Gaussian Discriminant Analysis

Data: $\{(x_i, y_i)\}_{i=1}^{N}$ $\qquad P(y|x) = \dfrac{P(x|y) \cdot P(y)}{P(x)}$

$x_i \in \mathbb{R}^p$, $y_i \in \{0,1\}$

$$\hat{y} = \arg\max_{y \in \{0,1\}} P(y|x) = \arg\max_{y} P(x|y) \cdot P(y)$$

where $$P(Y=y|X=x) = \frac{P(X=x|Y=y) \cdot P(Y=y)}{P(X=x)}$$

so $$\underset{\text{posterior}}{P(y|x)} \propto \underbrace{P(x|y)}_{\text{likelihood}} \underbrace{P(y)}_{\text{prior}}$$

$y \sim \text{Bernoulli}(\phi) \Rightarrow$

y	1	0
P	ϕ	$1-\phi$

$$\begin{cases} \phi^y, & y=1 \\ (1-\phi)^{1-y}, & y=0 \end{cases} \Rightarrow \phi^y \cdot (1-\phi)^{1-y}$$

$$\left.\begin{aligned} x|y=1 &\sim N(\mu_1, \Sigma) \\ x|y=0 &\sim N(\mu_2, \Sigma) \end{aligned}\right\} \Rightarrow N(\mu_1, \Sigma)^y \cdot N(\mu_2, \Sigma)^{1-y}$$

Log-likelihood: $\ell(\theta) = \log\left(\prod_{i=1}^{N} P(x_i, y_i)\right)$

$\theta = (\mu_1, \mu_2, \Sigma, \phi)$

$\hat{\theta} = \arg\max_{\theta} \ell(\theta)$

$$\begin{aligned} &= \sum_{i=1}^{N} \log\left(P(x_i|y_i) P(y_i)\right) \\ &= \sum_{i=1}^{N} \left[\log P(x_i|y_i) + \log P(y_i)\right] \\ &= \sum_{i=1}^{N} \left[\log N(\mu_1, \Sigma)^{y_i} \cdot N(\mu_2, \Sigma)^{1-y_i} + \log \phi^{y_i} (1-\phi)^{1-y_i}\right] \\ &= \sum_{i=1}^{N} \Big[\underbrace{\log N(\mu_1, \Sigma)^{y_i}}_{①} + \underbrace{\log N(\mu_2, \Sigma)^{1-y_i}}_{②} + \underbrace{\log \phi^{y_i} (1-\phi)^{1-y_i}}_{③}\Big] \end{aligned}$$

$y_i = 1$	N_1
$y_i = 0$	N_2

$N_1 + N_2 = N$

求解：1) 求 ϕ，对③：③ $= \sum\left[\log \phi^{y_i} + \log(1-\phi)^{1-y_i}\right] = \sum\left[y_i \log\phi + (1-y_i)\log(1-\phi)\right]$

$$\frac{\partial ③}{\partial \phi} = \sum\left(y_i \cdot \frac{1}{\phi} - (1-y_i)\frac{1}{1-\phi}\right) = 0$$

$$\Rightarrow \sum y_i \cdot (1-\phi) = \sum (1-y_i)\cdot\phi$$

$$\Rightarrow \sum y_i - \phi\sum y_i = \phi \cdot N - \phi\cdot\sum y_i$$

$$\Rightarrow \boxed{\hat{\phi} = \frac{1}{N}\sum y_i = \frac{N_1}{N}}$$

2) 求 μ_1：① $= \sum \log N(\mu_1, \Sigma)^{y_i}$

$$= \sum y_i \cdot \log\left[(2\pi)^{-\frac{p}{2}}|\Sigma|^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}(x_i-\mu_1)\cdot\Sigma^{-1}(x_i-\mu_1)\right\}\right]$$

$$\Rightarrow \mu_1 = \arg\max_{\mu_1} ① = \arg\max_{\mu_1} \sum y_i \cdot \left(-\frac{1}{2}(x_i-\mu_1)^T\Sigma^{-1}(x_i-\mu_1)\right) \quad (*)$$

$$(*) = -\frac{1}{2}\sum y_i (x_i-\mu_1)^T\Sigma^{-1}(x_i-\mu_1)$$

$$= -\frac{1}{2}\sum y_i (x_i^T\Sigma^{-1} - \mu_1^T\Sigma^{-1})(x_i-\mu_1)$$

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$$= -\frac{1}{2}\sum y_i \left(x_i^T\Sigma^{-1}x_i - \underbrace{\mu_1^T}_{1\times p}\underbrace{\Sigma^{-1}}_{p\times p}\underbrace{x_i}_{p\times 1} - x_i^T\Sigma^{-1}\mu_1 + \mu_1^T\Sigma^{-1}\mu_1\right)$$

$$\frac{\partial(*)}{\partial \mu_1} = -\frac{1}{2}\sum y_i\left(-2\Sigma^{-1}x_i + 2\Sigma^{-1}\mu_1\right) = 0$$

$$\Rightarrow \sum y_i(\mu_1 - x_i) = 0$$

$$\Rightarrow \sum y_i \mu_1 = \sum y_i x_i$$

$$\Rightarrow \boxed{\hat{\mu}_1 = \frac{\sum y_i x_i}{\sum y_i} = \frac{\sum_{i=1}^{N} y_i x_i}{N_1}}$$

3) 求 Σ: $\hat{\Sigma} = \arg\max_{\Sigma} ① + ②$

$$C_1 = \{x_i \mid y_i = 1, i = 1, \cdots, N\}$$

$$C_2 = \{x_i \mid y_i = 0, i = 1, \cdots, N\}$$

$$|C_1| = N_1, \quad |C_2| = N_2, \quad N_1 + N_2 = N$$

$$① + ② = \sum_{x_i \in C_1} \log N(\mu_1, \Sigma) + \sum_{x_i \in C_2} \log N(\mu_2, \Sigma)$$

$$\sum \log N(\mu, \Sigma) = \sum \log (2\pi)^{-\frac{p}{2}} \cdot |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x_i - \mu)^T \Sigma^{-1} (x_i - \mu)\}$$

$$= \sum \left(\log (2\pi)^{-\frac{p}{2}} - \frac{1}{2} \log |\Sigma| - \frac{1}{2} (x_i - \mu)^T \Sigma^{-1} (x_i - \mu) \right)$$

$$= \sum C - \frac{1}{2} N \cdot \log |\Sigma| - \frac{1}{2} \sum (x_i - \mu)^T \Sigma^{-1} (x_i - \mu)$$

x_i $p \times 1$

μ $p \times 1$

$$= -\frac{1}{2} N \log |\Sigma| - \frac{1}{2} N \cdot tr(S \cdot \Sigma^{-1}) + C$$

$$\sum tr((x_i - \mu)^T \Sigma^{-1} (x_i - \mu))$$

$$= \sum tr((x_i - \mu)^T (x_i - \mu) \cdot \Sigma^{-1})$$

$$= tr(\underbrace{\sum (x_i - \mu)^T (x_i - \mu)}_{N \cdot S} \cdot \Sigma^{-1})$$

$$= N \cdot tr(S \cdot \Sigma^{-1})$$

$$① + ② = -\frac{1}{2} N \log |\Sigma| - \frac{1}{2} N_1 \, tr(S_1 \Sigma^{-1}) - \frac{1}{2} N_2 tr(S_2 \cdot \Sigma^{-1}) + C$$

$$\frac{\partial (① + ②)}{\partial \Sigma} = -\frac{N}{2} \cdot \frac{1}{|\Sigma|} \cdot |\Sigma| \cdot \Sigma^{-1} - \frac{N_1}{2} \cdot S_1^T \cdot \Sigma^{-2} - \frac{N_2}{2} S_2^T \cdot \Sigma^{-2}$$

$$= 0$$

Tips:

$$\frac{\partial |A|}{\partial A} = |A| \cdot A^{-1}$$

$$\Rightarrow -N \cdot \Sigma^{-1} + N_1 \cdot S_1^T \Sigma^{-2} + N_2 \cdot S_2^T \cdot \Sigma^{-2} = 0$$

$$\frac{\partial \, tr(S \cdot \Sigma^{-1})}{\partial \Sigma} = -S^T \cdot \Sigma^{-2}$$

$$\Rightarrow N \Sigma - N_1 S_1 - N_2 S_2 = 0$$

$$\frac{\partial \, tr(AB)}{\partial A} = B^T$$

$$\Rightarrow \boxed{\hat{\Sigma} = \frac{1}{N}(N_1 S_1 + N_2 S_2)}$$ ★