Iteration and recursion

Due: Wednesday, January 23rd, 12:30. Deposit your paper copy, stapled and with your name and student ID on the front, in the designated assignment drop box. Upload the following files to Blackboard:

- `iteration.py` and `recursion.py` for Question 1;
- `steffensen.py` and `test_steffenson.py` for Question 2;
- a text file called `README` explaining any other files submitted. If you submit no other files, a `README` file is not needed.

Make sure all files submitted include a comment line with your name and student number, e.g.

```
# Peter Peterson 100456789
```

If you worked together with class mates on your code, and a substantial part of the code you submit coincides with theirs, you must list their names in a comment, e.g.

```
# Written in collaboration with Shawn Shawnson and Lea Leason.
```

Failure to do so may qualify your work as plagiarism.
Also, be a **good programmer** and include comments with a brief description of the functionality, input and output arguments and usage of each function or script. Add some comments that explain what steps are taken. Marks will be awarded or subtracted based on the readability and transparency of your code.
A discussion thread for this assignment is available. Pose your questions there before approaching the lecturer or TA.

Question 1 **50 marks**

Consider the nonlinear equation

$$f(x) = \sin(\pi x) - x^2 = 0$$

(**a**) Can you "solve the equation by hand", i.e. express the solution x^* explicitly in terms of elementary functions and the constant π?

(**b**) Show that the equation $f(x) = 0$ has the same solution(s) as $g(x) = x$ if

$$g(x) = \frac{1}{2\pi}\sin(\pi x) - \frac{1}{2\pi}x^2 + x$$

In fact, we can show that for any initial point in $(0, 2]$ the sequence $x_k = g(x_{k-1})$ converges to a unique solution x^*.

(**c**) Write a function that computes this sequence. Your function should:

- Take for input the an initial point, a maximal number of iterations and a tolerance for $|x_k - x_{k-1}|$ (the estimated error) and for $|f(x_k)|$ (the residual).
- Print to the commend window the list of iterates, stopping when either the maximal number of iterations is reached or the estimated error **and** the residual are below their respective tolerance.
- Output the approximate solution, its estimated error and its residual.

(**d**) If you used a `for` or `while` loop in (**c**), program a function with the same functionality using recursion (i.e. no explicit loops). If you used recursion in (**c**) then program a function with the same functionality using loops (i.e. no recursion).
Name your functions `iteration.m` for the version with a loop and `recursion.m` for the version without.

(**e**) What happens if you take $x_0 = 0$? What happens if you take $x_0 < 0$?

Question 2 **50 marks**

(**a**) Write a function that implements the following pseudo-code:
Input: f, f', x, ϵ, N.
Output: x^*.

1. Repeat N times:
   (a) Set $y_1 = x$.
   (b) Take one Newton step, starting from y_1. Call the result y_2.
   (c) Take one Newton step, starting from y_2. Call the result y_3.
   (d) Set
   $$x = y_1 - \frac{(y_2 - y_1)^2}{y_3 - 2y_2 + y_1}$$
   (e) Display $|f(x)|$.
   (f) If $|f(x)| < \epsilon$ print "converged!", break.
2. Output $x^* = x$.

This algorithm is called Steffensen's iteration.

(**b**) Write a sctipt to test your function on the problem

$$\exp(-x^2 + x) - \frac{1}{2}x = 1.0836 \quad \text{(with initial guess } x = 1)$$

Show that Newton iteration does not converge quadratically, but your new iterative algorithm does.