Conical Flow: Supersonic Flow Past Un-Yawed Cone Computer Project

Class: ARO 3111 – Gas Dynamics & Highspeed Aerodynamics

Section #02

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List of Symbols:

M = Mach Number

M_n = Normal Component of Mach Number

P = Pressure

ρ = Density

T = Temperature

γ = Specific Heat Ratio (1.4)

$\beta = \theta_s$ = Shock Angle

$\delta = \theta$ = Deflection Angle

θ_c = Cone Angle

∞ = Freestream

o = Stagnation values

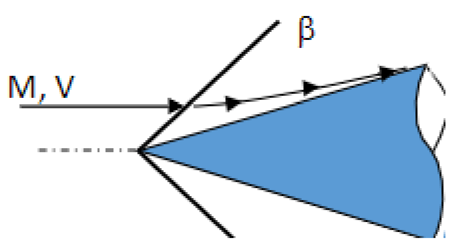
c = Surface of the cone

a = Speed of Sound (1089 ft/s)

V = Flow Velocity

V_θ = Normal Velocity Component

V_r = Radial Velocity Component



Summary of Key Equation

Taylor-Maccoll Equation: $\frac{dV_r}{d\theta} = V_\theta$; $\frac{d^2V_r}{d\theta^2} = \frac{V_\theta^2 V_r - \frac{\gamma-1}{2}(1-V_r^2-V_\theta^2)(2V_r+V_\theta \cot\theta)}{\frac{(\gamma-1)}{2}(1-V_r^2-V_\theta^2)-V_\theta^2}$

Initial Conditions at Station 2: $V_r = V_\infty \cos\beta$; $V_\theta = V_\infty \sin\theta_i [\frac{(\gamma-1)M_\infty^2 \sin^2\theta_i + 2}{(\gamma+1)M_\infty^2 \sin^2\theta_i}]$

Runge-Kutta Equations:

$$u_1 = V_r, \qquad \frac{du_1}{d\theta} = f_1$$

$$u_2 = \frac{dV_r}{d\theta}, \qquad \frac{du_2}{d\theta} = f_2$$

$$k_{1,1} = hf_1(t_i, w_{1,i}, w_{2,i}),$$
$$k_{1,2} = hf_2(t_i, w_{1,i}, w_{2,i}),$$

$$k_{2,1} = hf_1\left(t_i + \frac{h}{2}, w_{1,i} + \frac{k_{1,1}}{2}, w_{2,i} + \frac{k_{1,2}}{2}\right),$$
$$k_{2,2} = hf_2\left(t_i + \frac{h}{2}, w_{1,i} + \frac{k_{1,1}}{2}, w_{2,i} + \frac{k_{1,2}}{2}\right),$$

$$k_{3,1} = hf_1\left(t_i + \frac{h}{2}, w_{1,i} + \frac{k_{2,1}}{2}, w_{2,i} + \frac{k_{2,2}}{2}\right),$$
$$k_{3,2} = hf_2\left(t_i + \frac{h}{2}, w_{1,i} + \frac{k_{2,1}}{2}, w_{2,i} + \frac{k_{2,2}}{2}\right),$$

$$k_{4,1} = hf_1(t_i + h, w_{1,i} + k_{3,1}, w_{2,i} + k_{3,2}),$$
$$k_{4,2} = hf_2(t_i + h, w_{1,i} + k_{3,1}, w_{2,i} + k_{3,2}),$$

$$w_{1,i+1} = w_{1,i} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}),$$
$$w_{2,i+1} = w_{2,i} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}),$$

Flow Property Equations:

$$\frac{p_o}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}, \quad \frac{\rho_o}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}, \quad \frac{T_o}{T} = 1 + \frac{\gamma-1}{2}M^2$$

Executive Summary:

The objective of this computer project is to write a computer program that calculates the supersonic flow properties over an un-yawed cone. The solution includes the usage of the Taylor-Maccoll equation and the utilization of the 4^{th} – order Runge-Kutta method to solve for specific parameters.

This computer program uses initial conditions of Free Stream Mach (M_∞), Shockwave Angle (θ_s), and the gas constant (γ). It then follows by solving for the deflection angle using the θ-β-M relation. Once these parameters have been found, the initial conditions V_r & V_θ can be found using the geometry of V_∞. These initial conditions are then used in the 4^{th} – order Runge-Kutta numerical solver to solve for the radial and normal components of flow velocity with respect to a desired iteration of shockwave angles. These values are then used to calculate the Mach Numbers (M) that correspond to different shockwave angels. Lastly, the Mach numbers are used to calculate the flow properties in the conical flow region using planar shock relations.

The angle of the shock cone was unknown, but it was apparent that once the normal component (V_θ) of the flow velocity (V_∞), reaches a non-negative value, this means that the deflection angle is equivalent to the cone angle. In this specific case, the cone angle was found to be 7°. The plots of the flow properties (pressure, density, and temperature ratios) across the cone, showed that when $\theta_s < 12°$, the properties were constant at a value of 1. Once the shock wave passed 12°, each ratio increased at different values, but in increased the same fashion. The Mach number was constant up to 12°, then rapidly dropped once 12° had passed. Lastly, V_θ showed a linear increase throughout the change in shockwave angles.

These results from the Taylor-Maccoll computer program turned out to be accurate as it matched hand calculations of the same algorithm. After an oblique shock, the pressure, density, and temperature all increase so this satisfies the results from the computer program. As the shockwave angle decreases this will result in the Mach number decreasing.

In conclusion, the integration of the Taylor-Maccoll equations with the 4th-order Runge-Kutta solver presents a robust and reliable method for computing the flow properties over a cone. This approach solves for the normal and radial components of the flow velocity, which are essential for deriving the overall flow velocity. These calculated velocities lay the ground for calculations, offering a precise and dependable solution to understand and predict the behavior of supersonic flows over conical geometries.

Plots

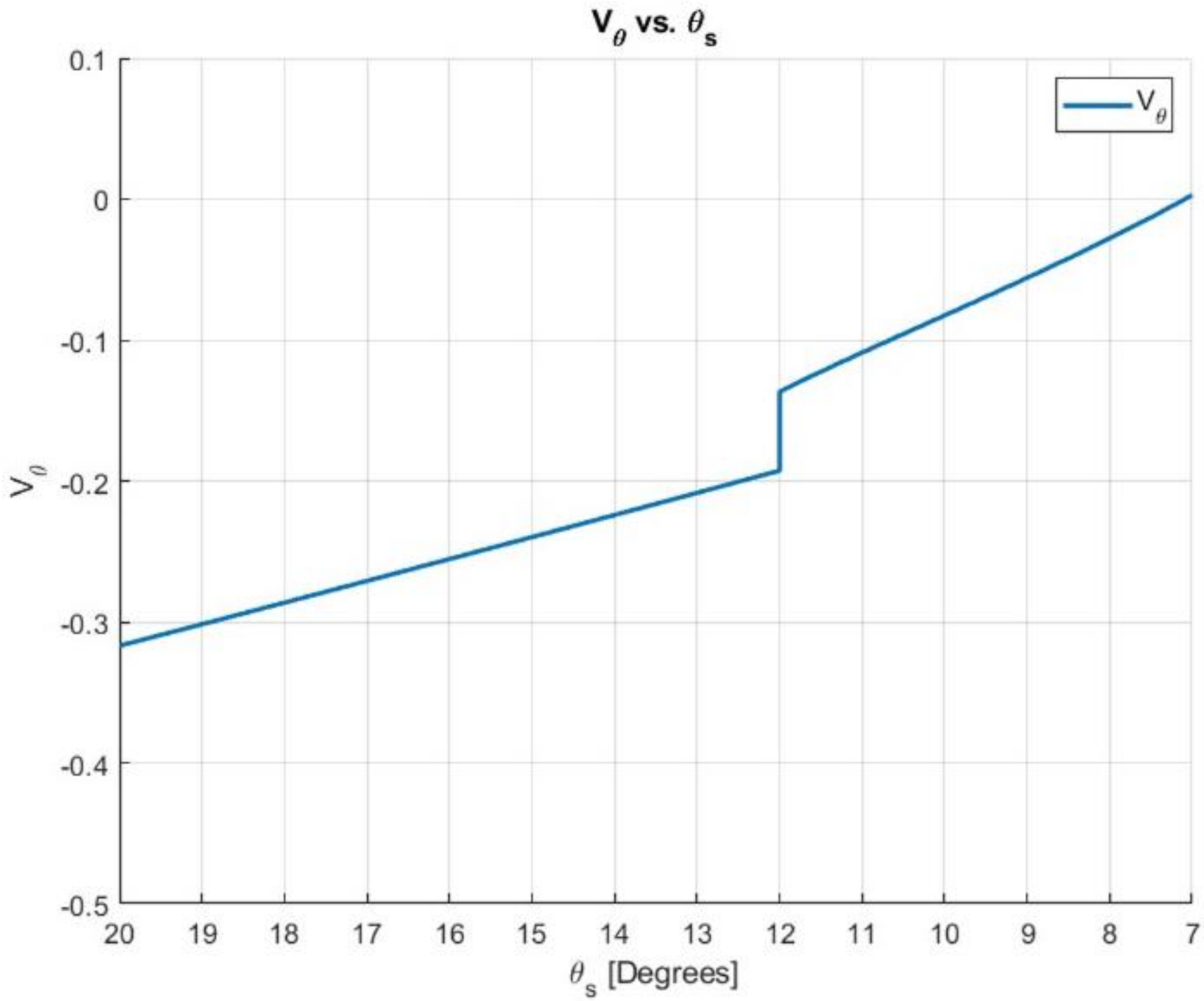


Figure 1: V_θ vs. θ_s

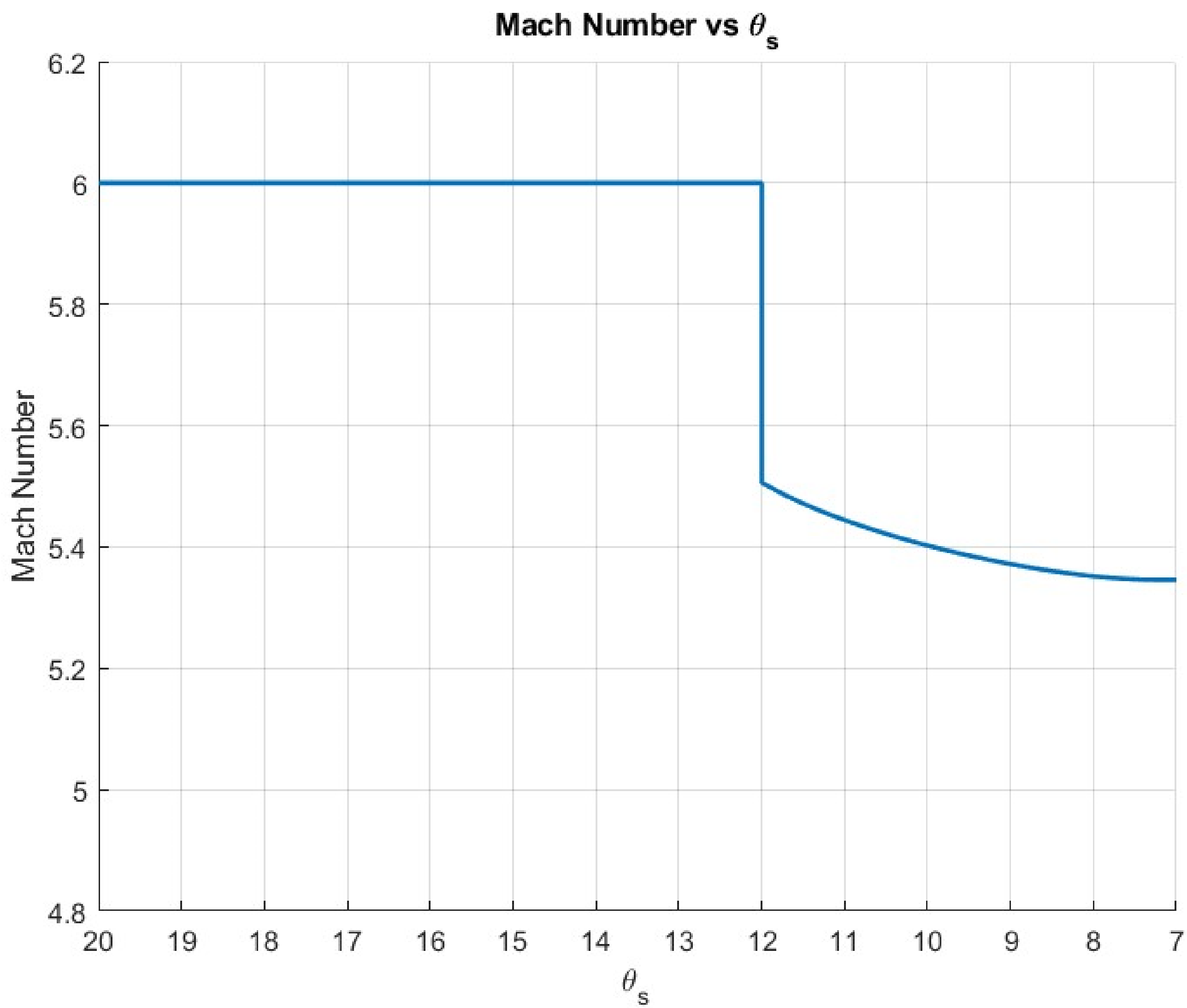


Figure 2: Mach Number vs. θ_s

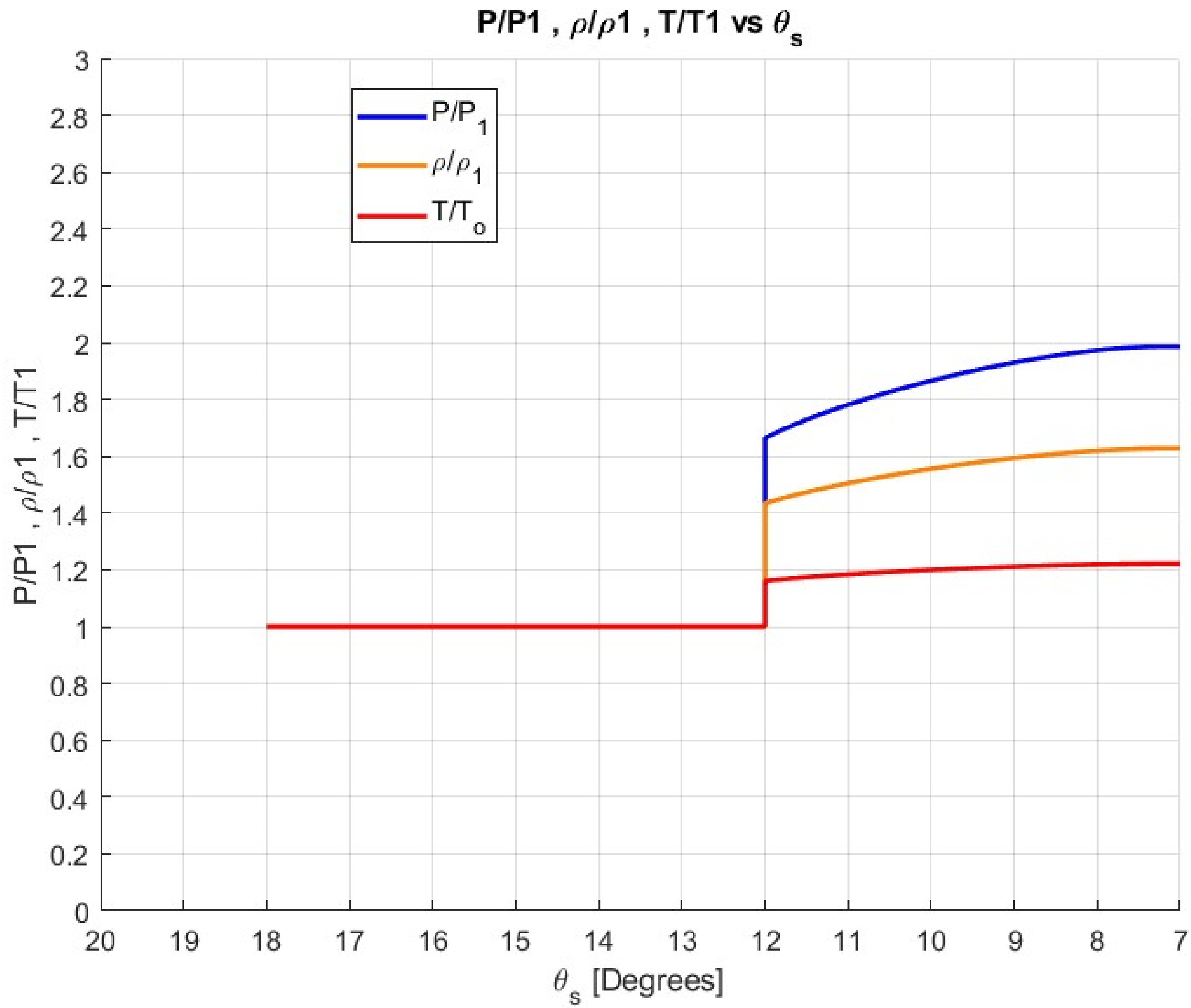


Figure 3: P/P_1 , ρ/ρ_1 , & T/T_1 vs. θ

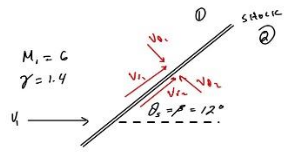
Table of Computed Data

Table 1: V_r, V_θ , Mach Number, and flow properties for various Shock Wave (θ_s) angles

θ_s	V_r	V_θ	Mach #	P/P_1	ρ/ρ_1	T/T_1
20	0.87083	-0.317	6	1	1	1
19	0.87623	-0.3017	6	1	1	1
18	0.88136	-0.2864	6	1	1	1
17	0.88623	-0.2709	6	1	1	1
16	0.89082	-0.2554	6	1	1	1
15	0.89514	-0.2399	6	1	1	1
14	0.89919	-0.2242	6	1	1	1
13	0.90297	-0.2085	6	1	1	1
12.1	0.90613	-0.1943	6	1	1	1
12	0.90647	-0.1927	5.50631	1	1	1
12	0.91657	-0.1368	5.49848	1.66413	1.43356	1.16083
11.9	0.9168	-0.1338	5.49115	1.67841	1.44234	1.16367
11.8	0.91703	-0.1309	5.48426	1.6919	1.45061	1.16634
11.7	0.91726	-0.1281	5.47773	1.70472	1.45845	1.16885
11.6	0.91748	-0.1252	5.47153	1.71694	1.46591	1.17124
11.5	0.9177	-0.1225	5.46561	1.72865	1.47305	1.17352
11.4	0.91791	-0.1197	5.45995	1.7399	1.47989	1.1757
11.3	0.91811	-0.117	5.45453	1.75074	1.48647	1.17779
11.2	0.91832	-0.1143	5.44931	1.76121	1.49281	1.17979
11.1	0.91851	-0.1116	5.44429	1.77133	1.49894	1.18173
11	0.91871	-0.1089	5.43945	1.78114	1.50486	1.18359
10.9	0.91889	-0.1063	5.43477	1.79066	1.5106	1.1854
10.8	0.91908	-0.1036	5.43025	1.7999	1.51616	1.18714
10.7	0.91926	-0.101	5.42588	1.80888	1.52157	1.18883
10.6	0.91943	-0.0983	5.42165	1.81763	1.52681	1.19047
10.5	0.9196	-0.0957	5.41755	1.82613	1.53192	1.19206
10.4	0.91976	-0.0931	5.41357	1.83442	1.53688	1.1936
10.3	0.91992	-0.0904	5.40972	1.84249	1.5417	1.1951
10.2	0.92008	-0.0878	5.40598	1.85036	1.5464	1.19656
10.1	0.92023	-0.0852	5.40235	1.85802	1.55097	1.19797
10	0.92038	-0.0825	5.39883	1.86549	1.55542	1.19934
9.9	0.92052	-0.0799	5.39542	1.87276	1.55975	1.20068

θ_s	V_r	V_θ	Mach #	P/P_1	ρ/ρ_1	T/T_1
9.8	0.92066	-0.0773	5.39212	1.87984	1.56396	1.20197
9.7	0.92079	-0.0746	5.38891	1.88674	1.56806	1.20323
9.6	0.92092	-0.072	5.38581	1.89345	1.57204	1.20445
9.5	0.92104	-0.0693	5.38281	1.89997	1.5759	1.20564
9.4	0.92116	-0.0666	5.37991	1.9063	1.57965	1.20678
9.3	0.92127	-0.064	5.37711	1.91244	1.58329	1.20789
9.2	0.92138	-0.0613	5.37441	1.9184	1.58681	1.20896
9.1	0.92149	-0.0586	5.37181	1.92415	1.59021	1.21
9	0.92159	-0.0559	5.36931	1.92972	1.59349	1.211
8.9	0.92168	-0.0531	5.36692	1.93508	1.59665	1.21196
8.8	0.92177	-0.0504	5.36463	1.94023	1.59969	1.21288
8.7	0.92186	-0.0476	5.36245	1.94517	1.6026	1.21376
8.6	0.92194	-0.0449	5.36038	1.94989	1.60537	1.2146
8.5	0.92201	-0.0421	5.35842	1.95439	1.60802	1.2154
8.4	0.92208	-0.0393	5.35658	1.95864	1.61052	1.21616
8.3	0.92215	-0.0364	5.35487	1.96266	1.61287	1.21687
8.2	0.92221	-0.0336	5.35327	1.96642	1.61508	1.21754
8.1	0.92227	-0.0307	5.35181	1.96991	1.61713	1.21815
8	0.92232	-0.0278	5.35049	1.97312	1.61901	1.21872
7.9	0.92236	-0.0249	5.3493	1.97603	1.62072	1.21923
7.8	0.92241	-0.022	5.34827	1.97863	1.62224	1.21969
7.7	0.92244	-0.019	5.3483	1.98091	1.62357	1.22009
7.6	0.92247	-0.016	5.3474	1.98283	1.6247	1.22043
7.5	0.9225	-0.0129	5.3467	1.98439	1.62561	1.2207
7.4	0.92252	-0.0099	5.3462	1.98555	1.62629	1.22091
7.3	0.92253	-0.0067	5.3458	1.98629	1.62672	1.22104
7.2	0.92254	-0.0036	5.3457	1.98659	1.6269	1.22109
7.1	0.92254	-0.0004	5.3458	1.9864	1.62679	1.22106
7	0.92254	0.00284	5.3457	1.9866	1.6269	1.22109

Appendix A: Hand Calculations of flow properties at station 2 and station C.



STATION ②

STATION Ⓒ at the cone

* Calculate $v_1/v_{max} = v_2/v_{max} = \left[1 + \frac{2}{(\gamma-1)M_1^2}\right]^{-0.5}$

$$= \left[1 + \frac{2}{(1.4-1)6^2}\right]^{-0.5} = 0.937$$

→ $v_1/v_{max} = 0.937 = V_{\infty_1}$

→ $v_2/v_{max} = 0.937 = V_{\infty_2}$

* Calculate v_{r_1}/v_{max} & v_{r_2}/v_{max} ⇒ $V_{\infty} \cos(\beta)$

↳ $v_{r_1}/v_{max} = V_{\infty_1}\cos\beta = 0.937\cos(12) = 0.9166$

↳ $v_{r_2}/v_{max} = V_{\infty_2}\cos\beta = 0.937\cos(12) = 0.9166$

→ $v_{r_1}/v_{max} = 0.9166$

→ $v_{r_2}/v_{max} = 0.9166$

* Calculate Ratios at Station 2 ; p_2/p_1 , ρ_2/ρ_1 , T_2/T_1 , p_{0_2}/p_{0_1}

$M_{n_1} = M_1 \sin(\beta) = 6\sin(12) = 1.2474$

$p_2/p_1 = 1 + \frac{2\gamma}{\gamma+1}(M_{n_1}^2 - 1)$ —— Eq. 4.9 JA

$$= 1 + \frac{2(1.4)}{1.4+1}(1.247^2 - 1) = 1.6489 \rightarrow p_2/p_1$$

$\rho_2/\rho_1 = \frac{(\gamma+1)M_{n_1}^2}{(\gamma-1)M_{n_1}^2+2}$ —— Eq. 4.8 JA

$$= \frac{(1.4+1)1.247^2}{(1.4-1)1.247^2+2} = 1.424 \rightarrow \rho_2/\rho_1$$

$T_2/T_1 = (p_2/p_1)(\rho_1/\rho_2)$ —— Eq. 4.11 JA

$$= 1.6489(1/1.424) = 1.158 \rightarrow T_2/T_1$$

$M_{n_2} = \left[\frac{M_{n_1}^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_{n_1}^2 - 1}\right]^{1/2}$ —— Eq. 4.10 JA

$$= \left[\frac{1.247^2 + \frac{2}{1.4-1}}{\frac{2(1.4)}{1.4-1}(1.247)^2 - 1}\right]^{1/2} = 0.814 \rightarrow M_{n_2}$$

θ-β-M Relation

↳ $\theta = \tan^{-1}\left[2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + 2\cos(2\beta)) + 2}\right)\right]$ —— Eq. 4.17 JA

$$= \tan^{-1}\left[2\cot(12)\left(\frac{6^2\sin^2(12) - 1}{6^2(1.4 + 2\cos(24)) + 2}\right)\right] = 3.511° \rightarrow \theta = \delta$$

$M_2 = \frac{M_{n_2}}{\sin(\beta-\delta)} = \frac{0.814}{\sin(12-3.511)} = 5.514 \rightarrow M_2$

↳ $p_{0_1}/p_1 = \left(1 + \frac{\gamma-1}{2}M_1^2\right)^{\gamma/(\gamma-1)}$ —— Eq 3.30 JA

$$= \left(1 + \frac{1.4-1}{2}6^2\right)^{1.4/(1.4-1)} = 1578.9 \rightarrow p_{0_1}/p_1$$

↳ $p_{0_2}/p_2 = \left(1 + \frac{\gamma-1}{2}M_2^2\right)^{\gamma/(\gamma-1)}$

$$= \left(1 + \frac{1.4-1}{2}5.514^2\right)^{1.4/(1.4-1)} = 944.7 \rightarrow p_{0_2}/p_2$$

↳ $p_{0_2}/p_{0_1} = (p_{0_2}/p_2)(p_1/p_{0_1})(p_2/p_1)$

$$= (944.7)(1/1578.9)(1.6489) = 0.986 \rightarrow p_{0_2}/p_{0_1}$$

$$\boxed{\begin{aligned} p_2/p_1 &= 1.6489 \\ \rho_2/\rho_1 &= 1.424 \\ T_2/T_1 &= 1.158 \\ p_{0_2}/p_{0_1} &= 0.986 \end{aligned}}$$

Appendix B: MATLAB Code for Taylor-Maccoll & Runge Kutta

```
% ARO - 3111 Gas Dynamics & Highspeed Aerodynamics
% Computer Assignment - Taylor-Maccoll Program
% Written By: Justin Millsap
% Date: 03/21/2024
% Tool Version: R2023b
% other .m files required: RungeKutta.m
% Other files required: NONE
% Reference: John D. Anderson - Modern Compressible Flow

tic
%% Nomenclature
%{
- Station 1 is the free stream
- Station 2 is right behind the shock
- Station c is at the surface of the cone
- Subscript o is stagnation values
- Cone angles is theta_c
- Conical shock angle is theta_s or beta
- Initial turning of the streamline as the flow crosses the shock from 1 to 2
  is delta which is < theta_c
%}
clc;
clear

%% Givens


theta_s = 12:-0.1:0; % Shock wave angle [deg]
theta_s = deg2rad(theta_s);
M_1 = 6;
gamma = 1.4;
beta = 12; % [deg]
beta = deg2rad(beta);


% Solve for M2 & Delta (flow deflection angle)
M_n1 = M_1 * sin(beta);

% Theta-Beta-M relation [Eq 4.17]
num_tbm = (M_1)^2 * (sin(beta))^2 - 1;
den_tbm = (M_1)^2 * (gamma + cos(2*beta)) + 2;
delta = atan(2 * cot(beta) * (num_tbm / den_tbm));

% Solve for M_n2
num_Mn2 = (M_n1)^2 + (2 / (gamma - 1));
den_Mn2 = ((2 * gamma) / (gamma - 1)) * (M_n1)^2 - 1;
M_n2 = sqrt(num_Mn2 / den_Mn2);

% Solve for M2
M_2 = M_n2 / sin(beta - delta);

% Solve for V_prime, V_r_prime, V_theta_prime
V_prime = (2 / ((gamma - 1) * (M_2)^2) + 1)^(-0.5);
```

```
V_theta_prime = sin(beta - delta) * V_prime*(-1);
V_r_prime = cos(beta - delta) * V_prime;

%%

[V_r , V_theta] = RungeKutta(V_r_prime, V_theta_prime , theta_s, gamma);


V_r = V_r(1:51);
V_theta = V_theta(1:51);

V = sqrt(V_r.^2 + V_theta.^2);
M = zeros(1,160);
 for i = 1:51
 M(i+108) = sqrt(2/((V(i)^-2 - 1)*(gamma-1)) );
 end

M(1:109) = M_1;
M(160) = 5.3457;
angles_for_plots_Mach = 22.9:-0.1:7;
angles_for_plots_Mach(109) = 12;

%% Ratios

% Pressure ratio (Isolating for P/P1)
P2overP1 = (1 + ((2*gamma)/(gamma+1))*(M_n1^2 -1) ) ;                    % Equation 3.30
PoverPo2 = (1+((gamma-1)/2).*M.^2).^(-gamma/(gamma-1));                  % Equation 3.30
P2overPo2 = (1+((gamma-1)/2).*M_2.^2).^(-gamma/(gamma-1));               % Equation 3.30
PoverP1 = (PoverPo2/P2overPo2)*P2overP1;

% Density ratio (Isolating for rho/rho1)

rho2overrho1 = (((gamma+1)* M_n1^2) / ((gamma-1) * (M_n1^2)+2)) ;
rhooverrhoo2 = (1 +((gamma-1)/2).*M.^2).^(-1/(gamma-1));                 % Equation 3.30
rho2overrhoo2 = (1 +((gamma-1)/2).*M_2^2).^(-1/(gamma-1));               % Equation 3.30
rhooverrho1 = (rhooverrhoo2/rho2overrhoo2)*rho2overrho1;

% Tempertuare ratio (Isolating for T/T1)
T = 1 + (gamma-1)/2.*M.^2;
T1 = 1+ (gamma -1)*M_1^2 / 2;
ToverT1 = T1./T;

figure(1);
set(gcf, 'Position', [100, 100, 800, 600]);
hold on;
plot(angles_for_plots_Mach, M(1:160), 'LineWidth', 2);
set(gca, 'XDir', 'reverse');
ylim([4.8 6.2]);
yticks(4.8:0.2:6.2);
xlim([7 20]);
xticks(7:1:20);
ylabel('Mach Number', 'FontSize', 14);
xlabel('\theta_s', 'FontSize', 14);
title('Mach Number vs \theta_s', 'FontSize', 16);
set(gca, 'FontSize', 12);
grid on;
```

```
hold off;


angles_for_plots_ratio = 18:-0.1:7;
rhooverrho1 = rhooverrho1(50:160);
PoverP1 = PoverP1(50:160);
ToverT1 = ToverT1(50:160);

PoverP1(1:60) = 1;
rhooverrho1(1:60) = 1;
ToverT1(1:60) = 1;

angles_for_plots_ratio(60) = 12;

figure(2);
set(gcf, 'Position', [100, 100, 800, 600]);
hold on;
plot(angles_for_plots_ratio, PoverP1, 'Color', 'b', 'LineWidth', 2);
plot(angles_for_plots_ratio, rhooverrho1, 'Color', '[1, 0.5, 0]', 'LineWidth', 2);
plot(angles_for_plots_ratio, ToverT1, 'Color', 'r', 'LineWidth', 2);
legend('P/P_1', '\rho/\rho_1', 'T/T_o', 'Location', 'best', 'FontSize', 12);
set(gca, 'XDir', 'reverse');
ylim([0 3]);
yticks(0:0.2:3);
xlim([7 20]);
xticks(7:1:20);
ylabel('P/P1 , \rho/\rho1 , T/T1', 'FontSize', 14);
xlabel('\theta_s [Degrees]', 'FontSize', 14);
title('P/P1 , \rho/\rho1 , T/T1 vs \theta_s', 'FontSize', 16);
set(gca, 'FontSize', 12);
grid on;
hold off;



angles_for_plots_Vtheta = [20 , 19 , 18 , 17 , 16 , 15 , 14 , 13 , 12.1 , 12]*pi/180;

Vtheta = -V_prime*sin(angles_for_plots_Vtheta);
Vr = V_prime*cos(angles_for_plots_Vtheta);
angles_for_plots_Vtheta = [angles_for_plots_Vtheta theta_s(1:51)]*180/pi;
V_theta = [Vtheta V_theta];
figure(3);
hold on;
set(gcf, 'Position', [100, 100, 800, 600]);
plot(angles_for_plots_Vtheta, V_theta, 'LineWidth', 2);
set(gca, 'XDir', 'reverse');
legend('V_\theta', 'FontSize', 12);
ylim([-0.5 0.1]);
yticks([-0.5:0.1:0.2]);
xlim([7 20]);
xticks(7:1:20);
ylabel('V_\theta', 'FontSize', 14);
xlabel('\theta_s [Degrees]', 'FontSize', 14);
title("V_\theta vs. \theta_s", 'FontSize', 16);
set(gca, 'FontSize', 12);
grid on;
```

```
hold off;



V_r = [Vr V_r];
M = M(110:160);
MachCalc = [6 6 6 6 6 6 6 6 6 6];
Mach = [MachCalc M];
ones = [1 1 1 1 1 1 1 1 1 1];
T_overT1 = [ones ToverT1(61:111)];
rho_over_rho1 = [ones rhooverrho1(61:111)];
P_overP1 = [ones PoverP1(61:111)];
toc

fprintf('θ_s | V_r | V_θ | Mach # | P/P_1 | ρ/ρ_1 | T/T_1\n');

fprintf('------------------------------------------------------------------------\n');

for i = 1:61
   fprintf('%7.1f | %5.3f | %7.3f | %6.2f | %5.3f | %9.3f | %5.3f\n', ...
         angles_for_plots_Vtheta(i), V_r(i), V_theta(i), Mach(i), P_overP1(i), rho_over_rho1(i), T_overT1(i));
end



%% Runge Kutta Function
function [V_r, V_theta] = RungeKutta(V_r_prime, V_theta_prime, theta_s, gamma)
%{
RungeKutta method that solves for V_r & V_theta.
The inputs are the initial conditions of V_r and V_theta (V_r_prime, V_theta_prime),
a vector of theta values (theta_s), and the specific heat ratio (gamma).
%}

h = -0.1 * pi / 180; % Convert step size to radians if theta_s is in degrees
V_r = zeros(1, length(theta_s)); % Initialize V_r
V_theta = zeros(1, length(theta_s)); % Initialize V_theta

% Set the initial conditions
V_r(1) = V_r_prime;
V_theta(1) = V_theta_prime;

for i = 1:(length(theta_s) - 1)

   f1 = @(theta_s, V_r, V_theta) V_theta;
   f2 = @(theta_s, V_r, V_theta) (V_theta^2 * V_r - (gamma - 1) / 2 * (1 - V_r^2 - V_theta^2) * (2 * V_r + V_theta
* cot(theta_s))) / ((gamma - 1) / 2 * (1 - V_r^2 - V_theta^2) - V_theta^2);

   k11 = h * f1(theta_s(i), V_r(i), V_theta(i));
   k12 = h * f2(theta_s(i), V_r(i), V_theta(i));

   k21 = h * f1(theta_s(i) + 0.5 * h, V_r(i) + 0.5 * k11, V_theta(i) + 0.5 * k12);
   k22 = h * f2(theta_s(i) + 0.5 * h, V_r(i) + 0.5 * k11, V_theta(i) + 0.5 * k12);

   k31 = h * f1(theta_s(i) + 0.5 * h, V_r(i) + 0.5 * k21, V_theta(i) + 0.5 * k22);
   k32 = h * f2(theta_s(i) + 0.5 * h, V_r(i) + 0.5 * k21, V_theta(i) + 0.5 * k22);
```

```
    k41 = h * f1(theta_s(i) + h, V_r(i) + k31, V_theta(i) + k32);
    k42 = h * f2(theta_s(i) + h, V_r(i) + k31, V_theta(i) + k32);

    V_r(i + 1) = V_r(i) + (1 / 6) * (k11 + 2 * k21 + 2 * k31 + k41);
    V_theta(i + 1) = V_theta(i) + (1 / 6) * (k12 + 2 * k22 + 2 * k32 + k42);
end

end
```