非平稳过程

No Stationary (Case by case)

① Cyclostationary (周期平稳)

$\forall T$. $R_Z(t,s) = R_Z(t+T, s+T)$ $\rightarrow R_Z(t-s)$

$\boxed{\exists T.\ R_Z(t,s) = R_Z(t+T, s+T)}$

Periodic $R_Z(t,s) = R_Z(t+hT, s+hT)$

{X_t}不平稳，但通过调整相位，使平稳

$\{X_t\}$ Cyclostationary T. Let $\underset{r.v}{\theta} \sim U(0,T)$ independent of $X(t)$

$\boxed{Y(t) = X(t+\theta) \text{ is w.s.s}}$

Conditional Expectation

Proof: $R_Y(t,s) = E(Y(t), Y(s)) = E(X(t+\theta)X(s+\theta))$ 此时出现两个随机变量 $(X(t), \theta)$

$= E_\theta(E_X(X(t+\theta)X(s+\theta)|\theta))$ 转为条件期望

X不是w.s.s

$= E_\theta(R_X(t+\theta, s+\theta))$ 积分换元

$= \frac{1}{T}\int_0^T R_X(t+\theta, \underbrace{s+\theta}_{\theta'}) d\theta$

$= \frac{1}{T}\int_s^{s+T} R_X(t-s+\theta', \theta') d\theta'$ 周期平稳

$= \frac{1}{T}\int_0^T R_X(t-s+\theta', \theta') d\theta'$ 性质

①定义新元，旧元换新元

$\theta' = s+\theta$

$\theta = \theta' - s$

$t+\theta = t+\theta'-s$

②处理Jacob: $d\theta' = d\theta$

③处理积分限: $\theta \in [0,T)$, $\theta' \in [s, T+s)$

① $E(Y|X)$ is r.v (依赖X)

不是普通期望(不是一个值)

X, Y 都是 r.v

$E(Y|X)$ $\int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy$

忽略条件期望本质

② $E(g(X,Y)|X)$

$E_X(E_Y(g(X,Y)|X)) = E_{XY}(g(X,Y))$ 重期望公式(条件期望的期望是无条件期望)

$$\int_{-\infty}^{+\infty} E(g(X,Y)\mid X) f_X(x)\,dx = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} g(x,y) f_{Y|X}(y|x)\,dy\right) f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\underline{f_{Y|X}(y|x)\cdot f_X(x)}\,dy\,dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

$f(x,y) = f(y|x)\cdot f(x)$

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e.g. $X_1,\cdots,X_n$ i.i.d $E(X_1+\cdots+X_n) = n\cdot E(X_1)$, 随机个随机变量的和

$\boxed{N \ r.v}$ independent of $\{X_t\}$, $E(X_1+\cdots+X_N)$ 两个 r.v

$E(X_1+\cdots+X_N)$

没有随机性

$= E[E(X_1+\cdots+X_N \mid N)]$ 当N在条件号 | 后面时，X_N 中的 N 没有随机性

$= E(N\cdot E(X_1)) = E(N)\cdot E(X_1) = N\cdot E(X_1)$

$E(Y g(X) \mid X) = g(X)\cdot E(Y\mid X)$ (*)

条件在X ⇒ g(X) 没有随机性

Mean Square Estimation $Y, X \to g(X) \to Y \Rightarrow \min_g E(Y-g(X))^2$

[均方估计]

X的某个形式去估计Y

$E(Y-g(X))^2 = E(Y-E(Y|X) + E(Y|X)-g(X))^2$

$= E\left(Y-E(Y|X)\right)^2 + (E(Y|X)-g(X))^2 + 2\underline{E[(Y-E(Y|X))(E(Y|X)-g(X))]}$ 0

全是随机变量 r.v.

条件期望：把一部分随机性先限制住简化

先对X取条件期望

只依赖于X

$E_X\left[\underline{E_Y[(Y-E(Y|X))\boxed{(E(Y|X)-g(X))}\mid X]}\right]$ (使用(*))

$[E(Y|X)-g(X)]\,\underline{E(Y-E(Y|X)\mid X)}$ 0

$E(Y-E(Y|X)\mid X) = E(Y|X) - E[E(Y|X)\mid X] = E(Y|X) - E(Y|X)E(1|X) = 0$

$$X(t)=\sum_{k=-\infty}^{+\infty}\alpha_k\,\phi(t-kT)$$

作用在幅度上(信息符号) information symbol

T → 码片长度

ϕ: 基带波形 (Baseband waveform)

"$E(\alpha_k\alpha_m)=R_\alpha(k-m)$ 给定的w.s.s条件"

$$R_X(t,s)=E(X(t),X(s))=E\Big(\sum_k\alpha_k\phi(t-kT)\Big)\Big(\sum_n\alpha_n\phi(s-nT)\Big)$$

$$=\sum_k\sum_n E(\alpha_k\alpha_n)\phi(t-kT)\phi(s-nT)$$

$$=\sum_k\sum_n R_\alpha(k-n)\phi(t-kT)\phi(s-nT)$$

$$\underset{\substack{\| \\ R_X(t,s)}}{R_X(s+T,t+T)}=\sum_{k=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\overset{k'-n'}{R_\alpha(k-n)}\,\phi(\overset{t-(1-k)T}{t+T-kT})\,\phi(\overset{s-(1-n)T}{s+T-nT})$$

k', n'

$$\frac{1}{T}\int_0^T R_X(t+\theta,s+\theta)d\theta=\frac{1}{T}\int_0^T\sum_k\sum_s R_\alpha(k-n)\,\phi(t+\theta-kT)\,\phi(s+\theta-kT)d\theta$$

$$=\frac{1}{T}\sum_k\sum_s R_\alpha(k-n)\int_0^T\phi(t+\theta-kT)\,\phi(\underbrace{s+\theta-nT}_{\theta'})d\theta$$

$$=\frac{1}{T}\sum_k\sum_s R_\alpha(k-n)\int_{s-nT}^{s-(n-1)T}\phi(\theta'-(k-n)T+t-s)\,\phi(\theta')\,d\theta'$$

Orthogonal Increment 正交增量 $X(t)$, $X(0)=0$

$\forall t_1 < t_2 \le t_3 \le t_4$. $\boxed{X(t_4)-X(t_3) \perp X(t_2)-X(t_1)}$, $t>s$

$R_X(t,s) = E(X(t)X(s)) = E\left((X(t)-X(s)+X(s))[X(s)-X(0)]\right)$

$= E(X^2(s)) = E[X^2(\min(t,s))]$ (相关函数不是依赖于两个时刻的差，而是两个时刻的最小值)

Assume $R_X(t,s) = g(\min(s,t))$

$\boxed{E((X(t_4)-X(t_3))(X(t_2)-X(t_1)))} = R_X(t_4,t_2) + R_X(t_3,t_1) - R_X(t_3,t_2) - R_X(t_4,t_1)$

$= g(t_2)+g(t_1)-g(t_2)-g(t_1) = \boxed{0}$

Brown Motion 非规则运动

def: ① $B(0)=0$

② Orthogonal Increment

③ $B(t)-B(s) \sim N(0,\sigma^2(t-s))$

$R_B(t,s) = E(B^2(\min(t,s))) = Var(B(\min(t,s)))$

$= \sigma^2 \min(s,t)$

$Y(t) = \frac{d}{dt}B(t)$

$R_Y(t,s) = E(Y(t)Y(s)) = E\left(\frac{d}{dt}B(t)\frac{d}{ds}B(s)\right)$

$= \frac{\partial^2}{\partial t \partial s} E(B(t)B(s)) = \frac{\partial^2}{\partial t \partial s} R_B(t,s)$

$= \frac{\partial}{\partial t \partial s} \sigma^2 \min(t,s)$

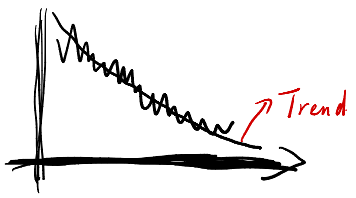
$\min(t,s) = \frac{1}{2}(t+s-|t-s|)$

white noise

$= -\frac{1}{2}\sigma^2 \frac{\partial^2}{\partial t \partial s}|t-s| = -\frac{1}{2}\sigma^2\frac{d}{ds}\mathrm{sgn}(t-s) = \boxed{\sigma^2 \delta(t-s)}$

Derivative $\Leftrightarrow$ High Pass

求导是高通滤波，去掉趋势，留下白噪声



Integral $\Leftrightarrow$ Low Pass

积分是低通滤波