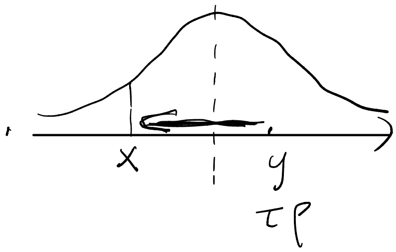
① 物理



粒子数量在时、空两维度上的分布

$f(x,t)$ Density of Certain Particle 粒子密度

τ时间内运动到 x 点的粒子数量

$$f(x,t+\tau)=\int_{-\infty}^{+\infty} f(x-y,t)\,\underbrace{\rho(\tau,y)}_{\text{ratio 比例}}\,dy \quad (\text{Diffusion Integral})$$

y 对应两侧的点

对τ做展开

$$\Rightarrow f(x,t)+\tau\frac{\partial f}{\partial t}+o(\tau)=\int_{-\infty}^{+\infty}\left(f(x,t)-y\frac{\partial f}{\partial x}+\frac{y^2}{2}\frac{\partial^2 f}{\partial x^2}+o(y^2)\right)\rho(\tau,y)dy$$

$$\Rightarrow f(x,t)+\tau\frac{\partial f}{\partial t}=\int_{-\infty}^{+\infty}\left(f(x,t)-y\frac{\partial f}{\partial x}+\frac{y^2}{2}\frac{\partial^2 f}{\partial x^2}\right)\rho(\tau,y)dy$$

$\rho(\tau,y)$ "Probability Density" $\underline{\int_{-\infty}^{+\infty}\rho(\tau,y)dy=1}$. $\rho(\tau,y)\geq 0$

$\underline{\int_{-\infty}^{+\infty} y\rho(\tau,y)\,dy=0}$ $\underline{\int_{-\infty}^{+\infty} y^2\rho(\tau,y)dy=D}$

两侧比例对称

$$\Rightarrow f(x,t)+\tau\frac{\partial f}{\partial t}=f(x,t)+\frac{D^2}{2}\frac{\partial^2 f}{\partial x^2} \qquad f(x,0)=\delta(x)$$

$$\Rightarrow \tau\cdot\frac{\partial f}{\partial t}=\frac{D^2}{2}\cdot\frac{\partial^2 f}{\partial x^2}$$ (Diffusion Equation) 二阶抛物线偏微分方程

$$\Rightarrow f(x,t)=\frac{1}{\sqrt{2\pi Dt}}\exp\left(-\frac{x^2}{2Dt}\right)$$ (扩散方程的解) 没有 Gaussian assumption 下求得

$N(0,Dt)$

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(2) Information Theory X. r.v 熵 $H(X)=\int_{-\infty}^{+\infty} f_X(x)\log_{X}(x)\,dx$ 最大熵微分

Maximum Entropy (Randomness) Uniform (finite interval) 平移不改变熵

(1) $(-\infty,+\infty)$ $E(X)=0$, $Var(X)=\sigma^2$

(2) $[0,+\infty)$ $E(X)=\mu$

(3) $[a,b]$

① $\max_{\boxed{f}} \underbrace{\left(-\int_{-\infty}^{+\infty} f(x)\log f(x)\,dx\right)}_{H(f)}$, s.t $\int_{-\infty}^{+\infty} x f(x)\,dx = 0$, $\int_{-\infty}^{+\infty} x^2 f(x)\,dx = \sigma^2$

函数　优化：泛函（算子）

Functional (Operator)

variational Method　$G(t) = H(f_0 + t\cdot g)$ $\Rightarrow$ $G(t) \leq G(0)$

变分方法　$t\in\mathbb{R}$　$\forall g$　$\Downarrow$

以函数为自变量 ⟶ 普通函数（可求导）⟶ 求导结果与优化函数相联系　$G'(0) = 0$

$f_0 = \arg\max_f H(f)$

$$G(t) = \int_{-\infty}^{+\infty} (f+tg)\log(f+tg)\,dx$$

$$L(t,\lambda_1,\lambda_2) = \int_{-\infty}^{+\infty} (f+tg)\log(f+tg)\,dx - \lambda_1\left(\int_{-\infty}^{+\infty} x(f+tg)\,dx\right) - \lambda_2\left(\int_{-\infty}^{+\infty} x^2(f+tg)\,dx - \sigma^2\right)$$

$$\frac{d}{dt}L(t,\lambda_1,\lambda_2) = \int_{-\infty}^{+\infty} g\log(f+tg)\,dx + \int_{-\infty}^{+\infty} g\,dx - \lambda_1\int_{-\infty}^{+\infty} xg\,dx - \lambda_2\int_{-\infty}^{+\infty} x^2 g\,dx$$

当 $t=0$ 时, $G'(0)=0$, $\frac{d}{dt}L(t,\lambda_1,\lambda_2) = \int_{-\infty}^{+\infty} g(\underbrace{\log f + 1 - \lambda_1 x - \lambda_2 x^2}_{0})\,dx = 0$, $\boxed{\forall g}$

$\Rightarrow \log f = \lambda_2 x^2 + \lambda_1 x - 1$ $\Rightarrow$ $\boxed{f = \exp(\lambda_2 x^2 + \lambda_1 x - 1)}$ 指数上二次型 $\sim N$

② $[0,+\infty)$

$$L(t,\lambda) = \int_0^{\infty} (f+tg)\log(f+tg)\,dx - \lambda_1\left(\int_0^{\infty} x(f+tg)\,dx - \mu\right) - \lambda_2\left(\int_0^{\infty}(f+tg)\,dx - 1\right)$$

$$\left.\frac{d}{dt}L(t,\lambda_1,\lambda_2)\right|_{t=0} = \int_0^{\infty} g(\log f(x) - \lambda_1 x - \lambda_2)\,dx \Rightarrow f(x) = c\cdot\exp(\lambda_1 x)$$ 单边无限是指数

③ $\left.\frac{d}{dt}L(t,\lambda)\right|_{t=0} = \int_a^b g(\log f(x) - \lambda)\,dx = 0 \Rightarrow f(x) = C$　Uniform 均匀

(3) Probability　Central Limit Theorem (CLT)　中心极限定理

$X_1, X_2, \cdots, X_n$, $E(X_k)=0$, $Var(X_k)=1$, $\frac{X_1+\cdots+X_n}{\sqrt{n}} \xrightarrow{n\to\infty} N(0,1)$

Characteristic Functions　X r.v　$\Phi_Z(w) = E(\exp(iwX))$

傅立叶反变换

$\Phi_X(w) = \int_{-\infty}^{+\infty} \exp(iwX) f_X(x)\,dx$　正定（概率密度）

$X_1 \sim f_{X_1}(x)$, $X_2 \sim f_{X_2}(x)$ independent . $X = X_1 + X_2 \sim f_{X_1} \circledast f_{X_2}$

$$F_X(x) = P(X \le x) = P(X_1+X_2 \le x)$$

$$\left(= \sum_{x_2} P(X_1+X_2 = x \mid X_2 = x_2)\, P(X_2 = x_2)\right)$$

$$= \int_{-\infty}^{+\infty} P(X_1+X_2 \le x \mid X_2 = x_2)\, f_{X_2}(x_2)\, dx_2$$

$$= \int_{-\infty}^{+\infty} \overset{F_{X_1}(x-x_2)}{\underline{P(X_1 \le x - x_2)}}\, f_{X_2}(x_2)\, dx_2$$

$$= \int_{-\infty}^{+\infty} F_{X_1}(x-x_2)\, f_{X_2}(x_2)\, dx_2$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X_1}(x-x_2)\, f_{X_2}(x_2)\, dx_2 \quad \text{（卷积）}$$

$$\Phi_X(w) = E(\exp(iwX)) = E(\exp(iw(X_1+X_2)))$$

$$= E(\exp(iwX_1)\exp(iwX_2))$$

$$= \Phi_{X_1}(w)\,\Phi_{X_2}(w)$$

$$\Phi_{\frac{X_1+\cdots+X_n}{\sqrt{n}}}(w) = E\left(\exp\left(iw\frac{X_1+\cdots+X_n}{\sqrt{n}}\right)\right) = E\left(\prod_{k=1}^{n} \exp\left(iw\frac{X_k}{\sqrt{n}}\right)\right)$$

$$= \prod_{k=1}^{n} E\left(\exp\left(iw\frac{X_k}{\sqrt{n}}\right)\right) = \prod_{k=1}^{n} \Phi_{X_k}\left(\frac{w}{\sqrt{n}}\right)$$

$$\left(1+\frac{\alpha}{n}+o\left(\frac{1}{n}\right)\right)^n \xrightarrow{n\to\infty} \exp(\alpha)$$

$$= \left(\Phi_{X_1}\left(\frac{w}{\sqrt{n}}\right)\right)^n = \left(1-\frac{w^2}{2n}+o\left(\frac{1}{n}\right)\right)^n \xrightarrow{n\to+\infty} \boxed{\exp\left(-\frac{w^2}{2}\right)}$$

傅立叶变换

$$\Longleftrightarrow \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

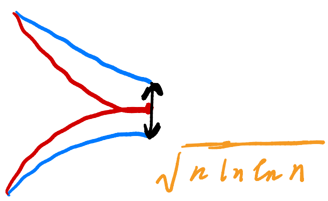
$N(0,1)$

$$\Phi_{X_1}\left(\frac{w}{\sqrt{n}}\right) = E\left(\exp\left(i\frac{w}{\sqrt{n}}X_1\right)\right) = E\left(1 + \overset{E(X)=0}{\underline{i\frac{w}{\sqrt{n}}X_1}} + \frac{1}{2}\left(i\frac{w}{\sqrt{n}}X_1\right)^2 + o\left(\frac{1}{n}\right)\right)$$

还有部分随机性

$$= 1 - \frac{w^2}{2n} + o\left(\frac{1}{n}\right)$$

$$\frac{X_1+\cdots+X_n}{n} \xrightarrow{n\to\infty} E(X_1)$$



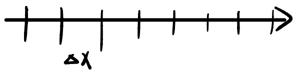
$$\Phi_{\frac{X_1+\cdots+X_n}{n}}(w) = \left(\phi_1\left(\frac{w}{n}\right)\right)^n$$

随机性消失

$$= \left(1 + i\frac{w}{n}\overset{\mu}{E(X_1)} + o\left(\frac{1}{n}\right)\right)^n$$

$$\xrightarrow{n\to\infty} \exp(iwE(X_1)) = \exp(iw\mu)$$

⊕ Stochastic Processes. Random Walk Symmetric One-Dimensional (一维对称随机游动)



$S_n = \sum_{k=1}^{n} X_n \qquad X_n \sim \begin{pmatrix} \Delta X & -\Delta X \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

independent

同分布

独立

$Var(X_k) = E(X_k^2)$

$= \frac{1}{2}\Delta X^2 + \frac{1}{2}(\Delta X)^2 = (\Delta X)^2$

$X(t) \quad \Delta t \quad t = n \cdot \Delta t \quad X(t) = \sum_{k=1}^{n} X_n$

$$X(t) = \sum_{k=1}^{n} X_k = \left(\sum_{k=1}^{n} \frac{X_k}{\Delta X}\right)\Delta X = \frac{\sum_{k=1}^{n}\left(\frac{X_k}{\Delta X}\right)}{\sqrt{n}} \cdot \sqrt{n} \cdot \Delta X$$

$$= \left(\frac{\sum \frac{X_k}{\Delta X}}{\sqrt{n}}\right)\sqrt{\frac{t}{\Delta t}} \cdot \Delta X = \frac{\sum \frac{X_k}{\Delta X}}{\sqrt{n}} \cdot \sqrt{t} \cdot \sqrt{\frac{\Delta X^2}{\Delta t}} \rightarrow N(0,1) \cdot \sqrt{Dt}$$

$$\Downarrow$$

$$N(0, Dt)$$

Let $\Delta t \to 0, \ \Delta X \to 0, \ \frac{(\Delta X)^2}{\Delta t} \longrightarrow D$

$$\tau \cdot \frac{\partial f}{\partial t} = \frac{D^2}{2} \cdot \frac{\partial^2 f}{\partial X^2}$$ (Diffusion Equation)

$$\Rightarrow f(x, t) = \frac{1}{\sqrt{2\pi D t}} \exp\left(-\frac{X^2}{2Dt}\right)$$ (扩散方程的解)

$N(0, Dt)$

$$\Rightarrow f \sim N(0, Dt)$$