Homework #3

Due: **Sunday, February 25, 2024 at 11:59 pm**

Turn in, on Canvas, a well organized and clear-to-read pdf file with your calculations, explanations, plots, as well as a ready-to-run .m/.slx files containing the MATLAB/Simulink codes (when applicable).

Problem 1

(a) Given a rotation matrix $R \in \mathbb{R}^{3\times 3}$, using Euler's formula, show that:

$$R(\alpha,\ \hat{\mathbf{a}}) = R(-\alpha,\ -\hat{\mathbf{a}}) = R(\alpha - 360°,\ \hat{\mathbf{a}}) = R(360° - \alpha,\ -\hat{\mathbf{a}}).$$

(b) If R is parametrized by the angles (Ω, i, ω) obtained through a 3-1-3 rotation sequence, show that the two singularities occur for $i = 0°$ and $i = 180°$.

Problem 2

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Problem 3

Create a single Simulink model with blocks that perform the following tasks:

Block 1. Input: attitude $(\alpha,\ \hat{\mathbf{a}})$; Output: rotation matrix $R(\alpha,\ \hat{\mathbf{a}}) \in \mathbb{R}^{3\times 3}$.

Block 2. Input: rotation matrix $R \in \mathbb{R}^{3\times 3}$; Output: attitude $(\alpha,\ \hat{\mathbf{a}})$, with $0 \le \alpha \le 180°$.

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Include the Simulink model with your submission. Models that won't compile/run properly will not receive any credit.

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I) SOLVE FOR $R(\alpha, \hat{a})$

USING EULER'S FORMULA AS FOLLOWS:

↳ $R(\alpha, \hat{a}) = \cos\alpha \cdot I + (1-\cos\alpha)\hat{a}\hat{a}^T - \sin\alpha[\hat{a}]_\times$

WHERE:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \hat{a}\hat{a}^T = \begin{bmatrix} a_x^2 & a_xa_y & a_xa_z \\ a_xa_y & a_y^2 & a_ya_z \\ a_xa_z & a_ya_z & a_z^2 \end{bmatrix} ; \quad [\hat{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

WE GET THE FOLLOWING:

$$R(\alpha,\hat{a}) = \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} + (1-\cos\alpha)\begin{bmatrix} a_x^2 & a_xa_y & a_xa_z \\ a_xa_y & a_y^2 & a_ya_z \\ a_xa_z & a_ya_z & a_z^2 \end{bmatrix} - \sin\alpha \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha + a_x^2 - a_x^2\cos\alpha & a_xa_y - a_xa_y\cos\alpha + a_z\sin\alpha & a_xa_z - a_xa_z\cos\alpha - a_y\sin\alpha \\ a_xa_y - a_xa_y\cos\alpha - a_z\sin\alpha & \cos\alpha + a_y^2 - a_y^2\cos\alpha & a_ya_z - a_ya_z\cos\alpha + a_x\sin\alpha \\ a_xa_z - a_xa_z\cos\alpha + a_y\sin\alpha & a_ya_z - a_ya_z\cos\alpha - a_x\sin\alpha & \cos\alpha + a_z^2 - a_z^2\cos\alpha \end{bmatrix}$$

$$\therefore \boxed{R(\alpha,\hat{a}) = \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha + a_x^2 - a_x^2\cos\alpha & a_xa_y - a_xa_y\cos\alpha + a_z\sin\alpha & a_xa_z - a_xa_z\cos\alpha - a_y\sin\alpha \\ a_xa_y - a_xa_y\cos\alpha - a_z\sin\alpha & \cos\alpha + a_y^2 - a_y^2\cos\alpha & a_ya_z - a_ya_z\cos\alpha + a_x\sin\alpha \\ a_xa_z - a_xa_z\cos\alpha + a_y\sin\alpha & a_ya_z - a_ya_z\cos\alpha - a_x\sin\alpha & \cos\alpha + a_z^2 - a_z^2\cos\alpha \end{bmatrix}}$$

II) $R(-\alpha, -\hat{a})$

$$= \begin{bmatrix} \cos(-\alpha) & 0 & 0 \\ 0 & \cos(-\alpha) & 0 \\ 0 & 0 & \cos(-\alpha) \end{bmatrix} + \begin{bmatrix} \cos(-\alpha) + (-a_x)^2 - (-a_x)^2\cos(-\alpha) & a_xa_y - a_xa_y\cos\alpha + a_z\sin\alpha & a_xa_z - a_xa_z\cos\alpha - a_y\sin\alpha \\ (-a_x)(-a_y) - (-a_xa_y)\cos(-\alpha) - (-a_z)\sin(-\alpha) & \cos\alpha + a_y^2 - a_y^2\cos\alpha & a_ya_z - a_ya_z\cos\alpha + a_x\sin\alpha \\ (-a_x)(-a_y) - (-a_x)(-a_y)\cos(-\alpha) + (-a_y\sin(-\alpha)) & a_ya_z - a_ya_z\cos\alpha - a_x\sin\alpha & \cos\alpha + a_z^2 - a_z^2\cos\alpha \end{bmatrix}$$

WHICH SIMPLIFIES TO: USING $\cos(-\alpha) = \cos(\alpha)$ & $\sin(-\alpha) = -\sin(\alpha)$

$$R(-\alpha,-\hat{a}) = \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha + a_x^2 - a_x^2\cos\alpha & a_xa_y - a_xa_y\cos\alpha + a_z\sin\alpha & a_xa_z - a_xa_z\cos\alpha - a_y\sin\alpha \\ a_xa_y - a_xa_y\cos\alpha - a_z\sin\alpha & \cos\alpha + a_y^2 - a_y^2\cos\alpha & a_ya_z - a_ya_z\cos\alpha + a_x\sin\alpha \\ a_xa_z - a_xa_z\cos\alpha + a_y\sin\alpha & a_ya_z - a_ya_z\cos\alpha - a_x\sin\alpha & \cos\alpha + a_z^2 - a_z^2\cos\alpha \end{bmatrix} = R(\alpha,\hat{a})$$

$$\boxed{\therefore R(-\alpha,\hat{a}) = R(\alpha,\hat{a})}$$

III) $R(\alpha - 360°, \hat{a})$

$*$ $\cos(\alpha - 360°) = \cos(\alpha)$ & $\sin(\alpha - 360°) = \sin(\alpha)$

$$= \begin{bmatrix} \cos(\alpha-360°) & 0 & 0 \\ 0 & \cos(\alpha-360°) & 0 \\ 0 & 0 & \cos(\alpha-360°) \end{bmatrix} + (1-\cos(\alpha-360))\begin{bmatrix} a_x^2 & a_xa_y & a_xa_z \\ a_xa_y & a_y^2 & a_ya_z \\ a_xa_z & a_ya_z & a_z^2 \end{bmatrix} - \sin(\alpha-360)\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$\leftarrow *$

$$= \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} + (1-\cos\alpha)\begin{bmatrix} a_x^2 & a_xa_y & a_xa_z \\ a_xa_y & a_y^2 & a_ya_z \\ a_xa_z & a_ya_z & a_z^2 \end{bmatrix} - \sin\alpha \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} = R(\alpha,\hat{a})$$

$$\boxed{\therefore R(\alpha-360,\hat{a}) = R(\alpha,\hat{a}) = R(-\alpha,-\hat{a})}$$

IV) $R(360^\circ - \alpha, -\hat{a})$ ** $\cos(360-\alpha) = \cos\alpha$ & $\sin(360-\alpha) = -\sin\alpha$

$$= \begin{bmatrix} \cos(360-\alpha) & 0 & 0 \\ 0 & \cos(360-\alpha) & 0 \\ 0 & 0 & \cos(360-\alpha) \end{bmatrix} + (1-\cos(360-\alpha)) \begin{bmatrix} \bar{a}_x^2 & \bar{a}_x \bar{a}_y & \bar{a}_x \bar{a}_z \\ \bar{a}_x \bar{a}_y & \bar{a}_y^2 & \bar{a}_y \bar{a}_z \\ \bar{a}_x \bar{a}_z & \bar{a}_y \bar{a}_z & \bar{a}_z^2 \end{bmatrix} - \sin(360-\alpha) \begin{bmatrix} 0 & \bar{a}_z & \bar{a}_y \\ \bar{a}_z & 0 & -\bar{a}_x \\ -\bar{a}_y & \bar{a}_x & 0 \end{bmatrix}$$

$\longleftarrow$ **

$$= \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} + (1-\cos\alpha) \begin{bmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{bmatrix} - \sin\alpha \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} = R(\alpha, \hat{a})$$

$$\boxed{\therefore \quad R(360-\alpha, -\hat{a}) = R(\alpha, \hat{a}) = R(-\alpha, -\hat{a}) = R(\alpha - 360, \hat{a})}$$

(b) If R is parametrized by the angles (Ω, i, ω) obtained through a 3-1-3 rotation sequence, show that the two singularities occur for $i = 0^\circ$ and $i = 180^\circ$.

$$R = R_3(\Omega) \cdot R_1(i) \cdot R_3(\omega)$$

WHERE

$$R_1(i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & \sin(i) \\ 0 & -\sin(i) & \cos(i) \end{bmatrix} \; ; \; R(\Omega) = \begin{bmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; R(\omega) = \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hookrightarrow R = R_3(\Omega) \cdot R_1(i) \cdot R_3(\omega) = \begin{bmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & \sin(i) \\ 0 & -\sin(i) & \cos(i) \end{bmatrix} \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\cos(i)\sin(\omega) & \cos(i)\cos(\omega) & \sin(i) \\ \sin(i)\sin(\omega) & -\cos(\omega)\sin(i) & \cos(i) \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \cos(\Omega)\cos(\omega) - \cos(i)\sin(\omega)\sin(\Omega) & \cos(\Omega)\sin(\omega) + \cos(i)\cos(\omega)\sin(\Omega) & \sin(\Omega)\sin(i) \\ -\cos(\omega)\sin(\Omega) - \cos(i)\cos(\Omega)\sin(\omega) & -\sin(\Omega)\sin(\omega) + \cos(i)\cos(\Omega)\cos(\omega) & \cos(\Omega)\sin(i) \\ \sin(i)\sin(\omega) & -\cos(\omega)\sin(i) & \cos(i) \end{bmatrix}$$

I) $i = 0^\circ$

BY COMPARISON:

- $r_{33} = \cos(i)$
- $\dfrac{r_{13}}{r_{23}} = \dfrac{\sin(\Omega)\sin(i)}{\cos(\Omega)\sin(i)} \xleftarrow{i=0^\circ} = \dfrac{\sin(\Omega)\sin(0)}{\cos(\Omega)\sin(0)}$
- $\dfrac{r_{31}}{r_{32}} = \dfrac{\sin(\omega)\sin(i)}{-\cos(\omega)\sin(i)} \xleftarrow{i=0^\circ} = \dfrac{\sin(\omega)\sin(0)}{-\cos(\omega)\sin(0)}$

$$\boxed{\sin(0) = 0 \longrightarrow \frac{0}{0} \therefore \text{ SINGULARITY}}$$

II) $i = 180^\circ$

BY COMPARISON:

- $r_{33} = \cos(i)$
- $\dfrac{r_{13}}{r_{23}} = \dfrac{\sin(\Omega)\sin(i)}{\cos(\Omega)\sin(i)} \xleftarrow{i=180^\circ} = \dfrac{\sin(\Omega)\sin(0)}{\cos(\Omega)\sin(0)}$
- $\dfrac{r_{31}}{r_{32}} = \dfrac{\sin(\omega)\sin(i)}{-\cos(\omega)\sin(i)} \xleftarrow{i=180^\circ} = \dfrac{\sin(\omega)\sin(0)}{-\cos(\omega)\sin(0)}$

$$\boxed{\sin(180) = 0 \longrightarrow \frac{0}{0} \therefore \text{ SINGULARITY}}$$

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```
% Inputs to Homework 3 Simulink Model
clear; clc;
% Inputs for Block 1

alpha = 104.47                  % Rotation Angles [ degrees ]
a = [-1 ; 1 ; -sqrt(3)]         % Axis of Rotation Versor

%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~%

% Inputs for Block 3

psi =  90             % Yaw [ Degrees ]
theta = 45             % Pitches [ Degrees ]
phi =   90       % Roll [ Degrees ]
```

See attached .m MATLAB script to run Simulink Model