PROBABILITY

Balaji J

Probablity Vs Statistics

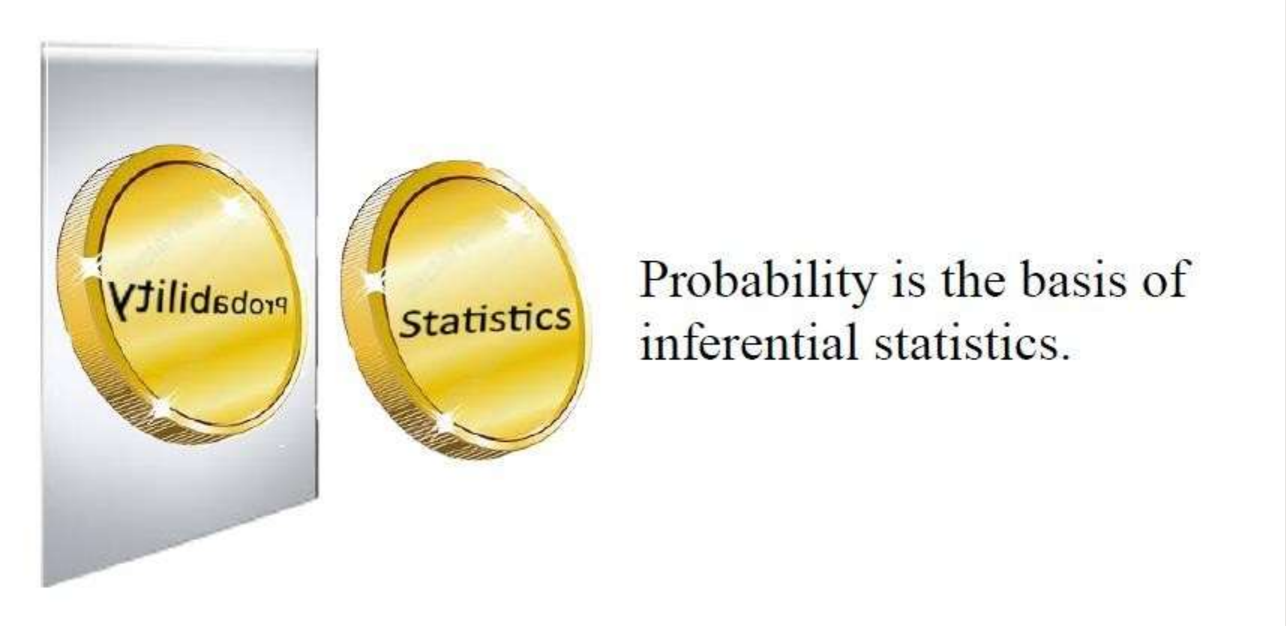
- Probability – Predict the likelihood of a future event
- Statistics – Analyze the past events

- Probability – What will happen in a given ideal world?
- Statistics – How ideal is the world?

WHAT IS PROBABILITY?

- Probability is a value between 0 and 1 that a certain event will occur

Probablity Vs Statistics



Probability - Applications

8 National Vital Statistics Reports, Vol. 54, No. 14, April 19, 2006

Table 1. Life table for the total population: United States, 2003

Age	Probability of dying between ages x to $x+1$ $q(_x)$	Number surviving to age x $l(_x)$	Number dying between ages x to $x+1$ $d(_x)$
0–1	0.006865	100,000	687
1–2	0.000469	99,313	47
2–3	0.000337	99,267	33
3–4	0.000254	99,233	25
4–5	0.000194	99,208	19
5–6	0.000177	99,189	18
6–7	0.000160	99,171	16

Insurance industry uses probabilities in actuarial tables for setting premiums and coverages.

EXAMPLE FOR PROBABLITY

- The probability that a fair coin will come up heads is 0.5
- Mathematically we write:

$$P(E_{heads}) = 0.5$$

Assigning Probabilities

Classical Method – *A priori* or Theoretical

Probability can be determined prior to conducting any experiment.

$$P(E) = \frac{\#\ of\ outcomes\ in\ which\ the\ event\ occurs}{total\ possible\ \#\ of\ outcomes}$$

Example: Tossing of a fair die

Assigning Probabilities

Empirical Method – *A posteriori* or Frequentist

Probability can be determined post conducting a thought experiment.

$$P(E) = \frac{\#\ of\ times\ an\ event\ occurred}{total\ \#\ of\ opportunities\ for\ the\ event\ to\ have\ occurred}$$

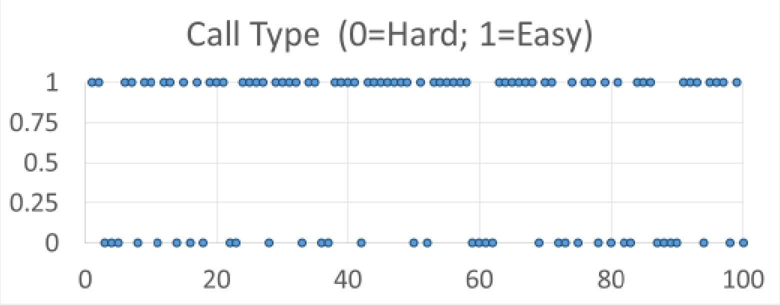
Example: Tossing of a weighted die…well!, even a fair die. The larger the number of experiments, the better the approximation.

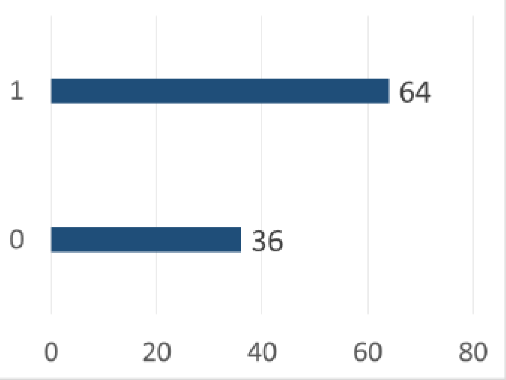
This is the most used method in statistical inference.

Assigning Probabilities

Empirical Method – *A posteriori* or Frequentist

100 calls handled by an agent at a call centre

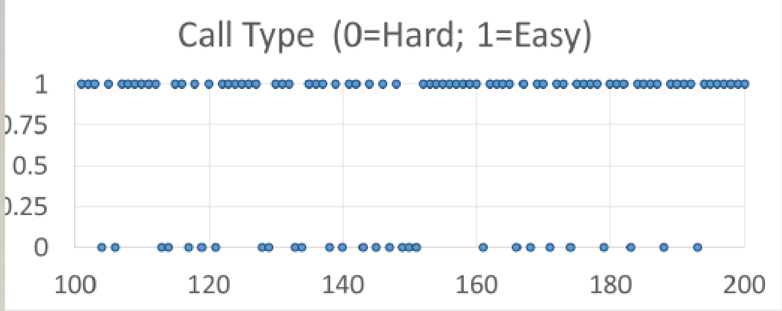


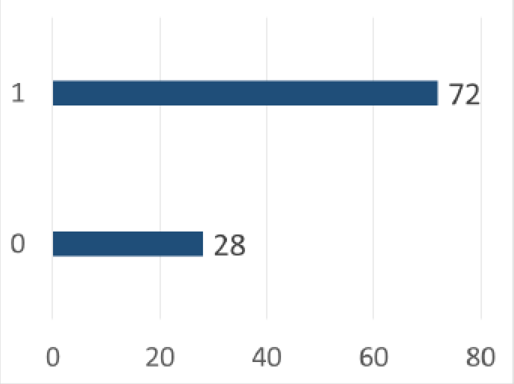


Assigning Probabilities

Empirical Method – *A posteriori* or Frequentist

Next 100 calls handled by an agent at a call centre

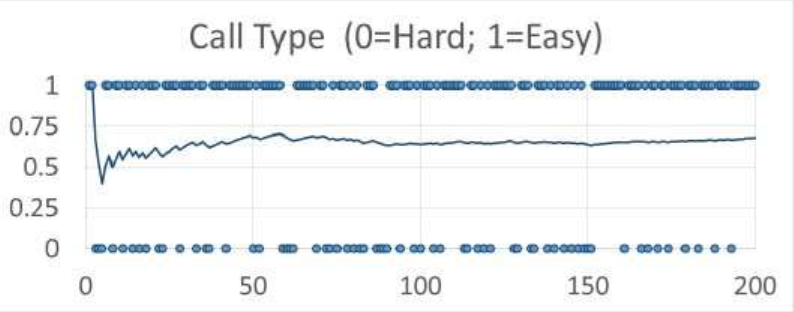


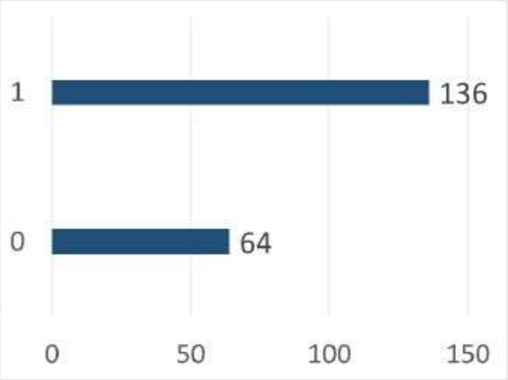


Assigning Probabilities

Empirical Method – *A posteriori* or Frequentist

Averages over the long run





P(easy) = 0.7

Assigning Probabilities

Empirical Method – *A posteriori* or Frequentist

Probability of having a monthly income of 1000 BHD is 10/23 = 0.43

INCOME(BHD)	FREQUENCY
100	10
345	1
1000	10
9833	2

Assigning Probabilities

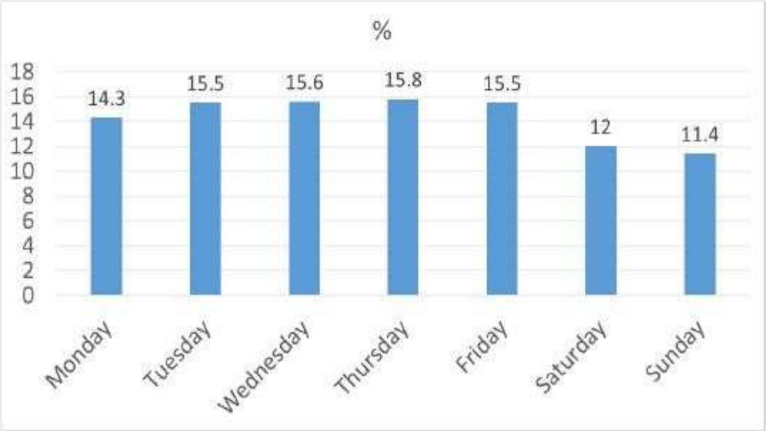
Subjective Method

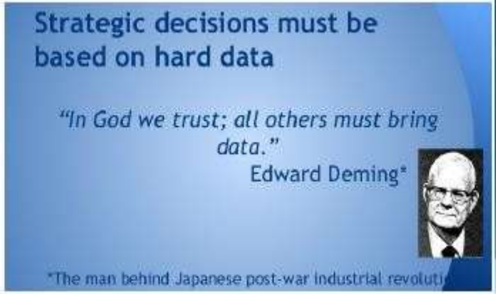
Based on feelings, insights, knowledge, etc. of a person.

What is the probability of rain tomorrow?

Assigning Probabilities

What is the probability of a baby being born on a Sunday?





Probability - Terminology

Sample Space – Set of all possible outcomes, denoted S.

Event – A subset of the sample space.

WHAT ISPROBABILITY?

- In the above “heads” example, the act of flipping a coin is called a trial.
- Over very many trials, a fair coin should come up “heads” half of the time.

TRIALS HAVE NO MEMORY!

- If a fair coin comes up tails 5 times in a row, the chance it will come up heads is *still* 0.5

- Each trial is independent of all others

EXPERIMENTS

- Each trial of flipping a coin can be called an experiment
- Each mutually independent outcome is called a simple event

SAMPLE SPACE

- The sample space is the sum of every possible simple event

EXAMPLE FOR SAMPLE SPACE

- Consider rolling a six-sided die
- One roll is an experiment
- The simple eventsare:

  $E_1=1$ $E_2=2$ $E_3=3$

  $E_4=4$ $E_5=5$ $E_6=6$

- Therefore, the sample space is:

  $S=\{E_1,E_2,E_3,E_4,E_5,E_6\}$

EXPERIMENTS

- The probability that a fair die will roll a six:
  The simple eventis:

  $E_6=6$(oneevent)

Total sample space:

$S = \{E_1 , E_2 , E_3 , E_4, E_5, E_6\}$ *(six possible outcomes)*

The probability:

P(Roll Six) = 1/6

PROBABILITY EXERCISE

- A company made a total of 50 trumpet valves
- It is determined that one of the valves was defective

- If three valves go into one trumpet, what is the probability that a trumpet has a defective valve?

PROBABILITY EXERCISE

1. Calculate the probability of having a defective valve:

$$P(E_{defectivevalve}) = \frac{1}{50} = 0.02$$

PROBABILITY EXERCISE

2. Calculate the probability of having a defective trumpet:

$$P(E_{defectivetrumpet}) = 3 \times P(E_{defectivevalve})$$

$$= 3 \times 0.02 = \mathbf{0.06}$$

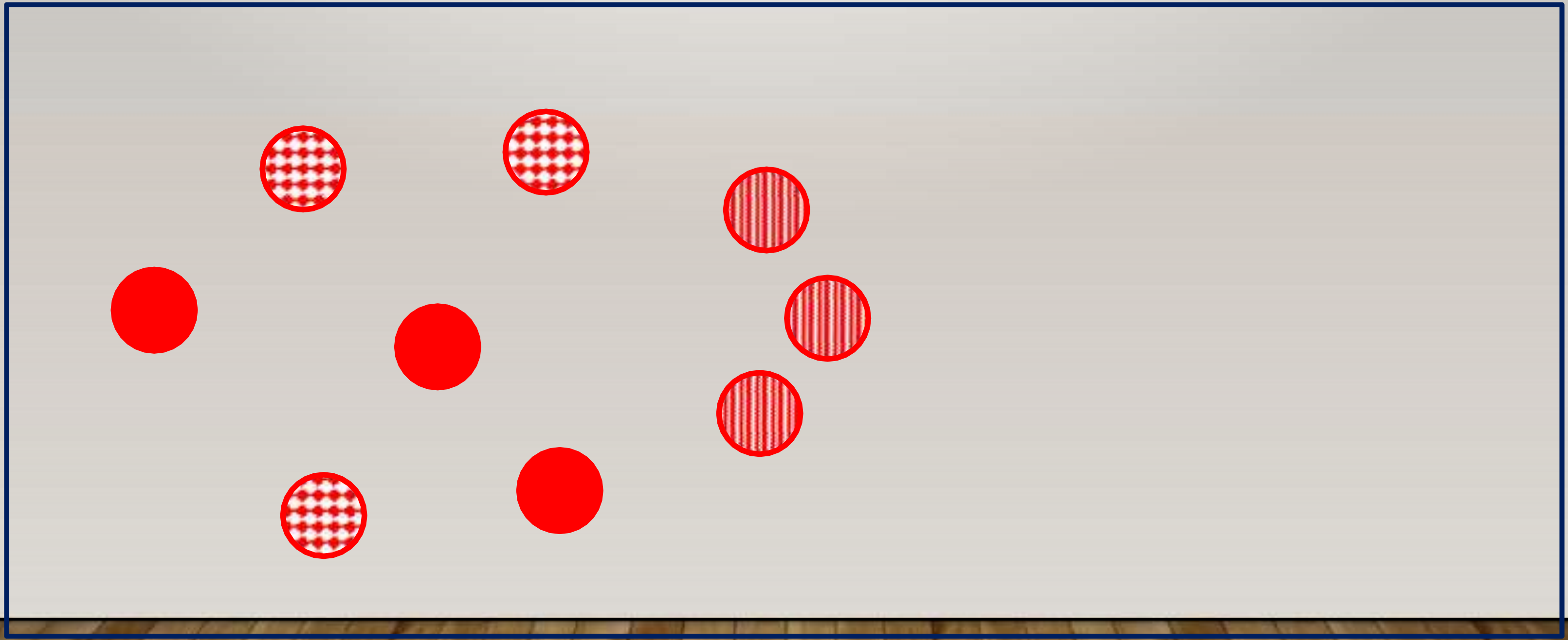
INTERSECTIONS, UNIONS &COMPLEMENTS

INTERSECTIONS

- In probability, an intersection describes the sample space where two events *both* occur.

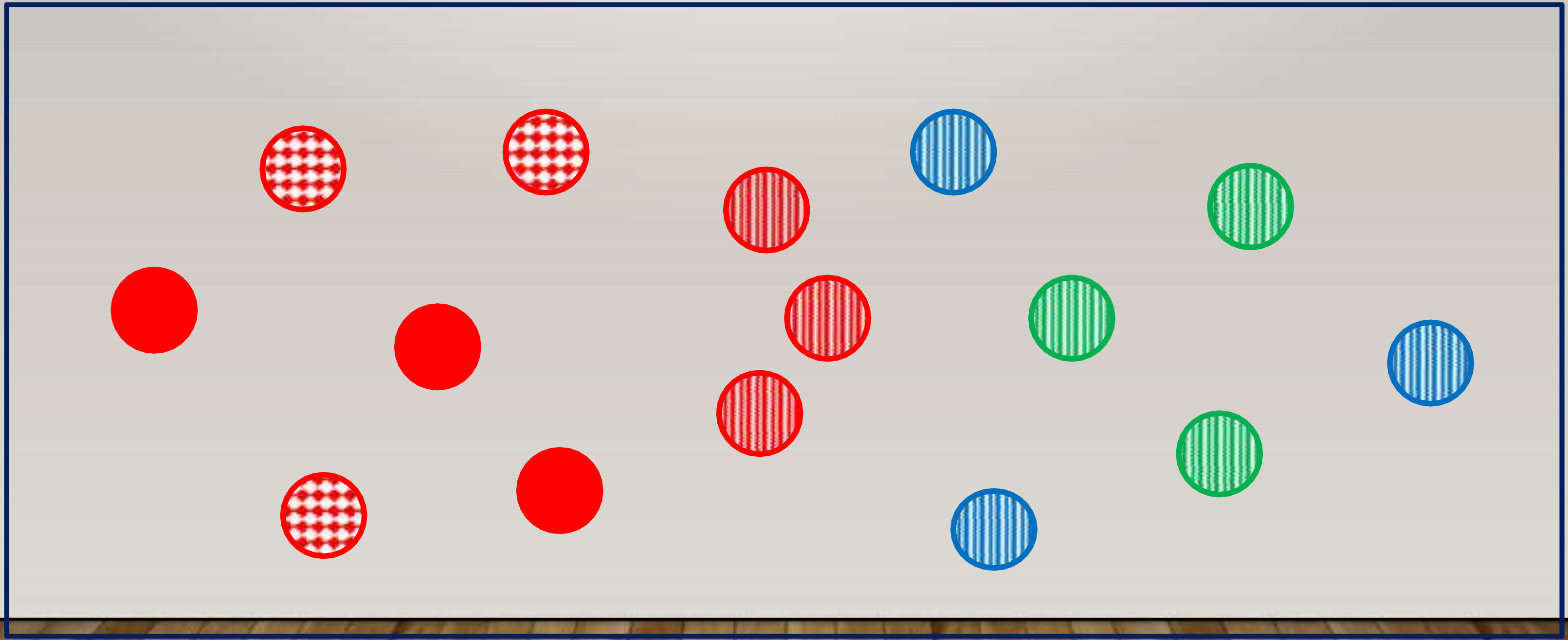
INTERSECTIONS

- 9 of the balls are red:



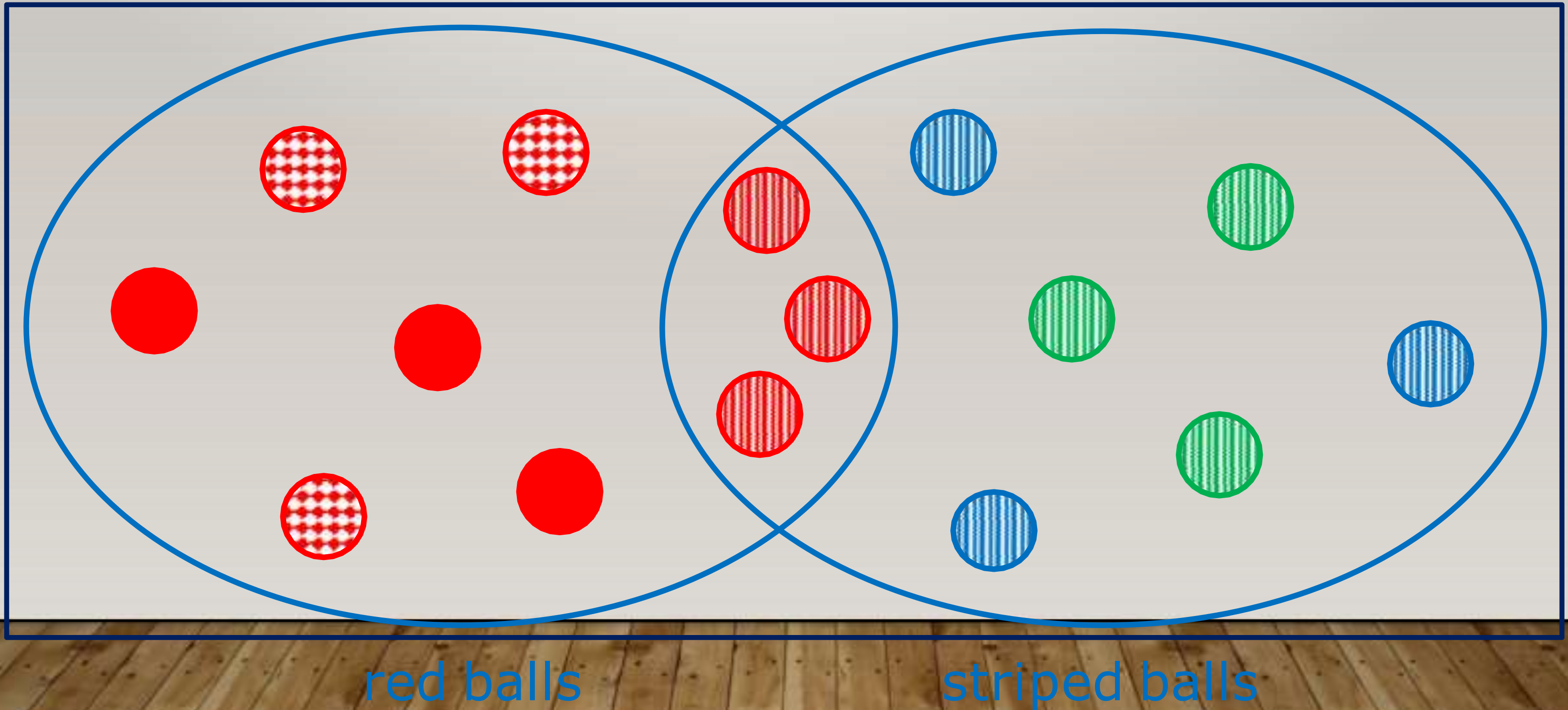
INTERSECTIONS

- 9 of the balls are striped:



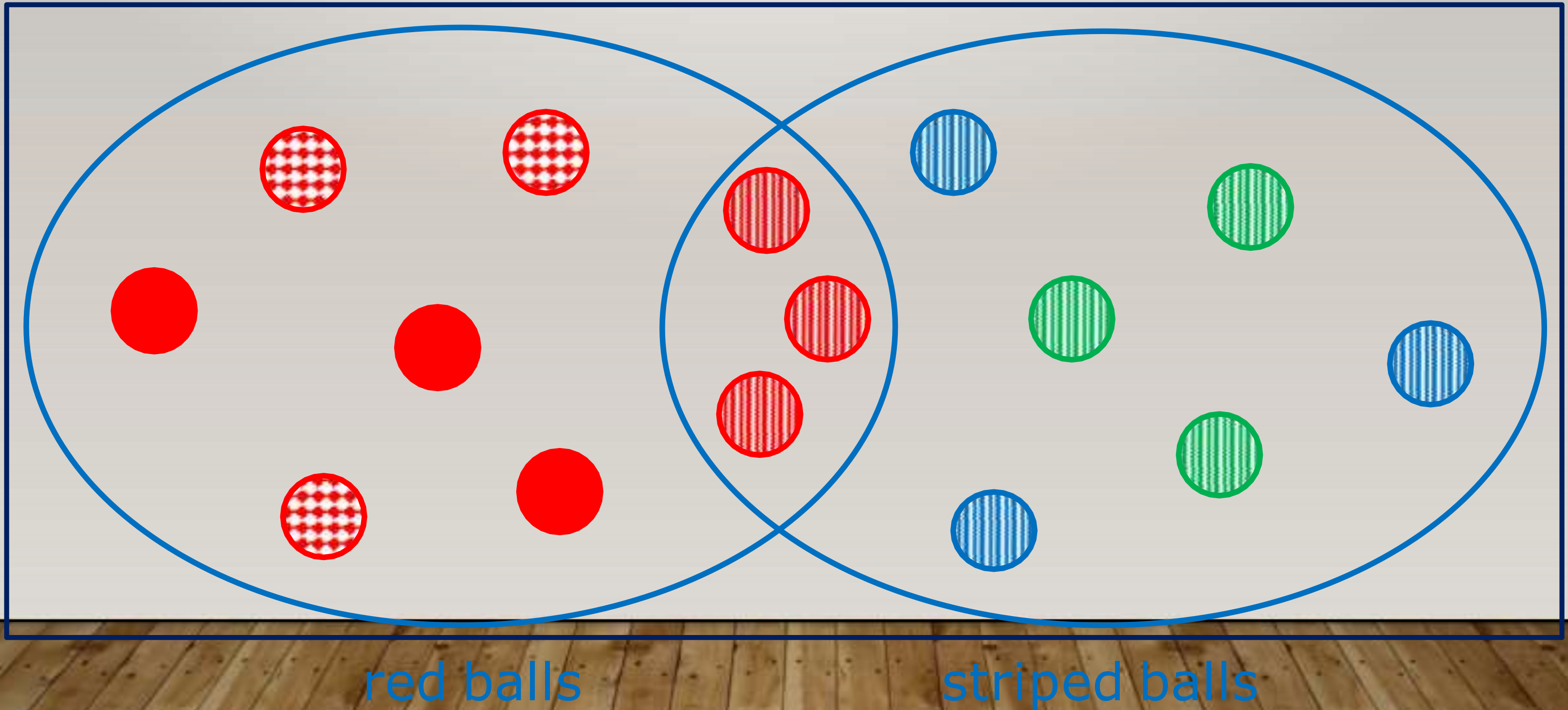
INTERSECTIONS

- 3 of the balls are both red and striped:

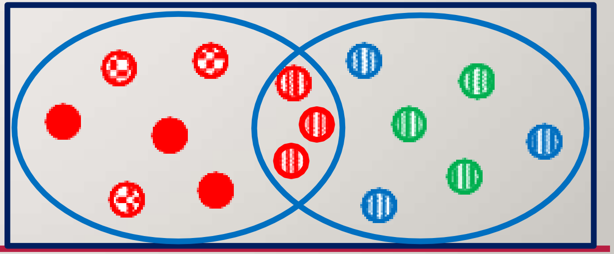


INTERSECTIONS

- What are the odds of a red, striped ball?



INTERSECTIONS



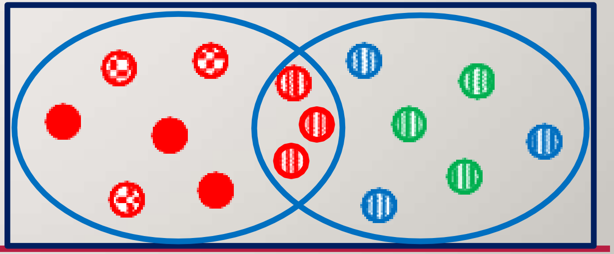
- If we assign A as the event of red balls, and B as the event of striped balls, the intersection of A *and* B is given as:

$$A \cap B$$

- Note that order doesn't matter:

$$A \cap B = B \cap A$$

INTERSECTIONS



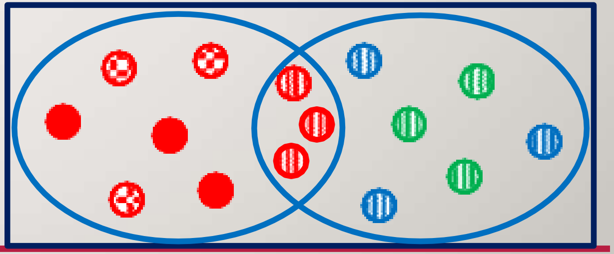
- The probability of A *and* B is given as

$$P(A \cap B)$$

- In this case:

$$P(A \cap B) = \frac{3}{15} = \mathbf{0.2}$$

UNIONS



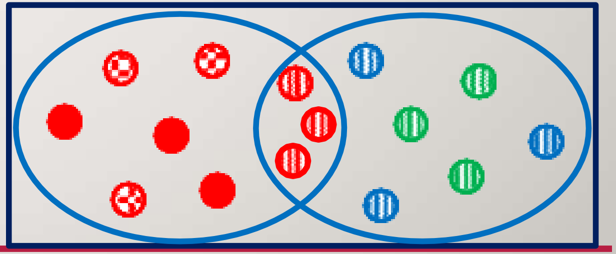
- The union of two events considers if A *or* B occurs, and is given as:

$$A \cup B$$

- Note again, order doesn't matter:

$$A \cup B = B \cup A$$

UNIONS



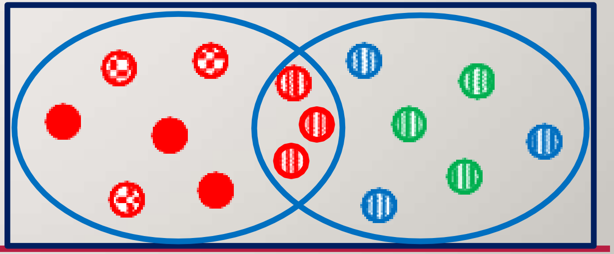
- The probability of A *or* B is given as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- In this case:

$$P(A \cup B) = \frac{9}{15} + \frac{9}{15} - \frac{3}{15} = \frac{15}{15} = \mathbf{1.0}$$

COMPLEMENTS



- The complement of an event considers everything outside of the event, given by:

$$\bar{A}$$

- The probability of *not* A is:

$$P(\bar{A}) = 1 - P(A) = \frac{15}{15} - \frac{9}{15} = \frac{6}{15} = \mathbf{0.4}$$

INDEPENDENT
&
DEPENDENT EVENTS

INDEPENDENT EVENTS

- An independent series of events occur when the outcome of one event has <u>no effect</u> on the outcome of another.

EXAMPLE FOR INDEPENDENT EVENT

- An example is flipping a fair coin twice
- The chance of getting heads on the second toss is independent of the result of the first toss.

INDEPENDENT EVENTS

- The probability of seeing two heads with two flips of a fair coin is:

$$P(H_1H_2) = P(H_1) \times P(H_2)$$

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

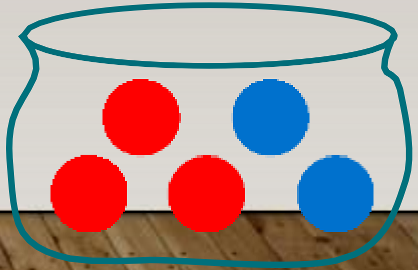
1st Toss	2nd Toss
H	H
H	T
T	H
T	T

DEPENDENT EVENTS

- A dependent event occurs when the outcome of a first event does affect the probability of a second event.
- A common example is to draw colored marbles from a bag *without replacement*.

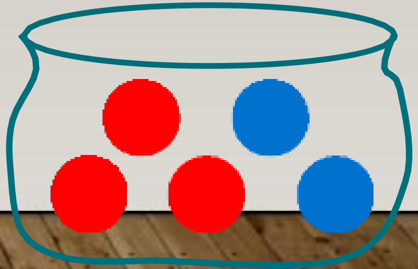
EXAMPLE FOR DEPENDENT EVENTS

- Imagine a bag contains 2 blue marbles and 3 red marbles.
- If you take two marbles out of the bag, what is the probability that they are both red?



EXAMPLE FOR DEPENDENT EVENTS

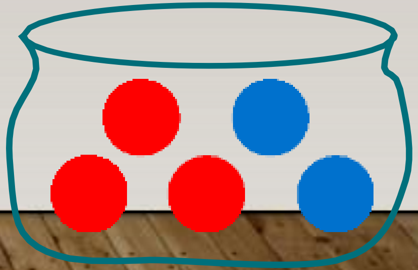
- Here the color of the first marble affects the probability of drawing a 2^{nd} red marble.



EXAMPLE FOR DEPENDENT EVENTS

- The probability of drawing a first red marble is easy:

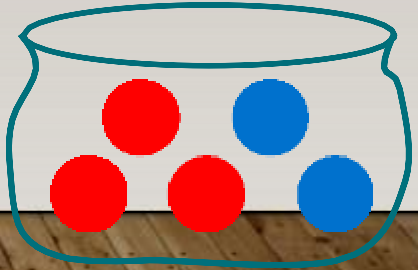
$$P(R_1) = \frac{3}{5}$$



EXAMPLE FOR DEPENDENT EVENTS

- The probability of drawing a second red marble *given that* the first marble was red is written as:

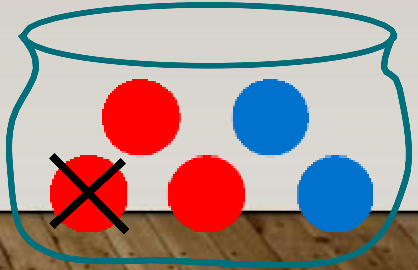
$$P(R_2|R_1)$$



EXAMPLE DEPENDENT EVENTS

- After removing a red marble from the sample set thisbecomes:

$$P(R_2|R_1) = \frac{2}{4}$$

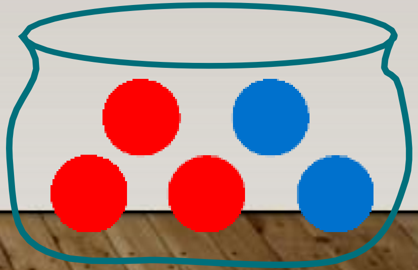


EXAMPLE FOR DEPENDENT EVENTS

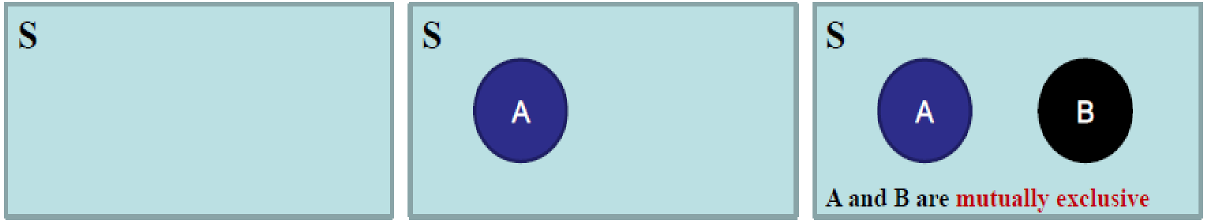
- So the probability of two red marbles is:

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2|R_1)$$

$$= \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \mathbf{0.3}$$



Probability - Rules



$$P(S) = 1$$

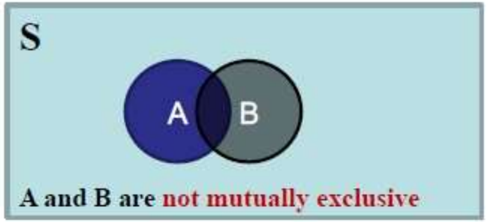
$$0 \leq P(A) \leq 1$$

$$P(A \text{ or } B) = P(A) + P(B)$$

Area of the rectangle denotes sample space, and since probability is associated with area, it cannot be negative.

Mutually Exclusive – If event A happens, event B cannot.

Probability - Rules



$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example

Event A – Customers who default on loans

Event B – Customers who are High Net Worth Individuals

Probability - Rules

Independent Events – Outcome of event B is not dependent on the outcome of event A.

Probability of customer B defaulting on the loan is not dependent on default (or otherwise) by customer A.

$$P(A \text{ and } B) = P(A) * P(B)$$

If the probability of getting an *easy* call is 0.7, what is the probability that the next 3 calls will be *easy*?

$$P(easy_1 \text{ and } easy_2 \text{ and } easy_3) = 0.7^3 = 0.343$$

Probability - Question

A basketball team is down by 2 points with only a few seconds remaining in the game. Given that:

- Chance of making a 2-point shot to tie the game = 50%
- Chance of winning in overtime = 50%
- Chance of making a 3-point shot to win the game = 30%

What should the coach do: go for 2-point or 3-point shot?

What are the assumptions, if any?

Probability - Types

Contingency table summarizing 2 variables, *Loan Default* and *Age*:

		Age			
		Young	**Middle-aged**	**Old**	**Total**
Loan Default	**No**	10,503	27,368	259	38,130
	Yes	3,586	4,851	120	8,557
	Total	14,089	32,219	379	46,687

Probability - Types

Convert it into probabilities:

		Age			
		Young	**Middle-aged**	**Old**	**Total**
Loan Default	**No**	0.225	0.586	0.005	0.816
	Yes	0.077	0.104	0.003	0.184
	Total	0.302	0.690	0.008	1.000

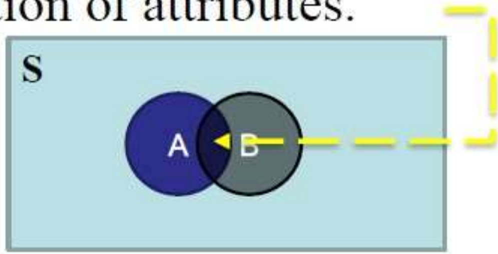
Probability - Types

Joint Probability

		Age			
		Young	Middle-aged	Old	Total
Loan Default	No	0.225	0.586	0.005	0.816
	Yes	0.077	0.104	0.003	0.184
	Total	0.302	0.690	0.008	1.000

Probability describing a combination of attributes.

P(Yes **and** Young) = 0.077

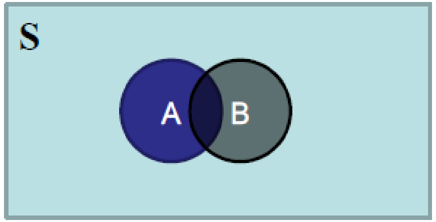


Probability - Types

Union Probability

		Age			
		Young	Middle-aged	Old	Total
Loan Default	No	0.225	0.586	0.005	0.816
	Yes	0.077	0.104	0.003	0.184
	Total	0.302	0.690	0.008	1.000

P(Yes **or** Young) = P(Yes) + P (Young) – P(Yes and Young) = 0.184 + 0.302 – 0.077 = 0.409



Probability - Types

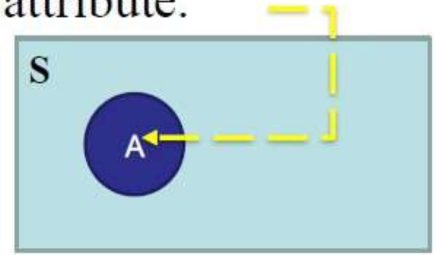
Marginal Probability

		Age			
		Young	Middle-aged	Old	Total
Loan Default	No	0.225	0.586	0.005	0.816
	Yes	0.077	0.104	0.003	0.184
	Total	0.302	0.690	0.008	1.000

Probability describing a single attribute.

P(No) = 0.816

P(Old) = 0.008



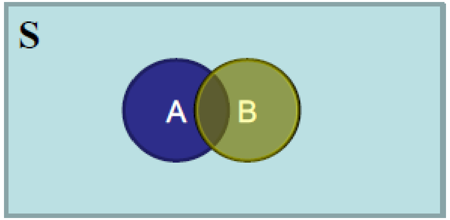
Probability - Types

Conditional Probability

		Age			
		Young	Middle-aged	Old	Total
Loan Default	No	0.225	0.586	0.005	0.816
	Yes	0.077	0.104	0.003	0.184
	Total	0.302	0.690	0.008	1.000

Probability of *A* occurring **given that** *B* has occurred.

The sample space is restricted to a single row or column.
This makes rest of the sample space irrelevant.



Probability - Types

Conditional Probability

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	Yes	0.077	0.104	0.003	0.184
	Total	0.302	0.690	0.008	1.000

What is the probability that a person will not default on the loan payment **given** she is middle-aged?

P(No | Middle-Aged) = 0.586/0.690 = 0.85

Note that this is the ratio of **Joint Probability** to **Marginal Probability**, i.e., $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$

P(Middle-Aged | No) =

Probability - Types

Conditional Probability

		Age			
		Young	Middle-aged	Old	Total
Loan Default	No	0.225	0.586	0.005	0.816
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Note that this is the ratio of **Joint Probability** to **Marginal Probability**, i.e., $P(A|B) = \frac{P(A\ and\ B)}{P(B)}$

P(Middle-Aged | No) = 0.586/0.816 = 0.72 (Order Matters)

Probability - Types

Conditional Probability – Visualizing using Probability Tables and Venn Diagrams

		Age			
		Young	Middle-aged	Old	Total
Loan Default	No	10,503	27,368	259	38,130
	Yes	3,586	4,851	120	8,557
	Total	14,089	32,219	379	46,687

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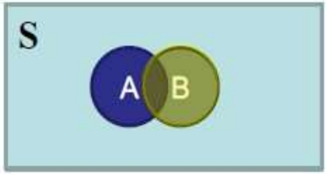
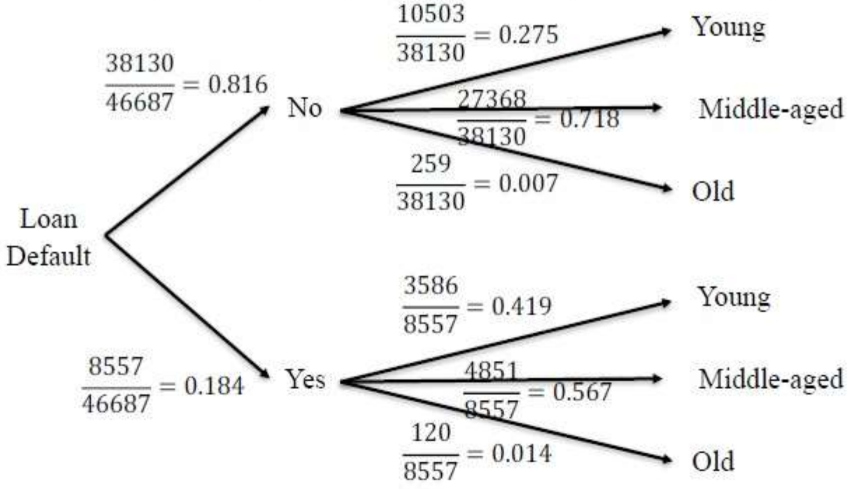


Probability - Types

Conditional Probability – Visualizing using Probability Trees

		Age (Numbers)				Age (Probabilities)			
		Young	Middle-aged	Old	Total	Young	Middle-aged	Old	Total
Loan Default	No	10,503	27,368	259	38,130	0.225	0.586	0.005	0.816
	Yes	3,586	4,851	120	8,557	0.077	0.104	0.003	0.184
	Total	14,089	32,219	379	46,687	0.302	0.690	0.008	1.000



Find

- P(Old and Yes)
- P(Yes and Old)
- P(Old)
- P(Yes)
- P(Old | Yes)
- P(Yes | Old)
- P(Young | No)

Probability - Types

Attention Check

Identify the type of probability in each of the below cases:

1. P(Old and Yes)
2. P(Yes and Old)
3. P(Old)
4. P(Yes)
5. P(Old | Yes)
6. P(Yes | Old)
7. P(Young | No)
8. P(Middle-aged or No)
9. P(Old or Young)

		Age (Probabilities)			
		Young	**Middle-aged**	**Old**	**Total**
Loan Default	**No**	0.225	0.586	0.005	**0.816**
	Yes	0.077	0.104	0.003	**0.184**
	Total	**0.302**	**0.690**	**0.008**	**1.000**

Probability - Types

Attention Check

Identify the type of probability in each of the below cases:

1. P(Old and Yes)
2. P(Yes and Old)
3. P(Old)
4. P(Yes)
5. P(Old | Yes)
6. P(Yes | Old)
7. P(Young | No)
8. P(Middle-aged or No)
9. P(Old or Young)

		Age (Probabilities)			
		Young	Middle-aged	Old	Total
Loan Default	No	0.225	0.586	0.005	**0.816**
	Yes	0.077	0.104	0.003	**0.184**
	Total	**0.302**	**0.690**	**0.008**	**1.000**

1 and 2: **Joint**; 3 and 4: **Marginal**; 5, 6 and 7: **Conditional**; 8 and 9: **Union**

CONDITIONAL PROBABILITY

CONDITIONAL PROBABILITY

- The idea that we want to know the probability of event A, *given* that event B has occurred, is conditional probability.
- This is written as $P(A|B)$

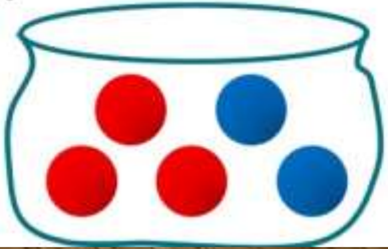
CONDITIONAL PROBABILITY

- Going back to dependent events, the probability of drawing two red marbles is:

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2|R_1)$$

- The conditional in this equation is:

$$P(R_2|R_1)$$



CONDITIONAL PROBABILITY

- Rearranging the formula gives:

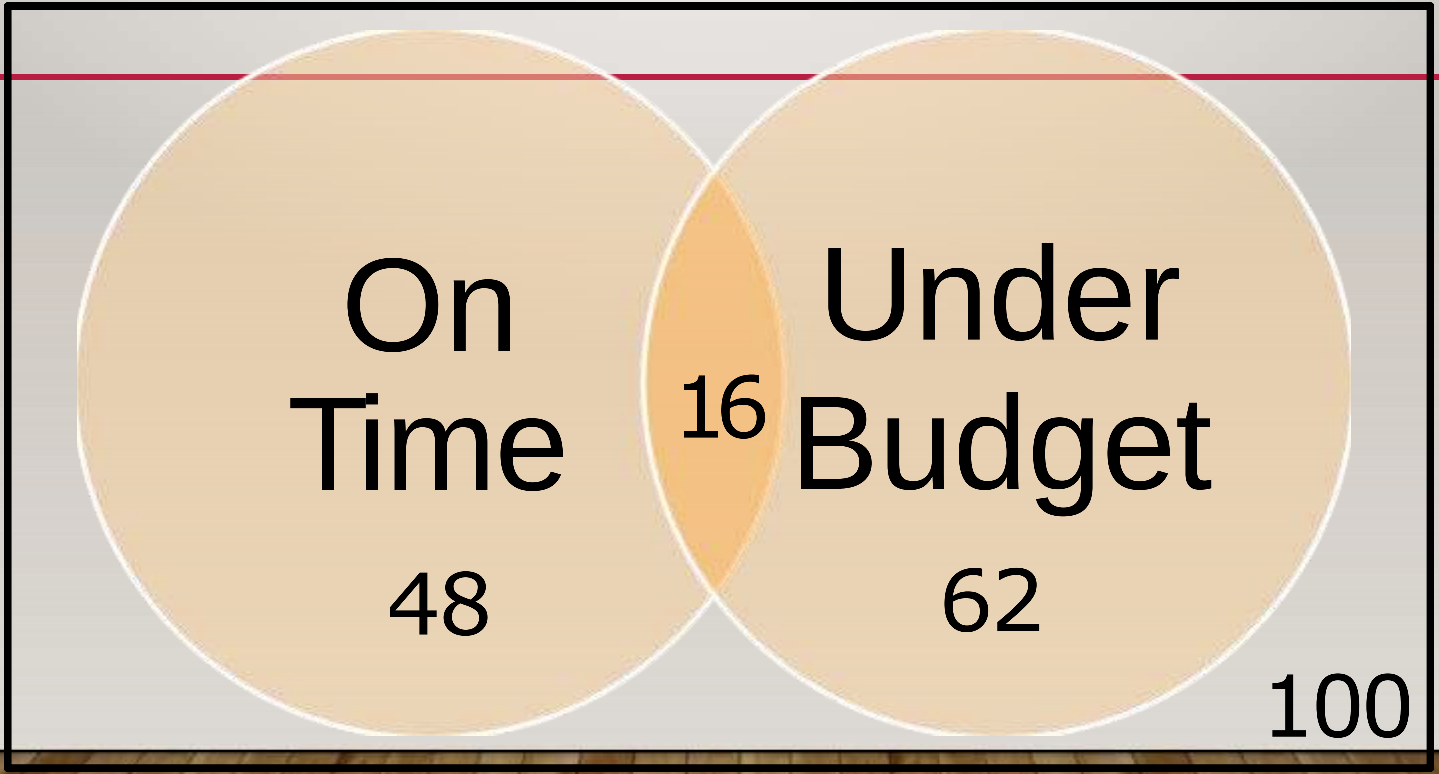
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- That is, the probability of A given B equals the probability of A and B divided by the probability of B

CONDITIONAL PROBABILITY EXERCISE

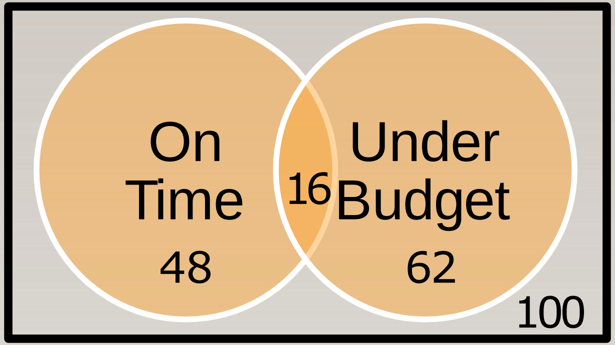
- A company finds that out of every 100 projects, 48 are completed on time, 62 are completed under budget, and 16are completed both on time and under budget.
- Given that a project is completed on time, what is the probability that it is under budget?

CONDITIONAL PROBABILITY EXERCISE



CONDITIONAL PROBABILITY EXERCISE

Given that a project is completed on time B, what is the probability that it is under budget A?



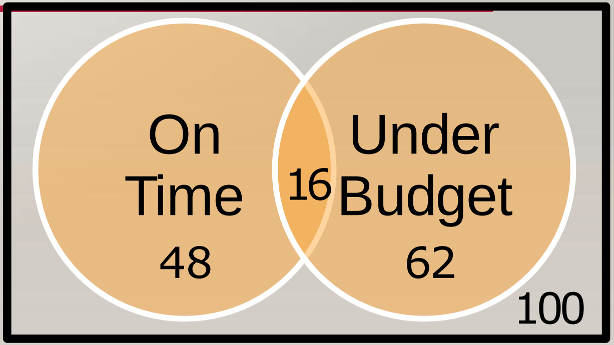
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{16}{48} = \mathbf{0.33}$$

ADDITION &MULTIPLICATION RULES

ADDITION RULE

- From our project example, what is the probability of a project completing on time *or* under budget?



- **Recall from the section on unions:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **This is the addition rule**

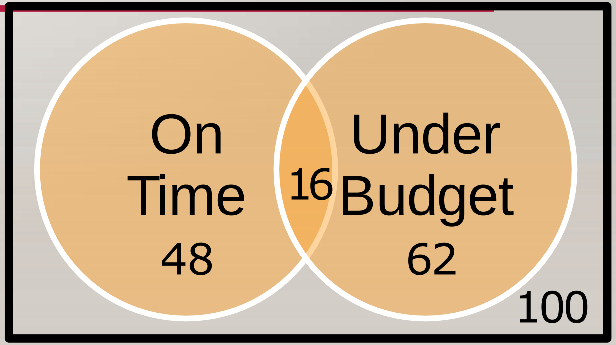
ADDITION RULE

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

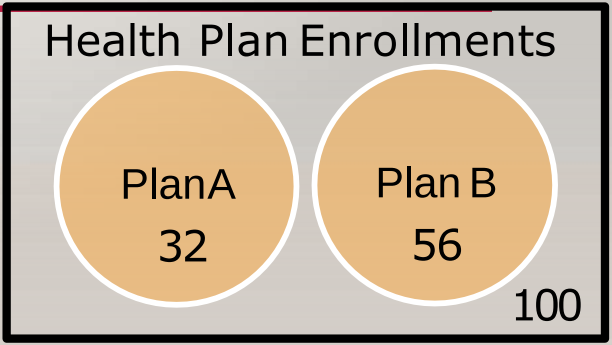
$$= \frac{48}{100} + \frac{62}{100} - \frac{16}{100}$$

$$= 0.48 + 0.62 - 0.16$$

$$= \mathbf{0.94}$$



ADDITION RULE FOR MUTUALLY EXCLUSIVE EVENTS



- When two events cannot both happen, they are said to be mutually exclusive.
- In this case, the addition rule becomes:

$$P(A \cup B) = P(A) + P(B) \cancel{-P(A \cap B)}$$

MULTIPLICATION RULE

- From the section on dependent events we saw that the probability of A and B is:

$$P(A \cap B) = P(A) \cdot P(B|A)$$

- This is the **multiplication rule**

BAYESTHEOREM

BAYESTHEOREM

- We've already seen conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ provided that } P(B) > 0$$

$$P(A \cap B) = P(A) \cdot P(B|A) \text{ provided that } P(A) > 0$$

BAYESTHEOREM

- We can then connect the two conditional probability formulas to get Bayes' Theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \textit{ provided that } P(A), P(B) > 0$$

BAYESTHEOREM

- Bayes Theorem is used to determine the probability of a *parameter*, given a certain event.
- The general formulais:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$