Engineering Mathematics

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1.

(a) $\frac{d^2y}{dx^2}\frac{dy}{dx}=1$ $\qquad u=\frac{dy}{dx}$ $\qquad y'(0)=0$

$\frac{du}{dx}u=1$

$\frac{u^2}{2}=x+c$ $\qquad \left(\frac{dy}{dx}\right)^2=2x+c'$ $\qquad c'=0$

$\frac{dy}{dx}=u=\pm\sqrt{2x}$

$dy=\pm\sqrt{2x}\,dx$

$y=\pm\frac{2\sqrt{2}}{3}x^{\frac{3}{2}}+c''$

(b)

$\frac{d^2y}{dx^2}=-3\frac{dy}{dx}y^2$

$\frac{dy}{dx}=u$

$\frac{d^2y}{dx^2}=\frac{d}{dx}u=\frac{dy}{dx}\frac{d}{dy}u=u\frac{du}{dy}$

$\frac{du}{dy}=-3y^2$

$du=-3y^2dy$

$u=-y^3+c$

$\frac{dy}{dx}=-y^3+c'$

$y'(1)=-2^{\frac{-3}{2}}$

$y(1)=2^{\frac{-1}{2}}$

$c'=0$

$y^{-3}dy=-dx$

$-\frac{1}{2}y^{-2}=-x+c''$

$c''=0$

$y^2=\frac{1}{2x}$ $\qquad y=(2x)^{\frac{-1}{2}}$

$=\frac{1}{\sqrt{2x}}$ #

(C) $\dfrac{d^2y}{dx^2} = e^y$

$u\dfrac{du}{dy} = e^y$

$\dfrac{u^2}{2} = e^y + c$

$u = \pm\sqrt{2e^y + c}$

$\dfrac{dy}{dx} = \pm\sqrt{2e^y + c}$

$y'(0) = \sqrt{2}$

$c = 0$

$\dfrac{1}{\sqrt{2e^y}}\,dy = dx$

$\dfrac{1}{\sqrt{2}}\int e^{-\frac{y}{2}}\,dy = \int dx = x + c'$

$-\sqrt{2}\,e^{\frac{-y}{2}} = x + c'$

$y(0) = 0$

$-\sqrt{2} = c'$

$y = -2\ln\left|\frac{-1}{\sqrt{2}}x + 1\right|$

2. (a)

$u(x,y) = X(x)\,Y(y)$

$x^2\,X'(x)\,Y(y) = y\,X(x)\,Y'(y) \quad \left(\div\,(X(x)\,Y(y))\right)$

$x^2\dfrac{X'(x)}{X(x)} = y\,\dfrac{Y'(y)}{Y(y)} = -\lambda$

$y' = X'(x) = -\lambda\,X(x)\,x^{-2}$

$\dfrac{dy}{y} = -\lambda x^{-2}\,dx$

$\ln y = \dfrac{\lambda}{x} + c$

$y = c_1 e^{\frac{\lambda}{x}}$

$z' = Y'(y) = -\lambda\,Y(y)\,y^{-1}$

$\dfrac{dz}{dy} = -\lambda\dfrac{z}{y}$

$\dfrac{1}{z}\,dz = -\lambda y^{-1}\,dy$

$\ln z = -\lambda\ln y + c'$

$z = c_2\,y^{-\lambda}$

$$u(x,y) = X(x)\,Y(y) = \sum_{\lambda} C_\lambda e^{\frac{a}{x}}\, y^{-\lambda}$$

(b)

$$\frac{\partial^2}{\partial x^2}u(x,y) = u(x,y) + \frac{\partial}{\partial y}u(x,y) \qquad 0<x<2, \quad y>0,$$

$$u(0,y) = u(2,y) = 0, \quad u(x,0) = \cos(\pi x)\sin(2\pi x)$$

$u(x,y) = X(x)\,Y(y)$ $\qquad X(0)=0 \quad X(2)=0 \qquad u(x,0) = \cos(\pi x)\sin(2\pi x)$

$X''(x)\,Y(y) = X(x)Y(y) + X(x)Y'(y) \qquad \div X(x)\,Y(y)$

$$\frac{X''(x)}{X(x)} = 1 + \frac{Y'(y)}{Y(y)} = -\lambda$$

$X''(x) + \lambda X(x) = 0 \qquad X(0)=0 \quad X(2)=0$

$Y'(y) + (\lambda+1)\,Y(y) = 0$

$\lambda = 0$

$X''(x) = 0$

$m^2=0 \quad m = 0,0$ $\qquad X(x) = C_1 + C_2 x \qquad C_1 = 0 \quad C_2 = 0$

$X(x) = 0$ trival solution

$\lambda = -\alpha^2 < 0$

$X''(x) - \alpha^2 X(x) = 0 \qquad \frac{e^{\alpha x}+e^{-\alpha x}}{2}$

$m^2 = \alpha^2 \quad m = \alpha, -\alpha$

$X(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x} = C_3 \sinh(\alpha x) + C_4 \cosh(\alpha x)$

$X(0) = 0 = C_4 \qquad X(2) = C_3 \frac{e^{2\alpha}-e^{-2\alpha}}{2} = 0 \qquad C_3 = 0 \qquad X(x) = 0$

trival solution

$\lambda = \alpha^2 > 0$

$m^2 + \alpha^2 = 0$ $\qquad X(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$

$m = \pm j\alpha$ $\qquad X(0) = c_2 = 0$ $\qquad X(2) = c_3 \sin(2\alpha) = 0$

$\sin(n\pi) = 0$

$\alpha = \frac{n\pi}{2}$

$Y'(y) + \left(\left(\frac{n\pi}{2}\right)^2 + 1\right) Y(y) = 0$

$m + \left(\left(\frac{n\pi}{2}\right)^2 + 1\right) = 0$ $\qquad Y(y) = c_3\, e^{-\left(\left(\frac{n\pi}{2}\right)^2+1\right)y}$

$$u(x,y) = \sum X(x)\,Y(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}x\right) e^{-\left(\left(\frac{n\pi}{2}\right)^2+1\right)y}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}x\right) = \cos\pi x \sin 2\pi x$$

$$= \frac{1}{2}\left(\sin 3\pi x + \sin \pi x\right)$$

$A_2 = A_6 = \frac{1}{2}$, $\forall n$ $n \neq 2, 6$ $A_n = 0$

$$u(x,y) = \frac{1}{2}\left(\sin(\pi x)\, e^{-(\pi^2+1)y} + \sin 3\pi x\, e^{-(9\pi^2+1)y}\right)$$

1c)

$$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0 \qquad 0 < x < 1, \quad 0 < y < 1,$$

$$u(0,y) = u(1,y) = u(x,0) = 0, \quad u(x,1) = 1 - 2|x - 1/2|$$

$u(x,y) = X(x)\,Y(y)$ $\qquad X(0) = 0 \quad X(1) = 0 \quad Y(0) = 0$

$u(x,1) = 1 - 2\,|x - \frac{1}{2}|$

$X''(x)Y(y) + X(x)Y''(y) = 0$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$$X''(x) + \lambda X(x) = 0$$

$$Y''(y) - \lambda Y(y) = 0$$

$\lambda = 0$ $\quad X(x) = c_2 x + c_1$ $\qquad X(0) = 0 \quad X(1) = 0$

$c_1, c_2 = 0$ trival solution

$\lambda = -\alpha^2 < 0$ $\quad m = \pm\alpha$

$$X(x) = c_1 \sinh(\alpha x) + c_2 \cosh(\alpha x)$$

$c_1, c_2 = 0$ trival solution

$\lambda = \alpha^2 > 0$

$$X(x) = c_1 \sin \alpha x + c_2 \cos \alpha x$$

$X(0) = c_2 = 0 \qquad X(1) = c_1 \sin\alpha = 0$

$\alpha = n\pi \qquad X(x) = c_1 \sin n\pi x$

$$Y''(y) - n\pi\, Y(y) = 0$$

$m^2 = n\pi \qquad m = \pm\sqrt{n\pi}$

$$Y(y) = c_3 \sinh(\sqrt{n\pi}\, y) + c_4 \cosh(\sqrt{n\pi}\, y)$$

$Y(0) = c_4 = 0$

$$u(x,y) = \sum X(x)Y(y) = \sum_{n=1}^{\infty} A_n \sinh(\sqrt{n\pi}\, y)\ \sin(n\pi x)$$

$$u(x,1) = 1 - 2\left|x - \tfrac{1}{2}\right| = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(\sqrt{n\pi})$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$A_n \sinh(\sqrt{n\pi}) = \int_{-1}^{1}\left(1 - 2\left|x - \tfrac{1}{2}\right|\right)\sin(n\pi x)\,dx$$

$$A_n = \frac{1}{\sinh\sqrt{n\pi}}\int_{-1}^{1}\left(1 - 2\left|x - \tfrac{1}{2}\right|\right)\sin(n\pi x)\,dx$$

$$A_n = \frac{2}{\sinh\sqrt{n\pi}}\left(\int_0^{\frac{1}{2}} 2x \sin n\pi x \, dx + \int_{\frac{1}{2}}^{1} (2-2x)\sin n\pi x \, dx\right)$$

$\int_{-1}^{1} \to 2\int_0^1$

$+$	$2x$	$\sin n\pi x$	$+$	$2-2x$	$\sin n\pi x$
$-$	2	$-\frac{\cos n\pi x}{n\pi}$	$-$	-2	$-\frac{\cos n\pi x}{n\pi}$
$+$	0	$-\frac{\sin n\pi x}{(n\pi)^2}$	$+$	0	$-\frac{\sin n\pi x}{(n\pi)^2}$

$$= \frac{2}{\sinh\sqrt{n\pi}}\left(\frac{4\sin\frac{n\pi}{2}}{(n\pi)^2}\right) = \frac{1}{\sinh\sqrt{n\pi}}\,\frac{8\sin\frac{n\pi}{2}}{(n\pi)^2}$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh(\sqrt{n\pi}\,y)\sin n\pi x$$

$$A_n = \frac{1}{\sinh\sqrt{n\pi}}\,\frac{8\sin\frac{n\pi}{2}}{(n\pi)^2} \qquad n=1,2,3,\dots$$

(d)

$$(x+1)\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}u(x,y) + \underline{\underline{\cos y}}$$ indep. of x

$$u(x,y) = X(x)Y(y) + \varphi(y)$$

$$(x+1)\,X'(x)Y(y) = X(x)Y'(y) + \underbrace{\varphi'(y) + \cos y}_{\varphi'(y)+\cos y = 0}$$

$$\frac{d\varphi(y)}{dy} = -\cos y$$

$$d\varphi(y) = -\cos y\, dy$$

$$\varphi(y) = -\sin y$$

$$\frac{(x+1)X'(x)}{X(x)} = \frac{Y'(y)}{Y(y)} = -\lambda$$

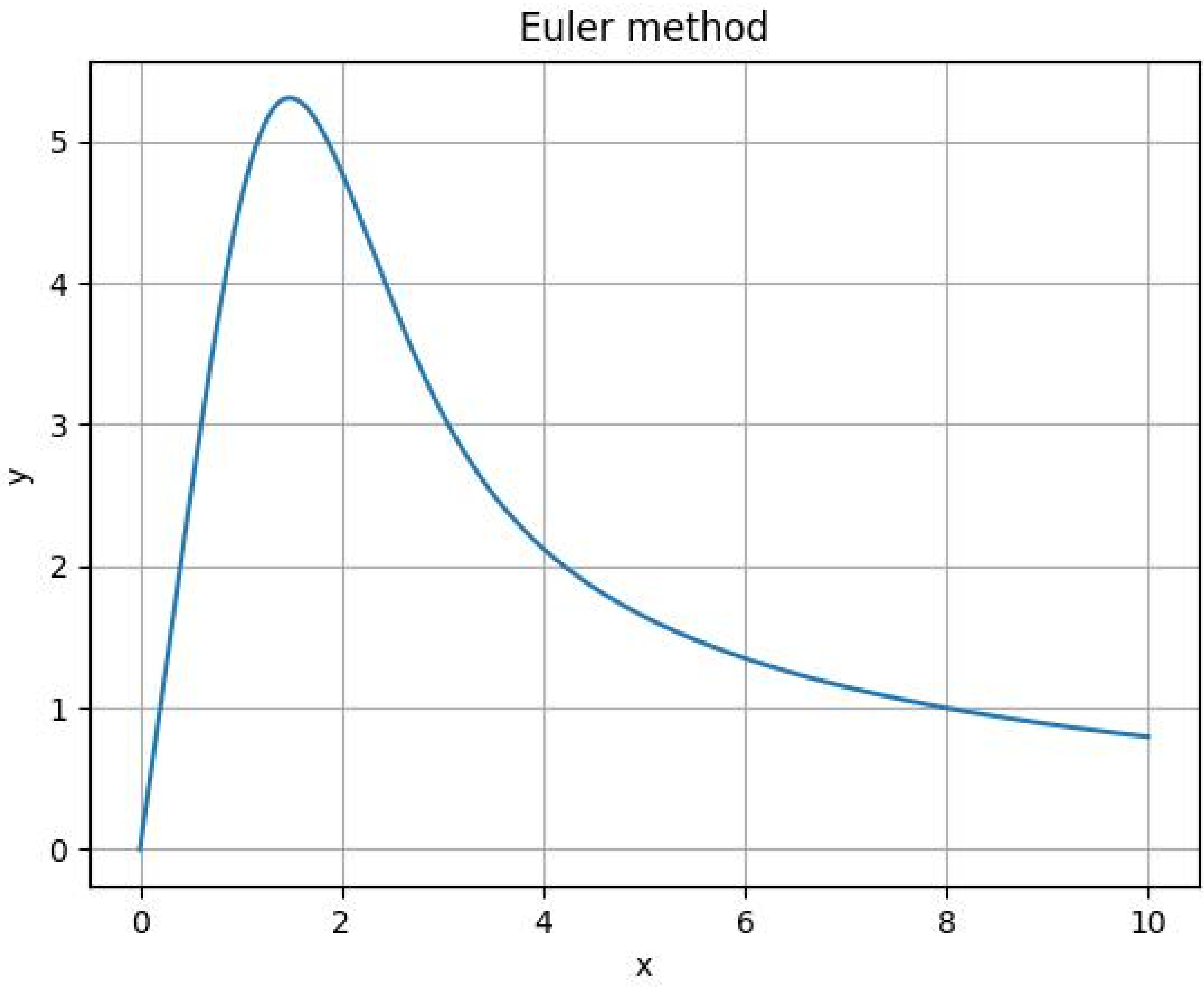
$$\frac{dY(y)}{dy} = -\lambda Y(y) \qquad \frac{1}{X(x)}\frac{dX(x)}{dx} = \frac{-\lambda}{x+1}$$

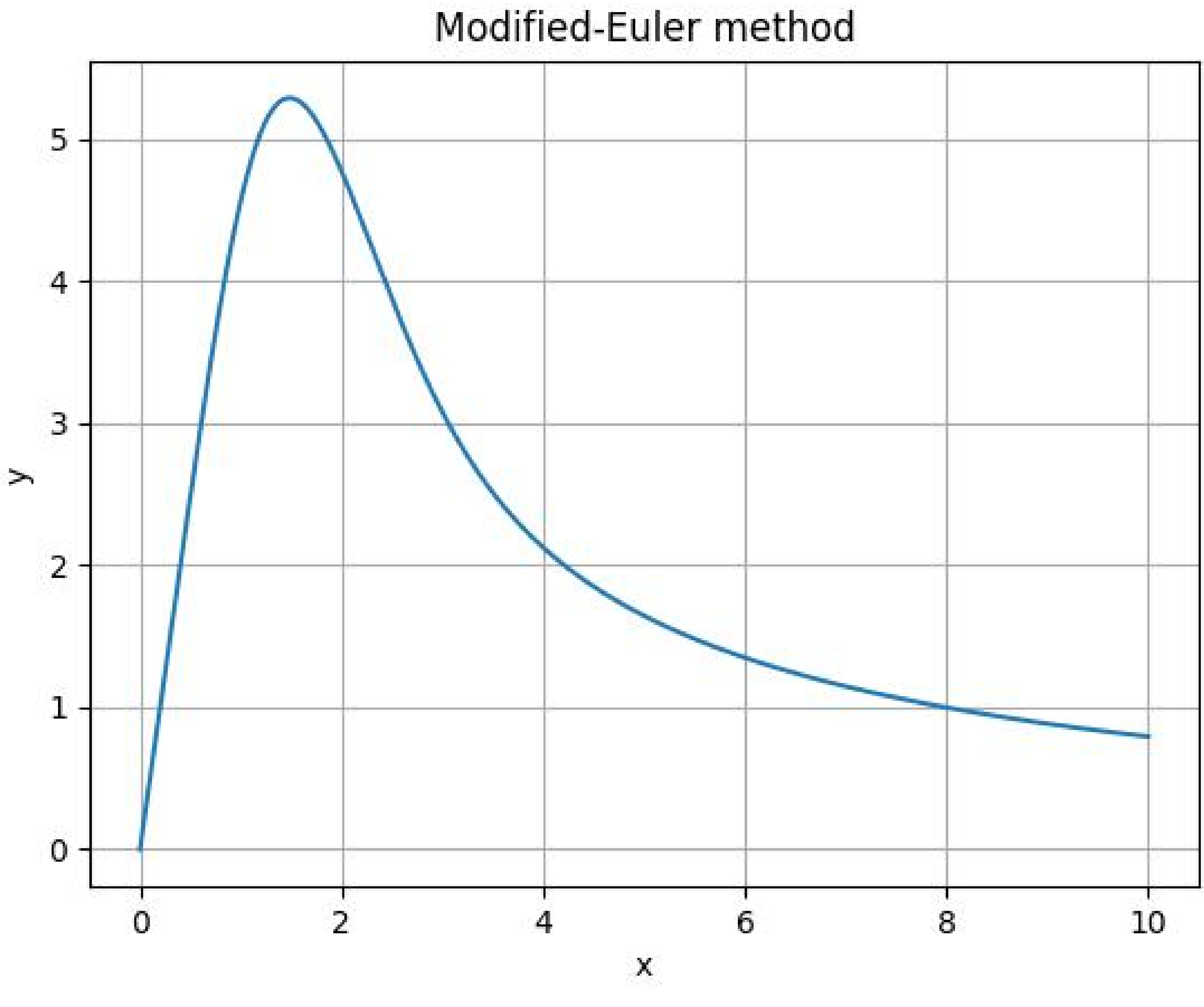
$$\frac{1}{Y(y)}dY(y) = -\lambda\, dy \qquad \ln X(x) = -\lambda\ln(x+1) + C$$

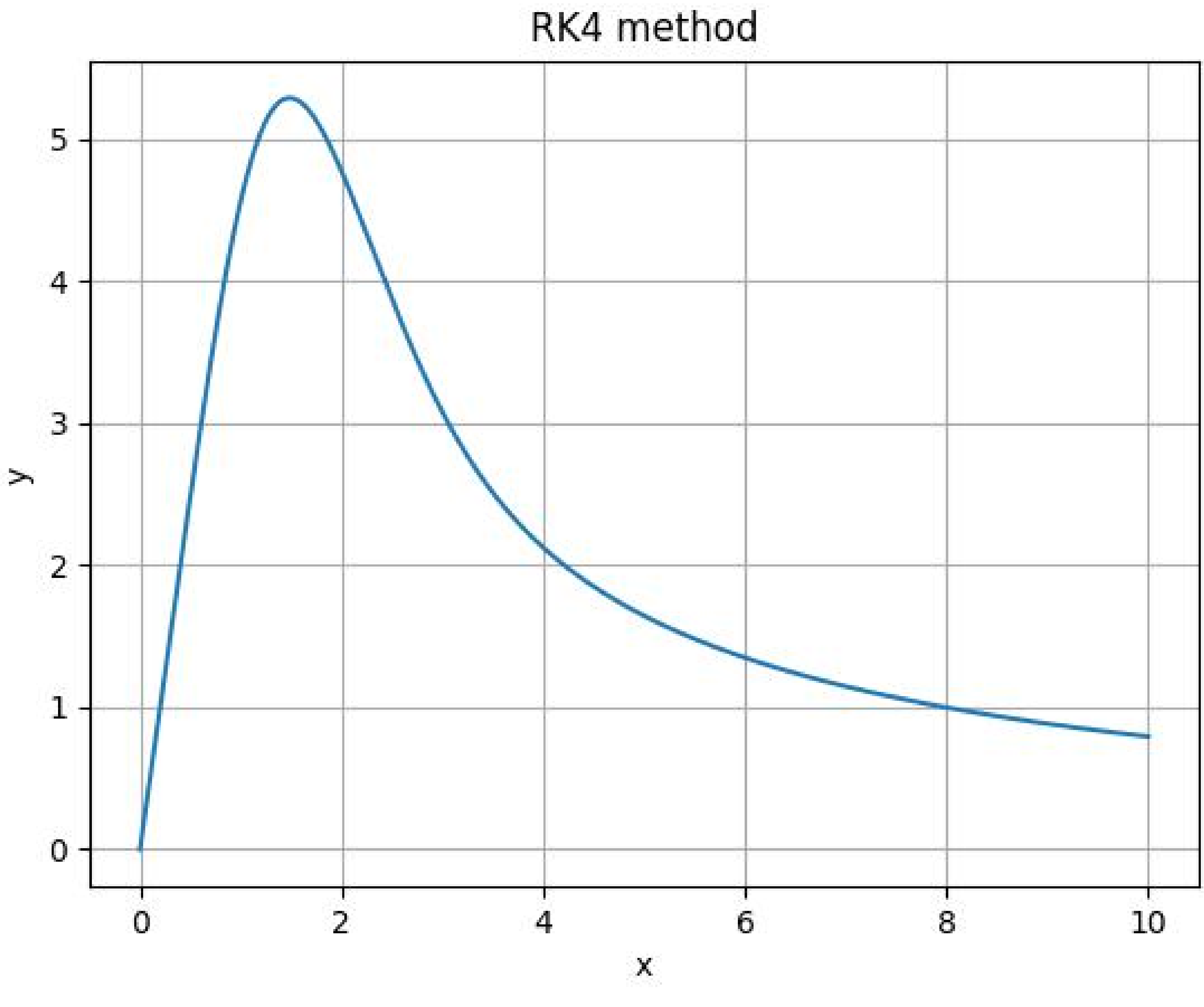
$$X(x) = c(x+1)^{-\lambda}$$

$$\ln Y(y) = -\lambda y + C$$

$$Y(y) = c' e^{-\lambda y} \qquad u(x,y) = \sum_{\lambda} C_\lambda (x+1)^{-\lambda} e^{-\lambda y}) - \sin y$$


Euler method
y
x
0
1
2
3
4
5
0
2
4
6
8
10


Modified-Euler method
y
x
0
1
2
3
4
5
0
2
4
6
8
10


RK4 method
y
x
0
1
2
3
4
5
0
2
4
6
8
10