EM-hw 2 b10901187 鍾闓凱

(1)

(a) $u(x,y,z) = X(x)Y(y)Z(z)$

$$X'(x)Y(y)Z(z) + y X(x)Y'(y)Z(z) + z^2 X(x)Y(y)Z'(z) = 0$$

$$\underbrace{\frac{X'(x)}{X(x)}}_{-\lambda} + \underbrace{y\frac{Y'(y)}{Y(y)}}_{-\mu} + \underbrace{z^2\frac{Z'(z)}{Z(z)}}_{\lambda+\mu} = 0$$

$$X'(x) + \lambda X(x) = 0$$

$m + \lambda = 0 \quad m = -\lambda \quad X(x) = c_1 e^{-\lambda x}$

$$y Y'(y) + \mu Y(y) = 0$$

$m + \mu = 0 \quad m = -\mu \quad Y(y) = c_2 y^{-\mu}$

$$z^2 Z'(z) - (\lambda+\mu) Z(z) = 0$$

$$Z'(z) - (\lambda+\mu) Z(z) z^{-2} = 0$$

$$\frac{dZ(z)}{dz} = (\lambda+\mu) Z(z) z^{-2}$$

$$\frac{1}{Z(z)} dZ(z) = (\lambda+\mu) z^{-2} dz$$

$$\ln|Z(z)| = -(\lambda+\mu) z^{-1} + C$$

$$Z(z) = c_3 e^{-(\lambda+\mu) z^{-1}}$$

$$u(x,y,z) = \sum_{\lambda}\sum_{\mu} A_{\lambda\mu}\, e^{-\lambda x}\, y^{-\mu}\, e^{-(\lambda+\mu) z^{-1}}$$

(b)

$$u(x,y,z) = \underbrace{v(x,y,z)}_{X(x)Y(y)Z(z)} + \psi(x) + \psi(y) + \psi(z)$$

$$X'(x)Y(y)Z(z) + X(x)Y'(y)Z(z) + X(x)Y(y)Z'(z) + \psi'(x) + \psi'(y) + \psi'(z) = x+y+z$$

$\psi'(x) = x \qquad \psi(x) = \frac{1}{2}x^2 + c_1$

$\psi'(y) = y \qquad \psi(y) = \frac{1}{2}y^2 + c_2$

$\psi'(z) = z \qquad \psi(z) = \frac{1}{2}z^2 + c_3$

$$\frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} + \frac{Z'(z)}{Z(z)} = 0$$

$-\lambda \qquad -\mu \qquad \lambda+\mu$

$X'(x) + \lambda X(x) = 0$

$m+\lambda = 0 \qquad m = -\lambda \qquad X(x) = c_4 e^{-\lambda x}$

$Y'(y) + \mu Y(y) = 0 \qquad Y(y) = c_5 e^{-\mu y}$

$m+\mu = 0 \qquad m = -\mu$

$Z'(z) - (\lambda+\mu) Z(z) = 0$

$m - (\lambda+\mu) = 0 \qquad m = \lambda+\mu \qquad Z(z) = c_6 e^{(\lambda+\mu) z}$

$$u(x,y,z) = \sum_{\lambda}\sum_{\mu} A_{\lambda\mu} e^{-\lambda x} e^{-\mu y} e^{(\lambda+\mu) z} + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + c'$$

(c)

$$u(x,y,t) = X(x)Y(y)T(t)$$

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = X(x)Y(y)T'(t)$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{T'(t)}{T(t)}$$

$$-\lambda \qquad -\mu \qquad -\lambda-\mu$$

$X(0)Y(y)T(t) = 0$

$X(0)=0 \quad X(2)=0 \quad Y(0)=0 \quad Y(2)=0$

$u(x,y,0) = (2x-x^2)(2y-y^2)$

$$X''(x) + \lambda X(x) = 0$$

$\lambda = 0 \quad X''(x) = 0 \quad X(x) = c_1 x + c_2 \quad c_1 = c_2 = 0$

$\lambda = -\alpha^2 < 0 \quad X''(x) - \alpha^2 X(x) = 0 \quad X(x) = c_1 \sinh(\alpha x) + c_2 \cosh(\alpha x) \quad c_2 = 0 \quad c_1 = 0$

$\lambda = \alpha^2 > 0 \quad X''(x) + \alpha^2 X(x) = 0 \quad m = \pm\alpha j \quad X(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$

$c_2 = 0$

$c_1 \sin(2\alpha) = 0$

$\alpha = \frac{n\pi}{2} \qquad \lambda = \frac{(n\pi)^2}{4}$

$X(x) = c_1 \sin(\frac{n\pi}{2}x)$

$$Y''(y) + \mu Y(y) = 0$$

same condition as $X(x)$

$Y(y) = c_2 \sin(\frac{m\pi}{2}y) \qquad \alpha = \frac{m\pi}{2} \qquad \mu = \frac{m^2\pi^2}{4}$

$T'(t) + (\lambda+\mu)T(t) = 0 \qquad m + \frac{(n^2+m^2)\pi^2}{4} = 0$

$$T(t) = c_3 e^{-\frac{(n^2+m^2)\pi^2}{4}t}$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi}{2}x \sin\frac{m\pi}{2}y \, e^{-\frac{(n^2+m^2)\pi^2}{4}t}$$

$$u(x,y,0) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi}{2}x\right)\sin\frac{m\pi}{2}y = (2x-x^2)(2y-y^2)$$

$$\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi}{2}x = \int_0^2 (2x-x^2)(2y-y^2)\sin\frac{m\pi}{2}y \, dy$$

$$A_{mn} = \int_0^2 \left(\int_0^2 (2x-x^2)(2y-y^2)\sin\frac{m\pi}{2}y\,dy\right)\sin\frac{n\pi}{2}x\,dx$$

$$= \frac{256}{m^3n^3\pi^6}(1-(-1)^m)(1-(-1)^n)$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\frac{n\pi}{2}x\sin\frac{m\pi}{2}y\, e^{-\frac{(n^2+m^2)\pi^2}{4}t}$$

2.

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

$\Theta(0)=0 \quad \Theta(\frac{\pi}{3})=0$

steady temperature $\rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

↳ should be periodic

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

$\lambda = 0 \quad \Theta(\theta) = c_1\theta + c_2 = 0$ trivial

$\lambda = -\alpha^2 < 0 \quad \Theta(\theta) = c_1\sinh\alpha\theta + c_2\cosh\alpha\theta \quad c_1 = c_2 = 0$

$\lambda = \alpha^2 > 0 \quad \Theta(\theta) = c_1\sin\alpha\theta + c_2\cos\alpha\theta \quad c_2 = 0$

$$c_1\sin\alpha\frac{\pi}{3} = 0$$

$\alpha\frac{\pi}{3} = n\pi \quad \alpha = 3n \quad \lambda = 9n^2$

$$\Theta(\theta) = c_1\sin 3n\theta$$

$$r^2R''(r) + rR'(r) - 9n^2R(r) = 0$$

$m(m-1) + m - 9n^2 = 0$

$m = \pm 3n$

$n=0 \quad R(r) = c_3 + \cancel{c_4\ln r}$

$n\in\mathbb{N} \quad R(r) = \cancel{c_3 r^{-3n}} + c_4 r^{3n}$

$r\to 0$

$\ln r \to -\infty$

$r^{-n} \to \infty$

$R(r) = C_3 \qquad n=0$

$R(r) = C_4 r^{3n} \qquad n \in N$

$$u(r,\theta) = R(r)\Theta(\theta) = \begin{cases} 0 & n=0 \\ A_n r^{3n} \sin 3n\theta & n \in N \end{cases}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{3n} \sin 3n\theta$$

$$u(1,\theta) = \sum_{n=1}^{\infty} A_n \sin 3n\theta = \sin 6\theta + \sin(12\theta)$$

$$A_2 = 1 \quad A_4 = 1$$

$$u(r,\theta) = r^6 \sin 6\theta + r^{12} \sin 12\theta$$

3.

(3) Solve the steady temperature $u(r, z)$ in a cylinder region where

$0 \le r \le 1, \quad 0 \le z \le 2,$

$u(1,z) = z$ for $0 < z < 1$, $\quad u(1,z) = 2 - z$ for $1 < z < 2$,

$u(r,0) = 0, \quad u(r,2) = 0 \quad 0 < r < 1$

Suppose that $u(r, z)$ is independent of θ. (10 scores)

$u(r,z) = R(r)Z(z) \qquad Z(0) = 0 \quad Z(2) = 0$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z) = 0$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} = -\frac{Z''(z)}{Z(z)} = -\lambda$$

$$Z''(z) - \lambda Z(z) = 0$$

$\lambda = 0 \qquad Z(z) = c_1 z + c_2 \Rightarrow 0$ trival

$\lambda = \alpha^2 > 0 \quad Z(z) = c_1 \sinh \alpha z + c_2 \cosh \alpha z = 0$ trival

$\lambda = -\alpha^2 < 0 \quad Z(z) = c_1 \sin \alpha z + c_2 \cos \alpha z$

$c_2 = 0$

$c_1 \sin 2\alpha = 0$

$2\alpha = n\pi \quad \alpha = \frac{n\pi}{2} \quad \lambda = -\frac{n^2\pi^2}{4}$

$Z(z) = c_1 \sin \frac{n\pi}{2} z$

compared to modified Bessel function (page 201)

$x^2y'' + xy' - (x^2 + v^2)y = 0$ solution：$c_1 I_v(x) + c_2 K_v(x)$

$v=0$ 的情形

$x^2y'' + xy' - (\alpha^2x^2 + v^2)y = 0$ solution：$c_1 I_v(\alpha x) + c_2 K_v(\alpha x)$

$I_v(x)$: modified Bessel function of the 1st kind

$K_v(x)$: modified Bessel function of the 2nd kind

$$R''(r) + \frac{1}{r}R'(r) - \frac{n^2\pi^2}{4}R(r) = 0$$

$$r^2R''(r) + rR'(r) - \frac{n^2\pi^2}{4}r^2R(r) = 0$$

$v = 0 \quad \alpha = \frac{n\pi}{2}$

$$R(r) = c_1 I_0(\alpha r) + \cancel{c_2 K_0}(\alpha r) \quad K_0(0) \to \infty$$

$$u(r,z) = R(r)Z(z) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{2} z \; I_0\left(\frac{n\pi}{2} r\right)$$

$$u(1,z) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{2} z \; I_0\left(\frac{n\pi}{2}\right)$$

u dv

$$A_n I_0\left(\frac{n\pi}{2}\right) = \int_0^1 z \sin \frac{n\pi}{2} z \, dz + \int_1^2 (2-z) \sin \frac{n\pi}{2} z \, dz$$

$$= -\frac{2z}{n\pi} \cos \frac{n\pi}{2} z \Big|_0^1 + \int_0^1 \frac{2}{n\pi} \cos \frac{n\pi}{2} z \, dz - \frac{2(2-z)}{n\pi} \cos \frac{n\pi}{2} z \Big|_1^2 - \int_1^2 \frac{2}{n\pi} \cos \frac{n\pi}{2} z \, dz$$

$$= \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} z \Big|_0^1 - \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} z \Big|_1^2$$

$$= \frac{8}{n^2\pi^2} \sin\left(\frac{n}{2}\pi\right)$$

$$A_n = \frac{8 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 I_0\left(\frac{n\pi}{2}\right)} \qquad u(r,\theta) = \sum_{n=1}^{\infty} \frac{8 I_0\left(\frac{n\pi r}{2}\right)}{n^2\pi^2 I_0\left(\frac{n\pi}{2}\right)} \sin \frac{n\pi z}{2} \sin \frac{n\pi}{2}$$

4.

$$\mathcal{L}_{t\to s}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right) = \mathcal{L}_{t\to s}\, 1$$

$$\frac{dU(x,s)}{dx} + sU(x,s) - \underbrace{u(x,0)}_{0} = \frac{1}{s}$$

$$\frac{dU(x,s)}{dx} + s\,U(x,s) = \frac{1}{s}$$

$m+s=0 \quad m=-s$

$U(x,s) = c_1 e^{-sx}$

$U(x,s) = c_2$

$sc_2 = \frac{1}{s} \quad c_2 = \frac{1}{s^2}$

$$U(x,s) = c_1 e^{-sx} + \frac{1}{s^2}$$

$$\mathcal{L}_{t\to s}(u(0,t)) = U(0,s) = c_1 + \frac{1}{s^2} = \frac{2}{s^3} + \frac{1}{s^2}$$

$$c_1 = \frac{2}{s^3}$$

$$U(x,s) = \frac{2}{s^3}e^{-sx} + \frac{1}{s^2}$$

$$u(x,t) = \mathcal{L}^{-1}_{s\to t}\left\{\frac{2}{s^3}e^{-sx} + \frac{1}{s^2}\right\}$$

$$= u(t-x)\,f(t-x) + t$$

$$= (t-x)^2\,u(t-x) + t$$

5.

(a) $\phi_0(x)=1 \qquad \phi_1(x)=x \qquad \phi_2(x)=x^2$

$$\psi_0(x)=\frac{\phi_0(x)}{\|\phi_0(x)\|}=\frac{1}{\sqrt{\int_0^4 1^2dx}}=\frac{1}{2}$$

$$g_1(x)=\phi_1(x)-\frac{1}{2}(\phi_1(x),\psi_0(x))=x-\frac{1}{2}\int_0^4\frac{1}{2}x\,dx=x-2$$

$$\psi_1(x)=\frac{x-2}{\sqrt{\int_0^4(x-2)^2dx}}=\frac{\sqrt{3}(x-2)}{4}$$

$$g_2(x)=\phi_2(x)-\frac{1}{2}(\phi_2(x),\psi_0(x))-\frac{\sqrt{3}(x-2)}{4}(\phi_2(x),\psi_1(x))$$

$$=x^2-\frac{1}{2}\int_0^4\frac{x^2}{2}dx-\frac{\sqrt{3}}{4}(x-2)\int_0^4 x^2\frac{\sqrt{3}}{4}(x-2)dx$$

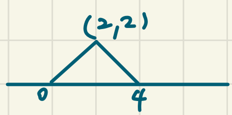
$$=x^2-\frac{16}{3}-\frac{3}{16}(x-2)\frac{64}{3}$$

$$=x^2-4x+\frac{8}{3}$$

$$\psi_2(x)=\frac{g_2(x)}{\|g_2(x)\|}=\frac{x^2-4x+\frac{8}{3}}{\sqrt{\int_0^4(x^2-4x+\frac{8}{3})^2dx}}=\frac{x^2-4x+\frac{8}{3}}{\frac{16}{3\sqrt{5}}}=\frac{3\sqrt{5}}{16}x^2-\frac{3\sqrt{5}}{4}x+\frac{\sqrt{5}}{2}$$

$$\left\{\frac{1}{2},\ \frac{\sqrt{3}}{4}(x-2),\ \frac{3\sqrt{5}}{16}x^2-\frac{3\sqrt{5}}{4}x+\frac{\sqrt{5}}{2}\right\}$$

(b) $\min(x,4-x)\rightarrow$



$$f(x)\cong f_2(x)=c_1P_0(x)+c_2P_1(x)+c_3P_2(x)$$

$$c_1=\int_0^2 x\frac{1}{2}dx+\int_2^4(4-x)\frac{1}{2}dx=1+1=2$$

$$c_2=\frac{\sqrt{3}}{4}\left(\int_0^2 x(x-2)dx+\int_2^4(4-x)(x-2)dx\right)=0$$

$-x^2+6x-8$

$-\frac{x^3}{3}+3x^2-8x$

$-\frac{64}{3}+\frac{8}{3}+\cancel{\frac{48-12-32}{16}}+16$

$$c_3 = \frac{3\sqrt{5}}{16}\left(\int_0^2 x\left(x^2-4x+\frac{8}{3}\right)dx + \int_2^4 (4-x)\left(x^2-4x+\frac{8}{3}\right)dx\right)$$

$x^3-4x^2+\frac{8}{3}x$

$-\frac{4}{3}$

$-1 \quad 4 \quad -\frac{8}{3}$

$4 \quad -16 \quad \frac{32}{3}$

$-\frac{4}{3}$

$$= -\frac{\sqrt{5}}{2}$$

$$f_2(x) = 1 - \frac{15}{32}x^2 + \frac{15}{8}x - \frac{5}{4}$$

$$= -\frac{15}{32}x^2 + \frac{15}{8}x - \frac{1}{4}$$

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Gram-Schmidt-Basis:
[0.2773501 0.2773501 0.2773501 0.2773501 0.2773501 0.2773501 0.2773501
 0.2773501 0.2773501 0.2773501 0.2773501 0.2773501 0.2773501]
[-4.44749590e-01 -3.70624658e-01 -2.96499727e-01 -2.22374795e-01
 -1.48249863e-01 -7.41249317e-02  4.56491668e-18  7.41249317e-02
  1.48249863e-01  2.22374795e-01  2.96499727e-01  3.70624658e-01
  4.44749590e-01]
[ 0.49168917  0.24584459  0.04469902 -0.11174754 -0.22349508 -0.2905436
 -0.31289311 -0.2905436  -0.22349508 -0.11174754  0.04469902  0.24584459
  0.49168917]
[-4.59933106e-01 -3.04327779e-18  2.50872603e-01  3.34496804e-01
  2.92684704e-01  1.67248402e-01 -6.85573854e-18 -1.67248402e-01
 -2.92684704e-01 -3.34496804e-01 -2.50872603e-01 -3.04327779e-18
  4.59933106e-01]
[ 0.37945799 -0.25297199 -0.36795926 -0.20697708  0.042162    0.24530617
  0.32196435  0.24530617  0.042162   -0.20697708 -0.36795926 -0.25297199
  0.37945799]
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Weighted-Gram-Schmidt-Basis:
[0.37796447 0.37796447 0.37796447 0.37796447 0.37796447 0.37796447
 0.37796447 0.37796447 0.37796447 0.37796447 0.37796447 0.37796447
 0.37796447]
[-8.01783726e-01 -6.68153105e-01 -5.34522484e-01 -4.00891863e-01
 -2.67261242e-01 -1.33630621e-01 -8.41120960e-18  1.33630621e-01
  2.67261242e-01  4.00891863e-01  5.34522484e-01  6.68153105e-01
  8.01783726e-01]
[ 1.12815215  0.68494952  0.32232919  0.04029115 -0.16116459 -0.28203804
 -0.32232919 -0.28203804 -0.16116459  0.04029115  0.32232919  0.68494952
  1.12815215]
[-1.26710519e+00 -3.72677996e-01  1.49071198e-01  3.72677996e-01
  3.72677996e-01  2.23606798e-01  3.15376320e-17 -2.23606798e-01
 -3.72677996e-01 -3.72677996e-01 -1.49071198e-01  3.72677996e-01
  1.26710519e+00]
[ 1.2117489  -0.11005083 -0.50444922 -0.38369074 -0.06840998  0.21236836
  0.32122946  0.21236836 -0.06840998 -0.38369074 -0.50444922 -0.11005083
  1.2117489 ]
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