1 Problem Statement

We will consider a spatial Poisson process on $\mathbb{R}^2$ with rate $\lambda = 1$, i.e. the number of points in region with area A follows a Poisson distribution with rate A. To model increasing larger samples sizes, we look at $[-\frac{t}{2}, \frac{t}{2}]^2$ for increasing larger t.

Problem. If we construct the nearest-neighbor graph on the points contained in $[-\frac{t}{2}, \frac{t}{2}]^2$, what is the expected number of edges that cross the y-axis?

For $N \sim \text{Po}(t^2)$, let $x_1, \ldots, x_N$ be the points in $[-\frac{t}{2}, \frac{t}{2}]^2$ and let $x_{n(i)}$ be $x_i's$ nearest neighbor. Define

$$X_i = \begin{cases} 1 & x_i \text{ and } x_{n(i)} \text{ belong to opposite halves, } x_i \neq x_{n(n(i))} \\ \frac{1}{2} & x_i \text{ and } x_{n(i)} \text{ belong to opposite halves, } x_i = x_{n(n(i))} \\ 0 & \text{otherwise} \end{cases}$$

Let T be the total number of crossings. Then

$$\mathbb{E}T = \mathbb{E}[\mathbb{E}[T|N]] = \mathbb{E}_N\left[\sum_{i=1}^{n} \mathbb{E}X_i\right] = \sum_{n=0}^{\infty} n\mathbb{E}X_1 \frac{t^{2n}e^{-t^2}}{n!} = \mathbb{E}X_1 \sum_{n=0}^{\infty} n \frac{t^{2n}e^{-t^2}}{n!} = t^2\mathbb{E}X_1$$

It remains to find $\mathbb{E}X_1 = P(X_1 = 1) + \frac{1}{2}P(X_1 = \frac{1}{2})$.

2 Finding $\mathbb{E}X_1$

2.1 Finding $P(X_1 = \frac{1}{2})$

For brevity, define

$$O = \mathbb{1}(x_1 \text{ and } x_{n(1)} \text{ belong to opposite halves})$$

$$D = \mathbb{1}(x_1 = x_{n(n(1))})$$

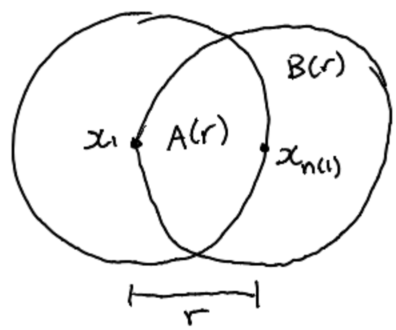
Then $P(X_1 = 1) = P(O = 1 \cap D = 1)$. We condition on the distance d between x_1 and the y-axis. Note $d \sim \text{Unif}([0, \frac{t}{2}])$.

$$\begin{aligned} P(X_1 = 1|d) &= P(O = 1 \cap D = 1|d) \\ &= P(D = 1|O = 1, d)P(O = 1||d) \\ &= \int_d^{\infty} P(D = 1|O = 1, d, r) f(r|O = 1, d)\, dr * P(O = 1|d), \end{aligned} \tag{1}$$

where $r = ||x_1 - x_{n(1)}||$ is the distance between x_1 and its nearest neighbor.

2.1.1 $P(D = 1|O = 1, d, r)$

Given $O = 1$, d, and r, we want to find the probability $x_1 = x_{n(n(1))}$. We condition on $r = ||x_1 - x_{n(1)}||$. Consider the following figure depicting the situation.



In order for $x_1 = x_{n(n(1))}$, i.e. x_1 is $x_{n(1)}$'s nearest neighbor, there cannot be a point contained in a circle of radius r round $x_{n(1)}$. Since we on condition r, we know there does not exist a point in the intersection of the two circles. Hence, $D = 1$ if and only no point lies in the region with area $B(r)$.

$$B(r) = \pi r^2 - r^2 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) = r^2 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right),$$

and thus,

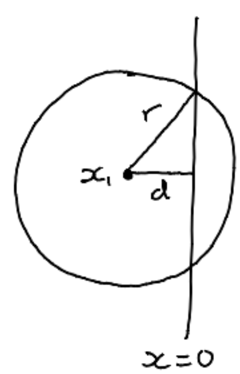
$$P(D = 1 | O = 1, d, r) = e^{-r^2 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right)}.$$

2.1.2 $f(r|O = 1, d)$

According to Bayes' rule,

$$f(r|O = 1, d) = \frac{P(O = 1|r, d) f(r|d)}{P(O = 1|d)}.$$

For $f(O = 1|r, d) f(r|d)$ consider the following situation.



$x_{n(1)}$ lies somewhere along the circle of radius r centered at x_1, which is a distance d away from the y-axis. $O = 1$ if and only if $x_{n(1)}$ lies on the arc on the other side of the y-axis.

Since $x_{n(1)}$ is equally likely to lie anywhere along the circle, this probability is equal to the ratio of the arc length to the circle's circumference,

$$P(O=1|r,d) = \frac{2\cos^{-1}\left(\frac{d}{r}\right) r}{2\pi r} = \frac{\cos^{-1}\left(\frac{d}{r}\right)}{\pi}.$$

r is independent of d. For all $r_0 \geq 0$,

$$P(r \leq r_0) = 1 - e^{-\pi r_0^2} \Rightarrow f(r) = 2\pi r e^{-\pi r^2}.$$

For $P(O=1|d)$, we condition on r again.

$$\begin{aligned} P(O=1|d) &= \int_d^\infty P(O=1|r,d) f(r|d)\, dr \\ &= \int_d^\infty \frac{\cos^{-1}\left(\frac{d}{r}\right)}{\pi} 2\pi r e^{-\pi r^2}\, dr \\ &= 2\int_d^\infty \cos^{-1}\left(\frac{d}{r}\right) r e^{-\pi r^2}\, dr \end{aligned} \tag{2}$$

Putting it all together,

$$\begin{aligned} f(r|O=1,d) &= \frac{P(O=1|r,d) f(r|d)}{P(O=1|d)} \\ &= \frac{\frac{\cos^{-1}\left(\frac{d}{r}\right)}{\pi} * 2\pi r e^{-\pi r^2}}{P(O=1|d)} \\ &= \frac{\cos^{-1}\left(\frac{d}{r}\right) * 2 r e^{-\pi r^2}}{P(O=1|d)} \end{aligned} \tag{3}$$

2.2 Plugging everything in

$$\begin{aligned} P(X_1=1|d) &= \int_d^\infty P(D=1|O=1,d,r) f(r|O=1,d)\, dr * P(O=1|d) \\ &= \int_d^\infty P(D=1|O=1,d,r) \frac{P(O=1|r,d) f(r|d)}{P(O=1|d)}\, dr * P(O=1|d) \\ &= \int_d^\infty P(D=1|O=1,d,r) P(O=1|r,d) f(r|d)\, dr \\ &= \int_d^\infty e^{-r^2\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)} \cos^{-1}\left(\frac{d}{r}\right) 2 r e^{-\pi r^2}\, dr \end{aligned} \tag{4}$$

Since $d \sim \text{Unif}([0, \frac{t}{2}])$, it follows that

$$\begin{aligned} P(X_1=1) &= \int_0^{t/2} P(X_1=1|d) \frac{2}{t}\, dd \\ &= \frac{4}{t}\int_0^{t/2}\int_d^\infty e^{-r^2\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)} \cos^{-1}\left(\frac{d}{r}\right) r e^{-\pi r^2}\, dr\, dd. \end{aligned} \tag{5}$$

2.3 Finding $P(X_1 = 1)$

Note,

$$P(X_1 = 1) = P(O = 1) - P(X_1 = 1).$$

It remains to solve for $P(O = 1)$. We condition on r and d:

$$\begin{aligned} P(O=1) &= \int_0^{t/2} P(O=1|d)\frac{2}{t}\,dd \\ &= \int_0^{t/2}\int_d^{\infty} P(O=1|d,r)f(r|d)\,dr\frac{2}{t}\,dd \\ &= \int_0^{t/2}\int_d^{\infty} \frac{\cos^{-1}\left(\frac{d}{r}\right)}{\pi}2\pi r e^{-\pi r^2}\,dr\frac{2}{t}\,dd \\ &= \frac{4}{t}\int_0^{t/2}\int_d^{\infty}\cos^{-1}\left(\frac{d}{r}\right) r e^{-\pi r^2}\,dr\,dd \end{aligned} \tag{6}$$

3 Solution

If we define

$$\varphi(t) = \frac{4}{t}\int_0^{t/2}\int_d^{\infty}\cos^{-1}\left(\frac{d}{r}\right) r e^{-\pi r^2}\,dr\,dd$$

$$\theta(t) = \frac{4}{t}\int_0^{t/2}\int_d^{\infty} e^{-r^2\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)}\cos^{-1}\left(\frac{d}{r}\right) r e^{-\pi r^2}\,dr\,dd$$

then

$$\mathbb{E}X_1 = (\varphi(t) - \theta(t)) - \frac{1}{2}\theta(t) = \varphi(t) - \frac{1}{2}\theta(t).$$

It follows that

$$\mathbb{E}T = t^2\mathbb{E}X_1 = t^2\left[\varphi(t) - \frac{1}{2}\theta(t)\right].$$

4 Generalizing to Higher Dimension

If we generalize to higher dimension, the following functions are required:

- $s_n(d,r) =$ the ratio of surface areas in an n-dimension hypersphere
- $f_n(r) =$ the density of distance to the nearest neighbor
- $c_n(r) =$ the volume of the crescent created by two overlapping n-dimensional hyperspheres

In two dimensions, these functions are

- $s_2(d,r) = \frac{2r\cos^{-1}\left(\frac{d}{r}\right)}{2\pi r}$,
- $f_2(r) = 2\pi r e^{-\pi r^2}$, and
- $c_2(r) = -r^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$.

Using these functions,

- $\varphi_n(t) = \frac{2}{t} \int_0^{t/2} \int_0^\infty s_n(d, r) f_n(r) \, dr \, dd$ and
- $\theta_n(t) = \frac{2}{t} \int_0^{t/2} \int_d^\infty e^{-c_n(r)} s_n(d, r) f_n(r) \, dr \, dd$.

4.1 s_n, f_n, and c_n in arbitrary dimension

For s_n, see *Concise Formulas for the Area and Volume of a Hyperspherical Cap.*

$$s_n(d, r) = \frac{1}{2} I_{1-\left(\frac{d}{r}\right)^2}\left(\frac{n-1}{2}, \frac{1}{2}\right),$$

where $I_z(z, b) = \frac{B_z(a,b)}{B(z,b)}$ is the regularized Beta function.

For $f_n(r)$,

$$F_n(r) = 1 - e^{-V_n(r)}$$

$$f_n(r) = \frac{\partial V_n(r)}{\partial r} e^{-V_n(r)},$$

where $V_n(r) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$ is the volume of a n-dimensional hypersphere with radius r.

For $c_n(r)$, we first calculate the volume of the intersection of two overlapping n-dimensional hyperspheres. This regions consists of two hyperspherical caps with $d = \frac{r}{2}$. According to *Concise Formulas for the Area and Volume of a Hyperspherical Cap*, each of these hyperspherical caps has volume

$$\frac{1}{2} V_n(r) I_{\frac{3}{4}}\left(\frac{n+1}{2}, \frac{1}{2}\right),$$

so the overlapping volume is

$$V_n(r) I_{\frac{3}{4}}\left(\frac{n+1}{2}, \frac{1}{2}\right).$$

Therefore, the crescent volume is

$$c_n(r) = V_n(r) - V_n(r) I_{\frac{3}{4}}\left(\frac{n+1}{2}, \frac{1}{2}\right) = V_n(r)\left[1 - I_{\frac{3}{4}}\left(\frac{n+1}{2}, \frac{1}{2}\right)\right].$$