# Factor Loading Recovery for Smoothed Tetrachoric Correlation Matrices

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# Introduction

# Item Factor Analysis

We often want to conduct exploratory factor analysis on binary response data

- The assumption of continuous outcomes required by the common linear factor model is violated when data are binary
- Tetrachoric correlation matrices (Brown & Benedetti, 1977; Divgi, 1979) are often used to estimate the correlations between the normally-distributed, continuous latent variables often assumed to underlie observed binary data

# Indefinite Tetrachoric Correlation Matrices

- Tetrachoric correlation matrices can be indefinite, particularly when computed from data sets with:
  - Few subjects
  - Many items
  - Extreme items (high factor loadings, extreme item difficulties)

# Proper Correlation Matrices

A correlation matrix, $\mathbf{R}_{p\times p}$ with elements $r_{ij} = r_{ji}$, $i, j \in \{1, \dots, p\}$, is a symmetric matrix that:

1. Has unit diagonal
2. Has all $|r_{ij}| \leq 1$
3. Is positive semidefinite (PSD), having eigenvalues $\lambda_i \geq 0$ $\forall\ i \in \{1, \dots, p\}$

# The Problem with Indefinite Correlation Matrices

$\mathbf{R}_{\text{tet}}$: The tetrachoric correlation matrix

$\mathbf{R}_{\text{Pop}}$: The population correlation matrix estimated by $\mathbf{R}_{\text{tet}}$

Problems:

- An indefinite $\mathbf{R}_{\text{tet}}$ is not in the set of possible $\mathbf{R}_{\text{Pop}}$ matrices
- Some multivariate analysis procedures do not work with indefinite correlation matrices (i.e., maximum likelihood factor analysis)
- Can lead to nonsensical interpretations (e.g., negative component variance in PCA)

A matrix smoothing algorithm is a procedure that modifies an indefinite correlation matrix to produce a correlation matrix that is at least PSD.

- The Higham Alternating Projections algorithm (APA; Higham, 2002)
- The Bentler-Yuan algorithm (BY; Bentler & Yuan, 2011)
- The Knol-Berger algorithm (KB; Knol & Berger, 1991)

# The Higham Alternating Projections Algorithm (2002)

Intuition: Find the closest PSD correlation matrix ($\mathbf{R}_{\text{APA}}$) to a given indefinite correlation matrix ($\mathbf{R}\_$) by iteratively projecting between two sets:

- $\mathcal{S}$: The set containing all possible $p \times p$ symmetric matrices that are PSD
- $\mathcal{U}$: The set containing all possible $p \times p$ symmetric matrices that have a unit diagonal

# The Higham Alternating Projections Algorithm (2002)

For symmetric matrix $\mathbf{A} \in \mathbb{R}^{p \times p}$, define two projection functions:

- $P_S(\mathbf{A}) = \mathbf{V}\mathrm{diag}(\max(\lambda_i, 0))\mathbf{V}'$: Project $\mathbf{A}$ onto $\mathcal{S}$ by replacing all negative eigenvalues with zero in the eigendecomposition.
- $P_U(\mathbf{A})$: Project $\mathbf{A}$ onto $\mathcal{U}$ by replacing the diagonal elements of $\mathbf{A}$ with ones.

Initialize $\mathbf{A}_0$ as the indefinite correlation matrix $\mathbf{R}_{\_}$. Repeat the operation

$$\mathbf{A}_{k+1} = P_U(P_S(\mathbf{A}_k))$$

until convergence occurs or the maximum number of iterations is exceeded.

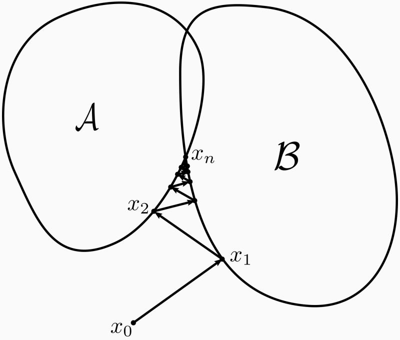





Figure 1: Simplified illustration of the method of alternating projections.

# The Bentler-Yuan Algorithm (2011)

Intuition: Shrink the correlations involving variables with minimum trace factor analysis (MTFA; Jamshidian & Bentler, 1998) estimated communalities $\geq 1$.

# The Bentler-Yuan Algorithm (2011)

$$\mathbf{R}_{\mathrm{BY}} = \Delta \mathbf{R}_0 \Delta + \mathbf{I}$$

$\mathbf{R}_0 = \mathbf{R}\_ - \mathbf{I}$
$\Delta^2$ is a diagonal matrix with elements $\delta_i^2$,

$$\delta_i^2 = \begin{cases} 1 & \text{if } h_i < 1 \\ k/h_i & \text{if } h_i \geq 1. \end{cases}$$

$k \in (0, 1)$ is a constant chosen by the user
$h_i$ is the MTFA communality estimate for the $i$th item

# The Knol-Berger Algorithm (1991)

Intuition: Replace all negative eigenvalues with a small positive constant in the eigenvalue decomposition and then scale the result to a correlation matrix.

# The Knol-Berger Algorithm (1991)

$$\mathbf{R}\_ = \mathbf{V}\Lambda\mathbf{V}'$$

$$\Lambda_+ = \mathsf{diag}[\mathsf{max}(\lambda_i, 0)],\ i \in \{1, \dots, p\}$$

$$\mathbf{R}_{\mathrm{KB}} = [\mathrm{dg}(\mathbf{V}\Lambda_+\mathbf{V}')]^{-1/2}\mathbf{V}\Lambda_+\mathbf{V}'[\mathrm{dg}(\mathbf{V}\Lambda_+\mathbf{V}')]^{-1/2}$$

# Example: Matrix Smoothing Algorithms

$$\mathbf{R}_{\_} = \begin{bmatrix} 1 & 0.48 & 0.64 & 0.48 & 0.65 & 0.83 \\ 0.48 & 1 & 0.52 & 0.23 & 0.68 & 0.75 \\ 0.64 & 0.52 & 1 & 0.60 & 0.58 & 0.74 \\ 0.48 & 0.23 & 0.60 & 1 & 0.74 & 0.80 \\ 0.65 & 0.68 & 0.58 & 0.74 & 1 & 0.80 \\ 0.83 & 0.75 & 0.74 & 0.80 & 0.80 & 1 \end{bmatrix}$$

Eigenvalues: (4.21, 0.77, 0.52, 0.38, 0.18, -0.06)

Communalities: (1.029, 1.122, 0.557, 1.299, 0.823, 0.997)

Variables 1, 2, and 4 have estimated communalities $> 1$.

# Example: Matrix Smoothing Algorithms

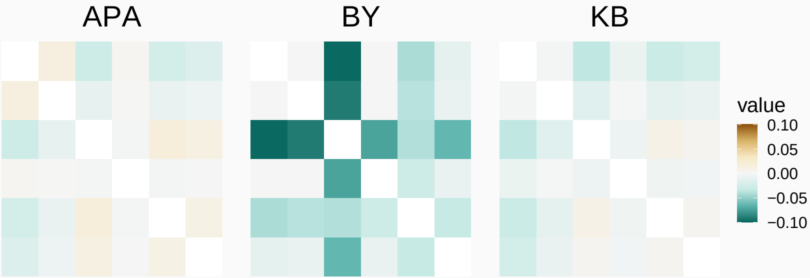



**Figure 2:** Differences between the elements of the $\mathbf{R}_{\mathrm{Sm}}$ and $\mathbf{R}_{\_}$ matrices for the Higham, Bentler-Yuan, and Knol-Berger algorithms.

# Common Factor Model

$$\mathbf{P} = \mathbf{F}\Phi\mathbf{F}' + \Theta^2 \tag{1}$$

- $\mathbf{P}$: $p \times p$ population correlation matrix
- $\mathbf{F}$: $p \times m$ factor loading matrix
- $\Phi$: $m \times m$ factor correlation matrix
- $\Theta^2$: $p \times p$ matrix of unique item variances

# Common Factor Model with Model Approximation Error

$$\mathbf{P} = \mathbf{F}\Phi\mathbf{F}' + \Theta^2 + \mathbf{W}\mathbf{W}' \tag{2}$$

- $\mathbf{P}$: $p \times p$ population correlation matrix
- $\mathbf{F}$: $p \times m$ factor loading matrix
- $\Phi$: $m \times m$ factor correlation matrix
- $\Theta^2$: $p \times p$ matrix of unique item variances
- $\mathbf{W}$: $p \times q$ minor factor loading matrix for the $q \gg p$ minor common factors

# Methods

- Major common factors: $m \in \{1, 3, 5, 10\}$
- Items per factor: $p/m \in \{5, 10\}$
- Subjects per item: $N/p \in \{5, 10, 15\}$
- Factor Loading: Loading $\in \{0.4, 0.6, 0.8\}$
- Model Error: $\upsilon_E \in \{0.0, 0.1, 0.3\}$
    - Proportion of uniqueness variance apportioned to minor common factors

Fully-crossed design with 216 unique conditions

## Simulation Procedure

For each of the 216 unique conditions, conduct 1,000 replications of the following steps:

1. Generate binary response data using Equation (1)
2. Compute the tetrachoric correlation matrix
3. If the matrix is PSD, next; Else, smooth using:
    - Higham (2002)
    - Bentler-Yuan (2011)
    - Knol-Berger (1991)
4. For each of the three smoothed correlation matrices and the unsmoothed matrix, estimate factor loadings using:
    - Principal Axes factor extraction (PA)
    - Ordinary Least Squares (OLS)
    - Maximum Likelihood (ML)

# $\mathrm{R}_{\mathbf{Pop}}$ Recovery Criterion

Given two $p \times p$ symmetric matrices, $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$,

$$\mathrm{D}_{\mathrm{s}}(\mathbf{A}, \mathbf{B}) = \sqrt{\sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \frac{\left(a_{ij} - b_{ij}\right)^2}{p(p-1)/2}}.$$

- $\mathbf{R}_{\mathrm{Sm}} \in \{\mathbf{R}_{-}, \mathbf{R}_{\mathrm{APA}}, \mathbf{R}_{\mathrm{BY}}, \mathbf{R}_{\mathrm{KB}}\}$
- $\mathbf{R}_{\mathrm{Pop}} = \mathbf{F}\Phi\mathbf{F}' + \Theta^2 + \mathbf{W}\mathbf{W}'$

Evaluate recovery of $\mathbf{R}_{\mathrm{Pop}}$ using $\mathrm{D}_{\mathrm{s}}(\mathbf{R}_{\mathrm{Pop}}, \mathbf{R}_{\mathrm{Sm}})$

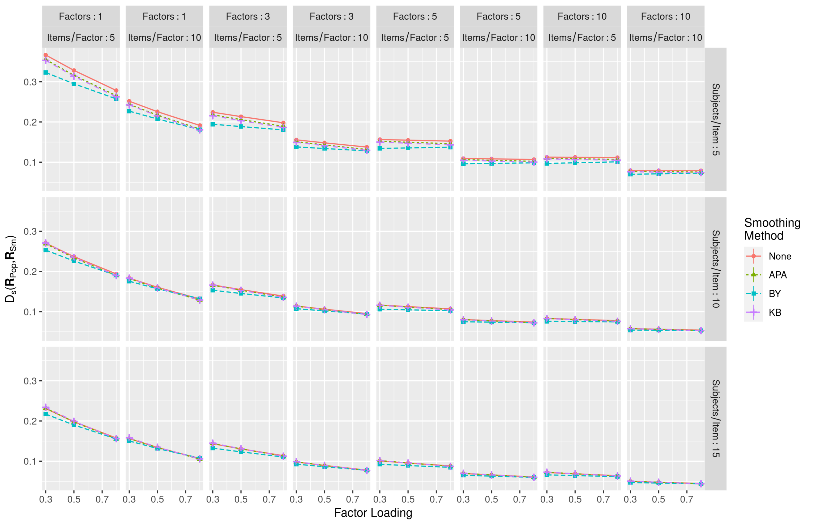
# $\mathbf{F}$ Recovery Criterion

Evaluate how well the factor loading matrix, $\mathbf{F}$, was recovered using:

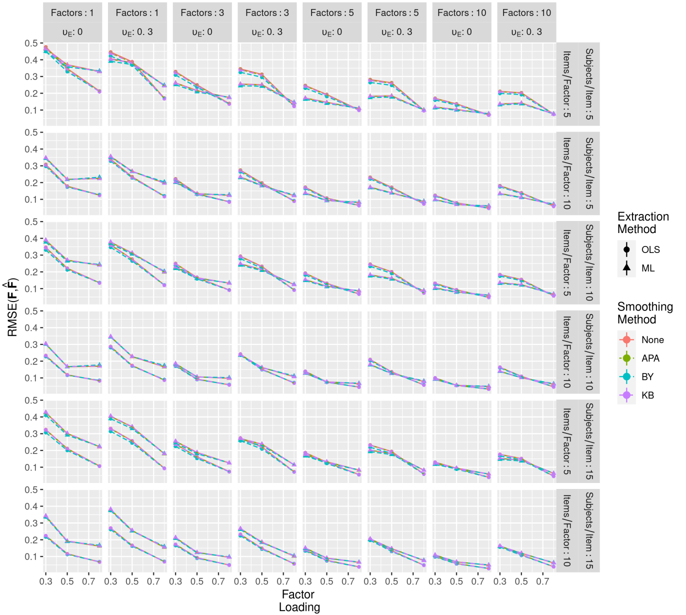
$$\mathrm{RMSE}(\mathbf{F}, \hat{\mathbf{F}}) = \sqrt{\sum_{i=1}^{p} \sum_{j=1}^{m} \frac{\left(f_{ij} - \widehat{f}_{ij}\right)^2}{pm}}$$

# Results

# $\mathrm{R_{Pop}}$ Recovery



# F Recovery



Factors : 1
Factors : 3
Factors : 5
Factors : 10
υE: 0
υE: 0. 3
Subjects/Item : 5
Subjects/Item : 10
Subjects/Item : 15
Items/Factor : 5
Items/Factor : 10
RMSE(F,F̂)
Factor Loading
0.1
0.2
0.3
0.4
0.5
0.3
0.5
0.7
Extraction Method
OLS
ML
Smoothing Method
None
APA
BY
KB

# Appendix

**Algorithm 1:** For an indefinite correlation matrix $\mathbf{R}\_$, find the nearest PSD correlation matrix

Initialize $\mathbf{S}_0 = 0$; $\mathbf{Y}_0 = \mathbf{R}\_$

**for** $k = 1, 2, \dots$ **do**

$\quad \mathbf{Z}_k = \mathbf{Y}_{k-1} - \mathbf{S}_{k-1}$

$\quad \mathbf{X}_k = P_S(\mathbf{Z}_k)$

$\quad \mathbf{S}_k = \mathbf{X}_k - \mathbf{Z}_k$

$\quad \mathbf{Y}_k = P_U(\mathbf{X}_k)$

**end**

The algorithm continues until convergence occurs or the maximum number of iterations is exceeded. If the algorithm converges, $\mathbf{R}_{\mathrm{APA}} = \mathbf{Y}_k$.

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# References ii

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