Option Pricing Functions to Accompany *Derivatives Markets*

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Contents

1 Introduction

This vignette is an overview to the functions in the *derivmkts* package, which was conceived as a companion to my book *Derivatives Markets* McDonald (2013). The material has an educational focus. There are other option pricing packages for R, but this package has several distinguishing features:

- function names (mostly) correspond to those in *Derivatives Markets*.

1

Function	Description
bscall	European call
bsput	European put
bsopt	European call and put and associated Greeks: delta, gamma, vega, theta, rho, psi, and elasticity
assetcall	Asset-or-nothing call
assetput	Asset-or-nothing put
cashcall	Cash-or-nothing call
cashput	Cash-or-nothing put

Table 1: Black-Scholes related option pricing functions

- vectorized Greek calculations are convenient both for individual options and for portfolios
- the `quincunx` function illustrates the workings of a quincunx (Galton board).
- binomial functions include a plotting function that provides a visual depiction of early exercise

2 Pricing functions and greeks

2.1 European Calls and Puts

Table 1 lists the Black-Scholes related functions in the package.[1] The functions `bscall`, `bsput`, and `bsopt` provide basic pricing of European calls and puts. There are also options with binary payoffs: cash-or-nothing and asset-or-nothing options. All of these functions are vectorized. The function `bsopt` by default provides option greeks. Here are some examples:

```
s <- 100; k <- 100; r <- 0.08; v <- 0.30; tt <- 2; d <- 0
bscall(s, k, v, r, tt, d)
[1] 24.02
bsput(s, c(95, 100, 105), v, r, tt, d)
[1]  7.488  9.239 11.188
```

2.2 Barrier Options

There are pricing functions for the following barrier options:[2]

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[1] See Black and Scholes (1973) and Merton (1973).

[2] See Merton (1973), p. 175, for the first derivation of a barrier option pricing formula and McDonald (2013), Chapter 14, for an overview.

- down-and-in and down-and-out barrier binary options
- up-and-in and up-and-out barrier binary options
- more standard down- and up- calls and puts, constructed using the barrier binary options

Naming for the barrier options generally follows the convention

`[u|d][i|o][call|put]`

which means that the option is up'' or down'', in'' or out'', and a call or put.[3] An up-and-in call, for example, would be denoted by `uicall`. For binary options, we add the underlying, which is either the asset or $1: cash:

`[asset|cash][u|d][i|o][call|put]`

```
H <- 115
bscall(s, c(80, 100, 120), v, r, tt, d)
[1] 35.28 24.02 15.88
## Up-and-in call
uicall(s, c(80, 100, 120), v, r, tt, d, H)
[1] 34.55 23.97 15.88
bsput(s, c(80, 100, 120), v, r, tt, d)
[1]  3.450  9.239 18.141
## Up-and-out put
uoput(s, c(80, 100, 120), v, r, tt, d, H)
[1] 2.328 5.390 9.070
```

2.3 Perpetual American Options

The functions `callperpetual` and `putperetual` price infinitely-lived American options.[4] The pricing formula assumes that all inputs (risk-free rate, volatility, dividend yield) are fixed. This is of course usual with the basic option pricing formulas, but it is more of a conceptual stretch for an infinitely-lived option than for a 3-month option.

In order for the option to have a determined value, the dividend yield on the underlying asset must be positive if the option is a call. If this is not true, the call is never exercised and the price is undefined.[5] Similarly, the risk-free rate must be positive if the option is a put.

---

[3]This naming convention differs from that in McDonald (2013), in which names are `callupin`, `callupout`, etc. Thus, I have made both names available for these functions.

[4]Merton (1973) derived the price of a perpetual American put.

[5]A well-known result (see Merton (1973)) is that a standard American call is never exercised before expiration if the dividend yield is zero and the interest rate is non-negative. A perpetual call with $\delta = 0$ and $r > 0$ would thus never be exercised. The limit of the option price as $\delta \to 0$ is s, so in this case the function returns the stock price as the option value.

By default, the perpetual pricing formulas return the price. By setting `showbarrier=TRUE`, the function returns both the option price and the stock price at which the option is optimally exercised (the "barrier"). Here are some examples:

```
s <- 100; k <- 100; r <- 0.08; v <- 0.30; tt <- 2; d <- 0.04
callperpetual(s, c(95, 100, 105), v, r, d)
[1] 44.71 43.82 43.00
callperpetual(s, c(95, 100, 105), v, r, d, showbarrier=TRUE)
$price
[1] 44.71 43.82 43.00

$barrier
[1] 338.6 356.4 374.2
```

2.4 Option Greeks

Options greeks are mathematical derivatives of the option price with respect to inputs; see McDonald (2013), Chapters 12 and 13, for a discussion of the greeks for vanilla options. Greeks for vanilla and barrier options can be computed using the `greeks` function, which is a wrapper for any pricing function that returns the option price and which uses the default naming of inputs.^[In this version of the package, I have two alternative functions that return Greeks:

- The `bsopt` function by default produces prices and Greeks for European calls and puts.
- The `greeks2` function takes as arguments the name of the pricing function and then inputs as a list.

These may be deprecated in the future. `greeks2` is more cumbersome to use but may be more robust. I welcome feedback on these functions and what you find useful.]

```
H <- 105
greeks(uicall(s, k, v, r, tt, d, H))
              uicall
Premium     18.719815
Delta        0.605436
Gamma        0.008011
Vega         0.480722
Rho          0.836133
Theta       -0.012408
Psi         -1.210530
Elasticity  3.234200
```

The value of this approach is that you can easily compute Greeks for spreads and custom pricing functions. Here are two examples. First, the value at time 0 of a prepaid contract that pays S_T^a at time T is given by the `powercontract()` function:

```
powercontract <- function(s, v, r, tt, d, a) {
    price <- exp(-r*tt)*s^a* exp((a*(r-d) + 1/2*a*(a-1)*v^2)*tt)
}
```

We can easily compute the Greeks for a power contract:

```
greeks(powercontract(s=40, v=.08, r=0.08, tt=0.25, d=0, a=2))
            powercontract
Premium          1634.936
Delta              81.747
Gamma               2.044
Vega                0.654
Rho                 4.087
Theta              -0.387
Psi                -8.175
Elasticity          2.000
```

Second, consider a bull spread in which we buy a call with a strike of k_1 and sell a call with a strike of k_2. We can create a function that computes the value of the spread, and then compute the greeks for the spread by using this newly-created function together with `greeks()`:

```
bullspread <- function(s, v, r, tt, d, k1, k2) {
    bscall(s, k1, v, r, tt, d) - bscall(s, k2, v, r, tt, d)
}
greeks(bullspread(39:41, .3, .08, 1, 0, k1=40, k2=45))
            bullspread_39 bullspread_40 bullspread_41
Premium         2.0020318     2.1551927     2.306e+00
Delta           0.1542148     0.1519426     1.487e-01
Gamma          -0.0017692    -0.0027545    -3.614e-03
Vega           -0.0080732    -0.0132218    -1.822e-02
Rho             0.0401235     0.0392251     3.793e-02
Theta          -0.0005476    -0.0003164    -8.246e-05
Psi            -0.0601438    -0.0607771    -6.099e-02
Elasticity      3.0041376     2.8200287     2.645e+00
```

The Greeks function is vectorized, so you can create vectors of greek values with a single call. This example plots, for a bull spread, the gamma as a function of the stock price; see Figure 1.
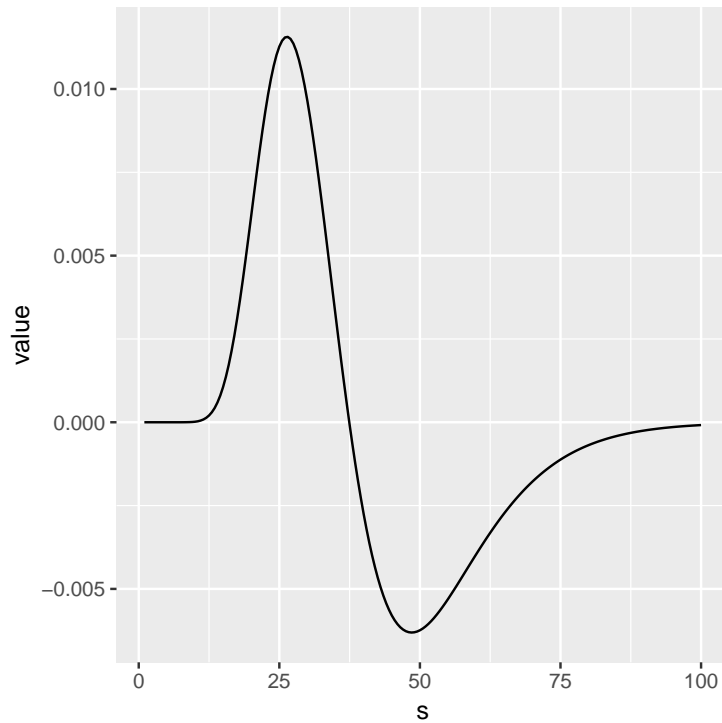
Figure 1: Gamma for a 40-45 bull spread.

```r
sseq <- seq(1, 100, by=0.5)
greeks(bullspread(sseq, .3, .08, 1, 0, k1=40, k2=45),
            initcaps = TRUE, long = TRUE) %>%
    filter(greek == 'Gamma' ) %>%
    ggplot(aes(x = s, y = value)) + geom_line()
```

This code produces the plots in Figure 2:

```r
k <- 100; r <- 0.08; v <- 0.30; tt <- 2; d <- 0
S <- seq(.5, 200, by=.5)
Call <- greeks(bscall(S, k, v, r, tt, d), long = TRUE)
Put <- greeks(bsput(S, k, v, r, tt, d), long = TRUE)
ggplot(rbind(Call, Put), aes(x = s, y = value, color = funcname )) +
    geom_line() + facet_wrap(~ greek, scales = 'free_y') +
    scale_color_discrete(name = 'Option', labels = c('Call','Put' )) +
    scale_x_continuous('Stock', breaks =c(0, 100, 200)  ) +
    scale_y_continuous('Value')
```
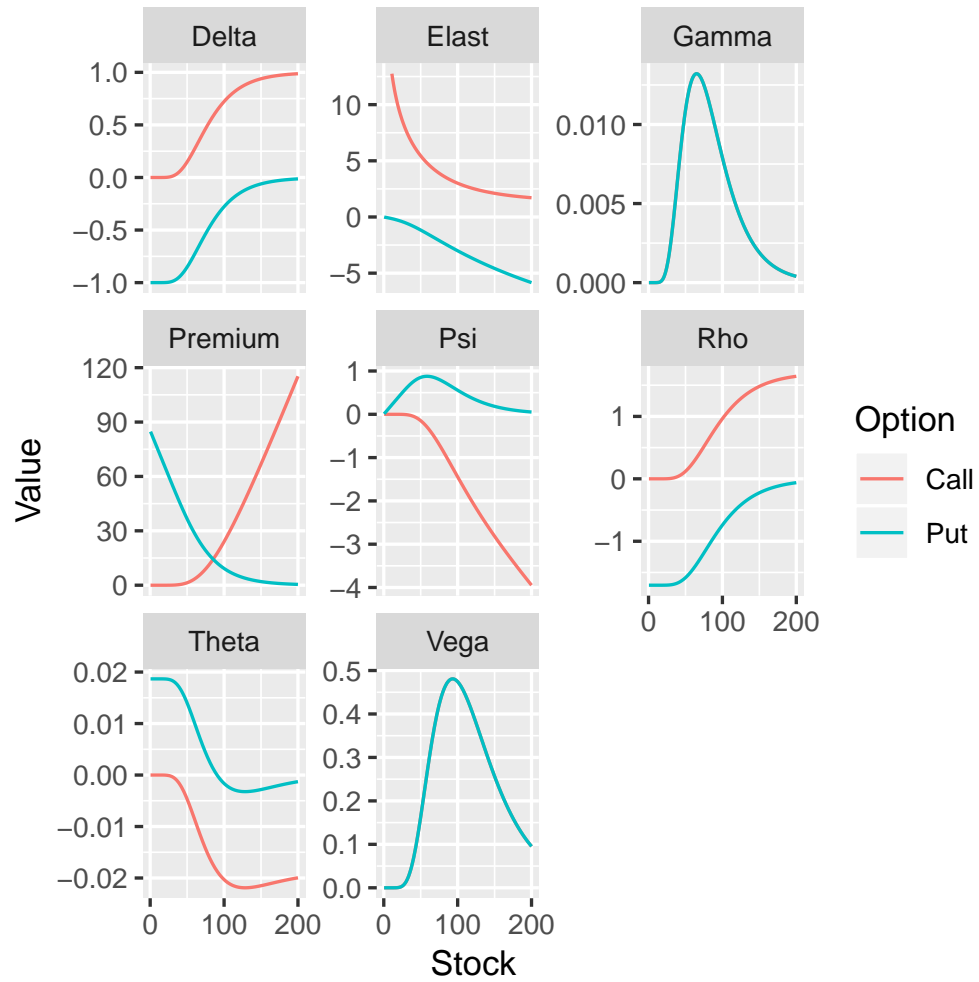
Figure 2: All option Greeks for a call and a put, computed using the greeks function

2.5   Binomial Pricing of European and American Options

There are two functions related to binomial option pricing:[6]

- **binomopt** computes prices of American and European calls and puts. The function has three optional parameters that control output:

  - `returnparams=TRUE` will return as a vector the option pricing inputs, computed parameters, and risk-neutral probability.

  - `returngreeks=TRUE` will return as a vector the price, delta, gamma, and theta at the initial node.

  - `returntrees=TRUE` will return as a list the price, greeks, the full stock price tree, the exercise status (`TRUE` or `FALSE`) at each node, and the replicating portfolio at each node.

- **binomplot** displays the asset price tree, the corresponding probability of being at each node, and whether or not the option is exercised at each node. This function is described in more detail in Section 4.2.

Here are examples of pricing, illustrating the default of just returning the price, and the ability to return the price plus parameters, as well as the price, the parameters, and various trees:

```
s <- 100; k <- 100; r <- 0.08; v <- 0.30; tt <- 2; d <- 0.03
binomopt(s, k, v, r, tt, d, nstep=3)
price
 20.8
binomopt(s, k, v, r, tt, d, nstep=3, returnparams=TRUE)
   price        s        k        v        r       tt        d    nstep
 20.7961 100.0000 100.0000   0.3000   0.0800   2.0000   0.0300   3.0000
       p       up       dn        h
  0.4391   1.3209   0.8093   0.6667
binomopt(s, k, v, r, tt, d, nstep=3, putopt=TRUE)
price
12.94
binomopt(s, k, v, r, tt, d, nstep=3, returntrees=TRUE, putopt=TRUE)
$price
price
12.94


$greeks
```

---

[6]See Cox, Ross, and Rubinstein (1979), Rendleman and Bartter (1979), and McDonald (2013), Chapter 11

```
    delta      gamma      theta
-0.335722   0.010614  -0.007599


$params
        s         k         v         r        tt         d     nstep         p
100.0000  100.0000    0.3000    0.0800    2.0000    0.0300    3.0000    0.4391
       up        dn         h
   1.3209    0.8093    0.6667


$oppricetree
      [,1]    [,2]    [,3]   [,4]
[1,] 12.94   3.816   0.000   0.00
[2,]  0.00  21.338   7.176   0.00
[3,]  0.00   0.000  34.507  13.49
[4,]  0.00   0.000   0.000  47.00


$stree
     [,1]    [,2]    [,3]    [,4]
[1,]  100  132.09  174.47  230.45
[2,]    0   80.93  106.89  141.19
[3,]    0    0.00   65.49   86.51
[4,]    0    0.00    0.00   53.00


$probtree
     [,1]    [,2]    [,3]     [,4]
[1,]    1  0.4391  0.1928  0.08464
[2,]    0  0.5609  0.4926  0.32441
[3,]    0  0.0000  0.3146  0.41445
[4,]    0  0.0000  0.0000  0.17650


$exertree
      [,1]  [,2]   [,3]   [,4]
[1,] FALSE FALSE FALSE  FALSE
[2,] FALSE FALSE FALSE  FALSE
[3,] FALSE FALSE  TRUE   TRUE
[4,] FALSE FALSE FALSE   TRUE


$deltatree
       [,1]     [,2]     [,3]
[1,] -0.3357  -0.1041   0.0000
[2,]  0.0000  -0.6471  -0.2419
[3,]  0.0000   0.0000  -0.9802
```

```
$bondtree
      [,1]  [,2]  [,3]
[1,] 46.51 17.56  0.00
[2,]  0.00 73.71 33.03
[3,]  0.00  0.00 94.81
```

2.6 Asian Options

There are analytical functions for valuing geometric Asian options and Monte Carlo routines for valuing arithmetic Asian options.[7] Be aware that the `greeks()` function at this time will not work with the Monte Carlo valuation for arithmetic Asian options. I plan to address this in a future release.[8]

2.6.1 Geometric Asian Options

Geometric Asian options can be valued using the Black-Scholes formulas for vanilla calls and puts, with modified inputs. The functions return both call and put prices with a named vector:

```
s <- 100; k <- 100; r <- 0.08; v <- 0.30; tt <- 2; d <- 0.03; m <- 3
geomavgpricecall(s, 98:102, v, r, tt, d, m)
[1] 14.01 13.56 13.12 12.70 12.28
geomavgpricecall(s, 98:102, v, r, tt, d, m, cont=TRUE)
[1] 10.952 10.498 10.058  9.632  9.219
geomavgstrikecall(s, k, v, r, tt, d, m)
[1] 9.058
```

2.6.2 Arithmetic Asian Options

Monte Carlo valuation is used to price arithmetic Asian options. For efficiency, the function `arithasianmc` returns call and put prices for average price and average strike options. By default the number of simulations is 1000. Optionally the function returns the standard deviation of each estimate

---

[7]See Kemna and Vorst (1990).

[8]As the functions are currently written, each invocation of the pricing function will start with a different random number seed, resulting in price variation that is due solely to random variation. Moreover, random number generation changes the random number seed globally. In a future release I hope to address this by saving and restoring the seed within the greeks function. For the curious, a Stackoverflow post discusses this issue.
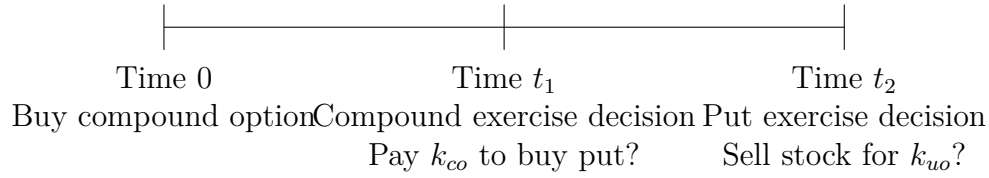
Figure 3: The timeline for a compound option: a call to buy a put. The compound option expires at time t_1 and the underlying asset is a put option that expires at time t_2. At time t_1, the owner decides whether to pay k_{co} to buy a put option which has time to expiration $t_2 - t_1$. At time t_2 the owner decides whether to exercise the put, selling the stock for the strike price of k_{uo}.

```
arithasianmc(s, k, v, r, tt, d, 3, numsim=5000, printsds=TRUE)
             Call    Put sd Call sd Put
Avg Price  13.930  8.272   21.61 11.701
Avg Strike  8.341  5.259   14.27  7.466
Vanilla    20.020 11.279   32.74 15.244
```

The function `arithavgpricecv` uses the control variate method to reduce the variance in the simulation. At the moment this function prices only calls, and returns both the price and the regression coefficient used in the control variate correction:

```
arithavgpricecv(s, k, v, r, tt, d, 3, numsim=5000)
 price   beta
13.958  1.046
```

2.7   Compound Options

A compound option is an option where the underlying asset is an option.[9] The terminology associated with compound options can be confusing, so it may be easiest to start with an example.

Figure 3 is a timeline for a compound option that is an option to buy a put. The compound option expires at t_1 and the put expires at t_2. The owner of the compound option only acquires the put if at time t_1 it is worth at least k_{co}, and only exercises the put if at time t_2 the stock price is less than k_{uo}.

---

[9]See Geske (1979) and McDonald (2013), Chapter 14.

2.7.1 Definition of a Compound Option

Based on the example, you can see that there are three prices associated with a compound option:

- The price of an underlying asset.
- The price of the underlying option, which is an option to buy or sell the underlying asset (we will refer to this as the price of the underlying option)
- The price of the compound option, which gives us the right to buy or sell the underlying option

The definition of a compound option therefore requires that we specify

- whether the underlying option is a put or a call

- whether the compound option is a put or a call

- the strike price at which you can exercise the underlying option (k_{uo})

- the strike price at which you can exercise the compound option (k_{co})

- the date at which you can exercise the compound option (first exercise date, t_1)

- the date at which you can exercise the underlying option expires, $t_2 > t_1$.

Given these possibilities, you can have a call on a call, a put on a call, a call on a put, and a put on a put. The valuation procedure require calculating the underlying asset price at which you are indifferent about acquiring the underlying option.

The price calculation requires computing the stock price above or below which you would optimally exercise the option at time t_1.

2.7.2 Examples

As an example, consider the following inputs for a call option to buy a call option:

```
s <- 100; kuo <- 95; v <- 0.30; r <-  0.08; t1 <- 0.50; t2 <- 0.75; d <- 0
kco <- 3.50

calloncall(s, kuo, kco, v, r, t1, t2, d, returnscritical=TRUE)
    price scritical
    13.12     88.68
```

With these parameters, after 6 months ($t_1 = 0.5$), the compound option buyer decides whether to pay \$3.50 to acquire a 3-month call on the underlying asset. (The volatility of the underlying asset is 0.3.) It will be worthwhile to pay the compound strike, \$3.50, as long as the underlying asset price exceeds 88.68.

Similarly, there is a put on the call, and a call and put on the corresponding put:

```
putoncall(s, kuo, kco, v, r, t1, t2, d, returnscritical=TRUE)
    price scritical
   0.5492   88.6800
callonput(s, kuo, kco, v, r, t1, t2, d, returnscritical=TRUE)
    price scritical
   3.425     98.298
putonput(s, kuo, kco, v, r, t1, t2, d, returnscritical=TRUE)
    price scritical
   1.384     98.298
```

2.8   Jumps and Stochastic Volatility

The `mertonjump` function returns call and put prices for a stock that can jump discretely.[10] A poisson process controls the occurrence of a jump and the size of the jump is lognormally distributed. The parameter `lambda` is the mean number of jumps per year, the parameter `alphaj` is the log of the expected jump, and `sigmaj` is the standard deviation of the log of the jump. The jump amount is thus drawn from the distribution

$$Y \sim \mathcal{N}(\alpha_J - 0.5\sigma_J^2, \sigma_J^2)$$

```
mertonjump(s, k, v, r, tt, d, lambda=0.5, alphaj=-0.2, vj=0.3)
   Call   Put
1 28.15 13.37
c(bscall(s, k, v, r, tt, d), bsput(s, k, v, r, tt, d))
[1] 24.025  9.239
```

2.9   Bonds

The simple bond functions provided in this version compute the present value of cash flows (`bondpv`), the IRR of the bond (`bondyield`), Macaulay duration (`duration`), and convexity (`convexity`).

```
coupon <- 8; mat <- 20; yield <- 0.06; principal <- 100;
modified <- FALSE; freq <- 2
price <- bondpv(coupon, mat, yield, principal, freq)
price
[1] 123.1
bondyield(price, coupon, mat, principal, freq)
```

---

[10]See Merton (1976).

```
[1] 0.06
duration(price, coupon, mat, principal, freq, modified)
[1] 11.23
convexity(price, coupon, mat, principal, freq)
[1] 170.3
```

3 Simulated prices

The function `simprice` provides a flexible way to generate prices that can be used for Monte Carlo or any other purpose.

3.1 Simulated price paths

A simple example is to generate 5 random sample paths of daily stock prices for a year; see Figure 4:

```
s0 <- 100; v <- 0.3; r <- 0.10; d <- 0; tt <- 1
trials <- 5; periods <- 365; set.seed(1)
s <- simprice(s0 = s0, v = v, r = r, tt = tt, d = d, trials = trials,
              periods = periods, jump = FALSE, long = TRUE)
ggplot(s, aes(x = period, y = price, color = trial, group = trial)) + geom_line()
```

We can do the same exercise adding jumps; see Figure 5.

```
s0 <- 100; v <- 0.3; r <- 0.10; d <- 0; tt <- 1
trials <- 5; periods <- 365; jump <- TRUE; lambda <- 2;
alphaj <- -0.1; vj <- 0.2; set.seed(1)
s <- simprice(s0 = s0, v = v, r = r, tt = tt, d = d, trials = trials,
              periods = periods, jump = jump, alphaj = alphaj,
              lambda = lambda, vj = vj, long = TRUE)
ggplot(s, aes(x = period, y = price, color = trial, group = trial)) + geom_line()
```
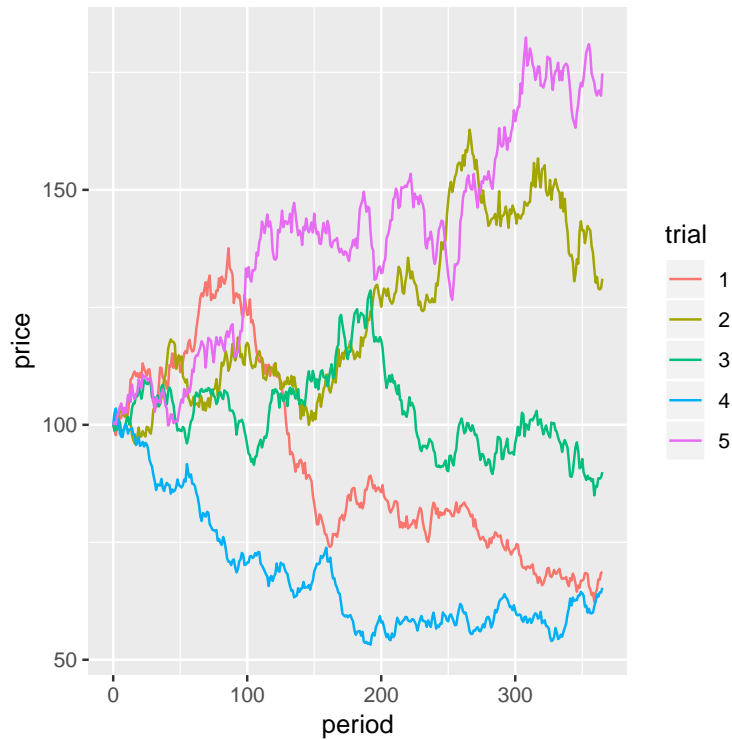
Figure 4: Five simulated paths for the same stock, no jumps.

3.2 Multiple correlated stocks

Here is an example using a three-asset covariance matrix, showing that the simulated series have the correct variance structure. We have three correlated stocks with standard deviations of 60%, 20%, and 45%., with the covariance matrix given by v. Compare the estimated covariance matrix with v:

```r
set.seed(1)
vi <- diag(3)
tt <- 2; periods <- tt*365
diag(vi) <- c(.6, .2, .45)
vc <- matrix(c(1, .4, -.3,
               .4, 1, .25,
               -.3, .25, 1), nrow = 3)
v <- vi %*% vc %*% vi
s <- simprice(s0 = s0, v = v, r = r, tt = tt, d = d, trials = 1,
              periods = periods, jump = FALSE, long = TRUE)

threestocks <- s %>%
    filter(trial == 1) %>%
```
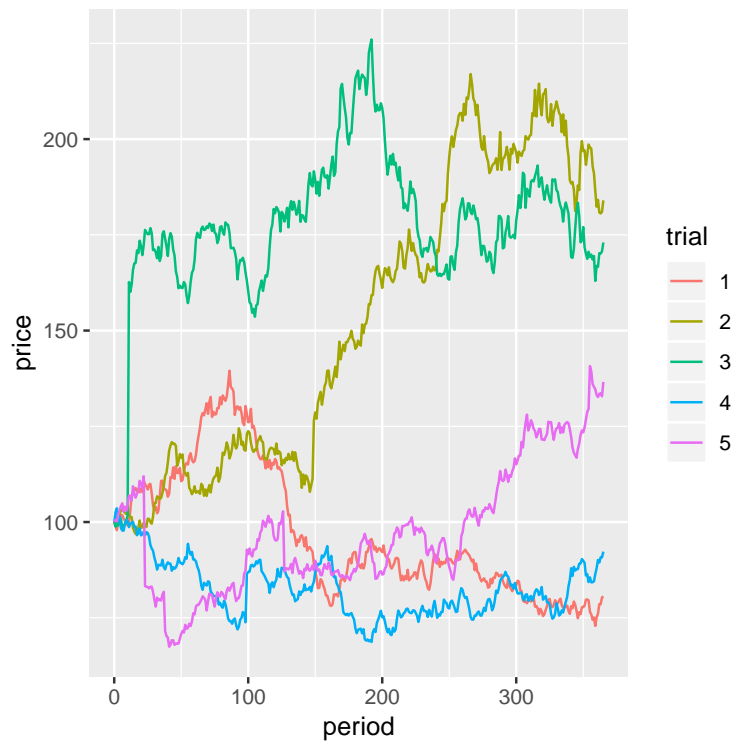
Figure 5: Five simulated paths for the same stock, which can jump.

```
    group_by(asset) %>%
    mutate(ret = log(price/lag(price)),
           row = row_number()) %>%
    select(asset, period, ret) %>%
    spread(key = asset, value = ret )

var(threestocks[2:4], na.rm = TRUE)*365
          1       2        3
1  0.37637 0.04590 -0.07944
2  0.04590 0.04163  0.02933
3 -0.07944 0.02933  0.23029
v
        [,1]    [,2]     [,3]
[1,]   0.360 0.0480 -0.0810
[2,]   0.048 0.0400  0.0225
[3,] -0.081 0.0225  0.2025
```

4 Functions with Graphical Output

Several functions provide visual illustrations of some aspects of the material.

4.1 Quincunx or Galton Board

The quincunx is a physical device the illustrates the central limit theorem. A ball rolls down a pegboard and strikes a peg, falling randomly either to the left or right. As it continues down the board it continues to strike a series of pegs, randomly falling left or right at each. The balls collect in bins and create an approximate normal distribution.

The quincunx function allows the user to simulate a quincunx, observing the path of each ball and watching the height of each bin as the balls accumulate. More interestingly, the quincunx function permits altering the probability that the ball will fall to the right.

Figure 6 illustrates the function after dropping 200 balls down 20 levels of pegs with a 70% probability that each ball will fall right:

```
par(mar=c(2,2,2,2))
quincunx(n=20, numballs=200, delay=0, probright=0.7)
```

4.2 Plotting the Solution to the Binomial Pricing Model

The `binomplot` function calls `binomopt` to compute the option price and the various trees, which it then uses in plotting:

The first plot, figure 7, is basic:

```
s0 <- 100; k <- 100; v <- 0.3; r <- 0.08; tt <- 2; d <- 0
binomplot(s0, k, v, r, tt, d, nstep=6, american=TRUE, putopt=TRUE)
```

The second plot, figure 8, adds a display of stock prices and arrows connecting the nodes.

```
binomplot(s0, k, v, r, tt, d, nstep=6, american=TRUE, putopt=TRUE,
    plotvalues=TRUE, plotarrows=TRUE)
```

As a final example, consider an American call when the dividend yield is positive and `nstep` has a larger value. Figure 9 shows the plot, with early exercise evident.

```
d <- 0.06
binomplot(s0, k, v, r, tt, d, nstep=40, american=TRUE)
```
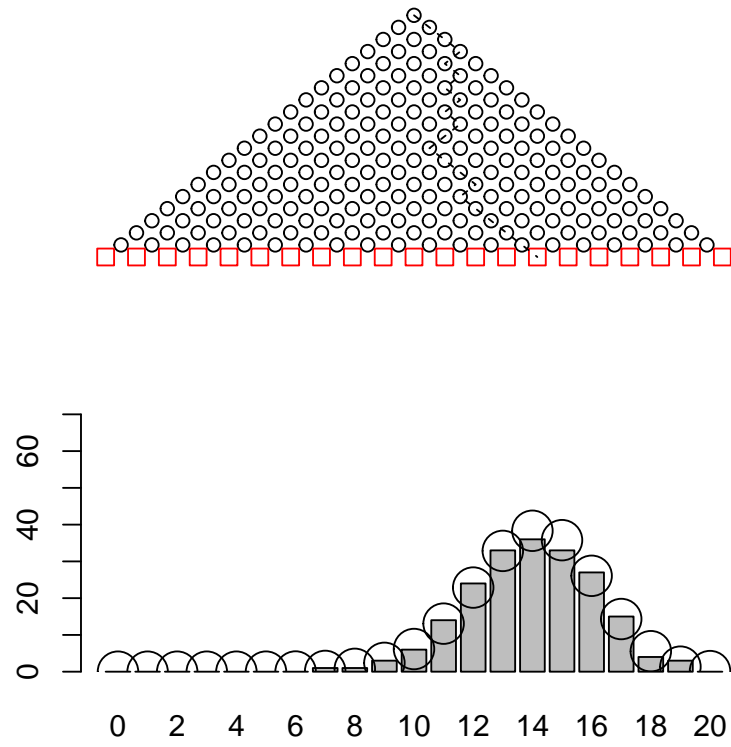
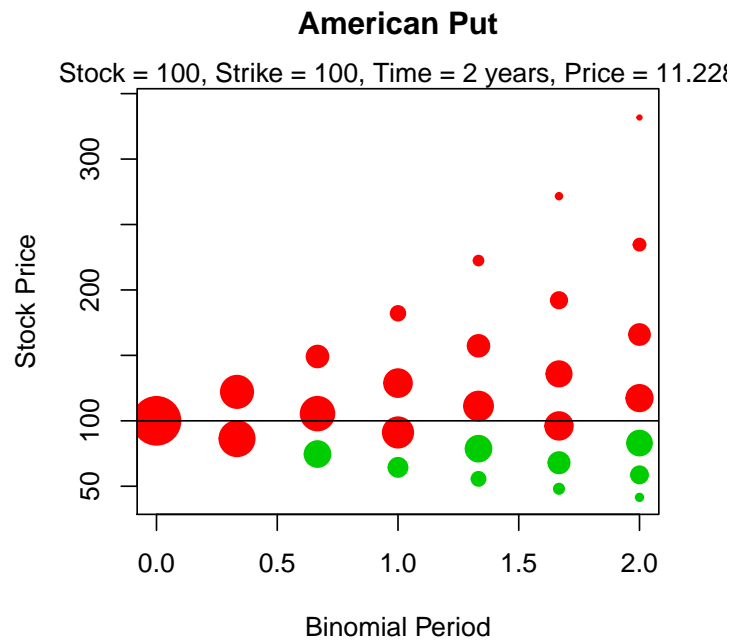Figure 6: Output from the Quincunx function

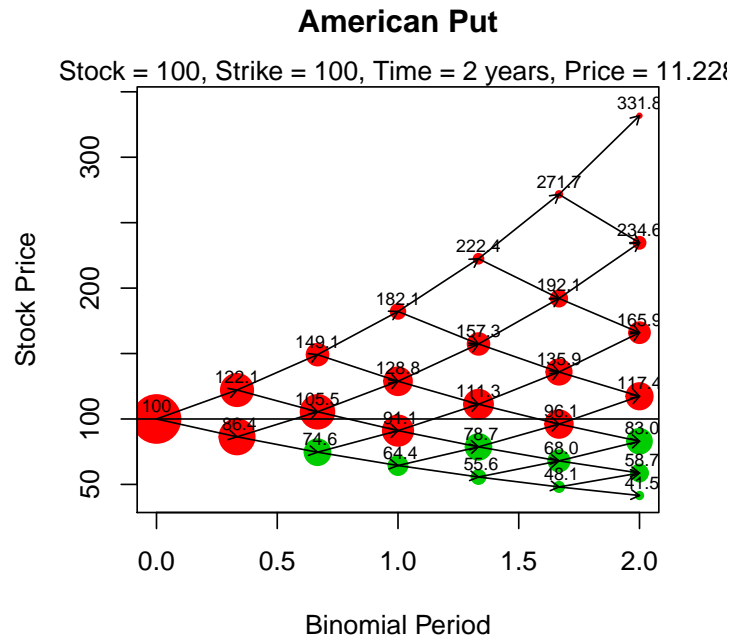Figure 7: Basic option plot showing stock prices and nodes at which the option is exercised.

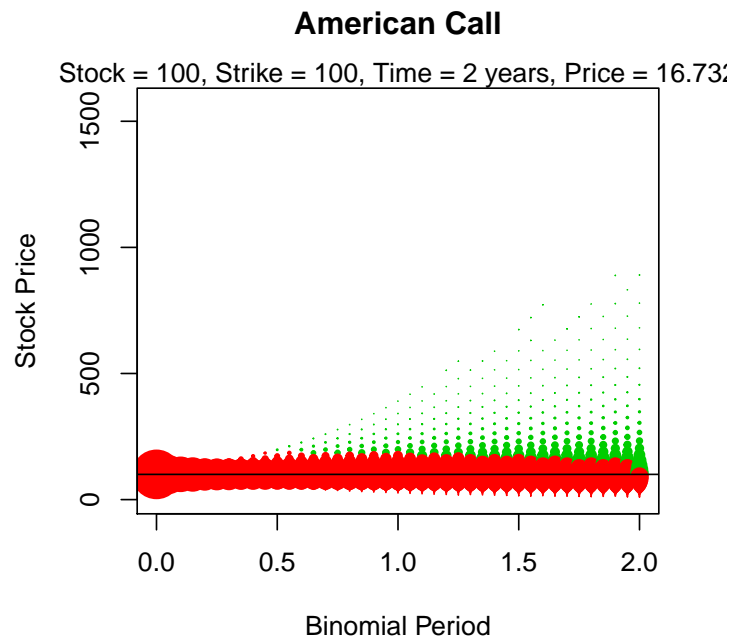Figure 8: Same plot as Figure 7 except that values and arrows are added to the plot.
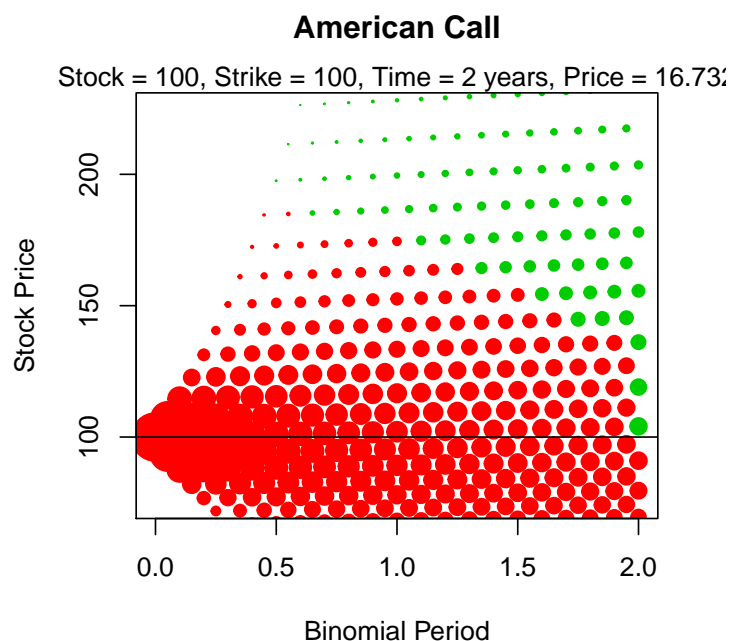

Figure 9: Binomial plot when nstep is 40.

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American Call

Stock = 100, Strike = 100, Time = 2 years, Price = 16.73.

Figure 10: Binomial plot when nstep is 40 using the argument ylimval to focus on a subset.

The large value of `nstep` creates a high maximum terminal stock price, which makes details hard to discern in the boundary region where exercise first occurrs. We can zoom in on that region by selecting values for `ylimval`; the result is in Figure 10.

```
d <- 0.06
binomplot(s0, k, v, r, tt, d, nstep=40, american=TRUE, ylimval=c(75, 225))
```

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