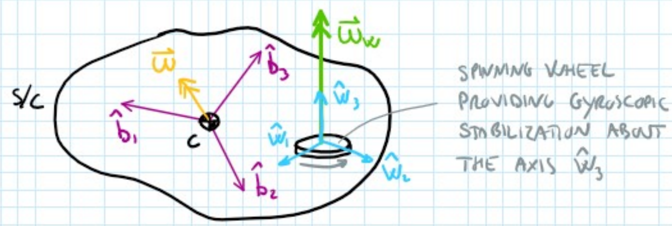
ARO 4090 - WEEK 12

- M. MAGGIA

Lecture 18

DUAL-SPIN STABILIZATION

- PASSIVE STABILIZATION TECHNIQUE
- IMPROVEMENT ON SINGLE-SPIN STABILIZATION APPROACH:
  - ↳ WE DO NOT REQUIRE THE ENTIRE S/C TO SPIN ABOUT THE MAJOR AXIS; THE GYROSCOPIC STABILIZATION IS PROVIDED BY A SPINNING WHEEL MOUNTED IN THE S/C WITH ITS SPIN AXIS ALIGNED WITH THE DESIRED AXIS WE WISH TO KEEP INERTIALLY FIXED.



  - ↳ THE NAME "DUAL-SPIN" COMES FROM THE FACT THAT THERE ARE 2 ANGULAR VELOCITIES INVOLVED

1) $\vec{\omega}$: ANG. VEL. OF S/C W.R.T. INERTIAL RF (PROPERLY : $\vec{\omega}^{B/I}$)

2) $\vec{\omega}_w$: ANG. VEL. OF WHEEL W.R.T. BODY-RF (PROPERLY : $\vec{\omega}^{W/B}$)

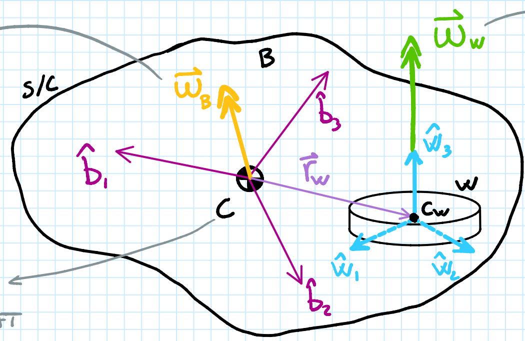
NOTICE THAT WE NEED 3 RF'S :

$\mathcal{I}$-RF WITH VERSORS $\hat{i}_1, \hat{i}_2, \hat{i}_3$ (NOT SHOWN IN FIG.)

$\mathcal{B}$-RF WITH VERSORS $\hat{b}_1, \hat{b}_2, \hat{b}_3$ (FIXED WITH S/C)

$\mathcal{W}$-RF WITH VERSORS $\hat{w}_1, \hat{w}_2, \hat{w}_3$ ($\hat{w}_3$ IS FIXED WITH S/C)

- ★ THE WHEEL IS ALSO KNOWN AS FLYWHEEL (FW) OR MOMENTUM WHEEL (MW)
- ★ FOR DUAL-SPIN STABILIZATION WE'LL CONSIDER ONLY 1 WHEEL SPINNING AT CONSTANT ANGULAR SPEED; HOWEVER, WE COULD MOUNT MORE THAN 1 RW WITH VARYING SPEEDS (IN THIS CASE, THE CONTROL WOULD BE ACTIVE, NOT PASSIVE. NORMALLY 4 RWs).
- ★ LET'S CONSIDER THE S/C AS THE SUM OF BODY (B) AND WHEEL (W)



$$(S/C) = (B) + (W)$$

$$\vec{H}^{C}_{S/C_{\mathcal{B}}} = \boxed{\vec{H}^{C}_{B_{\mathcal{B}}}} + \boxed{\vec{H}^{C}_{W_{\mathcal{B}}}}$$

ANG. MOMENTUM W.R.T. C

BODY

$$\underbrace{I^{C}_{B_{\mathcal{B}}}}_{\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}} \underbrace{\vec{\omega}^{\,B/I}_{\mathcal{B}}}_{\begin{bmatrix} p \\ q \\ r \end{bmatrix}} \; +$$

WHEEL

$$\underbrace{I^{C}_{W_{\mathcal{B}}}}_{} \; \underbrace{\vec{\omega}^{\,W/I}_{\mathcal{B}}}_{}$$

$$I^{C}_{W_{\mathcal{B}}} = \underbrace{I^{C_W}_{W_{\mathcal{B}}}}_{} - \underbrace{m_W}_{\text{MASS OF WHEEL}} \underbrace{\left[r_{W_{\mathcal{B}}}\right]^2_{\times}}_{}$$

$$\vec{\omega}^{\,W/I}_{\mathcal{B}} = \underbrace{\vec{\omega}^{\,W/\mathcal{B}}_{\mathcal{B}}}_{} + \underbrace{\vec{\omega}^{\,\mathcal{B}/I}_{\mathcal{B}}}_{\begin{bmatrix} p \\ q \\ r \end{bmatrix}}$$

$$\vec{\omega}^{\,W/\mathcal{B}}_{\mathcal{B}} = \underbrace{R_{\mathcal{B}W}}_{} \; \underbrace{\vec{\omega}^{\,W/\mathcal{B}}_{W}}_{\begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix}}$$

$R_{\mathcal{B}W}$: DEPENDS ON RELATIVE ORIENTATION OF WHEEL W.R.T. $\mathcal{B}$-RF AXES

$\vec{\omega}^{\,W/\mathcal{B}}_{W}$: ANG. VEL. OF WHEEL W.R.T. BODY, IN W-RF COORDS.

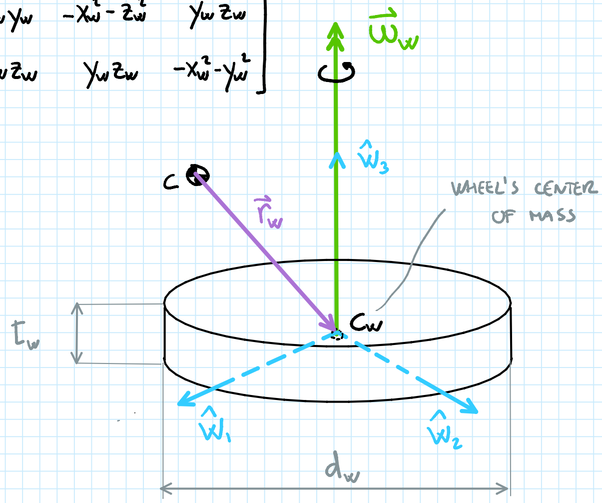
$$I^{C_W}_{W_{\mathcal{B}}} = R_{\mathcal{B}W} \cdot \underbrace{I^{C_W}_{W_W}}_{} \cdot R_{W\mathcal{B}}$$

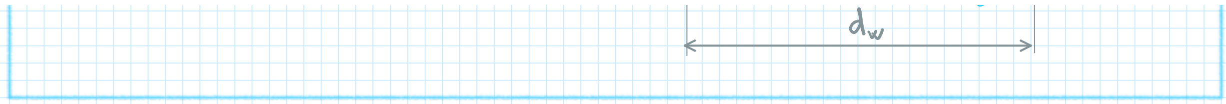
$$I^{C_W}_{W_W} = \begin{bmatrix} I_{tw} & 0 & 0 \\ 0 & I_{tw} & 0 \\ 0 & 0 & I_{aw} \end{bmatrix}$$

$$\left[r_{W_{\mathcal{B}}}\right]^2_{\times} = \begin{bmatrix} -y_w^2 - z_w^2 & x_w y_w & x_w z_w \\ x_w y_w & -x_w^2 - z_w^2 & y_w z_w \\ x_w z_w & y_w z_w & -x_w^2 - y_w^2 \end{bmatrix}$$

$$I_{aw} = \frac{1}{4} m_w d_w^2$$

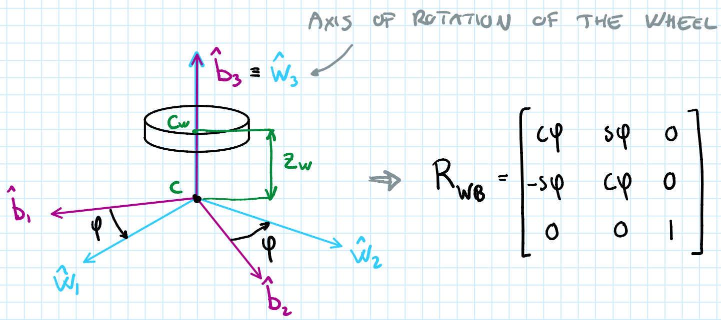
$$I_{tw} = \frac{1}{12} m_w \left(\frac{3}{4} d_w^2 + t_w^2 \right)$$





LET'S ASSUME

- $\hat{w}_3 \equiv \hat{b}_3 \quad \& \quad x_w = y_w = 0 \quad \& \quad \Omega = \text{CONST.}$



$$\Rightarrow \quad R_{WB} = \begin{bmatrix} c\varphi & s\varphi & 0 \\ -s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{BW} = R_{WB}^{T}$$

THUS :

$$I^{c}_{w_B} = \begin{bmatrix} I_{t_w} + m_w z_w^2 & 0 & 0 \\ 0 & I_{t_w} + m_w z_w^2 & 0 \\ 0 & 0 & I_{a_w} \end{bmatrix}$$

$$\vec{\omega}^{\,W/I}_{B} = \begin{bmatrix} p \\ q \\ r + \Omega \end{bmatrix}$$

$$\vec{H}^{c}_{S/C_B} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \begin{bmatrix} I_{t_w} + m_w z_w^2 & 0 & 0 \\ 0 & I_{t_w} + m_w z_w^2 & 0 \\ 0 & 0 & I_{a_w} \end{bmatrix} \begin{bmatrix} p \\ q \\ r + \Omega \end{bmatrix}$$

$$\frac{{}^{I}d\vec{H}^{c}_{S/C_B}}{dt} = \dot{\vec{H}}^{c}_{S/C_B} + \omega^{B/I}_{B} \wedge \vec{H}^{c}_{S/C_B} = \vec{M}^{c}_{B}$$

$$= \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} I_{t_w} + m_w z_w^2 & 0 & 0 \\ 0 & I_{t_w} + m_w z_w^2 & 0 \\ 0 & 0 & I_{a_w} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \left(\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \begin{bmatrix} I_{t_w} + m_w z_w^2 & 0 & 0 \\ 0 & I_{t_w} + m_w z_w^2 & 0 \\ 0 & 0 & I_{a_w} \end{bmatrix} \begin{bmatrix} p \\ q \\ r + \Omega \end{bmatrix} \right) = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

$$
+\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}\left(\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} + \begin{bmatrix} I_{tw}+m_w z_w^2 & 0 & 0 \\ 0 & I_{tw}+m_w z_w^2 & 0 \\ 0 & 0 & I_{aw} \end{bmatrix}\begin{bmatrix} p \\ q \\ r+\Omega \end{bmatrix}\right) = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}
$$

$$
\begin{aligned}
M_x &= (I_{xx} + I_{tw} + m_w z_w^2)\,\dot{p} + (I_{zz} - I_{yy})\,qr + (I_{aw} - I_{tw} - m_w z_w^2)\,q\,(r+\Omega) \\
M_y &= (I_{yy} + I_{tw} + m_w z_w^2)\,\dot{q} + (I_{xx} - I_{zz})\,rp + (I_{tw} + m_w z_w^2 - I_{aw})\,p\,(r+\Omega) \\
M_z &= (I_{zz} + I_{aw})\,\dot{r} + (I_{yy} - I_{xx})\,pq
\end{aligned}
$$

$$
\begin{aligned}
M_x &= (I_{xx} + I_{tw} + m_w z_w^2)\,\dot{p} + (I_{zz} + I_{aw} - (I_{yy} + I_{tw} + m_w z_w^2))\,qr + (I_{aw} - I_{tw} - m_w z_w^2)\,q\Omega \\
M_y &= (I_{yy} + I_{tw} + m_w z_w^2)\,\dot{q} + ((I_{xx} + I_{tw} + m_w z_w^2) - (I_{zz} + I_{aw}))\,pr + (I_{tw} + m_w z_w^2 - I_{aw})\,p\Omega \\
M_z &= (I_{zz} + I_{aw})\,\dot{r} + (I_{yy} - I_{xx})\,pq
\end{aligned}
$$

- IF THE RW IS SMALL/LIGHT W.R.T. THE BODY

$$
\begin{aligned}
I_{xx} &\approx \boxed{I_{xx}} + \boxed{I_{tw} + m_w z_w^2} \\
I_{yy} &\approx \boxed{I_{yy}} + \boxed{I_{tw} + m_w z_w^2} \\
I_z &\approx \boxed{I_{zz}} + \boxed{I_{aw}}
\end{aligned}
$$

BODY >> WHEEL

LET'S ALSO CALL

$$I_w := I_{aw} - I_{tw} - m_w z_w^2$$

THUS THE E.O.M BECOME

$$
\begin{aligned}
M_x &\approx I_{xx}\,\dot{p} + (I_{zz} - I_{yy})\,qr + I_w\,q\Omega \\
M_y &\approx I_{yy}\,\dot{q} + (I_{xx} - I_{zz})\,pr - I_w\,p\Omega \\
M_z &\approx I_{zz}\,\dot{r} + (I_{yy} - I_{xx})\,pq
\end{aligned}
$$

★ SINCE WE WANT TO STUDY THE STABILITY OF THE TORQUE-FREE MOTION $M_x = M_y = M_z = 0$

TORQUE-FREE MOTION OF DUAL-SPIN S/C (BODY + FLYWHEEL MOUNTED ON Z-AXIS)

$$
\boxed{
\begin{aligned}
&\boxed{\begin{aligned} &I_{xx}\,\dot{p} + (I_{zz} - I_{yy})\,qr \\ &I_{yy}\,\dot{q} + (I_{xx} - I_{zz})\,pr \\ &I_{zz}\,\dot{r} + (I_{yy} - I_{xx})\,pq \end{aligned}}
\boxed{\begin{aligned} &+ I_w\,q\Omega \\ &- I_w\,p\Omega \\ & \end{aligned}}
\begin{aligned} &= 0 \\ &= 0 \\ &= 0 \end{aligned}
\end{aligned}
}
$$

CONTRIBUTION TO DYNAMICS OF S/C WITHOUT WHEEL (SAME AS SINGLE-SPIN)

CONTRIBUTION TO DYNAMICS OF THE FLYWHEEL

WE, IN FACT, CHANGED THE S/C DYNAMICS.

$$\boxed{\begin{aligned} \dot{p} &= \frac{I_{yy} - I_{zz}}{I_{xx}} qr - \frac{I_w}{I_{xx}} q\Omega \\ \dot{q} &= \frac{I_{zz} - I_{xx}}{I_{yy}} pr + \frac{I_w}{I_{yy}} p\Omega \\ \dot{r} &= \frac{I_{xx} - I_{yy}}{I_{zz}} pq \end{aligned}}$$

NONLINEAR SYSTEM

- WE CAN CLEARLY SEE THAT THE STATE $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$ REPRESENTS AN EQUILIBRIUM POINT FOR THE DUAL-SPIN S/C TOO.

- LET'S ANALYZE THE STABILITY OF $\vec{x}_0 = [p_0, q_0, r_0] = [0, 0, n]^T$:

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} 0 & \frac{I_{yy} - I_{zz}}{I_{xx}} r_0 - \frac{I_w}{I_{xx}}\Omega & \frac{I_{yy} - I_{zz}}{I_{xx}} q_0 \\ \frac{I_{zz} - I_{xx}}{I_{yy}} r_0 + \frac{I_w}{I_{yy}}\Omega & 0 & \frac{I_{zz} - I_{xx}}{I_{yy}} p_0 \\ \frac{I_{xx} - I_{yy}}{I_{zz}} q_0 & \frac{I_{xx} - I_{yy}}{I_{zz}} p_0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} 0 & \frac{I_{yy} - I_{zz}}{I_{xx}} n - \frac{I_w}{I_{xx}}\Omega & 0 \\ \frac{I_{zz} - I_{xx}}{I_{yy}} n + \frac{I_w}{I_{yy}}\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

or, EQUIVALENTLY:

$$\begin{cases} \dot{\varepsilon}_x + \left(\sigma_x n + \frac{I_w}{I_{xx}}\Omega\right) \varepsilon_y = 0 \\ \dot{\varepsilon}_y + \left(\sigma_y n - \frac{I_w}{I_{yy}}\Omega\right) \varepsilon_x = 0 \\ \dot{\varepsilon}_z = 0 \end{cases}$$

(NOTICE THAT X-Y AND Z DYNAMICS ARE ONCE AGAIN DECOUPLED)

$\mathcal{L}$

$$\begin{cases} sE_x - \varepsilon_x(0) + \left(\sigma_x n + \frac{I_w}{I_{xx}}\Omega\right) E_y = 0 \\ sE_y - \varepsilon_y(0) + \left(\sigma_y n - \frac{I_w}{I_{yy}}\Omega\right) E_x = 0 \end{cases}$$

$$\begin{cases} sE_y - \mathcal{E}_y(0) + \left(\sigma_y n - \frac{I_w}{I_{yy}}\Omega\right)E_x = 0 \\ sE_z - \mathcal{E}_z(0) = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} s & \sigma_x n + \frac{I_w}{I_{xx}}\Omega & 0 \\ \sigma_y n - \frac{I_w}{I_{yy}}\Omega & s & 0 \\ 0 & 0 & s \end{bmatrix}}_{A} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \mathcal{E}_x(0) \\ \mathcal{E}_y(0) \\ \mathcal{E}_z(0) \end{bmatrix}$$

$$\det(A) = \boxed{s\left(s^2 - n^2\sigma_x\sigma_y + I_w\Omega\left(\frac{\sigma_x n}{I_{yy}} - \frac{\sigma_y n}{I_{xx}} + \frac{I_w\Omega}{I_{xx}I_{yy}}\right)\right)}$$ → CHARACTERISTIC POLYNOMIAL

$$s_1 = 0$$

$$s_{2,3} = \pm\sqrt{n^2\sigma_x\sigma_y - I_w\Omega\left(\frac{\sigma_x n}{I_{yy}} - \frac{\sigma_y n}{I_{xx}} + \frac{I_w\Omega}{I_{xx}I_{yy}}\right)}$$

- RECALL THAT FOR SINGLE-SPIN S/C $\sigma_x\sigma_y > 0 \Longrightarrow$ UNSTABLE EQUILIBRIUM

- HOWEVER, FOR DUAL-SPIN S/C, WE CAN FIND VALUES OF Ω SO THAT THE ARGUMENT OF THE SQUARE ROOT IS NEGATIVE, PRODUCING PURELY IMAGINARY POLES AND THUS CREATING NEUTRALLY STABLE EQUILIBRIUM.

FOR NEUTRAL STABILITY:

$$n^2\sigma_x\sigma_y - I_w\Omega\left(\frac{\sigma_x n}{I_{yy}} - \frac{\sigma_y n}{I_{xx}} + \frac{I_w\Omega}{I_{xx}I_{yy}}\right) < 0$$

$$-\frac{\Omega^2 I_w^2}{I_{xx}I_{yy}} - \Omega I_w n\left(\frac{\sigma_x}{I_{yy}} - \frac{\sigma_y}{I_{xx}}\right) + n^2\sigma_x\sigma_y < 0$$

$$\frac{\Omega^2 I_w^2}{I_{xx}I_{yy}} + \Omega I_w n\left(\frac{\sigma_x}{I_{yy}} - \frac{\sigma_y}{I_{xx}}\right) - n^2\sigma_x\sigma_y > 0$$

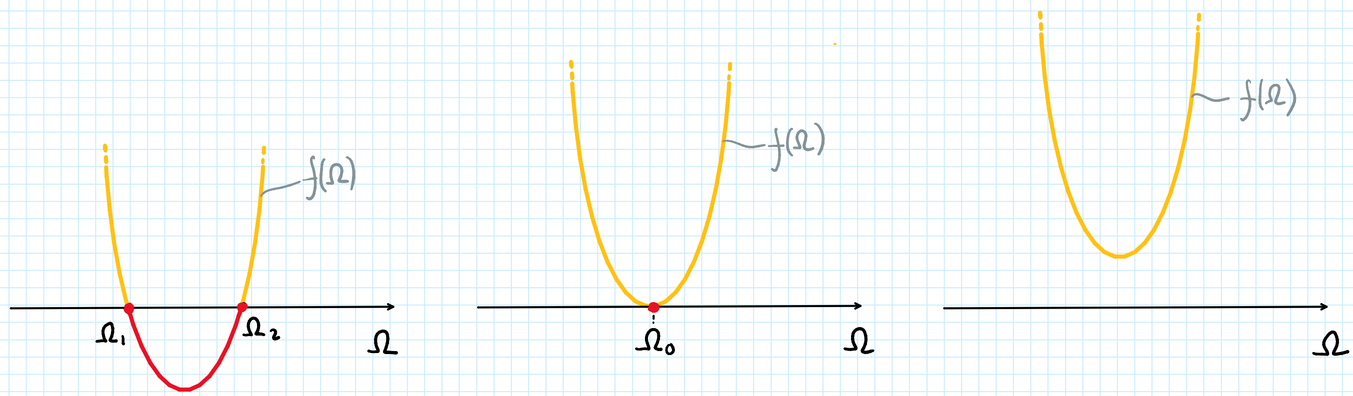
$$\Omega^2 + \Omega\frac{\cancel{I_w}\cdot I_{xx}I_{yy}}{I_w^{\cancel{2}}}n\left(\frac{\sigma_x}{I_{yy}} - \frac{\sigma_y}{I_{xx}}\right) - n^2\frac{\sigma_x\sigma_y I_{xx}I_{yy}}{I_w^2} > 0$$

$$\Omega^2 + \Omega\frac{n}{I_w}\left(\sigma_x I_{xx} - \sigma_y I_{yy}\right) - n^2\frac{\sigma_x\sigma_y I_{xx}I_{yy}}{I_w^2} > 0$$

$$\Omega^2 + \Omega \underbrace{\boxed{\frac{n}{I_w}\left(I_{zz}-I_{yy}+I_{zz}-I_{xx}\right)}}_{b} + \underbrace{\boxed{\frac{n^2}{I_w^2}\left(I_{zz}-I_{yy}\right)\left(I_{zz}-I_{xx}\right)}}_{c} > 0$$

REPRESENTS A PARABOLA

$$f(\Omega) = \Omega^2 + b\,\Omega + c > 0$$



CASE I		CASE II		CASE III	
N. STABLE	$\Omega < \Omega_1$ or $\Omega > \Omega_2$	N. STABLE	$\Omega \neq \Omega_0$	N. STABLE	$\forall \Omega$
UNSTABLE	$\Omega \in [\Omega_1, \Omega_2]$	UNSTABLE	$\Omega = \Omega_0$	UNSTABLE	ϕ
IF	$b^2 - 4c > 0$	IF	$b^2 - 4c = 0$	IF	$b^2 - 4c < 0$

$$b^2 - 4c = \frac{n^2}{I_w^2}\left(\cancel{4I_{zz}^2} + I_{xx}^2 + I_{yy}^2 - \cancel{4I_{zz}I_{xx}} - \cancel{4I_{zz}I_{yy}} + 2I_{xx}I_{yy} - \cancel{4I_{zz}^2} + \cancel{4I_{yy}I_{zz}} + \cancel{4I_{xx}I_{zz}} - 4I_{xx}I_{yy}\right)$$

$$= \frac{n^2}{I_w^2}\left(I_{xx}^2 + I_{yy}^2 - 2I_{xx}I_{yy}\right)$$

$$= \frac{n^2}{I_w^2}\left(I_{xx} - I_{yy}\right)^2 \geq 0$$

- IF $I_{xx} \neq I_{yy} \iff$ CASE I

$$\Omega_{1,2} = -\frac{b}{2} \mp \frac{1}{2}\sqrt{b^2-4c}$$

$$= \frac{n}{I_w}\left(\frac{I_{xx}+I_{yy}}{2} - I_{zz}\right) \mp \frac{|n(I_{xx}-I_{yy})|}{2I_w}$$

$$\overset{!}{=} \frac{n}{I_w}\left(\frac{I_{xx}+I_{yy}}{2} - I_{zz}\right) \mp \frac{|n(I_{xx}-I_{yy})|}{2I_w}$$

•• IF $I_{xx} = I_{yy} = I_t \Leftrightarrow$ CASE II (AXIALSYMMETRY)

$(I_{zz} = I_a)$

$$\Omega_0 = n\,\frac{I_t - I_a}{I_w}$$