Multi: $f(x) = x^T x$

$$\Rightarrow \frac{df(x)}{dx} = \, ? = \nabla f(x)$$

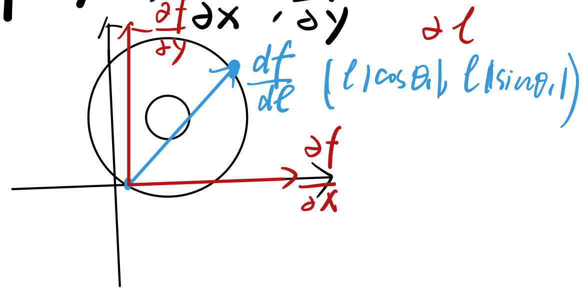
$\rightarrow$ 破坏整体性

pre: $$\frac{df(x)}{dx} = \nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \cdots, \frac{\partial f(x)}{\partial x_n} \right)$$

C^1: 一阶导连续可微

方向导数(Directional Derivative)

$$f(x,y) \Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial l}$$



$$\frac{df}{dl} = \|l\| \cos\theta \cdot \frac{\partial f}{\partial x} + \|l\| \sin\theta \cdot \frac{\partial f}{\partial y}$$

$$n元 \Rightarrow \frac{df}{dl} = Df(x)[l] = l \cdot \left(\cos\theta_1 \frac{\partial f}{\partial x_1} + \cos\theta_2 \cdot \frac{\partial f}{\partial x_2} + \cdots + \cos\theta_n \cdot \frac{\partial f}{\partial x_n} \right)$$

$$= \underbrace{[|l|\cos\theta_1, |l|\cos\theta_2, \cdots, |l|\cos\theta_n]}_{l^T} \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}}_{\nabla f(x)}$$

e.g. $f(x,y) = 3x + 2y$

$$\frac{\partial f}{\partial y} = 2 \triangleq g(x,y)$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 x_1} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \end{bmatrix} \begin{bmatrix} l\cos\theta_1 \\ l\cos\theta_2 \end{bmatrix}$$

$$
\begin{bmatrix} \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \cdots & \frac{\partial^2 f}{\partial x_n x_n} \end{bmatrix}
\begin{bmatrix} \vdots \\ l\cos\theta_n \end{bmatrix}
$$

第一行：$\frac{\partial^2 f}{\partial x_1 x_1} \cdot l\cos\theta_1 + \frac{\partial^2 f}{\partial x_1 x_2} l\cos\theta_2 + \cdots + \frac{\partial^2 f}{\partial x_1 x_n} \cdot l\cos\theta_n$

g $\frac{\partial f}{\partial x_1}$ 沿l方向的方向导数

$$D[g(x)][L] = \frac{dg}{dL}$$

$$= \frac{\partial g}{\partial x_1} \cdot l \cdot \cos\theta_1 + \cdots + \frac{\partial g}{\partial x_n} \cdot l \cdot \cos\theta_n$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial^2 f}{\partial x_1 x_1}$$

$\Rightarrow$ 第i行为 $\frac{\partial f}{\partial x_i}$ 的方向导数，$D(\nabla f(x))[l]$

例1：$f(x) = \frac{1}{2} x^T A x$ （二次型）

解：$\lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} = Df(x)[v]$

$$= \frac{1}{2}\left[(x^T + tv)^T A (x^T + tv) - x^T A x\right] / t$$

$$\lim_{t\to 0} \frac{1}{2}\left[t v^T A x + \underbrace{t^2 v^T A v}_{t} + t x^T A v\right] / t \quad (\to 0)$$

$$= \frac{1}{2}[v^T A x + x^T A v] = \underbrace{v^T A x}_{v^T \cdot \nabla f(x)}$$

$\because v$ is arbitrary

$\Rightarrow \nabla f(x) = Ax$

$$\frac{\nabla f(x+tv) - \nabla f(x)}{t} = Av = \nabla^2 f(x) \cdot v$$

$\Rightarrow \nabla^2 f(x) = A$

$D f(x)[v] = v^T \nabla f(x)$ $\langle v, \nabla f(x) \rangle$

$\Rightarrow D f(A)[d] = \langle d, \nabla f(A) \rangle = \langle \nabla f(A), d \rangle$

矩阵求导

$$f(A) \Rightarrow \left(\frac{df(A)}{dA}\right)_{ij} = \frac{df(A)}{dA_{ij}}$$

e.g. $f(A) = tr(AB)$

$\langle A, B \rangle = tr(A^T B)$

$$\frac{df(A)}{dA} = \nabla f(A) \ ?$$

$$\lim_{t \to 0} \frac{f(A+td) - f(A)}{t} = \frac{tr[(A+td)B] - tr(AB)}{t} = tr(dB)$$

$\langle B^T, d \rangle$

$$\Rightarrow \frac{d\,tr(AB)}{dA} = B^T = \nabla^2 f(A)$$

e.g.

$f(A) = \ln |A|$, $A \in S^n_{++}$ (对称正定阵) p.d.

sym 对称矩阵且特征值都大于零

$$\frac{d f(A)}{dA} = A^{-1}$$

$$\left(f(x) = \ln x, x > 0 \Rightarrow \frac{df(x)}{dx} = \frac{1}{x} \right)$$