Chapter 8 Smoothing Methods on Time Series

We now examine another areas of time series forecasting similar to weighted moving averages, often called *exponential smoothing methods*. One of the major distinctions between regressions methods on time series data and smoothing methods is that regressions give equal weight to all of the sample data whereas smoothing methods give decreasing weight to older, more distant data.

Chapter 8

An economist is an expert who will know tomorrow why the things he predicted yesterday didn't happen today.

– Evan Esar

Single Exponential Smoothing

A popular, widely used time series smoothing and forecasting method is ***single exponential smoothing*** (also called ***one-parameter exponential smoothing***). Exponential smoothing is a form of weighted moving average in that recent observations carry more weight than older observations.

Exponential smoothing is based on the concept that a new, one step-ahead, forecast can be determined from the ***previous forecast*** and the ***error of the previous forecast***. We illustrate this concept below.

Suppose we have the following series.

Period	Y_t	*Forecast*
1	354	
2	368	
3	329	
4		
5		

and suppose a one-step ahead forecast for period 4, made at period 3, is 352.5. We denote this by

$$\hat{Y}_3(1) = 352.5.$$

Period	Y_t	*Forecast*
1	354	

2	368	
3	329	
4		352.5
5		

Then suppose period 4's actual value is 389

Period	Y_t	*Forecast*
1	354	
2	368	
3	329	
4	389	352.5
5		

The forecast error is

$$e_3(1) = Y_4 - \hat{Y}_3(1)$$
$$= 389 - 352.5 = 36.5$$

We *underforecasted* by 36.5

Period	Y_t	*Forecast*	*Error*
1	354		
2	368		
3	329		
4	389	352.5	36.5
5			

Since we had underforecasted we will adjust upward our next forecast, based on the error of forecast. We adjust our next forecast upward by 20% of the forecast error; 20% of 36.5 = 7.3

Hence, the forecast for period 5 is the old forecast, 352.5, adjusted upwarded by 7.3,

Forecast for period 5

$$352.5 + 7.3 = 359.8$$
$$\hat{Y}_4(1) = 359.8$$

Period	Y_t	*Forecast*	*Error*
1	354		
2	368		
3	329		
4	389	352.5	36.5
5		359.8	

The actual value for period 5 is 350, so the one-step ahead period 5 forecast error is:

$$e_4(1) = \mathrm{Y}_5 - \hat{\mathrm{Y}}_4(1)$$
$$= 350 - 359.8$$
$$= -9.8$$

We *overforecasted* by 9.8

Period	Y_t	*Forecast*	*Error*
1	354		

2	368		
3	329		
4	389	352.5	36.5
5	350	359.8	- 9.8

The forecast for period 6 then is adjusted by 20% of the forecast error,
20% of -9.8 $=$ -1.96

Forecast for period 6

$$359.8 + (-1.96) = 357.8$$

$$\hat{Y}_5(1) = 357.8$$

And so on.

Each new forecast, $\hat{Y}_t(1)$, is based on the preceding forecast, $\hat{Y}_{t-1}(1)$, plus an adjustment for forecast error.

If we denote the adjustment factor by α, we have

$$\hat{Y}_t(1) = \hat{Y}_{t-1}(1) + \alpha e_{t-1}(1)$$

Algebraically, we re-arrange the last equation as, ***exponential smoothing***

Single Exponential Smoothing, one-step ahead forecast

$$\boldsymbol{\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha)\hat{Y}_{t-1}(1)} \qquad \textbf{8.1}$$

In the case of $\alpha = .2$, we have

Single Exponential Smoothing Forecast, one-step ahead forecast, with $\alpha = .2$

$$\hat{Y}_t(1) = .2Y_t + .8\hat{Y}_{t-1}(1)$$

The new forecast is the weighted average of the current observation Y_t and the previous forecast $\hat{Y}_{t-1}(1)$.

Returning to equation 8.1 we show why this is called *exponential smoothing*.

Since $\quad \hat{Y}_t(1) = \alpha \mathrm{Y}_t + (1 - \alpha)\hat{\mathrm{Y}}_{t-1}(1) \qquad$ 8.2

we shift back by one period

$$\hat{Y}_{t-1}(1) = \alpha \mathrm{Y}_{t-1} + (1 - \alpha)\hat{\mathrm{Y}}_{t-2}(1) \qquad 8.3$$

by substituting equation 8.2 in for $\hat{Y}_{t-1}(1)$ in equation 8.1 we have

$$\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha)\left(\alpha \mathrm{Y}_{t-1} + (1 - \alpha)\hat{\mathrm{Y}}_{t-2}(1)\right) \qquad 8.4$$

Rearranging

$$\hat{Y}_t(1) = \alpha \mathrm{Y}_t + \alpha(1 - \alpha)\mathrm{Y}_{t-1} + (1 - \alpha)^2\hat{\mathrm{Y}}_{t-2}(1) \qquad 8.5$$

Using equation 8.1 again as a format, $\hat{Y}_{t-2}(1)$ is written as

$$\hat{Y}_{t-2}(1) = \alpha \mathrm{Y}_{t-2} + (1 - \alpha)\hat{\mathrm{Y}}_{t-3}(1)$$

By substituting for $\hat{Y}_{t-2}(1)$ in equation (8.11) we have

$$\hat{Y}_t(1) = \alpha \mathrm{Y}_t + (1-\alpha)\alpha \mathrm{Y}_{t-1} + (1-\alpha)^2\left(\alpha \mathrm{Y}_{t-2} + (1-\alpha)\hat{\mathrm{Y}}_{t-3}(1)\right) \qquad 8.6$$

$$\hat{Y}_t(1) = \alpha \mathrm{Y}_t + \alpha(1-\alpha)\mathrm{Y}_{t-1} + \alpha(1-\alpha)^2\mathrm{Y}_{t-2} + (1-\alpha)^3\hat{\mathrm{Y}}_{t-3}(1) \qquad 8.7$$

And so on.

Since α is less than 1, α, $\alpha(1-\alpha)$, $\alpha(1-\alpha)^2 \ldots$ are exponentially decreasing weights applied to previous observations.

$$\hat{Y}_t(1) = \alpha \mathrm{Y}_t + \alpha(1-\alpha)\mathrm{Y}_{t-1} + \alpha(1-\alpha)^2\mathrm{Y}_{t-2} + (1-\alpha)^3\mathrm{Y}_{t-3}(1) + (1-\alpha)^4\hat{\mathrm{Y}}_{t-4}(1) + \ldots \qquad 8.8$$

In the case of $\alpha = .2$, we have

$$\hat{Y}_t(1) = .2Y_t + .16\,Y_{t-1} + .128Y_{t-2} + .1024Y_{t-3}(1) + \ldots \qquad 8.9$$

This is an infinite weighted moving average, and it can be shown that

$$\alpha + \alpha(1-\alpha) + \alpha(1-\alpha)^2 + \alpha(1-\alpha)^3 + \alpha(1-\alpha)^4 + \ldots = 1 \qquad 8.10$$

The attraction of exponential smoothing is that is not necessary to use equation 8.7. Rather equation 8.1

$$\hat{Y}_t(1) = \alpha \mathrm{Y}_t + (1-\alpha)\hat{Y}_{t-1}(1) \qquad 8.1$$

is equivalent to the infinite sum

$$\hat{Y}_t(1) = \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2Y_{t-2} + \alpha(1-\alpha)^3Y_{t-3} + \alpha(1-\alpha)^4Y_{t-4} + \ldots$$

With equation 8.1, all that is needed is the current observation and the previous forecaast to make a new forecast.

Starting Value for Exponential Smoothing

Exponential Smoothing thus only needs a starting value for forecast and a choice for α. For our choice for α we shall use $\alpha = .20$.

For a starting value, $\hat{Y}_0(1)$, we shall use the average of the 30 observations of data, $\overline{Y} = 356.53$. Thus, an exponential smoothing forecast for Period 2

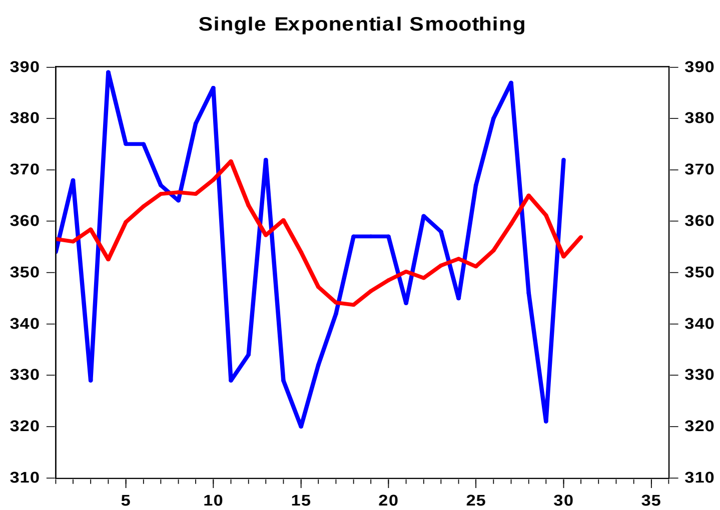
$$\hat{Y}_1(1) = .2Y_1 + (1-.2)\hat{Y}_0(1)$$
$$\hat{Y}_1(1) = .2(354) + .8(356.53) = 356.02$$

For Period 3 $\quad \hat{Y}_2(1) = .2(368) + .8(356.02) = 358.42$

And so on, recursively through all 30 observations of data.

Table 8.1

Period	*Actual*	*One-Step Ahead Forecast*
t	Y_t	$\hat{Y}_t(1)$
1	354	356.53
2	368	356.02
3	329	358.42
4	389	352.54
5	375	359.83
6	375	362.86
7	367	365.29
8	364	365.60
9	379	365.31
10	386	368.04
11	329	371.64
12	334	363.11
13	372	357.29
14	329	360.23
15	320	353.98
16	332	347.19
17	342	344.15
18	357	343.72
19	357	346.38
20	357	348.50
21	344	350.20
22	361	348.96
23	358	351.37
24	345	352.69
25	367	351.16
26	380	354.32
27	387	359.46
28	346	364.97
29	321	361.17
30	372	353.14
31		356.91



The Choice of α in Single Exponential Smoothing

In the Single Exponential Smoothing equation

$$\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha)\hat{Y}_{t-1}(1)$$

α is called the ***smoothing constant*** (the one-parameter) and is always between 1 and 0.

The one-parameter smoothing constant, $\boldsymbol{\alpha}$

$$\boxed{\mathbf{0 \leq \boldsymbol{\alpha} \leq 1}} \qquad 8.10$$

To understand the effect of different α's, let us consider the extreme values for α.

If $\alpha = 1$, then equation (8.1) becomes

$$\hat{Y}_t(1) = 1Y_t + (1 - 1)\hat{Y}_{t-1}(1)$$

$$\hat{Y}_t(1) = Y_t$$

This means that the forecast for the next period is simply the observation of the current period. This is called ***naive forecasting***, being the simplest forecast possible, in which we forecast next period entirely on the current period.

Time series which have major jumps or changes are best modelled by exponential smoothing in which the alpha is "large," around .4 to .5.

If $\alpha = 0$, then equation (8.1) becomes

$$\hat{Y}_t(1) = 0Y_t + (1 - 0)\hat{Y}_{t-1}(1)$$

$$\hat{Y}_t(1) = \hat{Y}_{t-1}(1)$$

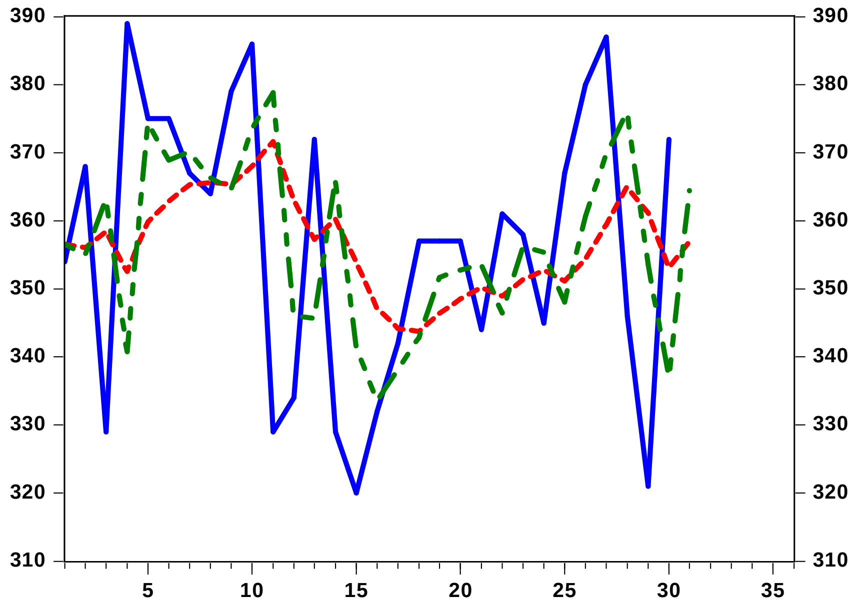
This means that the forecast for the next period is the previous period's forecast.

$$\hat{Y}_t(1) = \overline{Y}$$

in which the data varies slowly about some common mean.

Thus, for slowly changing data a low alpha from .05 to .2 is sufficient. For data that changes rapidly, a high alpha of .4 to .5 will adjust the new forecast adequately, as illustrated with Figure 8-2.

Single Exponential Smoothing with Higher Alpha Values



While a visual display of data will provide a good indication of the choice of α, the choice can be refined by computing the SSE for different α's.

The Initial Forecast Value

To forecast for Period 2 with exponential smoothing requires a forecast for Period 1, $\hat{Y}_0(1)$, i.e.

$$\hat{Y}_1(1) = \alpha Y_1 + (1 - \alpha)\hat{Y}_0(1)$$

If we set $\hat{Y}_0(1) = \overline{Y}$, (In this example, $\overline{Y} = 356.53$) we have an adequate starting value.

Recursively, we calculate

$$\hat{Y}_t(1) = .2Y_t + .8\hat{Y}_{t-1}(1)$$

So, starting with

$$\hat{Y}_0(1) = 356.53$$
$$\hat{Y}_1(1) = .2Y_1 + .8\hat{Y}_0(1)$$
$$\hat{Y}_2(1) = .2Y_2 + .8\hat{Y}_1(1)$$
$$\hat{Y}_3(1) = .2Y_3 + .8\hat{Y}_2(1)$$

and so on.

Table 8.1

		$\alpha = .2$		
Period	*Actual*	*Forecast*	*Error*	*Squared Error*
t	Y_t	$\hat{Y}_t(1)$	$e_t(1)$	$e_t^2(1)$
1	354	356.53	-2.53	6.40
2	368	356.02	11.98	143.42
3	329	358.42	-29.42	865.49
4	389	352.54	36.46	1,329.67
5	375	359.83	15.17	230.18
6	375	362.86	12.14	147.32
7	367	365.29	1.71	2.92
8	364	365.60	31.63	2.66
9	379	365.31	13.69	187.53
10	386	368.04	17.96	322.40
11	329	371.64	-42.64	1,817.80
12	334	363.11	-29.11	847.30
13	372	357.29	14.71	216.48
14	329	360.23	-31.23	975.28
15	320	353.98	-33.98	1,154.88
16	332	347.19	15.19	230.64
17	342	344.15	-2.15	4.62
18	357	343.72	13.28	176.37
19	357	346.38	10.62	112.88
20	357	348.50	8.50	72.24
21	344	350.20	6.20	38.45
22	361	348.96	12.04	144.95
23	358	351.37	6.63	43.98
24	345	352.69	-7.69	59.21
25	367	351.16	15.84	25.04
26	380	354.32	25.68	659.23
27	387	359.46	27.54	758.47
28	346	364.97	-18.97	359.77
29	321	361.17	-40.17	1,613.96
30	372	353.14	18.86	355.72
31		356.91		

$$\mathrm{SSE} = \sum e_t^2(1) = 13{,}131$$

The sum of the squared errors, *SSE*, is 13,131. For different values of α the *SSE* will change so that *SSE* is a function of α.

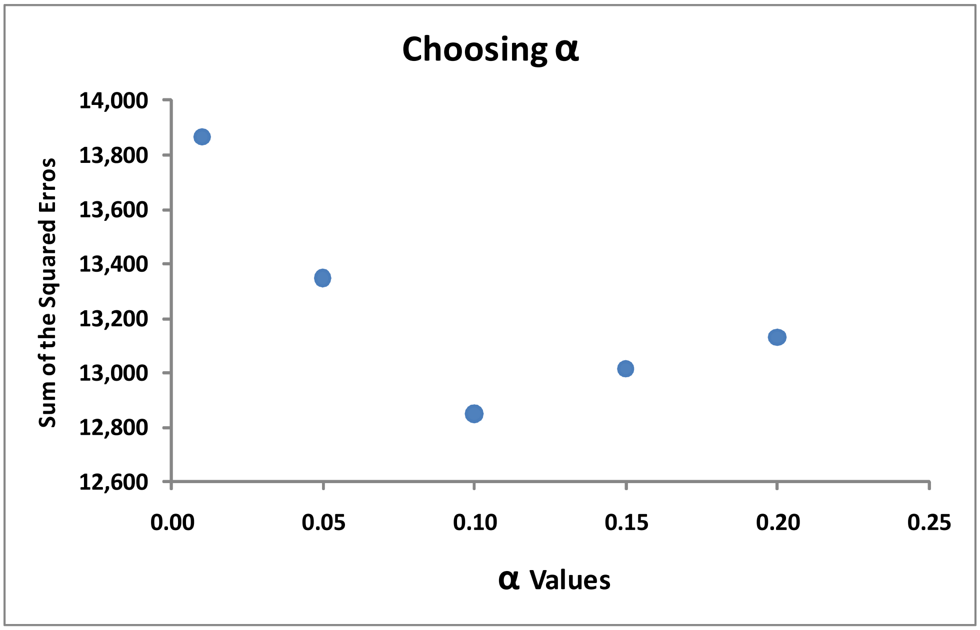
$$\boldsymbol{SSE(\alpha) = \sum e_t^2(1)} \qquad 8.11$$

If we recursively construct the one-step ahead forecasts with $\alpha = .15$, we get $SSE = 13{,}104$. With $\alpha = .05$, $SSE = 13{,}348$. Table 8.36 lists different values for and the corresponding *SSE*.

Table 8.2

α	*SSE*
.01	13,867
.05	13,348
.10	12,850
.15	13,014
.20	13,131

Figure 8-3 is a simple chart of these numbers.



The table and chart suggest that an alpha of about .10 would fit this data best, and therefore would be used for forecasting.

Forecasting and Confidence Intervals when using Single Exponential Smoothing

Forecasting for more than one-step ahead

Single exponential smoothing is also only a one-step ahead forecasting procedure, so that multi-period forecasts are just the repeats of the one-step ahead forecast. On Table 8.10 above the last forecast, for period 31 is 356.91, $\hat{Y}_{30}(1) = 356.91$. This will be the repeated forecast for periods 32, 33, 34, and so on.

Confidence Intervals of Forecast for Single Exponential Smoothing

For each step-ahead that is forecasted there is a slightly different formula for the variance of forecast.

The one step ahead forecast variance $\sigma^2_{\hat{Y}_t(1)}$ is just the *MSE*, σ^2_ϵ, where

One-step ahead forecast variance

$$\boldsymbol{\sigma^2_\epsilon = \frac{SSE}{n-(k+1)}} \tag{8.12}$$

One-step ahead forecast variance for single exponential smoothing

$$\boldsymbol{\sigma^2_{\hat{Y}_t(1)} = \sigma^2_\epsilon} \tag{8.13}$$

Here, $k = 1$, since we are considering only α (the one-parameter) exponential smoothing.

Hence,

$$\sigma^2_{\hat{Y}_t(1)} = \sigma^2_\epsilon = \frac{SSE}{n-(k+1)}$$

$$\sigma^2_{\hat{Y}_t(1)} = \frac{13{,}131}{30-(1+1)}$$

$$\sigma^2_{\hat{Y}_t(1)} = 468.96$$

$$\sigma_{\hat{Y}_t(1)} = \sqrt{468.96} = 21.66$$ the standard error of forecast

Hence, with 28 degrees of freedom, a 95 percent confidence interval around $\hat{Y}_{30}(1) = 356.91$ is

$$\hat{Y}_{30}(1) \pm (2.048)(\sigma_{\hat{Y}_t(1)})$$

$$356.91 \pm (2.048)(21.66)$$

$$356.91 \pm 44.35$$

Thus a 95 percent confidence interval of forecast for the one-step ahead exponential smoothing forecast is

$$312.56 \leq \hat{Y}_{30}(1) \leq 401.26$$

Two-step ahead forecast variance:

Two-step ahead forecast variance for single exponential smoothing

$$\boldsymbol{\sigma^2_{\hat{Y}_t(2)} = \sigma^2_\epsilon\left(1 + \alpha^2\right)} \tag{8.14}$$

In this example, $\sigma^2_\epsilon = 468.96$, $\alpha = .2$

$$\sigma^2_{\hat{Y}_t(2)} = \sigma^2_\epsilon\left(1 + \alpha^2\right)$$

$$\sigma^2_{\hat{Y}_t(2)} = 468.96\left(1 + .2^2\right)$$

$$\sigma^2_{\hat{Y}_t(2)} = 468.96\left(1.04\right) = 487.72$$

The standard error of forecast, 2 steps ahead

$$\sigma_{\hat{Y}_t(2)} = \sqrt{487.72} = 22.08$$

The corresponding 95 percent confidence interval around $\hat{Y}_{30}(2) = 356.91$ is

$$\hat{Y}_{30}(2) \pm (2.048)(\sigma_{\hat{Y}_t(2)})$$

$$356.91 \pm (2.048)(22.08)$$

$$356.91 \pm 45.23$$

Thus a 95 percent confidence interval of forecast for the two-step ahead exponential smoothing forecast is

$$311.68 \leq \hat{Y}_{30}(2) \leq 402.14$$

Three-step ahead forecast variance:

Three-step ahead forecast variance for single exponential smoothing

$$\boldsymbol{\sigma^2_{\hat{Y}_t(3)} = \sigma^2_\epsilon\left(1 + 2\alpha^2\right)} \tag{8.15}$$

Three-step ahead forecast standard error for single exponential smoothing

$$\boldsymbol{\sigma_{\hat{Y}_t(3)} = \sqrt{\sigma^2_\epsilon\left(1 + 2\alpha^2\right)}} \tag{8.16}$$

The corresponding 95 percent confidence interval of forecast calculation.

$$\hat{Y}_{30}(3) \pm (2.048)(\sigma_{\hat{Y}_t(3)})$$

$$\sigma_{\hat{Y}_t(3)} = \sqrt{468.96\left(1 + (2 \times .2^2)\right)}$$

$$\sigma_{\hat{Y}_t(3)} = \sqrt{468.96\left(1.08\right)}$$

$$\sigma_{\hat{Y}_t(3)} = 22.51$$

$$\hat{Y}_{30}(3) \pm (2.048)(\sigma_{\hat{Y}_t(3)})$$

$$356.91 \pm (2.048)(22.51)$$
$$356.91 \pm 46.09$$

Thus a 95 percent confidence interval of forecast for the three-step ahead exponential smoothing forecast is

$$310.83 \leq \hat{Y}_{30}(3) \leq 403.00$$

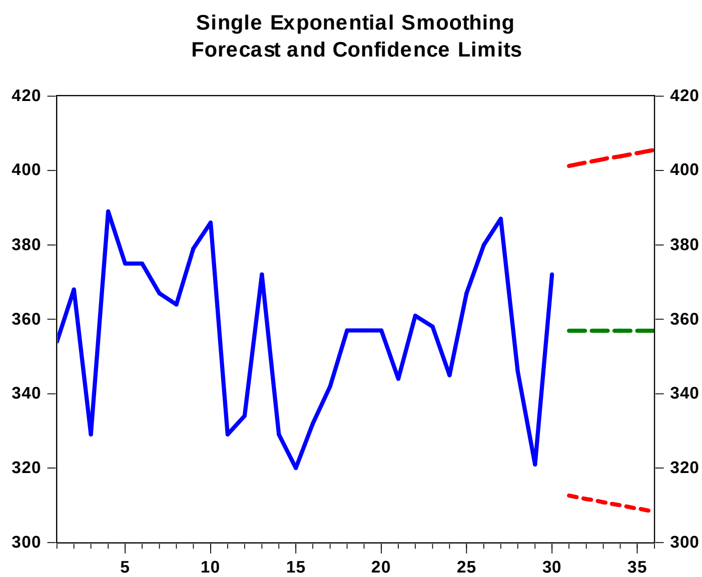
And so on.

ℓ-step ahead forecast variance for single exponential smoothing

$$\boldsymbol{\sigma^2_{\hat{Y}_t(\ell)} = \sigma^2_\epsilon\left(1 + (\ell - 1)\alpha^2\right)} \tag{8.17}$$

Table 8.3

			Confidence Interval	
Period	***Actual***	***Forecast***	***Lower* 95%**	***Upper* 95%**
⋮	⋮	⋮	⋮	⋮
28	346	364.97		
29	321	361.17		
30	372	353.14		
31		356.91	312.56	401.26
32		356.91	311.68	402.14
33		356.91	310.82	403.00
34		356.91	309.97	403.85
35		356.91	309.14	404.68
36		356.91	308.33	405.50

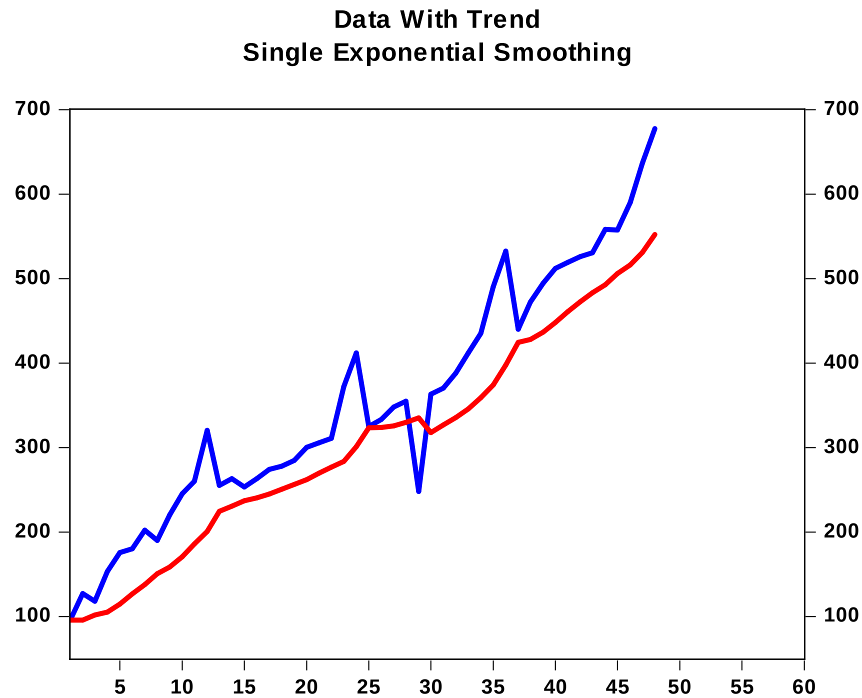


Given the ease of updating and minimum of data storage required, exponential smoothing is widely used by business, especially if large numbers of forecasts must be made frequently, such as inventory levels of many items.

Exponential smoothing can also be automatic. Once a *SSE* minimization program has been written, any time series can be automatically forecasted with exponential smoothing. Being so automatic is considered an advantage, but it can also be deceptively incorrect in that all of the time series are treated alike.

Double Exponential Smoothing

The preceding discussion of single exponential smoothing has assumed that the time series has a constant mean or slowly changing mean, data with no trend. If the series has addative trend, single exponential smoothing will follow the trend, but tends to lag the data series, as illustrated by Figure 8-4.



The method of ***Double Exponential Smoothing*** takes into account the trend of the series. To distinguish between single and double exponential smoothing, we shall denote them by

single exponential smoothing:

$$\hat{Y}_t^{[1]}(1) = \alpha Y_t + (1 - \alpha)\hat{Y}_{t-1}^{[1]}(1)$$

double exponential smoothing:

$$\hat{Y}_t^{[2]}(1) = \alpha \hat{Y}_{t-1}^{[1]}(1) + (1 - \alpha)\hat{Y}_{t-1}^{[2]}(1)$$

Thus, double exponential smoothing is a smoothing of single exponential smoothing. Double exponential smoothing is a smoothing of a smoothing. (It is very smooth.)

In other words, we take a data series and smooth it with single exponential smoothing. Then the smoothed series is smoothed again. As our example we shall we consider the Sales data of Table 8.3 and Figure 8-5.

Table 8.4

Period	*Year* 1	*Period*	*Year* 2	*Period*	*Year* 3	*Period*	*Year* 4
1	95.5	13	255.0	25	325.0	27	440.0
2	127.0	14	263.0	26	333.0	38	472.0
3	118.0	15	253.0	27	348.0	39	494.5
4	153.0	16	263.0	28	355.0	40	512.0
5	175.5	17	274.0	29	248.0	41	519.0
6	180.0	18	278.0	30	363.0	42	526.0
7	202.0	19	284.5	31	370.0	43	530.5
8	190.0	20	300.0	32	388.0	44	558.0
9	220.5	21	305.5	33	412.0	45	557.5
10	245.0	22	310.5	34	435.0	46	590.0
11	260.0	23	372.0	35	490.5	47	636.5
12	320.0	24	412.0	36	532.5	48	677.5

Using an $\alpha = .2$ we have,

single exponential smoothing:

$$\hat{Y}_t^{[1]}(1) = .2Y_t + .8\hat{Y}_{t-1}^{[1]}(1)$$

double exponential smoothing:

$$\hat{Y}_t^{[2]}(1) = .2\hat{Y}_{t-1}^{[1]}(1) + .8\hat{Y}_{t-1}^{[2]}(1)$$

Because the data has trend, for a starting value we shall use the simple average of the first five observations, $\hat{Y}_0(1) = 133.8$.

Table 8.5

Period	*Actual*	*Single Exponential Smoothing*	*Double Exponential Smoothing*
1	95.5	133.8	133.8
2	127.0	126.1	133.8
3	118.0	126.3	132.3
4	153.0	124.6	131.1
5	175.5	130.3	129.8
6	180.0	139.4	129.9
7	202.0	147.5	131.8
8	190.0	158.4	134.9
9	220.5	164.7	139.6
10	245.0	175.9	144.6
11	260.0	189.7	150.9
12	320.0	203.8	158.6
13	255.0	227.0	167.7
14	263.0	232.6	179.5
15	253.0	238.7	190.1
16	263.0	241.5	199.9
17	274.0	245.8	208.2
18	278.0	251.5	215.7
19	284.5	256.8	222.9
20	300.0	262.3	229.7
21	305.5	269.9	236.2

22	310.5	277.0	242.9
23	372.0	283.7	249.7
24	412.0	301.4	256.5
25	325.0	323.5	265.5
26	333.0	323.8	277.1
27	348.0	325.6	286.4
28	355.0	330.1	294.3
29	248.0	335.1	301.4
30	363.0	317.7	308.2
31	370.0	326.7	310.1
32	388.0	335.4	313.4
33	412.0	345.9	317.8
34	435.0	359.1	323.4
35	490.5	374.3	330.6
36	532.5	397.5	339.3
37	440.0	424.5	351.0
38	472.0	427.6	365.7
39	494.5	436.5	378.1
40	512.0	448.1	389.7
41	519.0	460.9	401.4
42	526.0	472.5	413.3
43	530.5	483.2	425.2
44	558.0	492.7	436.8
45	557.5	505.7	447.9
46	590.0	516.1	459.5
47	636.5	530.9	470.8
48	677.5	552.0	482.8
49		577.1	496.7

For a one-step ahead forecast, we use

Double Exponential Smoothing, one-step ahead forecast

$$\boxed{\hat{Y}_t(1) = 2\hat{Y}_t^{[1]}(1) - \hat{Y}_t^{[2]}(1) \qquad 8.18}$$

Examples

A one-step ahead forecast for period 2 is

$$\hat{Y}_1(1) = 2\hat{Y}_1^{[1]}(1) - \hat{Y}_1^{[2]}(1) = 2(126.1) - 133.8 = 118.5$$

A one-step ahead forecast for period 3 is

$$\hat{Y}_2(1) = 2\hat{Y}_2^{[1]}(1) - \hat{Y}_2^{[2]}(1) = 2(126.3) - 132.3 = 120.4$$

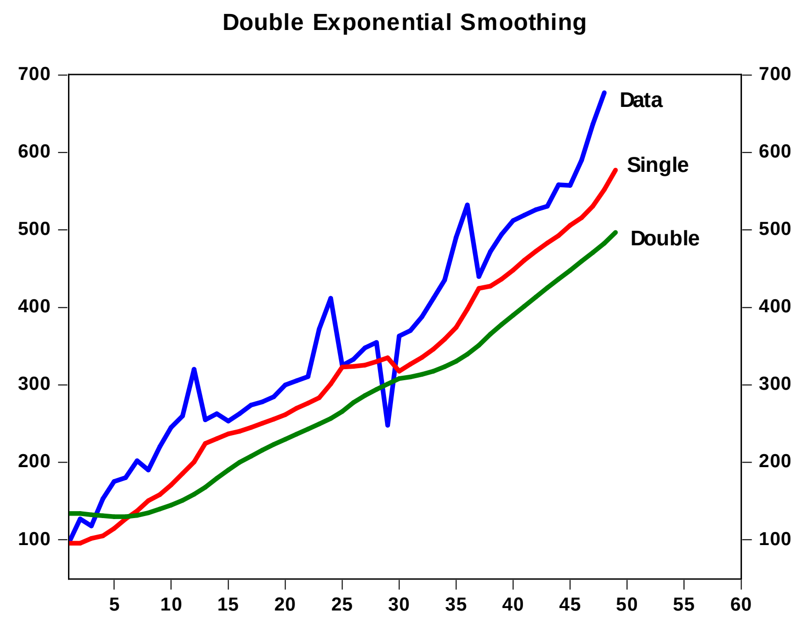
$$\vdots$$

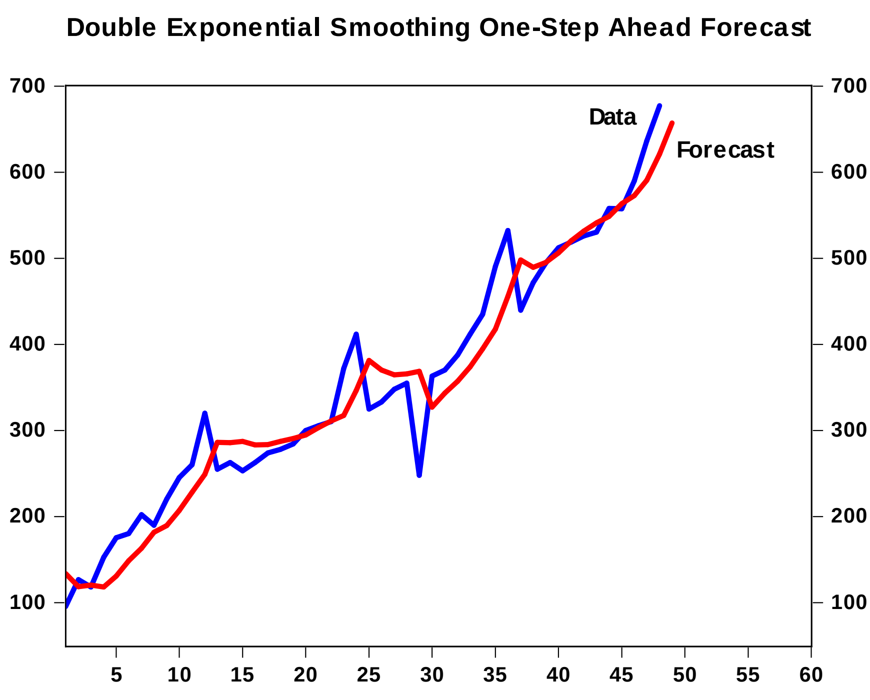
And so on, to a one-step ahead forecast for period 49

$$\hat{Y}_{48}(1) = 2\hat{Y}_{48}^{[1]}(1) - \hat{Y}_{48}^{[2]}(1) = 2(577.1) - 496.7 = 657.2$$

Table 8.6

Period	*Actual*	*Single Exponential Smoothing*	*Double Exponential Smoothing*	*One-Step Ahead Forecast*
1	95.5	133.8	133.8	133.8
2	127.0	126.1	133.8	118.5
3	118.0	126.3	132.3	120.4
4	153.0	124.6	131.1	118.2
5	175.5	130.3	129.8	130.8
6	180.0	139.4	129.9	148.8
7	202.0	147.5	131.8	163.2
8	190.0	158.4	134.9	181.8
9	220.5	164.7	139.6	189.8
10	245.0	175.9	144.6	207.1
11	260.0	189.7	150.9	228.5
12	320.0	203.8	158.6	248.9
13	255.0	227.0	167.7	286.3
14	263.0	232.6	179.5	285.7
15	253.0	238.7	190.1	287.2
16	263.0	241.5	199.9	283.2
17	274.0	245.8	208.2	283.5
18	278.0	251.5	215.7	287.2
19	284.0	256.8	222.9	290.7
20	300.0	262.2	229.7	294.8
21	305.5	269.8	236.2	303.4
22	310.5	276.9	242.9	311.0
23	372.0	283.6	249.7	317.6
24	412.0	301.3	256.5	346.1
25	325.0	323.4	265.4	381.4
26	333.0	323.8	277.0	370.5
27	348.0	325.6	286.4	364.8
28	355.0	330.1	294.2	365.9
29	248.0	335.1	301.4	368.7
30	363.0	317.7	308.1	327.2
31	370.0	326.7	310.0	343.4
32	388.0	335.4	313.4	357.4
33	412.0	345.9	317.8	374.0
34	435.0	359.1	323.4	394.8
35	490.5	374.3	330.5	418.0
36	532.5	397.5	339.3	455.8
37	440.0	424.5	350.9	498.1
38	472.0	427.6	365.7	489.6
39	494.5	436.5	378.1	494.9
40	512.0	448.1	389.7	506.5
41	519.0	460.9	401.4	520.3
42	526.0	472.5	413.3	531.7
43	530.5	483.2	425.1	541.3
44	558.0	492.7	436.8	548.6
45	557.5	505.7	447.9	563.5
46	590.0	516.1	459.5	572.7
47	636.5	530.9	470.8	590.9
48	677.5	552.0	482.8	621.2
49		577.1	496.7	657.2





We see how the one-step ahead double exponential smoothing method tracks the data much closer than single exponential smoothing.

Forecasting and Confidence Intervals when using Double Exponential Smoothing

Double Exponential Smoothing Forecast

$$\hat{Y}_{T+1}(\ell) = \hat{\beta}_0 + \hat{\beta}_1 \ell \qquad 8.19$$

where $$\hat{\beta}_0 = \left(2\hat{Y}_T^{[1]}(1) - \hat{Y}_T^{[2]}(1)\right) \qquad 8.20$$

and $$\hat{\beta}_1 = \left(\frac{\alpha}{1-\alpha}\right)\left(\hat{Y}_T^{[1]}(1) - \hat{Y}_T^{[2]}(1)\right) \qquad 8.20$$

Note, however, that $\hat{\beta}_0$ and $\hat{\beta}_1$ are not fixed parameters in that they are updated with each new observation.

Table 8.6

Period	*Actual*	*Single Exponential Smoothing*	*Double Exponential Smoothing*	*One-Step Ahead Forecast*
1	95.5	133.8	133.8	133.8
2	127.0	126.1	133.8	118.5
3	118.0	126.3	132.3	120.4
4	153.0	124.6	131.1	118.2
5	175.5	130.3	129.8	130.8
6	180.0	139.4	129.9	148.8
7	202.0	147.5	131.8	163.2
8	190.0	158.4	134.9	181.8
9	220.5	164.7	139.6	189.8
10	245.0	175.9	144.6	207.1
11	260.0	189.7	150.9	228.5
12	320.0	203.8	158.6	248.9
13	255.0	227.0	167.7	286.3
14	263.0	232.6	179.5	285.7
15	253.0	238.7	190.1	287.2
16	263.0	241.5	199.9	283.2
17	274.0	245.8	208.2	283.5
18	278.0	251.5	215.7	287.2
19	284.0	256.8	222.9	290.7
20	300.0	262.2	229.7	294.8
21	305.5	269.8	236.2	303.4
22	310.5	276.9	242.9	311.0
23	372.0	283.6	249.7	317.6
24	412.0	301.3	256.5	346.1
25	325.0	323.4	265.4	381.4
26	333.0	323.8	277.0	370.5
27	348.0	325.6	286.4	364.8
28	355.0	330.1	294.2	365.9
29	248.0	335.1	301.4	368.7
30	363.0	317.7	308.1	327.2
31	370.0	326.7	310.0	343.4
32	388.0	335.4	313.4	357.4
33	412.0	345.9	317.8	374.0
34	435.0	359.1	323.4	394.8

35	490.5	374.3	330.5	418.0
36	532.5	397.5	339.3	455.8
37	440.0	424.5	350.9	498.1
38	472.0	427.6	365.7	489.6
39	494.5	436.5	378.1	494.9
40	512.0	448.1	389.7	506.5
41	519.0	460.9	401.4	520.3
42	526.0	472.5	413.3	531.7
43	530.5	483.2	425.1	541.3
44	558.0	492.7	436.8	548.6
45	557.5	505.7	447.9	563.5
46	590.0	516.1	459.5	572.7
47	636.5	530.9	470.8	590.9
48	677.5	552.0	482.8	621.2
49		577.1	496.7	657.5

Using the numbers in Table 8.7 we have

$$\hat{\beta}_0 = \left(2\hat{Y}_{48}^{[1]}(1) - \hat{Y}_{48}^{[2]}(1)\right) = \left(2 \cdot 577.1 - 496.7\right) = 657.5$$

$$\hat{\beta}_1 = \left(\frac{\alpha}{1-\alpha}\right)\left(\hat{Y}_{48}^{[1]}(1) - \hat{Y}_{48}^{[2]}(1)\right) = \left(\frac{.2}{1-.2}\right)\left(577.1 - 496.7\right) = 20.10$$

$$\hat{Y}_{T+1}(\ell) = \hat{\beta}_0 + \hat{\beta}_1 \ell$$

$$\hat{Y}_{49}(\ell) = 657.5 + 20.10\ell$$

Notice the slight adjustment in the notation. Since single and double exponential smoothing provide one-step ahead forecasts, we have an initial forecast for period 49. Thus, when $\ell = 0$,

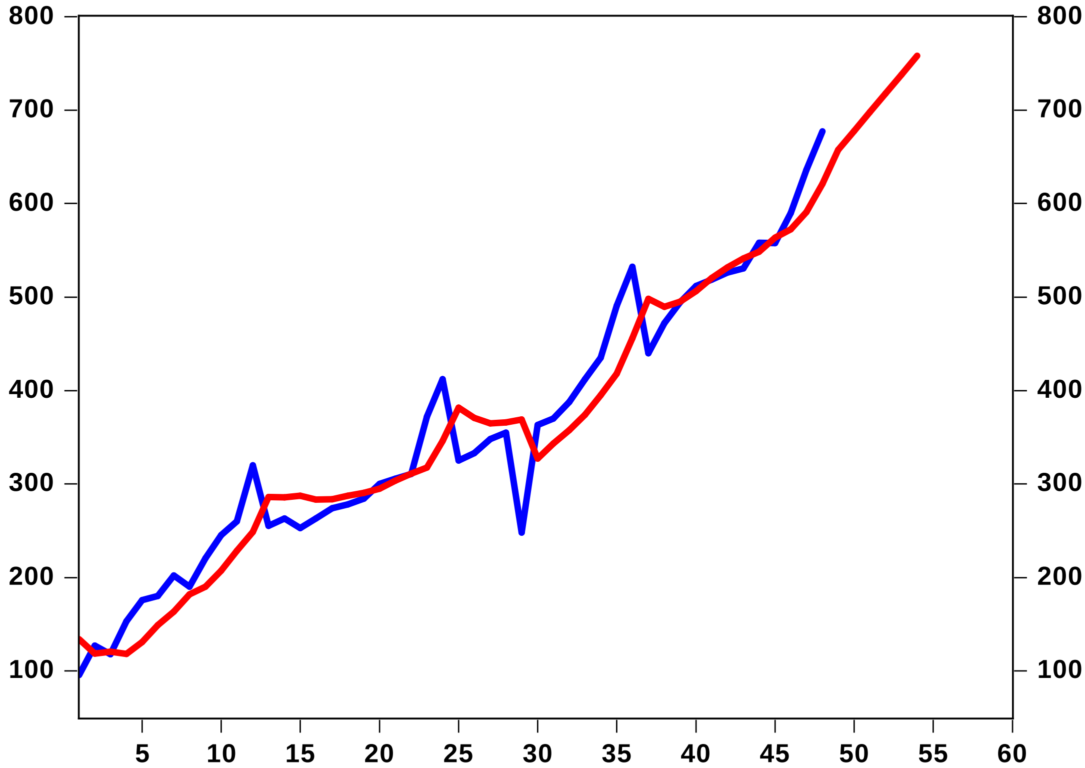
$$\hat{Y}_{49}(0) = 657.5$$

which is the last one-step ahead forecast using double exponential smoothing. To forecast for periods 50, 51, 52, 53, 54, ... , we use $\ell = 1, 2, 3, 4, 5, \ldots$.

Table 8.8

Period	Actual	Single Exponential Smoothing	Double Exponential Smoothing	Forecast	
⋮	⋮	⋮	⋮	⋮	
45	557.5	505.7	447.9	563.5	
46	590.0	516.1	459.5	572.7	
47	636.5	530.9	470.8	590.9	
48	677.5	552.0	482.8	621.2	
49		577.1	496.7	657.5	$\ell = 0$
50				677.6	$\ell = 1$
51				697.7	$\ell = 2$
52				717.8	$\ell = 3$
53				737.9	$\ell = 4$
54				758.0	$\ell = 5$

Double Exponential Smoothing with Forecast



Confidence Intervals of Forecast for Double Exponential Smoothing

For each step-ahead that is forecasted there is a slightly different formula for the variance of forecast.

One-step ahead forecast variance

As before with single exponential smoothing, the one step ahead forecast $\sigma^2_{\hat{Y}_t(1)}$ is just the *MSE*, σ^2_ϵ,

where $\sigma^2_\epsilon = \frac{SSE}{n-(k+1)}$

One-step ahead forecast variance for double exponential smoothing

$$\boldsymbol{\sigma^2_{\hat{Y}_{t+1}(0)} = \sigma^2_\epsilon} \qquad 8.21$$

Here, $k = 2$, since we are using α twice (the two-parameter) exponential smoothing.

Hence,

$$\sigma^2_{\hat{Y}_{t+1}(0)} = \sigma^2_\epsilon = \frac{SSE}{n-(k+1)}$$

$$\sigma^2_{\hat{Y}_{t+1}(0)} = \frac{107{,}323}{48-(2+1)}$$

$$\sigma^2_{\hat{Y}_{t+1}(0)} = 2{,}384.96$$

$\sigma_{\hat{Y}_{t+1}(0)} = \sqrt{2{,}384.96} = 48.84$ the standard error of forecast

Hence, with 45 degrees of freedom, a 95 percent confidence interval around $\hat{Y}_{49}(0) = 657.5$ is

$$\hat{Y}_{49}(0) \pm (1.96)\left(\sigma_{\hat{Y}_{t+1}(0)}\right)$$

$$641.4 \pm (1.96)(48.84)$$
$$641.45 \pm 95.73$$

Thus a 95 percent confidence interval of forecast for the one-step ahead double exponential
smoothing forecast is

$$545.72 \le \hat{Y}_{49}(0) \le 737.18$$

We list several of the multi-period forecast variances so that the pattern becomes apparent.

Two-step ahead forecast variance for double exponential smoothing

$$\boldsymbol{\sigma^2_{\hat{Y}_{t+1}(1)} = \sigma^2_\epsilon\left(1 + (2\alpha)^2\right)} \qquad 8.21$$

Three-step ahead forecast variance for double exponential smoothing

$$\boldsymbol{\sigma^2_{\hat{Y}_{t+1}(2)} = \sigma^2_\epsilon\left(1 + (2\alpha)^2 + (2\alpha + \alpha^2)^2\right)} \qquad 8.22$$

Four-step ahead forecast variance for double exponential smoothing

$$\boxed{\sigma^2_{\hat{Y}_{t+1(3)}} = \sigma^2_{\epsilon}\left(1 + (2\alpha)^2 + (2\alpha + \alpha^2)^2 + (2\alpha + 2\alpha^2)^2\right)} \quad 8.22$$

Five-step ahead forecast variance for double exponential smoothing

$$\boxed{\sigma^2_{\hat{Y}_{t+1(4)}} = \sigma^2_{\epsilon}\left(1 + (2\alpha)^2 + (2\alpha + \alpha^2)^2 + (2\alpha + 2\alpha^2)^2 + (2\alpha + 3\alpha^2)^2\right)} \quad 8.23$$

$$\vdots$$

The general formula for ℓ steps ahead forecast variance:

ℓ-step ahead forecast variance for double exponential smoothing

$$\sigma^2_{\hat{Y}_{t+1}(\ell-1)} = \sigma^2_\epsilon\Big(1 + (2\alpha)^2 + (2\alpha + \alpha^2)^2 + (2\alpha + 2\alpha^2)^2 + \ldots + (2\alpha + (\ell - 2)\alpha^2)^2\Big) \qquad 8.24$$

In this example, with $\alpha = .2$, we have

1-step ahead: $\sigma^2_{\hat{Y}_{49}(0)} = 2{,}384.94$

2-steps ahead: $\sigma^2_{\hat{Y}_{49}(1)} = 2{,}384.94\Big(1 + (.4)^2\Big) = 2{,}766.53$

3-steps ahead: $\sigma^2_{\hat{Y}_{49}(2)} = 2{,}384.94\Big(1 + (.4)^2 + (.44)^2\Big) = 3{,}228.25$

$\vdots$

The corresponding 95 percent confidence intervals are

1-step ahead $641.4 \pm (1.96)\sqrt{2{,}384.94}$

641.4 ± 95.73

Thus a 95 percent confidence interval of forecast for the one-step ahead double exponential smoothing forecast is

$$545.72 \leq \hat{Y}_{49}(0) \leq 737.18$$

2-steps ahead $657.5 \pm (1.96)\sqrt{2{,}766.53}$

657.5 ± 103.09

Thus a 95 percent confidence interval of forecast for the two-step ahead double exponential smoothing forecast is

$$554.4 \leq \hat{Y}_{49}(1) \leq 760.6$$

3-steps ahead $673.6 \pm (1.96)\sqrt{3{,}228.25}$

673.6 ± 111.4

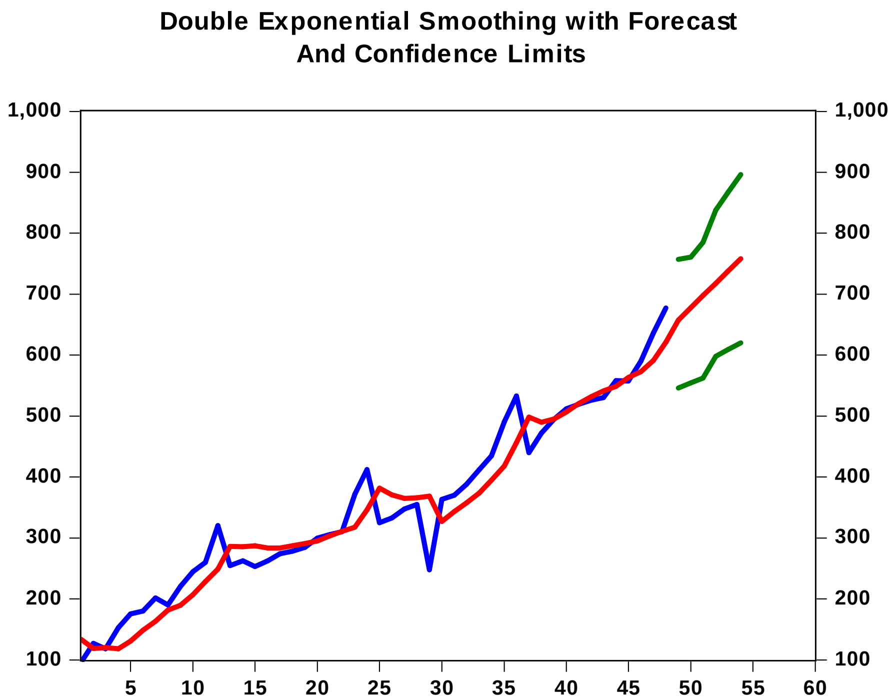
Thus a 95 percent confidence interval of forecast for the three-step ahead double exponential smoothing forecast is

$$562.2 \leq \hat{Y}_{49}(2) \leq 785.0$$

Table 8.6

			95 Percent Confidence Intervals	
Period	***Actual***	***Forecasts***	***Lower***	***Upper***
$\vdots$	$\vdots$	$\vdots$		
45	557.5	563.5		
46	590.0	572.7		
47	636.5	590.9		
48	677.5	621.2		

49	657.5	545.7	757.2
50	677.6	554.4	760.6
51	697.7	562.2	785.0
52	717.8	597.9	837.7
53	737.9	609.0	866.8
54	758.0	620.0	896.1



Updating the Forecasts

When a new observation is recorded, as in this case Y_{49}, then the next set of forecasts, (for periods 50, . . . ,55) will be updated as a result of this new information. There will be new parameters, $\hat{\beta}_0$ and $\hat{\beta}_1$ for the equation

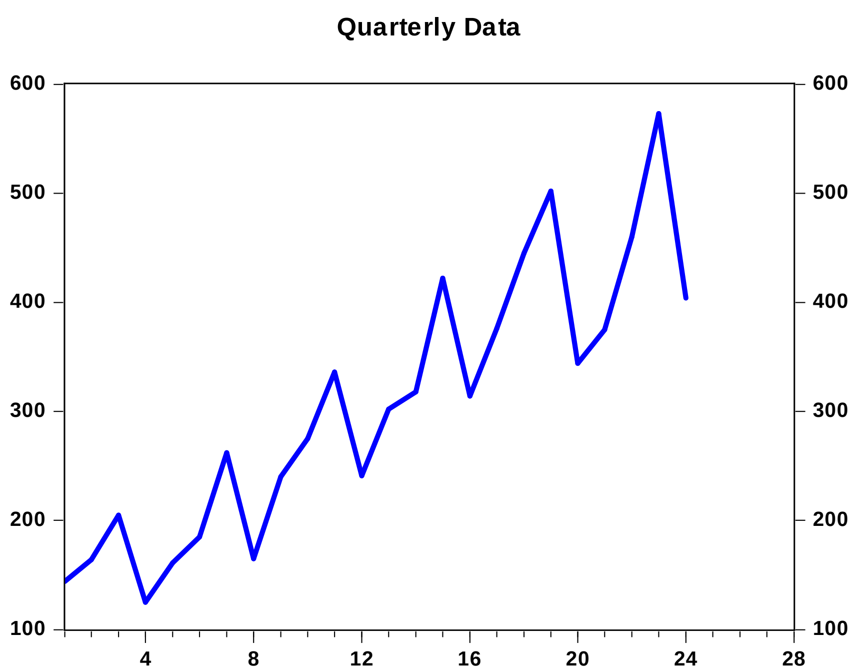
$$\hat{Y}_{t+1}(\ell) = \hat{\beta}_0 + \hat{\beta}_1 \ell$$

Winter's Method

The data graphed below is quarterly data with addative trend and seasonality.

Table 8.7

Obs.	*Quarter*	*Data*	*Obs.*	*Quarter*	*Data*
1	1	144	13	1	302
2	2	164	14	2	318
3	3	205	15	3	422
4	4	125	16	4	314
5	1	161	17	1	376
6	2	185	18	2	445
7	3	262	19	3	502
8	4	165	20	4	344
9	1	240	21	1	375
10	2	275	22	2	460
11	3	336	23	3	573
12	4	241	24	4	404



Winter's Method uses a form of Single Exponential Smoothing, and in addition uses a Seasonal Index and a Trend Adjustment. Both the Seasonal Index and Trend Adjustment are smoothed in the fashion of Exponential Smoothing.

We begin the process by setting an initial single exponential smoothing value for period 5, using the Actual from period 5, $Y_5 = 161$. We shall denote this value as $\hat{Y}_5 = 161$.

Next we determine the first four Seasonal Indices, S_1, S_2, S_3, S_4. We shall take the average of the first *five* periods of data, to insure a full seasonal cycle.

$$\overline{Y} = \frac{Y_1 + Y_2 + Y_3 + Y_4 + Y_5}{5} = \frac{144 + 164 + 205 + 125 + 161}{5} = 159.80$$

Then the 159.80 is divided into the first four observations to produce the first set of estimated Seasonal Indices

$$S_1 = \frac{144}{159.80} = 0.9011$$

$$S_2 = \frac{164}{159.80} = 1.0263$$

$$S_3 = \frac{205}{159.80} = 1.2829$$

$$S_4 = \frac{125}{159.80} = 0.7822$$

The initial *Trend Adjustment* T_5 value is the average of the average of the first four seasonal changes. What we mean by this is that the change from Period 1 to Period 5 is a annual change from Quarter 1 to Quarter 1:

$$\textit{Annual Change: Quarter } 1 = Y_5 - Y_1 = 161 - 144 = 17$$

so the *Average Annual Change* is:

$$\textit{Average Annual Change: Quarter } 1 = \frac{Y_5 - Y_1}{4} = \frac{161 - 144}{4} = 4.25$$

We repeat this for all four quarters and then take the average of these four numbers. In summary

$$T_5 = \frac{1}{4}\left(\frac{Y_5 - Y_1}{4} + \frac{Y_6 - Y_2}{4} + \frac{Y_7 - Y_3}{4} + \frac{Y_8 - Y_4}{4} \right) = 8.4375$$

Now we are in a position to compute the one-step ahead forecast for Period 6, which we denote: $\hat{Y}_5(1)$

$$\hat{Y}_5(1) = (Y_5 + T_5)S_2$$

We multiply by S_2 since that is the seasonal index for Quarter 2 and Period 6 occurs in Quarter 2.

$$\hat{Y}_5(1) = (Y_5 + T_5)S_2 = (161.0 + 8.4375) \times 1.0263 = 173.8937$$

The Updating Procedure of Winters Method

The updating procedure is similar to single exponential smoothing. Recall from single exponential smoothing that

$$\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha)\hat{Y}_{t-1}(1).$$

In this setting we update $\hat{Y}_t$ in similar fashion but we also update Seasonality S_t and Trend T_t

Winters Method: Updating $\hat{Y}_t$

$$\boldsymbol{\hat{Y}_t = \alpha_1\left(\frac{Y_t}{S_{t-4}}\right) + (1-\alpha_1)\left(\hat{Y}_{t-1} + T_{t-1}\right)} \qquad 8.25$$

S_t and T_t are also updated in similar fashion.

Winters Method: Updating S_t

$$\boldsymbol{S_t = \alpha_2\left(\frac{Y_t}{\hat{Y}_t}\right) + (1-\alpha_2)S_{t-4}} \qquad 8.25$$

Winters Method: Updating T_t

$$\boldsymbol{T_t = \alpha_3\left(\hat{Y}_t - \hat{Y}_{t-1}\right) + (1-\alpha_3)T_{t-1}} \qquad 8.25$$

The One-step Ahead Forecasts use these updating formulas.

Winters Method: One-step Ahead Forecast, $\hat{Y}_t(1)$

$$\boldsymbol{\hat{Y}_t(1) = \left(\hat{Y}_t + T_t\right) S_{(t+1)-4}} \qquad 8.26$$

Notice that Winters Method requires three smoothing parameters, α_1, α_2, α_3. We use parameters that are succesively lower in magnitude because each updated term is successively smoother. In the Table 8.29 below

$\alpha_1 = .4 \qquad \alpha_2 = .3 \qquad \alpha_3 = .1$

so that the updating formulas are

$$\hat{Y}_t = .4\left(\frac{Y_t}{S_{t-4}}\right) + .6\left(\hat{Y}_{t-1} + T_{t-1}\right)$$

$$S_t = .3\left(\frac{Y_t}{\hat{Y}_t}\right) + .7S_{t-4}$$

$$T_t = .1\left(\hat{Y}_t - \hat{Y}_{t-1}\right) + .9T_{t-1}$$

For example, updating for Period 6.

We begin by updating $\hat{Y}_5$ to get $\hat{Y}_6$

$$\hat{Y}_6 = .4\left(\frac{Y_6}{S_2}\right) + .6\left(\hat{Y}_5 + T_5\right) = .4\left(\frac{185}{1.0263}\right) + .6\left(161 + 8.4375\right) =$$
173.7674

Using the new $\hat{Y}_6$ we update S_6

$$S_6 = .3\left(\frac{Y_6}{\hat{Y}_6}\right) + .7S_2 = .3\left(\frac{185}{173.7674}\right) + .7(1.0263) = 1.0378$$

Using the new $\hat{Y}_6$ we update T_6

$$T_6 = .1\left(\hat{Y}_6 - \hat{Y}_5\right) + .9T_5 = .1\left(173.7674 - 161.0\right) + .9(8.4375) = 8.8705$$

We now compute a one-step ahead forecast for Period 7

$$\hat{Y}_6(1) = \left(\hat{Y}_6 + T_6\right) S_3 = \left(173.7674 + 8.8705\right) \times 1.2829 = 234.2976$$

And so on.

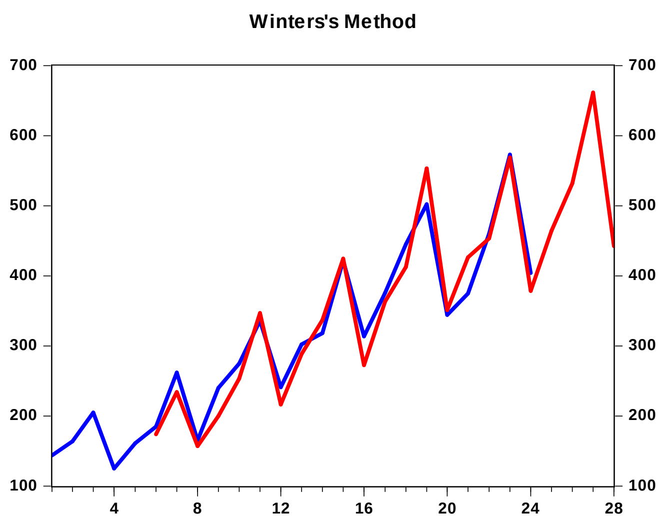
Notice that S_3 is the seasonal multiplier for the Period 7 one-step ahead forecast. Period 7 forecast occurs in Quarter 3, so we need the latest Quarter 3 multiplier. The S_6 seasonal multiplier will be used when we determine a one-step ahead forecast for Period 10. The Period 10 forecast occurs in Quarter 2, at that point S_6 will be the most recently updated Quarter 2 multiplier.

One step ahead forecast for Period 10

$$\hat{Y}_9(1) = \left(\hat{Y}_9 + T_9\right) S_6 = \left(232.21 + 11.84\right) \times 1.04 = 253.27$$

Table 8.8

Obs. t	*Quarter* Q_t	*Data* Y_t	*Single Exponential Smoothing* $\hat{Y}_t$ $\alpha_1 = .4$	*Seasonal Index* S_t $\alpha_2 = .3$	*Trend Adjustment* T_t $\alpha_3 = .1$	*One-Step Ahead Forecast* $\hat{Y}_t(1)$
1	1	144		0.90		
2	2	164		1.03		
3	3	205		1.28		
4	4	125		0.78		
5	1	161	161.00	0.93	8.44	
6	2	185	173.77	1.04	8.87	173.89
7	3	262	191.28	1.31	9.73	234.30
8	4	165	204.98	0.79	10.13	157.24
9	1	240	232.21	0.96	11.84	200.22
10	2	275	252.42	1.05	12.68	253.27
11	3	336	261.74	1.30	12.34	347.00
12	4	241	286.62	0.80	13.60	216.26
13	1	302	305.75	0.97	14.15	288.70
14	2	318	312.71	1.04	13.43	336.95
15	3	422	325.39	1.30	13.36	424.42
16	4	314	359.35	0.83	15.42	272.55
17	1	376	380.00	0.98	15.94	363.32
18	2	445	408.33	1.06	17.18	412.72
19	3	502	409.76	1.28	15.60	553.17
20	4	344	421.94	0.82	15.26	351.07
21	1	375	416.09	0.95	13.15	426.47
22	2	460	431.69	1.06	13.39	453.54
23	3	573	446.46	1.28	13.53	568.61
24	4	404	472.51	0.83	14.78	378.26
25	1					464.48
26	2					531.85
27	3					661.23
28	4					442.40



PROBLEMS AND QUESTIONS

Constant Mean Data

8.1

Listed below is a time series of 24 observations.

Plot the data and determine the mean and variance of the the data set.

Period	Observation	Period	Observation
1	95	13	118
2	100	14	86
3	87	15	86
4	123	16	112
5	90	17	85
6	96	18	101
7	75	19	135
8	78	20	120
9	106	21	76
10	104	22	115
11	89	23	90
12	83	24	92

One-step ahead forecasts

8.1 Determine a set of one-step ahead forecasts of the data, using

a. Three period moving average forecasts.

b. Five period moving average forecasts.

c. Four period moving average forecasts.

d. Three period weighted moving average forecasts; weights .6, .3 and .1

8.2 Determine a set of one-step ahead forecasts of the data, using

a. Single exponential smoothing forecasts with $\alpha = .1$.

b. Single exponential smoothing forecasts with $\alpha = .2$.

c. Single exponential smoothing forecasts with $\alpha = .7$.

8.3 Using the one-step ahead forecasts of Problem 8.5, determine the forecasts for period 25 in a., b., and c. above.

8.4 Determine the MSE of each of the one-step ahead forecasts in Problem 8.5
Which is the better forecast? Explain.

Data with Trend

8.5

Listed below is a time series of 24 observations.

Plot the data and determine the mean and variance of the the data set.

Period	Observation	Period	Observation
1	60	13	89
2	70	14	120
3	85	15	134
4	60	16	121
5	88	17	93
6	68	18	113
7	106	19	125
8	75	20	136

9	86	21	142
10	124	22	117
11	122	23	132
12	87	24	141

8.6 Determine the one-step ahead forecasts using

a. Three period moving average

b. Weighted moving average with weights .5, .3, .2.

What are the forecasts for period 25 in each case above?

What is the MSE in each case above?

8.7 Determine the one-step ahead forecasts using

a. Single exponential smoothing with $\alpha = .3$

b. What is the forecast for period 25?

c. What is the MSE of this forecasting model?

8.8 Determine the one-step ahead forecasts using

a. Double exponential smoothing with $\alpha = .3$.

b. What is the forecast for period 25?

c. What is the forecast for periods 26 to 30?

d. Determine the 95% confidence intervals of forecast for periods 25 to 30.

e. What is the MSE of the forecasting model?