

## *Chapter 8*

*November 9, 2017*

We now examine another area of time series forecasting similar to weighted moving averages, called *exponential smoothing methods*. One of the major distinctions between regressions methods on time series data and smoothing methods is that regressions give equal weight to all of the sample data whereas smoothing methods give decreasing weight to older, more distant data.

“An economist is an expert who will know tomorrow why the things he predicted yesterday didn’t happen today.”

— Evan Esar

### ***Single Exponential Smoothing***

A popular, widely used time series smoothing and forecasting method is *single exponential smoothing* (also called *one-parameter exponential smoothing*). Exponential smoothing is a form of weighted moving average in that recent observations carry more weight than older observations.

Exponential smoothing is based on the concept that a new, one step-ahead forecast can be determined from the *previous forecast* and the *error of the previous forecast*. We illustrate this concept below.

Suppose we have the following series:

Period	Actual( $Y_t$ )	Forecast
1	354	
2	368	
3	329	

And, suppose a one-step ahead forecast for period 4, made at period 3, is 352.5.  $\hat{Y}_3(1) = 352.5$ .

Period	Actual( $Y_t$ )	Forecast
1	354	
2	368	
3	329	
4	NA	352.5

Then suppose period 4's actual value is 389.

Period	Actual( $Y_t$ )	Forecast
1	354	
2	368	
3	329	
4	389	352.5

So, the forecast error is 36.5.

$$\varepsilon_3(1) = Y_4 - \hat{Y}_3(1)$$

$$36.5 = 389 - 352.5$$

In otherwords, we *underforecast* by 36.5.

Period	Actual( $Y_t$ )	Forecast	Error
1	354		
2	368		
3	329		
4	389	352.5	36.5

Since we underforecasted, we will adjust upward our next forecast, based on the error of our previous forecast. We adjust our next forecast upward by 20% of the forecast error. 20% of 36.5 is 7.3.

Hence, the forecast for period 5 is the old forecast of 352.5 adjusted upwarded by 7.3.

$$352.5 + 7.3 = 359.8$$

$$\hat{Y}_4(1) = 359.8$$

Period	Actual( $Y_t$ )	Forecast	Error
1	354		
2	368		
3	329		
4	389	352.5	36.5
5	NA	359.8	NA

The actual value for period 5 is 350, so the one-step ahead period 5 forecast error is

$$\varepsilon_4(1) = Y_5 - \hat{Y}_4(1)$$

$$-9.8 = 350 - 359.8$$

In otherwords, we *overforecast* by 9.8.

Period	Actual( $Y_t$ )	Forecast	Error
1	354		
2	368		
3	329		
4	389	352.5	36.5
5	350	359.8	9.8

The forecast for period 5 then is adjusted by 20% of the forecast error, 20% of  $-9.8$  is  $-1.96$ .

The Forecast for period 6 is:

$$359.8 + (-1.96) = 357.8$$

$$\hat{Y}_5(1) = 357.8$$

Each new forecast,  $\hat{Y}_t(1)$ , is based on the preceding forecast,  $\hat{Y}_{t-1}(1)$ , plus an adjustment for forecast error.

If we denote the adjustment factor by  $\alpha$ , we have:

$$\hat{Y}_t(1) = \hat{Y}_{t-1}(1) + \alpha \varepsilon_{t-1}(1)$$

Algebraically, we re-arrange the last equation as the formal representation of *exponential smoothing*.

### ***Single Exponential Smoothing, one-step ahead forecast***<sup>1</sup>

<sup>1</sup> Formal Notation, equation 8.1

In the case of  $\alpha = 0.2$ , we have

$$\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha) \hat{Y}_{t-1}(1)$$

$$\hat{Y}_t(1) = 0.2Y_t + 0.8\hat{Y}_{t-1}(1)$$

The new forecast is the weighted average of the current observation  $Y_t$  and the previous forecast  $\hat{Y}_{t-1}$ .

Returning to equation 8.1 displayed in the right margin, we illustrate why this approach is called *exponential smoothing*.

Since  $\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha) \hat{Y}_{t-1}(1)$ , we shift back by one period and get  $\hat{Y}_{t-1}(1) = \alpha Y_{t-1} + (1 - \alpha) \hat{Y}_{t-2}(1)$ .<sup>2</sup>

<sup>2</sup> Equation 8.2

By substituting equation 8.2 in for  $\hat{Y}_{t-1}(1)$  in equation 8.1 we have:

$$\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha) \left( \alpha Y_{t-1} + (1 - \alpha) \hat{Y}_{t-2}(1) \right)$$

Which we can then rearrange to get:

$$\hat{Y}_t(1) = \alpha Y_t + \alpha(1 - \alpha)Y_{t-1} + (1 - \alpha)^2 \hat{Y}_{t-2}(1)$$

Using equation 8.1 again as a format,  $\hat{Y}_{t-2}(1)$  is written as

$$\hat{Y}_{t-2}(1) = \alpha Y_{t-2} + (1 - \alpha) \hat{Y}_{t-3}(1)$$

By substituting for  $\hat{Y}_{t-2}(1)$  in equation (8.1) we have:

$$\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha) \alpha Y_{t-1} + (1 - \alpha)^2 \left( \alpha Y_{t-2} + (1 - \alpha) \hat{Y}_{t-3}(1) \right)$$

Which can then be rearranged as:

$$\hat{Y}_t(1) = \alpha Y_t + \alpha(1 - \alpha)Y_{t-1} + \alpha(1 - \alpha)^2 Y_{t-2}(1) + \alpha(1 - \alpha)^3 \hat{Y}_{t-3}(1)$$

And so on.

Since  $\alpha$  is less than 1,  $\alpha, \alpha(1 - \alpha), \alpha(1 - \alpha)^2, \dots$  are exponentially decreasing weights applied to previous observations.

$$\hat{Y}_t(1) = \alpha Y_t + \alpha(1 - \alpha)Y_{t-1} + \alpha(1 - \alpha)^2 Y_{t-2}(1) + \alpha(1 - \alpha)^3 \hat{Y}_{t-3}(1) + \alpha(1 - \alpha)^4 \hat{Y}_{t-4}(1) + \dots$$

In the case of  $\alpha = 0.2$ , we have

$$\hat{Y}_t(1) = 0.2Y_t + 0.16Y_{t-1} + 0.128Y_{t-2}(1) + 0.1024Y_{t-3}(1) + 0.08192\hat{Y}_{t-4}(1) + \dots$$

This is an infinite weighted moving average, shown that:

$$1 = \alpha + \alpha(1 - \alpha) + \alpha(1 - \alpha)^2 + \alpha(1 - \alpha)^3 + \alpha(1 - \alpha)^4 + \dots$$

The attraction of exponential smoothing is that it is not necessary to use the last equation (8.7), but rather the first (8.1)

$$\hat{Y}_t(1) = \alpha Y_t + (1 - \alpha)\hat{Y}_{t-1}(1)$$

as it is equivalent to the infinite sum

$$\hat{Y}_t(1) = \alpha Y_t + \alpha(1 - \alpha)Y_{t-1} + \alpha(1 - \alpha)^2Y_{t-2}(1) + \alpha(1 - \alpha)^3\hat{Y}_{t-3}(1) + \alpha(1 - \alpha)^4\hat{Y}_{t-4}(1) + \dots$$

With Single Exponential Smoothing, represented by equation 8.1, all that is needed to make a new forecast is the current observation and previous forecast.

*Starting Value for Exponential Smoothing*

Period	Actual	Forecast
1	354	356.533
2	368	356.027
3	329	358.421
4	389	352.537
5	375	359.830
6	375	362.864
7	367	365.291
8	364	365.633
9	379	365.306
10	386	368.045
11	329	371.636
12	334	363.109
13	372	357.287
14	329	360.230
15	320	353.984
16	332	347.187
17	342	344.150
18	357	343.720
19	357	346.376
20	357	348.501
21	344	350.200
22	361	348.960
23	358	351.368
24	345	352.695
25	367	351.156
26	380	354.325
27	387	359.460
28	346	364.968
29	321	361.174
30	372	353.139
31	NA	356.911

Exponential Smoothing just needs a starting value for forecast and a choice of  $\alpha$ . For our choice of  $\alpha$ , we shall use  $\alpha = 0.20$

For a starting value,  $Y_1(1)$ , we shall use the average of the 30 observations of data,  $\bar{Y} = 356.53$ .

Thus, an exponential smoothing forecast for Period 2:

$$\hat{Y}_1(1) = 0.2Y_1 + (1 - 0.2)\hat{Y}_0(1)$$

$$\hat{Y}_1(1) = 0.2(368) + 0.8(356.53) = 356.02$$

For Period 3:

$$\hat{Y}_2(1) = 0.2(368) + 0.8(356.02) = 358.42$$

And so on, recursively through all 30 observations of data.

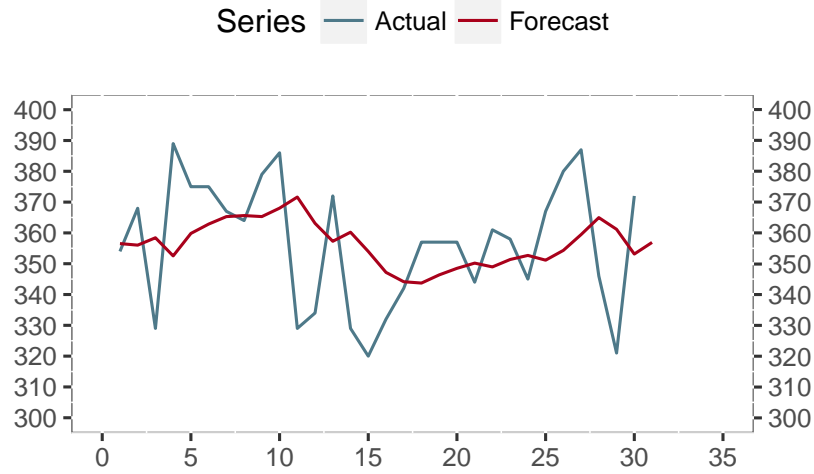


Figure 1: Single Exponential Smoothing.