ACF | W.S.S |

Binary Relation X, Y Joint Distribution $f_{X,Y}(x,y)$

$Y \sim kX$, Metric (Distance) $(E|X-Y|^2)^{\frac{1}{2}}$

$E(Y-kX)^2$

$= E|Y|^2 + k^2 E|X|^2 - 2k\boxed{E(XY)}$ $\Rightarrow$ Correlation

inner production $\Rightarrow$ Angle

① independence $\Leftrightarrow$ Uncorrelated (Orthogonal) $E(XY)=0$ $E(XY)=E(X)E(Y)$

$\hookrightarrow E[(X-E(X))(Y-E(Y))]$

$\theta \sim U(0,2\pi)$, $X=\cos\theta$. $Y=\sin\theta$, $\boxed{X^2+Y^2=1}$ 不独立

$E(X) = \int_{-\infty}^{+\infty} \cos\theta f_\theta(\theta) d\theta = \int_0^{2\pi} \cos\theta \cdot \frac{1}{2\pi} d\theta = \frac{1}{2\pi} \sin\theta \Big|_0^{2\pi} = 0$,

$E(Y) = 0$

$E(XY) = \int_{-\infty}^{+\infty} \cos\theta \sin\theta f_\theta(\theta) d\theta = \int_0^{2\pi} \cos\theta \sin\theta \cdot \frac{1}{2\pi} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \cdot d(-\cos\theta)$

$= \frac{1}{2\pi}\left(-\frac{1}{2}\cos^2\theta\right)\Big|_0^{2\pi} = 0$

$|E(XY)| \leq (E(X^2)E(Y^2))^{\frac{1}{2}}$ 柯西不等式 $(\sum_k x_k y_k) \leq (\sum_k x_k^2 \sum_k y_k^2)^{\frac{1}{2}}$ 内积

$|\langle X, Y\rangle| \leq (\langle x,x\rangle\langle y,y\rangle)^{\frac{1}{2}}$ $\int f(x) g(x) dx \leq (\int f^2(x) \int g^2(x))^{\frac{1}{2}}$

$0 \leq g(\alpha) = \langle \alpha X + Y, \alpha X + Y\rangle = \alpha^2\langle X,X\rangle + 2\cdot\alpha\langle X,Y\rangle + \langle Y,Y\rangle$ quadratic function

$\Delta = 4\langle X,Y\rangle^2 - 4\langle X,X\rangle\langle Y,Y\rangle \leq 0$ $\square$

discriminant

② $(X_1, \cdots, X_n)^T = X$ $\binom{n}{2} = \frac{n(n-1)}{2}$ Correlation Matrix

$E(X \cdot X^T) = R_X$, $R_X(i,j) = E(X_i X_j) = E(X_j X_i) = R_X(j,i)$ $R_X = R_X^T$

$\forall z \in \mathbb{R}^n$, $z^T R_X z = z^T E(XX^T) z = E(z^T X X^T z) = E(X^T z)^2 \geq 0$

正定 确定矢量

$X(t)$ Random Function $X(\omega, t)$ $\omega \in \Omega$ 样本点 Sample path (样本轨道) 给定ω时得到

$R_X(t,s) = E(X(t)X(s))$ (Auto) Correlation Function (自相关函数) ACF

$$\begin{cases} ① \ R_X(t,s) = R_X(s,t) & R_X(-\tau) = R_X(\tau) \\ ② \ R_X(t,t) \geq 0 & R_X(0) \geq 0 \\ ③ \ |R_X(t,s)| \leq (R_X(t,t) \cdot R_X(s,s))^{\frac{1}{2}} & |R_X(\tau)| \leq R_X(0) \end{cases}$$

Stationary (invariance)

Wide Sense (w.s.s) ① $E(X(t)) \equiv m$

宽平稳

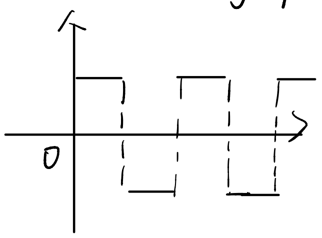
② $R_X(t+T, s+T) = R_X(t,s), \ \forall T$

$= R_X(t-s)$ ⇒ 只依赖于两个随机变量的差值（宽平稳核心性质：二元变一元）

相关函数有时延不变性（时间平移不变性） shifted invariance

Example:

② Random Telegraph Signal $X(t)$



$N(s) \sim P(\lambda s)$, $N(s)$ 跳变次数 $\quad X(0) \sim \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$P(N(s)=k) = \frac{(\lambda s)^k}{k!} \cdot \exp(-\lambda s)$$

① $$E(X(t)) = \sum_k k P(X(t)=k) = 1 \cdot \overset{\frac{1}{2}}{P(X(t)=1)} + (-1) \cdot \overset{\frac{1}{2}}{P(X(t)=-1)} = 0$$

$$P(X(t)=1) = \underbrace{P(X(t)=1 \mid X(0)=-1)}_{} \cdot \underset{\frac{1}{2}}{P(X(0)=-1)} + P(X(t)=1 \mid X(0)=1) \cdot \underset{\frac{1}{2}}{P(X(0)=1)} = \frac{1}{2}$$

时间0到时间t 跳变次数是奇数

$$P(X(t)=1 \mid X(0)=-1) \Leftrightarrow P(N(t) \text{ is odd})$$

$$P(N(t) \text{ is odd}) \Leftrightarrow \sum_{k \text{ is odd}} \frac{(\lambda t)^k}{k!} \exp(-\lambda t) = \boxed{\frac{1}{2}(1-\exp(-2\lambda t))}$$

$$P(N(t) \text{ is even}) = \sum_{k \text{ is even}} \frac{(\lambda t)^k}{k!} \exp(-\lambda t) = \frac{1}{2}(1+\exp(-2\lambda t))$$

$$\exp(\lambda t) = \sum_k \frac{(\lambda t)^k}{k!} \exp(-\lambda t) \qquad \exp(-\lambda t) = \sum_k \frac{(-\lambda t)^k}{k!} = \sum_k (-1)^k \cdot \frac{(\lambda t)^k}{k!}$$

$$\frac{1}{2}[\exp(\lambda t) - \exp(-\lambda t)] \cdot \exp(-\lambda t) = \frac{1}{2}(1-\exp(-2\lambda t))$$

② $$E(X(t)X(s)) = 1 \cdot P[X(t)X(s)=1] + (-1) \cdot P[X(t) \cdot X(s) = -1]$$
$= R_X(t,s)$

t,s 时刻同号 ⟺ 跳变偶数次 ⟺ t-s 是偶数

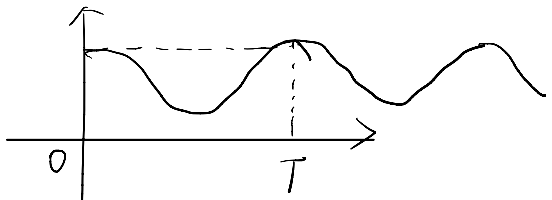
$$= 1 \cdot \frac{1}{2}(1+\exp(-2\lambda(t-s))) + (-1) \cdot \frac{1}{2}(1-\exp(-2\lambda(t-s)))$$

$$= \exp(-2\lambda(t-s))$$

$$R_X(\tau) = \exp(-2\lambda|\tau|)$$

$X(t)$ W.S.S. $\quad R_X(0)=R_X(T) \Rightarrow R_X(T+\tau) + R_X(\tau)$

局部特性能决定全局特性



Proof:

Consider $E|X(t)-X(t+T)|^2=0$

$$E|X(t)-X(t+T)|^2 = \underbrace{E(X^2(t))}_{R_X(0)} + \underbrace{E(X^2(t+T))}_{R_X(0)} - 2\underbrace{E(X(t)X(t+T))}_{R_X(T)}$$

$$= 2(R_X(0)-R_X(T)) = 0$$

$$|R_X(\tau)-R_X(\tau+T)| = |E(X(0)X(\tau)) - E(X(0)X(\tau+T))|$$

$$= |E(X(0)(X(\tau)-X(\tau+T)))| \le E(|X(0)|\cdot|X(\tau)-X(\tau+T)|)$$

$$\le \left[E[X^2(0)]\cdot E(|X(\tau)-X(\tau+T)|^2)\right]^{\frac{1}{2}}$$

- $X(t)$ W.S.S $\quad R_X(\tau)$ continuous at 0

$\Rightarrow R_X(t)$ continuous everywhere

Lec3. 深入相关函数 ACF

Transformer 关注相关 (二阶性质)

$X(t)$ Stochastic Processes $R_X(t,s) = E(X(t)X(s))$

$R_X(t,s) = R_X(t-s)$ W.S.S $R_X(\tau)$

Positive Definition Functions

$$\boxed{f(x)\ \text{p.d} \iff \forall n, \forall x_1,\cdots,x_n, \underbrace{(f(x_i-x_j))_{ij}}_{\text{矩阵}}\ \text{is p.d}}$$

$R_X(\tau)$ $R_X(i,j) = R_X(t_i - t_j) = E(X(t_i)X(t_j))$, $X = (X(t_1),\cdots,X(t_n))^T$

$R_X = E(XX^T) \geq 0$

Correlation Function $\Longleftrightarrow$ Positive Definite Function (一一对应) (相关函数 $\Longleftrightarrow$ 正定函数)

判断正定性:

Bochner: $f(x)$ is p.d $\iff \underbrace{\int_{-\infty}^{\infty} f(x)\exp(iwx)\,dx}_{F(w)} \geq 0$

☆ (函数正定 $\Leftrightarrow$ 其傅里叶变换为正)

充分性

"$\Leftarrow$" $F(w) = \int_{-\infty}^{\infty} f(x)\exp(-iwx)\,dx \Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{F(w)}_{\geq 0}\boxed{\exp(iwx)}\,dw$ [傅里叶反变换]

$\exp(iwx)$ is p.d $\forall n.\ x_1,\cdots,x_n$, $R = (\exp(iw(x_i-x_j)))_{ij}$

复指数正定

$\forall z = (z_1,\cdots,z_n)^T \in \mathbb{C}^n$, $z^H\cdot R\cdot z = \sum_{i=1}^{n}\sum_{j=1}^{n} R_{ij} z_i \bar{z}_j$ 展开二次型，与矩阵形式等价

复矢量

$= \sum_i\sum_j \exp(iw(x_i-x_j)) z_i\bar{z}_j$

$= \left(\sum_i \exp(iwx_i) z_i\right)\left(\sum_j \exp(-jwx_j)\bar{z}_j\right)$

$= \left(\sum_i \exp(iwx_i) z_i\right)\left(\overline{\sum_j \exp(jwx_j) z_j}\right)$

$= \left|\sum_i \exp(iwx_i) z_i\right|^2 \geq 0$ $\Rightarrow$ 复指数正定

② $\forall f_1,\cdots,f_n \geq 0$, $\boxed{\sum_{i=1}^{n} f_i \exp(iw_i x)}$ is p.d (正定函数正系数的线性组合正定)

③ Continuous $\int_{-\infty}^{\infty} \underbrace{F(w)}_{\geq 0}\exp(iwx)\,dw = 2\pi f(x)$ is p.d

$\Longrightarrow$" Fix n. Choose $x_1, \cdots, x_n$, $(f(x_i - x_j))_{ij}$.

$(f(x)$ p.d$)$

Choose $z = (\exp(i\omega x_1), \cdots, \exp(i\omega x_n))$

$$0 \le z^H R z = \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i - x_j)\exp(i\omega x_i)\overline{\exp(j\omega x_j)}$$ $\left[\overline{\exp(j\omega x_j)} = \exp(-j\omega x_j)\right]$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i - x_j)\exp(i\omega(x_i - x_j))$$ 【求和逼近积分】

$$\rightarrow \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-s)\exp(j\omega(t-s))\,dt\,ds$$

积分换元

①新元换旧元

$$\begin{cases} u = t - s \\ v = t + s \end{cases} \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

差Jacob行列式绝对值

$$dt\,ds = \left|\det\left(\frac{\partial(t,s)}{\partial(u,v)}\right)\right| du\,dv = \left|\frac{1}{\det\left(\frac{\partial(u,v)}{\partial(t,s)}\right)}\right| du\,dv = \frac{1}{2}\,du\,dv$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)\exp(j\omega u)\frac{1}{2}\,du\,dv$$

$$= \frac{1}{T}\iint_{\Diamond} f(u)\exp(j\omega u)\frac{1}{2}\,du\,dv$$

$$= \frac{1}{T}\left(\int_{-T}^{0}\int_{-u-T}^{u+T} + \int_{0}^{T}\int_{u-T}^{-u+T}\right) f(u)\exp(j\omega u)\frac{1}{2}\,dv\,du$$

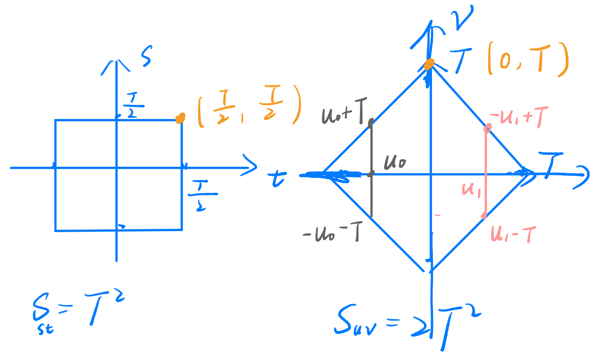
$$= \frac{1}{T}\int_{-T}^{T}\int_{|u|-T}^{-|u|+T} f(u)\exp(j\omega u)\frac{1}{2}\,dv\,du$$

$$= \frac{1}{T}\int_{-T}^{T}(2T - 2|u|) f(u)\exp(j\omega u)\frac{1}{2}\,du$$

$$= \frac{1}{T}\int_{-T}^{T}(T - |u|) f(u)\exp(j\omega u)\,du$$

$$= \int_{-T}^{T}\left(1 - \frac{|u|}{T}\right) f(u)\exp(j\omega u)\,du$$

$$\xrightarrow{T\to\infty} \int_{-\infty}^{+\infty} f(u)\exp(j\omega u)\,du$$



$S_{uv} = 2S_{st}$

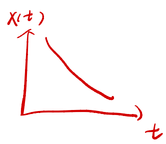
① Periodic Signal $X(t) = X(t+T)$, $\exists\, T > 0$ Fourier Series

$$X(t) = \sum_k \alpha_k \exp(i\omega_k t) \quad \omega_k = \frac{2k\pi}{T}, \quad \alpha_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega_k t)\,dt$$

② Non-Periodic Signal. $T \to \infty$ [没有周期 $\Leftrightarrow$ 周期无穷大] $\frac{2\pi}{T} \to 0$

$$\frac{1}{2\pi}\sum_k\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} X(s)\exp(-j\omega_k s)\,ds\right)\cdot\exp(j\omega_k t)\cdot\frac{2\pi}{T}$$ Random

$\int_{-\infty}^{+\infty}|X(t)|\,dt < \infty$



与平稳性相矛盾

$$\xrightarrow{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{X}(\omega)\exp(j\omega t)\,d\omega = X(t)$$

$$\int_{-\infty}^{+\infty} X(s)\exp(-j\omega s)\,ds = \hat{X}(\omega)$$

Fourier Transform

$$\lim_{T\to\infty} \frac{1}{T} E\left|\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega t)\,dt\right|^2$$ (Short Time 短时傅里叶变换)

Physical

$$= \frac{1}{T} E\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega t)\,dt\right)\left(\overline{\int_{-\infty}^{+\infty} X(s)\exp(-j\omega s)\,ds}\right)$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} E\left(X(t)\overline{X(s)}\right)\exp(-j\omega(t-s))\,dt\,ds$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_X(t-s)\exp(-j\omega(t-s))\,dt\,ds$$

$$\xrightarrow{T\to\infty} \int_{-\infty}^{\infty} R_X(\tau)\exp(-j\omega\tau)\,d\tau$$

正定函数(傅里叶函数正)