{标量}}$$

$$\Rightarrow S_w \cdot w = \frac{w^T \cdot S_w \cdot w}{w^T \cdot S_b \cdot w} \cdot S_b \cdot w$$

$$\Rightarrow w \propto \underbrace{S_w^{-1} \cdot S_b \cdot w}_{(\bar{x}_1 - \bar{x}_2) (\bar{x}_1 - \bar{x}_2)^T \cdot w}$$

$$\propto \boxed{S_w^{-1} \cdot (\bar{x}_1 - \bar{x}_2)}$$

$S_w$  对称,  $S_w^{-1} \propto I$

$$w \propto (\bar{x}_1 - \bar{x}_2)$$