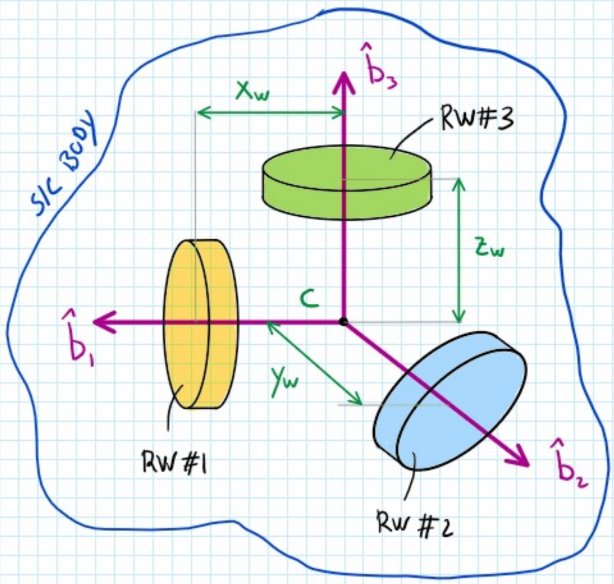
ARO 4090 - WEEK 15

- M. MAGGIA

Lecture 23

Example: S/C IN CIRCULAR LEO WITH 3 RW'S (1 PER BODY-FIXED RF AXIS)



ASSUMPTIONS:

- S/C BODY INERTIA MATRIX $I_B^C = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$
- RW#1 $\quad I_{aw_1} \gg I_{tw_1} + m_{w_1} x_w^2 \Rightarrow I_{wx} \approx I_{aw_1}$
- RW#2 $\quad I_{aw_2} \gg I_{tw_2} + m_{w_2} y_w^2 \Rightarrow I_{wy} \approx I_{aw_2}$
- RW#3 $\quad I_{aw_3} \gg I_{tw_3} + m_{w_3} z_w^2 \Rightarrow I_{wz} \approx I_{aw_3}$

EQUATIONS OF MOTION OF S/C LINEARIZED ABOUT $\psi = \theta = \phi = \dot{\psi} = \dot{\theta} = \dot{\phi} = 0$

$$
\begin{aligned}
&\underbrace{I_{xx}\ddot{\phi} + \omega_0^2 (I_{yy} - I_{zz})\phi - \omega_0 (I_{xx} - I_{yy} + I_{zz})\dot{\psi}}_{\text{S/C BODY}} \underbrace{+ I_{wx}\dot{\Omega}_x}_{\text{RW \#1}} \underbrace{- I_{wy}\Omega_y(\dot{\psi} + \omega_0\phi)}_{\text{RW \#2}} \underbrace{+ I_{wz}\Omega_z(\dot{\theta} - \omega_0)}_{\text{RW \#3}} = M_x \\
&I_{yy}\ddot{\theta} + I_{wx}\Omega_x(\dot{\psi} + \omega_0\phi) + I_{wy}\dot{\Omega}_y - I_{wz}\Omega_z(\dot{\phi} - \omega_0\psi) = M_y \\
&I_{zz}\ddot{\psi} + \omega_0^2 (I_{yy} - I_{xx})\psi + \omega_0 (I_{xx} - I_{yy} + I_{zz})\dot{\phi} - I_{wx}\Omega_x(\dot{\theta} - \omega_0) + I_{wy}\Omega_y(\dot{\phi} - \omega_0\psi) + I_{wz}\dot{\Omega}_z = M_z
\end{aligned}
$$

S/C BODY — RW #1 — RW #2 — RW#3

WHERE THE EXTERNAL TORQUE IS GIVEN BY:

$$
\begin{aligned}
M_x &= M_{gx} + M_{cx} + M_{dx} = -3\omega_0^2 (I_{yy} - I_{zz})\phi + K_{Px}(\phi_{com} - \phi) + K_{Dx}(\dot{\phi}_{com} - \dot{\phi}) + K_{Ix}\int_0^t (\phi_{com} - \phi)d\tau + M_{dx} \\
M_y &= M_{gy} + M_{cy} + M_{dy} = -3\omega_0^2 (I_{xx} - I_{zz})\theta + K_{Py}(\theta_{com} - \theta) + K_{D}(\dot{\theta}_{com} - \dot{\theta}) + K_{Iy}\int_0^t (\theta_{com} - \theta)d\tau + M_{dy} \\
M_z &= M_{gz} + M_{cz} + M_{dz} = 0 + K_{Pz}(\psi_{com} - \psi) + K_{D}(\dot{\psi}_{com} - \dot{\psi}) + K_{I}\int_0^t (\psi_{com} - \psi)d\tau + M_{dz}
\end{aligned}
$$

GG TORQUE (CAN BE BROUGHT TO LHS)

STATE FEEDBACK

RATE FEEDBACK

PID CONTROLLER

DISTURBANCE TORQUE (GG EXCLUDED)

$\{ \phi_{com}, \theta_{com}, \psi_{com}$: COMMAND (DESIRED) EULER ANGLES

(GG EXCLUDED)

$\phi_{com}, \theta_{com}, \psi_{com}$: COMMAND (DESIRED) EULER ANGLES

$\dot\phi_{com}, \dot\theta_{com}, \dot\psi_{com}$: COMMAND (DESIRED) ANGULAR RATES (EQUAL TO ZERO IF S/C IS TO MAINTAINED A FIXED ATTITUDE)

SHOULD BE MAINTAINED SMALL B/C OF THE LINEARIZATION ASSUMPTIONS

★ PITCH MOTION CONTROL

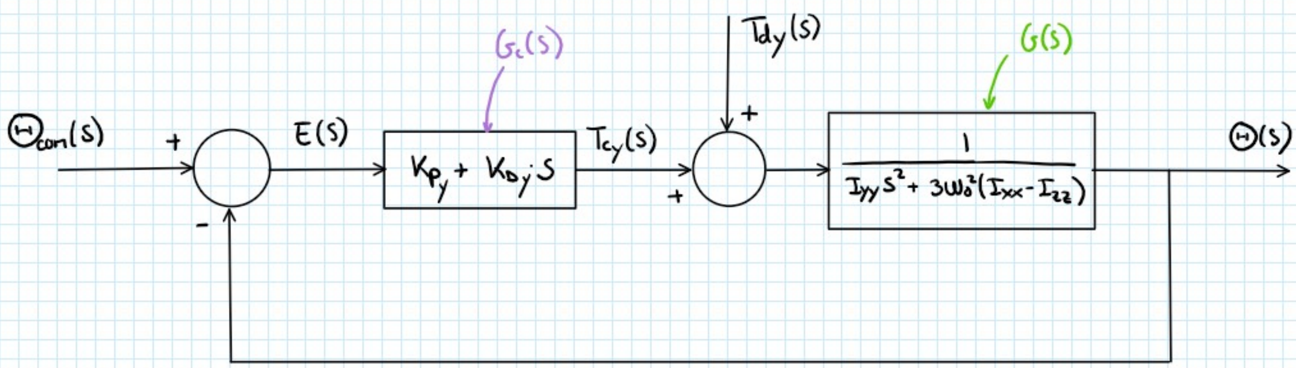
ASSUMPTIONS : 1 MW ABOUT y-AXIS (CONSTANT SPEED $\Rightarrow \dot\Omega = 0$)

PD CONTROLLER

$$I_{yy}\ddot\theta + 3\omega_o^2(I_{xx}-I_{zz})\theta = \underbrace{\boxed{K_{p_y}\theta_e + K_{D_y}\dot\theta_e}}_{M_{cy}(t)} + M_{dy}(t) \quad \text{where} \quad \theta_e = \theta_{com} - \theta$$

$\mathcal{L}$ (ZERO IC's)

$$I_{yy}s^2\Theta(s) + 3\omega_o^2(I_{xx}-I_{zz})\Theta(s) = \underbrace{\boxed{(K_{p_y} + sK_{D_y})E(s) + T_{dy}(s)}}_{T_{cy}(s)} \quad \text{where} \quad E(s) = \Theta_{com}(s) - \Theta(s)$$



- CLOSED-LOOP TRANSFER FUNCTION $(T(s))$ (when $T_{dy}(s) = 0$)

OUTPUT

INPUT

$$\left.\frac{\Theta(s)}{\Theta_{com}(s)}\right|_{T_{dy}(s)=0} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

$$K_{p_y} + K_{D_y}s$$

INPUT

$$= \frac{\dfrac{K_{Py}+K_{Dy}s}{I_{yy}s^2+3\omega_0^2(I_{xx}-I_{zz})}}{1+\dfrac{K_{Py}+K_{Dy}s}{I_{yy}s^2+3\omega_0^2(I_{xx}-I_{zz})}}$$

$$T(s) = \frac{K_{Dy}s+K_{Py}}{I_{yy}s^2+K_{Dy}s+3\omega_0^2(I_{xx}-I_{zz})+K_{Py}}$$

⚠ RECALL THAT THE C.L. TF IS EQUIVALENT TO THE OUTPUT OF THE SYSTEM WHEN THE INPUT IS AN IMPULSE (IF $\Theta_{com}(s)=1 \Rightarrow \Theta(s)=T(s)$)

FOR STABILITY, THE POLES OF $T(s)$ MUST HAVE NEGATIVE REAL PARTS:

POLES : s s.t. $\underbrace{I_{yy}}_{a}s^2+\underbrace{K_{Dy}}_{b}s+\underbrace{3\omega_0^2(I_{xx}-I_{zz})+K_{Py}}_{c}=0$

$$p_{1,2} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac}$$

$$(s-p_1)(s-p_2) = s^2+2\zeta\omega_n s+\omega_n^2$$

$$s^2-(p_1+p_2)s+p_1p_2 = s^2+2\zeta\omega_n s+\omega_n^2$$

EQUATING COEFFICIENTS:

s^2 : $1 = 1$

s^1 : $-p_1-p_2 = 2\zeta\omega_n \Rightarrow \frac{b}{2a}-\cancel{\frac{1}{2a}\sqrt{b^2-4ac}}+\frac{b}{2a}+\cancel{\frac{1}{2a}\sqrt{b^2-4ac}} = 2\zeta\omega_n$

$$\zeta\omega_n = \frac{b}{2a} = \frac{K_{Dy}}{2I_{yy}}$$

$$\boxed{K_{Dy} = 2I_{yy}\zeta\omega_n}$$

5° : $p_1 p_2 = \omega_n^2 \Rightarrow \left(-\frac{b}{2a} + \frac{1}{2a}\sqrt{\Delta}\right)\left(-\frac{b}{2a} - \frac{1}{2a}\sqrt{\Delta}\right) = \omega_n^2$ $\quad \Delta := b^2 - 4ac$

$$\frac{b^2}{4a^2} - \frac{\Delta}{4a^2} = \omega_n^2$$

$$\frac{\cancel{b^2} - \cancel{b^2} + \cancel{4a}c}{\cancel{4}a^{\cancel{2}}} = \omega_n^2$$

$$\omega_n^2 = \frac{c}{a} = \frac{3\omega_0^2 (I_{xx} - I_{zz}) + K_{py}}{I_{yy}}$$

$$\boxed{K_{py} = \omega_n^2 I_{yy} - 3\omega_0^2 (I_{xx} - I_{zz})}$$

REQUIREMENTS EXAMPLE:

PERCENT OVERSHOOT: $PO = 100\, e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \leq PO_{REQ.}$

$$\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \leq \ln \frac{PO_{REQ}}{100} = A \qquad 0 < PO_{REQ} < 100$$

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} \geq -A > 0$$

$$\frac{\pi^2\zeta^2}{1-\zeta^2} \geq A^2$$

$$\pi^2\zeta^2 \geq A^2 - A^2\zeta^2$$

$$(\pi^2 + A^2)\zeta^2 \geq A^2$$

$$\zeta \geq \frac{|A|}{\sqrt{\pi^2 + A^2}} \Rightarrow \boxed{\zeta \geq \frac{\left|\ln \frac{PO_{REQ}}{100}\right|}{\sqrt{\pi^2 + \left(\ln \frac{PO_{REQ}}{100}\right)^2}} = \zeta_{REQ}}$$

$\varphi = \mathrm{acos}(\zeta)$

OR

$$\boxed{\varphi \leq \mathrm{acos}\left[\frac{\left|\ln \frac{PO_{REQ}}{100}\right|}{\sqrt{\pi^2 + \left(\ln \frac{PO_{REQ}}{100}\right)^2}}\right] = \varphi_{REQ.}}$$

SETTLING TIME :

$$T_s = \frac{4}{\zeta\omega_n} \le T_{s_{REQ}} \Rightarrow \boxed{-\zeta\omega_n \le -\frac{4}{T_{s_{REQ}}} = -(\zeta\omega_n)_{REQ.}}$$

STEADY-STATE ERROR TO STEP DISTURBANCE

$$e_{ss} = \lim_{s\to 0} s\cdot E(s) \le e_{ss\,REQ.} \qquad \text{for} \quad T_{dy}(s) = \frac{\overbrace{B_y}^{\text{CONSTANT}}}{s} \quad \& \quad \Theta_{con} = 0$$

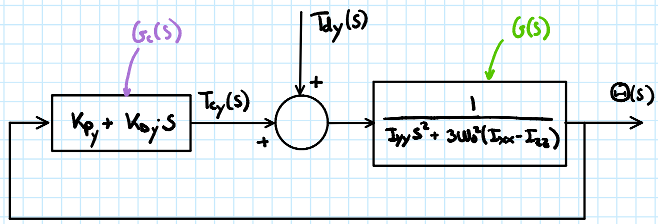
$$E(s) = \Theta_{con}(s) - \Theta(s) = -\Theta(s)$$

$$= -\lim_{s\to 0} s\,\Theta(s)$$

$$\Theta(s) = \left[G_c(s)\,\Theta(s) + T_{dy}(s)\right]G(s)$$

$$= G_c(s)\,G(s)\,\Theta(s) + T_{dy}(s)\,G(s)$$

$$= \frac{T_{dy}(s)\,G(s)}{1 - G_c(s)\,G(s)}$$



$$= -\lim_{s\to 0} s \, \frac{\dfrac{B_y}{s(I_{yy}s^2 + 3\omega_0^2(I_{xx}-I_{zz}))}}{1 - \dfrac{K_{p_y} + K_{D_y}s}{I_{yy}s^2 + 3\omega_0^2(I_{xx}-I_{zz})}}$$

$$= -\lim_{s\to 0} \cancel{s} \, \frac{B_y}{\cancel{s}\left(I_{yy}s^2 - K_{D_y}s + 3\omega_0^2(I_{yx}-I_{zz}) - K_{p_y}\right)}$$

$$= \frac{B_y}{K_{p_y} - 3\omega_0^2(I_{xx}-I_{zz})} \le e_{ss\,REQ.}$$

$$K_{p_y} - 3\omega_0^2(I_{xx}-I_{zz}) \ge \frac{B_y}{e_{ss\,REQ.}}$$

$$K_{p_y} \ge \frac{B_y}{e_{ss\,REQ}} + 3\omega_0^2(I_{xx}-I_{zz})$$

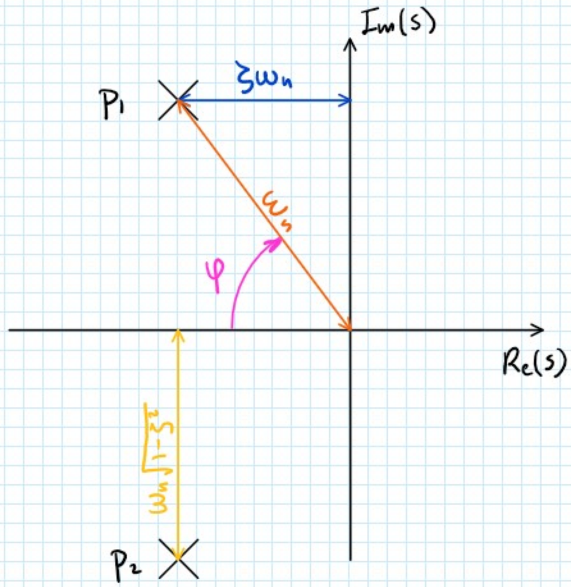
RECALL THAT $K_{p} = \omega_n^2 I_{yy} - 3\omega_0^2(I_{xx}-I_{zz})$

RECALL THAT $K_{p_y} = \omega_n^2 I_{yy} - 3\omega_0^2 (I_{xx} - I_{zz})$

$$\omega_n^2 I_{yy} - 3\omega_0^2 (I_{xx} - I_{zz}) \geq \frac{B_y}{e_{ss_{REQ}}} + 3\omega_0^2 (I_{xx} - I_{zz})$$

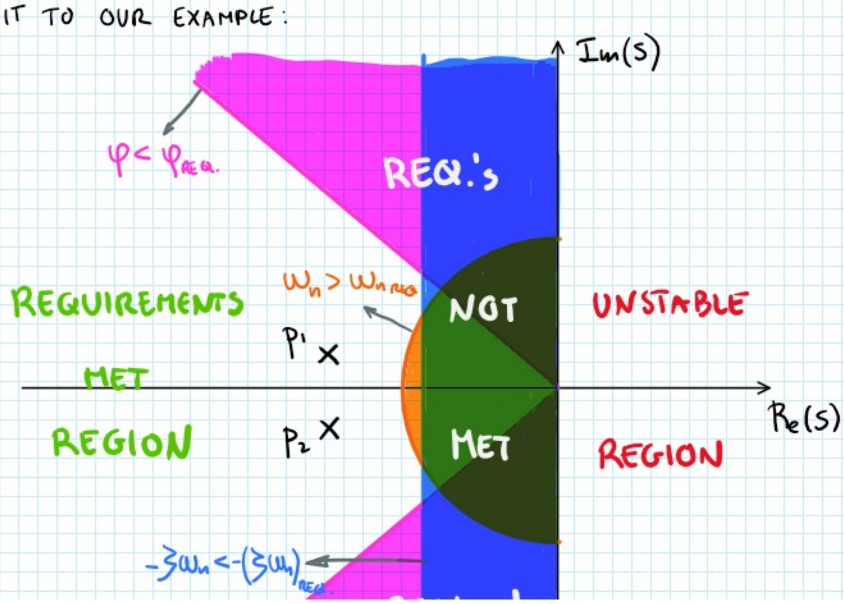
$$\boxed{\omega_n \geq \sqrt{\frac{B_y}{I_{yy} e_{ss_{REQ}}} + 6\omega_0^2 \frac{(I_{xx} - I_{zz})}{I_{yy}}} = \omega_{n_{REQ}}.}$$

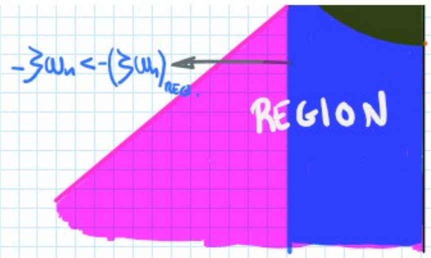
RECALL THAT, FOR $0 \leq \zeta \leq 1$



- × $p_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$
- $Re(p_1) = Re(p_2) = -\zeta\omega_n$
- $Im(p_1) = -Im(p_2) = \omega_n\sqrt{1-\zeta^2}$
- $|p_1| = |p_2| = \sqrt{Re(p_1)^2 + Im(p_1)^2} = \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)} = \sqrt{\cancel{\zeta^2\omega_n^2} + \omega_n^2 - \cancel{\zeta^2\omega_n^2}} = \omega_n$
- $\varphi = acos\left(\frac{\zeta\cancel{\omega_n}}{\cancel{\omega_n}}\right) = acos(\zeta)$

APPLYING IT TO OUR EXAMPLE:





YOU CAN CHOOSE ANY 2 POLES IN THE REGION THAT MEET THE REQUIREMENTS

★ ROLL-YAW MOTION CONTROL

ASSUMPTIONS : 1 MW ABOUT y-AXIS $\left(h_s = I_{w_y}\Omega_y\right)$

PD CONTROLLER

$\phi_{con} = \psi_{con} = 0$

$$\left\{\begin{aligned} &I_{xx}\ddot{\phi} + 4\omega_0^2(I_{yy}-I_{zz})\phi - \omega_0(I_{xx}-I_{yy}+I_{zz})\dot{\psi} - h_s(\dot{\psi}+\omega_0\phi) = -K_{p_x}\phi - K_{D_x}\dot{\phi} + M_{d_x} \\ &I_{zz}\ddot{\psi} + \omega_0^2(I_{yy}-I_{xx})\psi + \omega_0(I_{xx}-I_{yy}+I_{zz})\dot{\phi} + h_s(\dot{\phi}-\omega_0\psi) = -K_{p_z}\psi - K_D\dot{\psi} + M_{d_z} \end{aligned}\right.$$

$\mathcal{L}$ (ZERO I.C.'s)

$$\left\{\begin{aligned} &I_{xx}\Phi(s)s^2 + 4\omega_0^2(I_{yy}-I_{zz})\Phi(s) - \omega_0(I_{xx}-I_{yy}+I_{zz})\Psi(s)s - h_s\Psi(s)s - h_s\omega_0\Phi(s) + K_{p_x}\Phi(s) + K_{D_x}\Phi(s)s = T_{d_x}(s) \\ &I_{zz}\Psi(s)s^2 + \omega_0^2(I_{yy}-I_{xx})\Psi(s) + \omega_0(I_{xx}-I_{yy}+I_{zz})\Phi(s)s + h_s\Phi(s)s - h_s\omega_0\Psi(s) + K_{p_z}\Psi(s) + K_{D_z}\Psi(s)s = T_{d_z}(s) \end{aligned}\right.$$

$$\underbrace{\begin{bmatrix} I_{xx}s^2 + K_{D_x}s + 4\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 + K_{p_x} & -\left[\omega_0(I_{xx}-I_{yy}+I_{zz}) + h_s\right]s \\ \left[\omega_0(I_{xx}-I_{yy}+I_{zz}) + h_s\right]s & I_{zz}s^2 + K_{D_z}s + \omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 + K_{p_z} \end{bmatrix}}_{A_{CON}} \begin{bmatrix} \Phi(s) \\ \Psi(s) \end{bmatrix} = \begin{bmatrix} T_{d_x}(s) \\ T_{d_z}(s) \end{bmatrix}$$

- IF UNCONTROLLED : $K_{p_x} = K_{D_y} = K_{p_z} = K_{D_z} = 0$

$$\begin{bmatrix} I_{xx}s^2 + 4\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 & -\left[\omega_0(I_{xx}-I_{yy}+I_{zz}) + h_s\right]s \\ \left[\omega_0(I_{xx}-I_{yy}+I_{zz}) + h_s\right]s & I_{zz}s^2 + \omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 \end{bmatrix} \begin{bmatrix} \Phi(s) \\ \Psi(s) \end{bmatrix} = \begin{bmatrix} T_{d_x}(s) \\ T_{d_z}(s) \end{bmatrix}$$

A

$A_{UNC.}$

STABILITY : FIND POLES OF SYSTEM $\Longleftrightarrow$ s s.t. $\det(A_{UNC.}) = 0$

$$\det(A_{UNC}) = as^4 + bs^2 + c$$

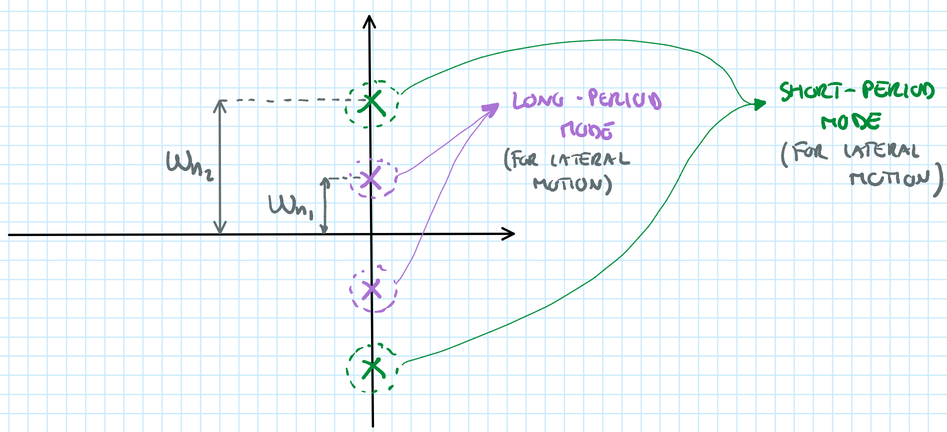
with

$$a = I_{xx} I_{zz}$$

$$b = \left(4\omega_0^2(I_{yy} - I_{zz}) - h_s\omega_0\right)I_{zz} + \left(\omega_0^2(I_{yy} - I_{zz}) - h_s\omega_0\right)I_{xx} + \left(\omega_0\left(I_{xx} - I_{yy} + I_{zz}\right)\right)^2$$

$$c = \left(4\omega_0^2(I_{yy} - I_{zz}) - h_s\omega_0\right)\left(\omega_0^2(I_{yy} - I_{zz} - h_s\omega_0\right)$$

AT BEST THE SYSTEM IS MARGINALLY STABLE. YOU SHOULD FIND THE VALUES OF THE MOMENTUM BIAS (h_s) SO THAT THE UNCONTROLLED SYSTEM IS M. STABLE, THAT IS, THE 4 POLES ARE ON THE IMAGINARY AXIS.



- WITH CONTROL :

$$\begin{bmatrix} \Phi(s) \\ \Psi(s) \end{bmatrix} = \frac{1}{\det(A_{CON})} \left[\begin{array}{c|c} I_{zz}s^2 + K_{D_z}s + \omega_0^2(I_{yy} - I_{zz}) - h_s\omega_0 + K_{P_z} & \left[\omega_0(I_{xx} - I_{yy} + I_{zz}) + h_s\right]s \\ \hline -\left[\omega_0(I_{xx} - I_{yy} + I_{zz}) + h_s\right]s & I_{xx}s^2 + K_{D_x}s + 4\omega_0^2(I_{yy} - I_{zz}) - h_s\omega_0 + K_{P_x} \end{array}\right] \begin{bmatrix} T_{d_x}(s) \\ T_{d_z}(s) \end{bmatrix}$$

$$\det(A_{CON}) = as^4 + bs^3 + cs^2 + ds + e$$

with

$$a = I_{xx} I_{zz}$$

$$b = \boxed{I_{zz} K_{Dx} + I_{xx} K_{Dz}}$$

$$c = \left(4\omega_0^2 (I_{yy} - I_{zz}) - h_s \omega_0\right) I_{zz} + \left(\omega_0^2 (I_{yy} - I_{zz}) - h_s \omega_0\right) I_{xx} + \left(\omega_0 \left(I_{xx} - I_{yy} + I_{zz}\right)\right)^2 + \boxed{K_{Dx} K_{Dz}}$$

$$d = \boxed{\left(4\omega_0^2 (I_{yy} - I_{zz}) - h_s \omega_0 + K_{Px}\right) K_{Dz} + \left(\omega_0^2 (I_{yy} - I_{zz}) - h_s \omega_0 + K_{Pz}\right) K_{Dx}}$$

$$e = \left(4\omega_0^2 (I_{yy} - I_{zz}) - h_s \omega_0 + \boxed{K_{Px}}\right)\left(\omega_0^2 (I_{yy} - I_{zz} - h_s \omega_0 + \boxed{K_{Pz}}\right)$$

$\boxed{}$: NEW TERMS W.R.T. UNCONTROLLED SYSTEM

WE CANNOT CALCULATE THE POLES BY HAND, SO LET'S USE ROUTH-HURWITZ CRITERION TO STUDY THE SYSTEM STABILITY :

s^4	a	c	e
s^3	b	d	0
s^2	f	e	
s^1	g		
s^0	e		

→ ROUTH ARRAY

$$\boxed{f} = \frac{bc - ad}{b}$$

$$\boxed{g} = \frac{\boxed{f} \cdot d - be}{\boxed{f}}$$

FOR ASYMPT. STABILITY ALL COEFFICIENTS OF ROUTH ARRAY MUST BE POSITIVE

$$\begin{cases} a > 0 \ ✓ \\ b > 0 \ ✓ \\ f > 0 \\ g > 0 \\ e > 0 \end{cases}$$

$a > 0$, $b > 0$: ALREADY MET B/C $I_{xx}, I_{zz}, K_{Dx}, K_{Dy} > 0$

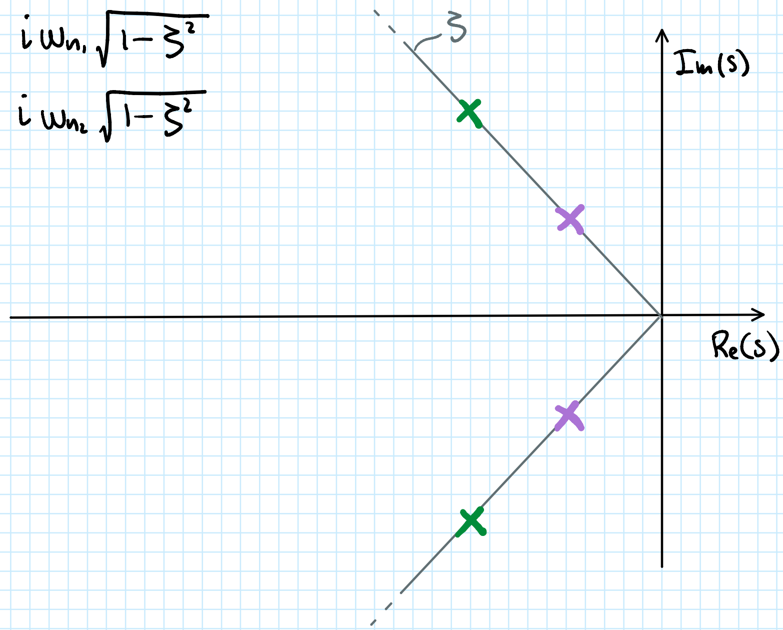
$f > 0$, $g > 0$, $e > 0$ ⟹ YOU'LL FIND CONDITIONS ON $K_{Px}, K_{Dx}, K_{Pz}, K_{Dz}$

REQUIREMENTS EXAMPLE :

- WE WANT ALL 4 POLES TO HAVE THE SAME DAMPING RATIO $\zeta_1 = \zeta_2 = \zeta$

$$P_{1,2} = -\zeta\omega_{n_1} \pm i\,\omega_{n_1}\sqrt{1-\zeta^2}$$

$$P_{3,4} = -\zeta\omega_{n_2} \pm i\,\omega_{n_2}\sqrt{1-\zeta^2}$$



$$a(s-p_1)(s-p_2)(s-p_3)(s-p_4) = as^4 + bs^3 + cs^2 + ds + e$$

$$a(s^2 + 2\zeta\omega_{n_1}s + \omega_{n_1}^2)(s^2 + 2\zeta\omega_{n_2}s + \omega_{n_2}^2) = as^4 + bs^3 + cs^2 + ds + e$$

$$as^4 + a2\zeta(\omega_{n_1}+\omega_{n_2})s^3 + a(\omega_{n_1}^2 + 4\zeta^2\omega_{n_1}\omega_{n_2} + \omega_{n_2}^2)s^2 + a2\zeta\,\omega_{n_1}\omega_{n_2}(\omega_{n_1}+\omega_{n_2})s + a\omega_{n_1}^2\omega_{n_2}^2 = \ldots$$

EQUATING COEFFICIENTS :

s^4 : $a = a$ ✓

s^3 : $a2\zeta(\omega_{n_1}+\omega_{n_2}) = b = I_{zz}K_{Dx} + I_{xx}K_{Dz}$

s^2 : $a(\omega_{n_1}^2 + 4\zeta^2\omega_{n_1}\omega_{n_2} + \omega_{n_2}^2) = c = \left(4\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0\right)I_{zz} + \left(\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0\right)I_{xx} + \left(\omega_0(I_{xx}-I_{yy}+I_{zz})\right)^2 + K_{Dx}K_{Dz}$

s^1 : $a2\zeta\omega_{n_1}\omega_{n_2}(\omega_{n_1}+\omega_{n_2}) = d = \left(4\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 + K_{Px}\right)K_{Dz} + \left(\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 + K_{Pz}\right)K_{Dx}$

s^0 : $a\omega_{n_1}^2\omega_{n_2}^2 = e = \left(4\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 + K_{Px}\right)\left(\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 + K_{Pz}\right)$

EQUATIONS : 4

UNKNOWNS : 7 $\quad \omega_{n_1}, \omega_{n_2}, h_s, K_{Px}, K_{Dx}, K_{Pz}, K_{Dz}$

- STEADY-STATE YAW ERROR FOR STEP DISTURBANCES

$$\underbrace{e_{ss\psi}}_{\text{STEADY-STATE YAW ERROR}} = \lim_{t\to\infty}\big(\underbrace{\psi_{com}}_{=0} - \psi(t)\big) = -\lim_{t\to\infty}\psi(t) \overset{\text{FVT}}{=} -\lim_{s\to 0} s\,\Psi(s)$$

STEADY-STATE YAW ERROR

$= 0$

FVT

STEP DISTURBANCES $T_{d_x}(s) = \frac{B_x}{s}$, $T_{d_z}(s) = \frac{B_z}{s}$

$$\Psi(s) = \frac{1}{\det(A_{con})}\left(-\left[\omega_0(I_{xx}-I_{yy}+I_{zz})+h_s\right]\cancel{s}\,\frac{B_x}{\cancel{s}} + \left(I_{xx}s^2 + K_{D_x}s + 4\omega_0^2(I_{yy}-I_{zz}) - h_s\omega_0 + K_{P_x}\right)\frac{B_z}{s}\right)$$

$$e_{ss_\psi} = \lim_{s\to 0} \frac{sB_x\left[\omega_0(I_{xx}-I_{yy}+I_{zz})+h_s\right] - \left(I_{xx}s^2+K_{D_x}s+4\omega_0^2(I_{yy}-I_{zz})-h_s\omega_0+K_{P_x}\right)B_z}{as^4+bs^3+cs^2+ds+e}$$

$$= -\frac{\left(4\omega_0^2(I_{yy}-I_{zz})-h_s\omega_0+K_{P_x}\right)B_z}{e}$$

$$= -\frac{\cancel{\left(4\omega_0^2(I_{yy}-I_{zz})-h_s\omega_0+K_{P_x}\right)}B_z}{\cancel{\left(4\omega_0^2(I_{yy}-I_{zz})-h_s\omega_0+K_{P_x}\right)}\left(\omega_0^2(I_{yy}-I_{zz})-h_s\omega_0+K_{P_z}\right)}$$

$$e_{ss_\psi} = \frac{-B_z}{\omega_0^2(I_{yy}-I_{zz})-h_s\omega_0+K_{P_z}} \leq e_{ss_{\psi_{REQ.}}}$$

(WE COULD REPEAT THE PROCESS TO FIND e_{ss_ϕ})