Quantum Interior Point Methods (QIPMs) with Iterative Refinement for Linear and Semidefinite Optimization

Exponentially Improved Complexity

Tamás Terlaky[1]
Brandon Augustino[1] Mohammadhossein Mohammadisiahroudi[1]
Ramin Fakhimi[1] Giacomo Nannicini[2] Zeguan Wu[1] Luis Zuluaga[1]

[1]Quantum Computing and Optimization Lab, Department of Industrial & Systems Engineering, Lehigh University

[2]Department of Industrial and Systems Engineering, University of Southern California

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Semidefinite Optimization

Let

- $b \in \mathbb{R}^m$
- matrices $A_1, \ldots, A_m, C \in \mathcal{S}^n$

Then, the primal-dual Semidefinite Optimization (SDO) pair is given by:

$$z_P = \inf_X \left\{ \mathrm{tr}\left(CX\right) : \mathrm{tr}\left(A_i X\right) = b_i, \ \forall i \in [m], X \succeq 0 \right\}$$

$$z_D = \sup_{y,S} \left\{ b^\top y : \sum_{i=1}^m y_i A_i + S = C, \ S \succeq 0, y \in \mathbb{R}^m \right\}$$

where

- $[m] = \{1, \ldots, m\}$
- $S = C - \sum_{i \in [m]} y_i A_i \succeq 0$ is the slack matrix of the dual problem
- $\mathcal{S}^n$ is the cone of $n \times n$ symmetric matrices
- We assume that the matrices $A_1, \ldots, A_m$ are linearly independent

Semidefinite Optimization

- SDOs can be used to study the properties of convex optimization problems
- SDOs cover a wide range of continuous and combinatorial optimization applications
  - Maximum eigenvalue problem
  - Matrix norm minimization
  - Structural optimization
- Approximations to NP-hard problems
  - Goemans and Williamson SDO formulation for MAXCUT
  - Lovász ϑ-number
  - AC-OPF SDO relaxations

Prevailing classical solution methodology: *Interior Point Methods*

Quantum algorithms for SDO

Generally, classical algorithms for SDO are categorized into two classes

1. Algorithms with **polylogarithmic** dependence on:
   - The inverse of the precision to which we solve the SDO problem
   - The size of the inscribed ellipsoid
2. Those that depend **polynomially** on these quantities, but exhibit an advantage with respect to n and m

Not Polynomial time!

More natural to categorize *quantum* algorithms for SDO according to how they attempt to achieve speedups

Classical Interior Point Method: Overview

- Iteration complexity $O(\sqrt{n}\log\frac{1}{\epsilon})$.
- In each iteration: solve a linearization of the perturbed optimality conditions known as the *Newton linear system*
- Solving this system is expensive, it requires $\mathcal{O}(n^6)$ arithmetic operations in the worst case

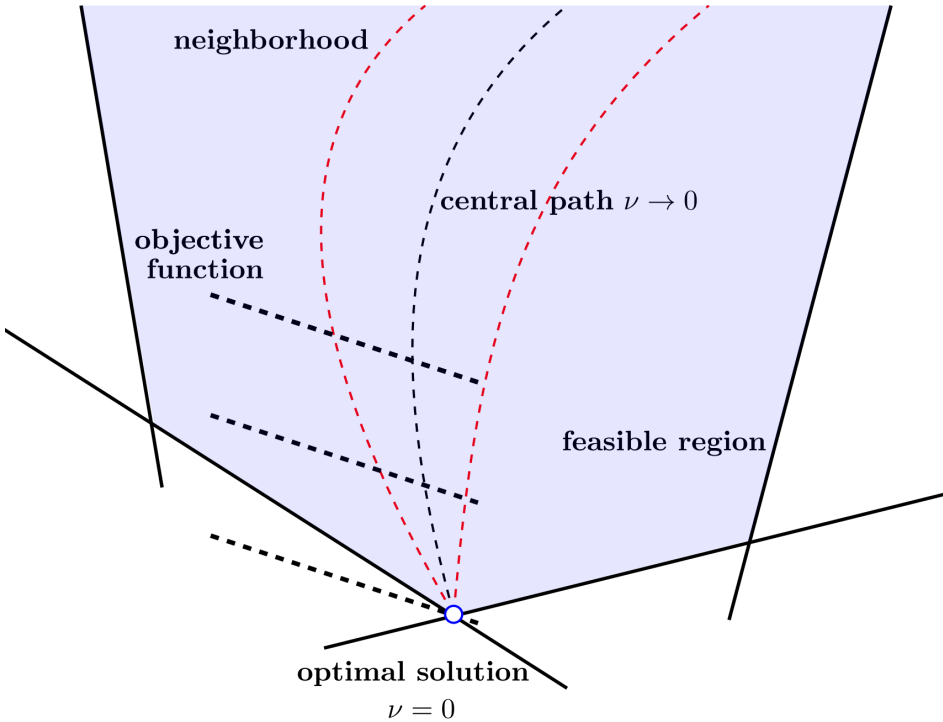


Figure: Interior Point Method

Could SDO problems be solved more efficiently in a Quantum setting?

Quantum Matrix Multiplicative Weights Methods

QMMUs quantize an algorithm from Arora and Kale using a clever interpretation of the primal variables

Trace normalized PSD matrices can be represented using *Gibbs states:*

$$\rho \leftarrow \frac{\exp(-H)}{\mathrm{tr}(\exp(-H))}$$

where H is the *Hamiltonian*, and

$$\exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots,$$

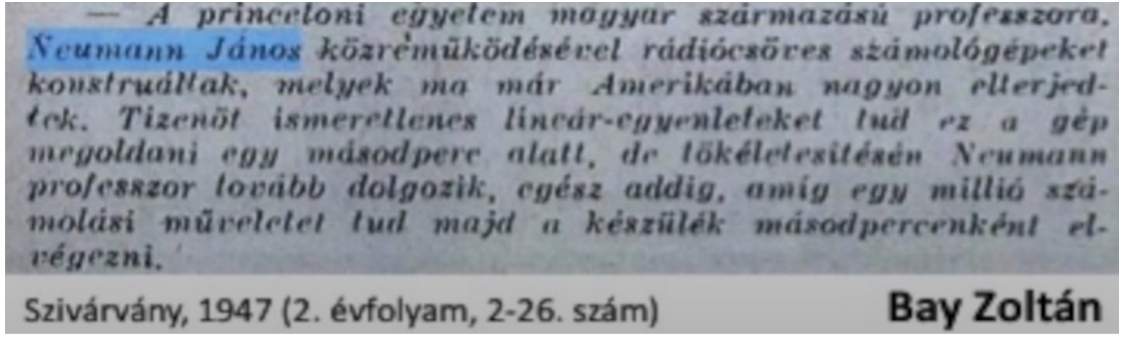
Current state of the art (van Apeldoorn and Gilyén (2018)):

$$\widetilde{\mathcal{O}}_{n,R,r,\frac{1}{\epsilon}}\left(\left(\sqrt{m}+\sqrt{n}\frac{Rr}{\epsilon}\right)s\left(\frac{Rr}{\epsilon}\right)^4\right)$$

where $\mathrm{tr}\,(X) \leq R$, $\|y\|_1 \leq r$.

Our HU variant with Iterative Refinement -> Exponentially Improved Complexity

Quantum linear algebra



— A princetoni egyetem magyar származású professzora, Neumann János közreműködésével rádiócsöves számológépeket konstruáltak, melyek ma már Amerikában nagyon elterjedtek. Tizenöt ismeretlenes lineár-egyenleteket tud ez a gép megoldani egy másodperc alatt, de tökéletesítésén Neumann professzor tovább dolgozik, egész addig, amíg egy millió számolási műveletet tud majd a készülék másodpercenként elvégezni.

Szivárvány, 1947 (2. évfolyam, 2-26. szám) **Bay Zoltán**

For continuous optimization

For solving real world optimization problems

Efficient, reliable
Quantum Linear Algebra
is needed!

Data flow in hybrid classic-quantum algorithms:

Classic → QRAM → QSolve → QTomography → Classic solution

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Notes on Quantum Linear Algebra

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Need to solve Newton systems – "accurately, fast"

$$Mz = \sigma \quad M|z\rangle = |\sigma\rangle$$

Algorithm	Complexity
Factorization (e.g. Cholesky)	$\mathcal{O}(p^3)$
Conjugate Gradient	$\mathcal{O}(pd\sqrt{\kappa}\log(\frac{1}{\epsilon}))$
HHL + QTA	$\mathcal{O}(\text{polylog}(p)\frac{d^2\kappa^2\|\|\sigma\|\|}{\|\|M\|\|\epsilon}) \times \mathcal{O}(\frac{p\|\|\sigma\|\|}{\|\|M\|\|\epsilon})$
VTAA-HHL + QTA	$\mathcal{O}(\text{polylog}(p)\frac{d^2\kappa\|\|\sigma\|\|}{\|\|M\|\|\epsilon}) \times \mathcal{O}(\frac{p\|\|\sigma\|\|}{\|\|M\|\|\epsilon})$
QLSA (Wossnig, et al. 2018) + QTA	$\mathcal{O}(\text{polylog}(p)\frac{\kappa\|\|\sigma\|\|}{\|\|M\|\|\epsilon}) \times \mathcal{O}(\frac{p\|\|\sigma\|\|}{\|\|M\|\|\epsilon})$
QLSA (Childs, et al. 2017) + QTA	$\mathcal{O}(\text{polylog}(\frac{p\kappa\|\|\sigma\|\|}{\|\|M\|\|\epsilon})d\kappa) \times \mathcal{O}(\frac{p\|\|\sigma\|\|}{\|\|M\|\|\epsilon})$
QLSA (Carrera, et al. 2020) + QTA	$\mathcal{O}(\text{polylog}(\frac{p\kappa\|\|\sigma\|\|}{\|\|M\|\|\epsilon})\kappa) \times \mathcal{O}(\frac{p\|\|\sigma\|\|}{\|\|M\|\|\epsilon})$
QLSA (Chakraborty, et al. 2018) + QTA	$\mathcal{O}(\text{polylog}(\frac{p\|\|\sigma\|\|}{\epsilon})\kappa\|\|M\|\|_F) \times \mathcal{O}(\frac{p\|\|\sigma\|\|}{\epsilon})$

with tomography

$*\kappa$ is the condition number of M,
d is maximum number of non-zero elements in each row and column in M,
p is the number of rows/columns of M, and
ϵ is the error of Linear Equation Solver.

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Quantum Interior Point Methods

- Kerenidis and Prakash (2018) made the first effort at a quantum interior point method
- Use quantum random access memory (QRAM) and block encodings to solve the Newton linear system
- Small neighborhood IPM:

$$\mathcal{N}_F(\gamma) = \left\{(X, y, S) \in \mathcal{P}^0 \times \mathcal{D}^0 : \left\| X^{1/2} S X^{1/2} - \nu I \right\|_F \leq \gamma \nu \right\}$$

- They posit a worst cast running time of

$$\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\xi},\frac{1}{\epsilon}} \left(\frac{n^{2.5}}{\xi^2} \mu \kappa^3 \log \frac{1}{\epsilon} \right)$$

  for SDPs
  - The term $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\xi}} \left(\frac{n^2 \kappa^2}{\xi^2} \right)$ comes from a tomography subroutine
  - $\mu \leq n$ and κ are factors corresponding to the QLSA
- For solving LPs, the running time is

$$\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\xi},\frac{1}{\epsilon}} \left(\frac{n^{1.5}}{\xi^2} \mu \kappa^3 \log \frac{1}{\epsilon} \right)$$

Quantum Interior Point Methods

Critical issues:

1. Overlook the crucial issue of symmetrizing the Newton linear system
   - Updates to the solution will not be symmetric
2. Convergence analysis implicitly assumes primal and dual feasibility satisfied *exactly*
   - Due to errors from QLSAs & Tomography, can't be guaranteed without further safeguards
3. Not a polynomial time algorithm.
   - Needed; $\xi << \epsilon$ and $\kappa >> \frac{1}{\epsilon^2}$
   - Polynomial dependence on κ, $\frac{1}{\epsilon^2}$.

Quantum Interior Point Methods

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Specifically: need

$$\mathrm{tr}\left(\Delta X \Delta S\right) = 0$$

will not hold! Not guaranteed to converge.

The Newton Linear System

For $\nu > 0$, assuming *interior point condition*, and linear independence of the matrices $A^{(i)}$, the central path is the solution set of equation system

$$\begin{aligned} \operatorname{tr}(A_i X) &= b_i \ \forall i \in [m],\ X \succ 0 \\ \sum_{i \in [m]} y_i A_i - S &= C,\ S \succ 0 \\ XS &= \nu I, \end{aligned} \tag{1}$$

- Linearizing the central path eqns gives the Newton linear system:

$$\begin{aligned} X\Delta S + \Delta X\, S &= \sigma\nu I - XS \\ \Delta S \in L \quad \Delta X &\in L^{\perp}. \end{aligned}$$

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This system has no symmetric solution!

The Newton Linear System

While ΔS will be symmetric, there is no symmetric ΔX that solves

$$X\Delta S + \Delta X\ S = \sigma\nu I - XS$$

The Newton Linear System

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Clear upon re-writing in terms of ΔX:

$$\Delta X = \underbrace{\sigma\nu S^{-1} - X}_{\text{symmetric}} - \underbrace{X\Delta S S^{-1}}_{\text{not symmetric}}$$

After symmetrizing, we can use QLSA to solve the linear system:

$$\Delta X \in L^{\perp}$$
$$\Delta S \in L$$
$$H_P(\Delta X S + \Delta S\ X) = \sigma\nu I - H_P(XS)$$

Symmetrizing the Newton System

- Symmetrization is a linear transformation:

$$H_P(M) = \frac{1}{2}\left[PMP^{-1} + P^{-T}M^TP^T\right].$$

  for a given invertible matrix P
- The Alizadeh-Haeberly-Overton (AHO) direction is given by

$$P = I$$

- The Nesterov-Todd (NT) direction is given by

$$P = W^{-1/2}$$

  where

$$\begin{aligned} W &= S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2} \\ &= X^{-1/2}(X^{1/2}SX^{1/2})^{1/2}X^{-1/2} \end{aligned}$$

- Additionally there is the so called HKM direction for which

$$P = S^{1/2}$$

The Newton Linear System

Due to the nonnegligible errors introduced by QLSA & Tomography, $(\Delta y, \Delta X, \Delta S)$ can only satisfy

$$\begin{pmatrix} 0 & \mathcal{A} & 0 \\ \mathcal{A}^\top & 0 & \mathcal{I} \\ 0 & \mathcal{E} & \mathcal{F} \end{pmatrix} \begin{pmatrix} \Delta y \\ \Delta X \\ \Delta S \end{pmatrix} = \begin{pmatrix} \xi_p \\ \xi_d \\ \sigma\nu I - H_p(XS) + \xi_c \end{pmatrix}$$

where (ξ_p, ξ_d, ξ_c) are errors introduced from state tomography

- No guarantee ΔX and ΔS coming from orthogonal subspaces

Consequence: Feasible IPMs convergence analysis cannot be applied

- Note: regardless if problem is linear, conic, semidefinite, etc.

New Approach: Development of an Inexact, but *Feasible* IPM for SDO

Inexact-Feasible QIPMs for SDO

- We cannot avoid using tomography in each iteration;
- Instead, we seek to recover a feasible-QIPM framework
- From primal and dual feasibility: $\Delta X \in \mathrm{Null}(\mathcal{A})$ and $\Delta S \in \mathcal{R}(\mathcal{A})$
  - $\mathrm{Null}(\mathcal{A}) \equiv$ nullspace of $\mathcal{A}_s$
  - $\mathcal{R}(\mathcal{A}) \equiv$ rowspace of $\mathcal{A}_s$
- Then we can set:

$$\mathbf{svec}(\Delta X) = \mathbf{svec}(Q_2 \Delta z)$$
$$\mathbf{svec}(\Delta S) = \mathbf{svec}(-\mathcal{A}_s^\top \Delta y)$$

A new approach: Inexact-Feasible QIPMs

- Hence, we can solve the quantum Newton system:

$$\begin{bmatrix} \mathcal{E}Q_2 & \mathcal{F}(-\mathcal{A}_s^\top) \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta y \end{bmatrix} = \mathbf{svec}(\sigma\nu I - H_p(XS)) \qquad \text{(OSS)}$$

- We allow for some error in satisfying the complementarity equation
- Error need to be proportional to a $0 < \beta < 1$ fraction of ν.
- Regardless of QLSA errors $(\Delta y, \Delta X, \Delta S)$ always primal-dual feasible

$$\mathcal{A}_s\mathbf{svec}(\Delta X) = \mathcal{A}_s Q_2 \Delta z = (\mathcal{A}_s Q_2)\Delta z = 0,$$
$$\mathcal{A}_s^\top \Delta y + \mathcal{I}\mathbf{svec}(\Delta S) = \mathcal{A}_s^\top \Delta y + (-\mathcal{A}_s^\top \Delta y) = \mathcal{A}_s^\top(\Delta y - \Delta y) = 0$$

Feasible IPM analysis can be recovered!

Inexact-Feasible Quantum Interior Point Method

Algorithm Inexact-Feasible Quantum Interior Point Method

Input: $\epsilon, \delta > 0$; $\sigma = 1 - \delta/\sqrt{n}$; $\beta, \gamma \in (0, 1)$

Output: An ϵ-optimal primal dual pair (X, y, S)

Choose $(X^{(0)}, y^{(0)}, S^{(0)}) \in \mathcal{N}_F(\gamma)$

Set $\nu^{(0)} = \frac{X^{(0)} \bullet S^{(0)}}{n}$

Compute bases for $\mathcal{R}(\mathcal{A}_s) = \mathcal{A}_s^\top$ and $\mathrm{Null}(\mathcal{A}_s) = Q_2$

while $\nu > \epsilon$ **do**

1. $\nu^{(k)} \leftarrow \frac{\mathrm{tr}(X^{(k)} S^{(k)})}{n}$
2. Compute matrices $(X^{(k)})^{-1}$ and $(S^{(k)})^{-1}$, $P^{(k)}$ classically.
3. Using block-encodings, solve Newton system to construct inexact search direction $|\Delta z^{(k)} \circ \Delta y^{(k)}\rangle$.
4. Obtain classical estimate $\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}$ of $\Delta z^{(k)}, \Delta y^{(k)}$ using vector state tomography.
5. Use classical estimate $\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}$ to obtain classical estimate $\overline{\Delta X}^{(k)}, \overline{\Delta S}^{(k)}$ of $\Delta X^{(k)}, \Delta S^{(k)}$
6. Update current solution

$$X^{(k+1)} \leftarrow X^{(k)} + \overline{\Delta X}^{(k)}, \; S^{(k+1)} \leftarrow S^{(k)} + \overline{\Delta S}^{(k)} \text{ and } y^{(k+1)} \leftarrow y^{(k)} + \overline{\Delta y}^{(k)}$$

$$k \leftarrow k + 1$$

end

Inexact-Feasible QIPMs for SDO: Running Time

Theorem (Augustino et al. (2021))

Let the problem data be stored in a quantum RAM (QRAM). When applied to SDO problems involving $n \times n$ matrices and $m = \mathcal{O}(n^2)$ constraints, the IF-QIPM requires at most

$$\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\epsilon}}\left(n^{3.5}\frac{\kappa^2}{\epsilon}\right)$$

QRAM accesses and $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\epsilon}}\left(n^{4.5}\right)$ arithmetic operations, where κ is an upper bound on the condition number

- Quadratic speedup in terms of n compared to classical IPMs
- This work:
  - Available on arXiv: https://arxiv.org/pdf/2112.06025.pdf P
  - Published in *Quantum*
- However, dependence on κ, ϵ^{-1} suggests no speedup overall
  - Dependence on κ is particularly problematic for IPMs
  - $\kappa \to \infty$ as we approach optimality (more on this later)

Iterative Refinement

Iterative Refinement is a classical computing technique that uses low-precision arithmetic to construct high-accuracy solutions to

- Linear systems of equations
- Linear Optimization (LO) problems
- Mixed-Integer Optimization problems

Seemingly, high-adaptability to quantum computing

- Particularly for quantum SDO solvers, which offer potential for speedups in low-precision regime

Iterative Refinement for linear systems

Let $M \in \mathbb{R}^{d \times d}$, $w \in \mathbb{R}^d$

- We want to solve $Mv = w$
- Define $r^{(k)} = w - Mv^{(k)}$ is the residual w.r.t. current solution
- In each iteration, we call an oracle O_{LS} solve the refining system

$$Mu^{(k)} = \eta^{(k)} r^{(k)},$$

  where $\eta^{(k)} = \|r^{(k)}\|^{-1}$
- $u^{(k)}$ can be used to reduce the residual at iteration k
- Use $u^{(k)}$ to update our solution to the linear system $Mv = w$ as:

$$v^{(k+1)} \leftarrow v^{(k)} + \frac{1}{\eta^{(k)}} u^{(k)}$$

Iteration complexity to reach $\|r\| \leq \zeta$: $\mathcal{O}\left(\log\left(\frac{1}{\zeta}\right)\right)$

If O_{LS} is a *quantum* oracle, then we need to account for normalization

Iterative Refinement for linear systems

Algorithm Iterative Refinement for LSPs

Input: Error tolerances $0 < \zeta \ll \xi < 1$, bound on norm of solution θ

Output: A ζ-precise solution v to $\frac{M}{\theta} v = \frac{w}{\theta}$

Normalize the data $(\widetilde{M}, \tilde{w}) \leftarrow \theta^{-1}(M, w)$

Initialize: $x^{(0)} \leftarrow 0$, $r^{(0)} \leftarrow \tilde{w}$, $\eta^{(0)} \leftarrow 1$, $k \leftarrow 0$

while $\|r\| > \zeta$ **do**

1. $\bar{u}^{(k)} \leftarrow$ **solve** $(\widetilde{M}, \eta^{(k)} r^{(k)})$ using $O_{LS}(\xi)$
2. $\tilde{u}^{(k+1)} \leftarrow \frac{\|\eta^{(k)} r^{(k)}\|}{\|\widetilde{M} \bar{u}^{(k)}\|} \bar{u}^{(k)}$
3. Update solution: $v^{(k+1)} \leftarrow v^{(k)} + \frac{1}{\eta^{(k)}} \tilde{u}^{(k)}$
4. Update residual $r^{(k+1)} \leftarrow \tilde{w} - \widetilde{M} x^{(k+1)}$
5. Update scaling factor $\eta^{(k+1)} \leftarrow \|r^{(k+1)}\|^{-1}$
6. $k \leftarrow k + 1$

end

Here, we define O_{LS} as the combined use of QLSA + State Tomography

Iterative Refinement for linear systems: Main results

Theorem (Mohammadisiahroudi et al. (2022))

Let $v^{(k)}$ and $\eta^{(k)}$, $k = 0, 1, \ldots$ be the sequence of solutions and scaling factors produced by Algorithm 2. Then for all $k \geq 0$,

(a) $\eta^{(k)} \geq \frac{1}{\xi^k}$,

(b) $\left\| \tilde{w} - \widetilde{M} v^{(k)} \right\| \leq \xi^k$.

Corollary (Mohammadisiahroudi et al. (2022))

Let $0 < \zeta \ll \xi$, and $\eta^{(0)} = 1$. Then, Algorithm 2 terminates in at most $\mathcal{O}\left(\log\left(\frac{1}{\zeta}\right)\right)$ iterations.

Iterative Refinement for linear systems: Complexity

Theorem (Mohammadisiahroudi et al. (2022))

Let the problem data be stored in a quantum RAM (QRAM), and ξ be the precision used for the oracle O_{LS} in every iteration. Then, each Algorithm 2 requires at most

$$\mathcal{O}\left(d\kappa_M \cdot \text{polylog}\left(d, \kappa_M\right)\right),$$

QRAM accesses and $\mathcal{O}(ds)$ classical arithmetic operations.

Corollary (Mohammadisiahroudi et al. (2022))

Let the problem data be stored in a QRAM, and fix $\xi \in (0,1)$. Then, setting $\zeta = \frac{\epsilon}{\theta}$, Algorithm 2 obtains an ϵ-precise solution of a linear system $Mv = w$ with at most

$$\mathcal{O}\left(d\kappa_M \cdot \text{polylog}\left(d, \kappa_M, \theta, \epsilon^{-1}\right)\right)$$

QRAM accesses and $\mathcal{O}\left(ds \cdot \text{polylog}\left(d, \kappa_M, \theta, \epsilon^{-1}\right)\right)$ arithmetic operations.

Inexact-Feasible QIPM with Iterative Refinement

Algorithm Inexact-Feasible Quantum Interior Point Method

Input: $\epsilon, \delta > 0$; $\sigma = 1 - \delta/\sqrt{n}$; $\beta, \gamma \in (0,1)$

Output: An ϵ-optimal primal dual pair (X, y, S)

Choose $(X^{(0)}, y^{(0)}, S^{(0)}) \in \mathcal{N}_F(\gamma)$

Set $\nu^{(0)} = \frac{X^{(0)} \bullet S^{(0)}}{n}$

Compute bases for $\mathcal{R}(\mathcal{A}_s) = \mathcal{A}_s^\top$ and $\text{Null}(\mathcal{A}_s) = Q_2$

while $\nu > \epsilon$ **do**

1. $\nu^{(k)} \leftarrow \frac{X^{(k)} \bullet S^{(k)}}{n}$
2. Compute matrices $(X^{(k)})^{-1}$ and $(S^{(k)})^{-1}$, $P^{(k)}$ classically.
3. Obtain classical estimate $\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}$ of $\Delta z^{(k)}, \Delta y^{(k)}$ using IR for the LSP
4. Use classical estimate $\overline{\Delta z}^{(k)}, \overline{\Delta y}^{(k)}$ to obtain classical estimate $\overline{\Delta X}^{(k)}, \overline{\Delta S}^{(k)}$ of $\Delta X^{(k)}, \Delta S^{(k)}$
5. Update current solution

$$X^{(k+1)} \leftarrow X^{(k)} + \overline{\Delta X}^{(k)}, \; S^{(k+1)} \leftarrow S^{(k)} + \overline{\Delta S}^{(k)} \text{ and } y^{(k+1)} \leftarrow y^{(k)} + \overline{\Delta y}^{(k)}$$

$$k \leftarrow k + 1$$

end

An IF-QIPM with "inner" Iterative Refinement

If we treat this algorithm as a Newton system subroutine, the complexity of the IF-QIPM for SDO improves to:

$$\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\epsilon}}\left(n^{2.5}\kappa\right)$$

QRAM accesses and $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\epsilon}}\left(n^{4.5}\right)$ arithmetic operations

- $\mathcal{O}\left(\frac{n\kappa}{\epsilon}\right)$ fewer QRAM accesses than the algorithm from earlier
- *Exponential* speedup

We will adapt IR techniques to SDO to further mitigate dependence on κ

The relationship between κ and optimality

Consider the LO setting

- Newton system coefficient matrix can be written as

$$A(XS^{-1})A^\top$$

  where $X = \mathrm{diag}(x)$ and $S = \mathrm{diag}(s)$

- In particular,

$$XS^{-1} = \mathrm{diag}\left(\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n}\right)$$

- By complementary slackness, some x_i growing large, while $s_i \to 0$ (and vise-versa)
- We have

$$\kappa = \mathcal{O}\left(\kappa_A \frac{1}{\nu^2}\right)$$

The relationship between κ and optimality

- For IPMs, we always have $\nu > \epsilon$ (due to the stopping criteria)
- This implies

$$\kappa = \mathcal{O}\left(\kappa_A \frac{1}{\nu^2}\right) = \mathcal{O}\left(\kappa_A \frac{1}{\epsilon^2}\right)$$

Hence, if ϵ were simply a fixed constant (e.g., $\epsilon = 10^{-2}$), we would have

$$\kappa = \mathcal{O}\left(\kappa_A\right)$$

Iterative Refinement for SDO

Algorithm Iterative Refinement for SDO using QIPMs

Input: Problem data $A_1, \ldots, A_m, C \in \mathcal{S}^n$, $b \in \mathbb{R}^m$, Error tolerances $0 < \zeta \ll \epsilon < 1$

Output: A ζ-optimal primal-dual solution (X, y, S) to the SDO problem $(A_1, \ldots, A_m, b, C)$

Initialize: $X^{(0)} \leftarrow 0$, $y^{(0)} \leftarrow 0$, $\eta^{(0)} \leftarrow 1$, $\varepsilon^{(0)} \leftarrow n$, $k \leftarrow 0$, $\bar{b} \leftarrow b$, $\bar{C} \leftarrow C$

while $\varepsilon > \zeta$ **do**

1. $(\overline{X}, \bar{y}) \leftarrow$ **solve** $(A_1, \ldots, A_m, \eta^{(k)}\bar{b}, \eta^{(k)}\bar{C})$ using $O_{\mathsf{SDO}}(\epsilon)$
2. Update solution

$$X^{(k+1)} \leftarrow X^{(k)} + \frac{1}{\eta^{(k)}}\overline{X}, \quad y^{(k+1)} \leftarrow y^{(k)} + \frac{1}{\eta^{(k)}}\bar{y}$$

3. Update refining problem data

$$\bar{b}_i^{(k+1)} \leftarrow b_i - \mathrm{tr}(A_i X^{(k+1)}), \quad \bar{C} \leftarrow C - \sum_{i=1}^{m} y_i^{(k+1)} A_i$$

4. Compute residual $\varepsilon^{(k+1)} = \frac{\mathrm{tr}\left(X^{(k+1)}\bar{C}\right)}{n}$
5. Update scaling factor $\eta^{(k+1)} = \frac{1}{\varepsilon^{(k+1)}}$
6. $k \leftarrow k + 1$

end

Putting everything together

Treat the IF-QIPM that uses IR for LSPs as an SDO subroutine that is called in fixed precision

- One can show that for any strictly feasible (X, y, S), the system (OSS) has a condition number bound

$$\kappa = \mathcal{O}\left(\kappa_T \frac{1}{\nu}\right) = \mathcal{O}\left(\kappa_T \frac{1}{\epsilon}\right)$$

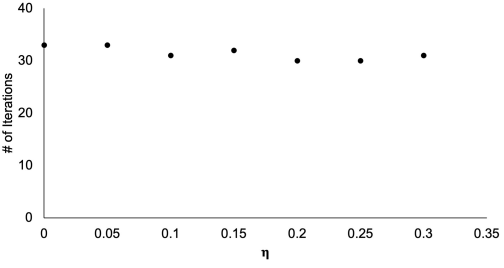
- In the context of the IR scheme (using fixed precision ξ)
- Here,

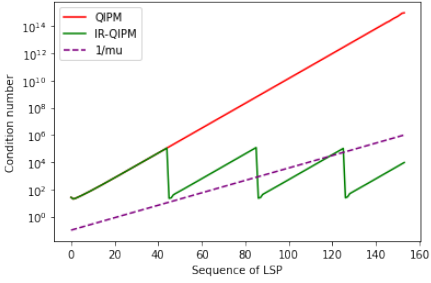
$$T = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B}_{\text{Null}} \end{pmatrix}$$

where $\mathcal{B}_{\text{Null}}$ is a basis for $\text{Null}(\mathcal{A})$

Numerical Experiments

- Developed a Python package for LO solver: https://github.com/QCOL-LU/QIPM.





Main result

Theorem

When applied to SDO problems involving $n \times n$ matrices and $m = \mathcal{O}(n^2)$ constraints, the IF-QIPM using "inner" and "outer" iterative refinement has an overall running time of

$$\widetilde{\mathcal{O}}_{n,\kappa_T,\frac{1}{\epsilon}}\left(n^{2.5}\kappa_T\right)$$

QRAM accesses and $\widetilde{\mathcal{O}}_{n,\kappa_T,\frac{1}{\epsilon}}\left(n^{4.5}\right)$ arithmetic operations, where κ_T is an upper bound on the condition number of a matrix that is dependent only on input data.

When applied to LO problems, the QIPM requires at most

$$\widetilde{\mathcal{O}}_{n,\kappa_T,\frac{1}{\epsilon}}\left(n^{1.5}\kappa_T\right)$$

QRAM accesses and $\widetilde{\mathcal{O}}_{n,\kappa_T,\frac{1}{\epsilon}}\left(n^{2.5}\right)$ arithmetic operations.

Classically new too! Same complexity using CG as linear system solver.

Complexity Result for IR

Theorem (Mohammadisiahroudi et al. (2023))

IR has quadratic convergence toward the optimal solution set of the SDO problem.

$$X^{(k+1)} \bullet S^{(k+1)} \leq \epsilon (X^{(k)} \bullet S^{(k)})^2$$

Corollary (Mohammadisiahroudi et al. (2023))

IR obtains an $\tilde{\epsilon}$-optimal solution to the primal-dual SDO in at most

$$\mathcal{O}\left(\log\log\left(\frac{1}{\tilde{\epsilon}}\right)\right)$$

- Challenge:
  - At each IR step we need an interior solution for the refining problem
  - One choice is $(\eta^{(k)} X^{(k)}, 0, \eta^{(k)} S^{(k)})$
  - As η^k grows, the initial complementarity increases
  - Cost may increase at each IR iteration

Conclusion

By combining IR techniques, we obtain:

- An exponential speedup over QIPMs for SDO currently in the literature
- Truly polynomial time QIPMs for SDO and LO
- Improve best classic complexity bound

Next steps (ongoing):

- Generalize techniques to other cones of interest
- Apply to other quantum SDO solving frameworks