# ARO 4090 - WEEK 13

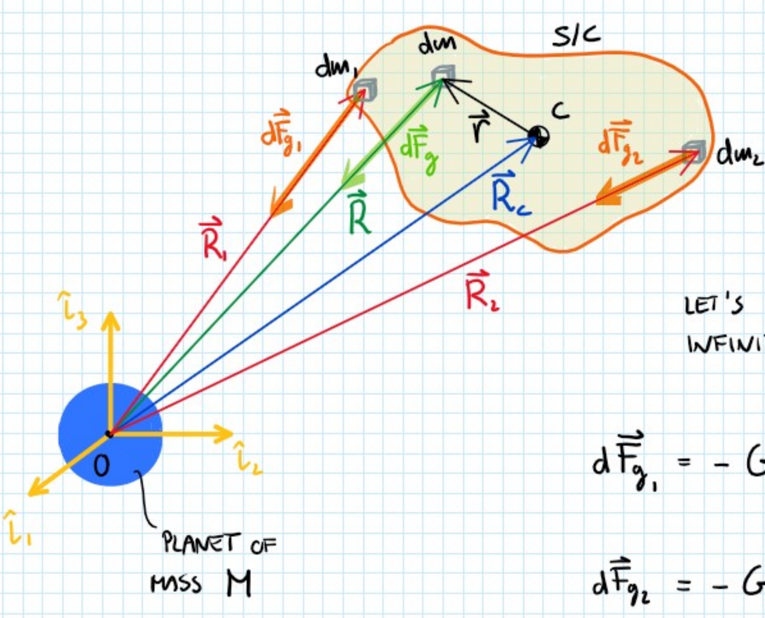
- M. MAGGIA

## Lectures 20-21

## GRAVITY-GRADIENT STABILIZATION

- THE FACT THAT THE S/C'S MASS IS DISTRIBUTED AND NOT CONCENTRATED IN A SINGLE POINT CAUSES DIFFERENT PARTS OF THE S/C TO BE SUBJECTED TO DIFFERENT GRAVITATIONAL FORCES. SINCE THIS DIFFERENCE IS CONTINUOUS, IT IS REFERRED TO AS GRADIENT. THE GRAVITATIONAL FORCE GRADIENT NATURALLY CAUSES A TORQUE.

- THE GRAVITY GRADIENT TORQUE:
  - ↳ CAN BE CONSIDERED A PERTURBATION TO OTHERWISE TORQUE-FREE MOTION.
  - ↳ CAN BE EXPLOITED TO STABILIZE A S/C.
  - ↳ IS STRONGER THE CLOSER THE S/C IS TO THE PLANET.




LET'S CONSIDER FOR INSTANCE TWO INFINITESIMAL MASSES $dm_1$ AND $dm_2$

$$d\vec{F}_{g_1} = -G\,\frac{dm_1 \cdot M}{R_1^3} \cdot \vec{R}_1$$ GRAV. FORCE ON $dm_1$

$$d\vec{F}_{g_2} = -G\,\frac{dm_2 \cdot M}{R_2^3}\,\vec{R}_2$$ GRAV. FORCE ON $dm_2$

$(dm_1 = dm_2)$

$R_1 = \|\vec{R}_1\|$, $R_2 = \|\vec{R}_2\|$

SINCE $R_1 < R_2 \Rightarrow \|d\vec{F}_{g,1}\| > \|d\vec{F}_{g,2}\|$

FOR A GENERIC INFINITESIMAL MASS $dm$, THE GRAVITATIONAL FORCE IS

$$d\vec{F}_g = -\mu \frac{dm}{R^3} \cdot \vec{R}$$

($\mu = G \cdot M$ IS THE GRAVITATIONAL PARAMETER IN THE RESTRICTED 2-BODY PROBLEM)

FROM FIGURE:

$$\vec{R} = \vec{R}_c + \vec{r}$$

$$R = \|\vec{R}_c + \vec{r}\|$$

$$R^{-3} = \|\vec{R}_c + \vec{r}\|^{-3}$$

SINCE $r \ll R_c$, WE CAN USE A TRUNCATED TAYLOR SERIES EXPANSION TO LINEARIZE $\|\vec{R}\|^{-3}$ ABOUT $\vec{R}_c$.

$$f(\vec{x}) \simeq f(\vec{x}_0) + \left.\frac{\partial f}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_0} (\vec{x} - \vec{x}_0)$$

WITH

$$\vec{x} = \vec{R}$$

$$\vec{x}_0 = \vec{R}_c$$

$$f(\vec{x}) = \|\vec{x}\|^{-3} = (\vec{R} \bullet \vec{R})^{-\frac{3}{2}}$$

$$\|\vec{R}\|^{-3} \simeq \|\vec{R}_c\|^{-3} - \left[\frac{3}{2}(\vec{R} \bullet \vec{R})^{-\frac{5}{2}} (\vec{R}^T + \vec{R}^T)\right]_{\vec{R}=\vec{R}_c} \cdot (\vec{R} - \vec{R}_c)$$

$$\simeq \frac{1}{R_c^3} - \frac{3}{\cancel{2}}\left[\frac{1}{R_c^5} \cancel{2}\vec{R}_c^T\right]\vec{r}$$

$$\simeq \frac{1}{R_c^3}\left(1 - 3\frac{\vec{R}_c^T \vec{r}}{R_c^2}\right)$$

THUS WE CAN RE-WRITE $d\vec{F}_g$ AS

$$d\vec{F}_g = -\mu \frac{dm}{R^3} \cdot \vec{R}$$

$$\boxed{d\vec{F}_g \simeq -\mu \frac{\vec{R}_c + \vec{r}}{R_c^3}\left(1 - \frac{3\vec{R}_c^T \vec{r}}{R_c^2}\right) dm}$$

TORQUE GENERATED BY $d\vec{F}_g$ WITH RESPECT TO THE CENTER OF MASS $C$ IS

$$d\vec{M}_g^c = \vec{r} \wedge d\vec{F}_g$$

$$\vec{M}_g^c = \int_{S/C} \vec{r} \wedge d\vec{F}_g$$

$$= \int_{S/C} \vec{r} \wedge -\mu \frac{\vec{R}_c + \vec{r}}{R_c^3}\left(1 - \frac{3\vec{R}_c^T\cdot\vec{r}}{R_c^2}\right) dm$$

$$= -\frac{\mu}{R_c^3}\int_{S/C} \vec{r} \wedge \left(\vec{R}_c + \vec{r} - \frac{3\vec{R}_c^T\vec{r}}{R_c^2}\vec{R}_c - \frac{3\vec{R}_c^T\vec{r}}{R_c^2}\vec{r}\right) dm$$

$$= -\frac{\mu}{R_c^3}\int_{S/C}\left(1 - \frac{3\vec{R}_c^T\vec{r}}{R_c^2}\right)\vec{r}\wedge\vec{R}_c \, dm$$

$$= -\frac{\mu}{R_c^3}\int_{S/C}\vec{r}\wedge\vec{R}_c \, dm + \frac{3\mu}{R_c^5}\int_{S/C}\vec{R}_c^T\cdot\vec{r}\,(\vec{r}\wedge\vec{R}_c)\, dm$$

$$= -\frac{\mu}{R_c^3}\underbrace{\boxed{\int_{S/C}\vec{r}\, dm}}_{=0}\wedge\vec{R}_c - \frac{3\mu}{R_c^5}\int_{S/C}(\vec{R}_c\wedge\vec{r})\,\vec{r}^T\vec{R}_c\, dm$$

From Lec.11

$$= -3\frac{\mu}{R_c^5}\int_{S/C}[\vec{R}_c]_\times\vec{r}\,\vec{r}^T\vec{R}_c\, dm$$

$$= -3\frac{\mu}{R_c^5}[\vec{R}_c]_\times\int_{S/C}\vec{r}\,\vec{r}^T dm\cdot\vec{R}_c$$

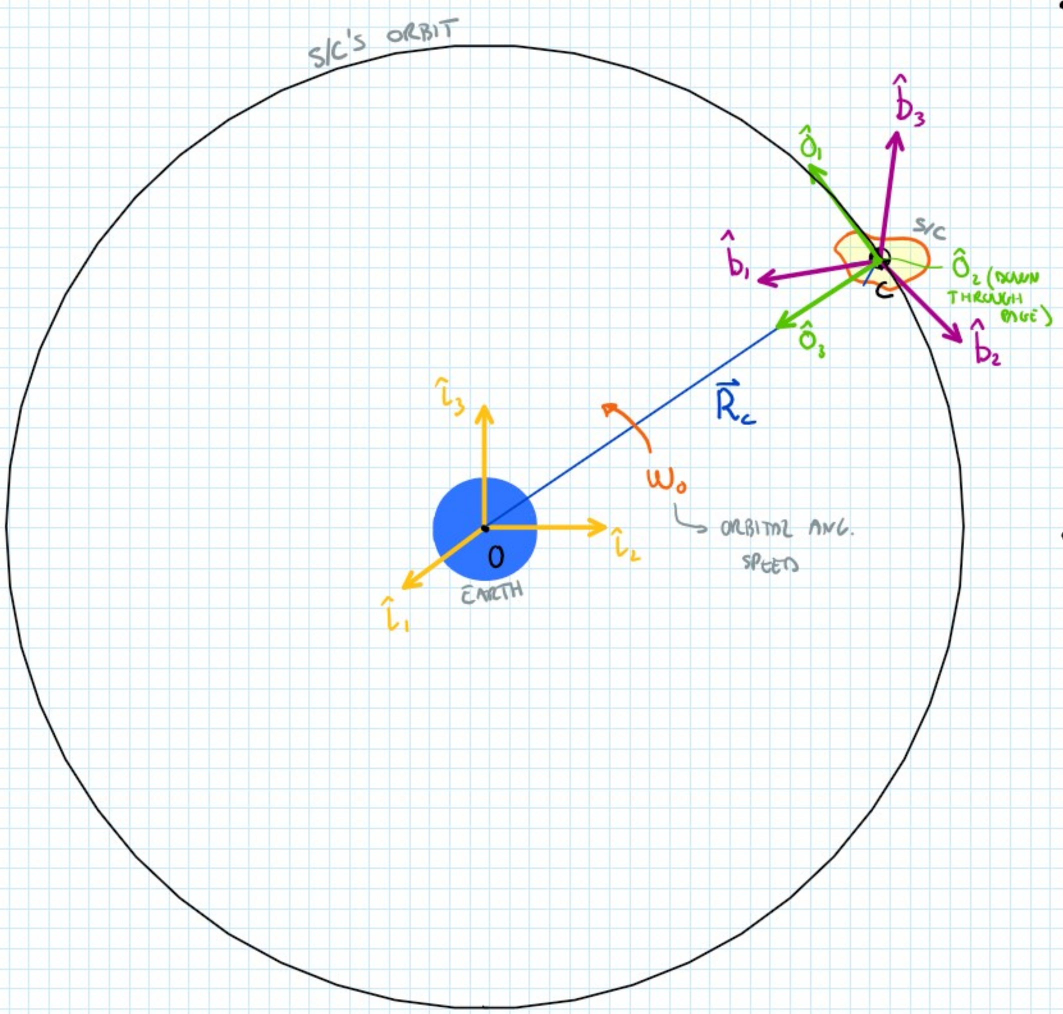
WE COULD SHOW THAT $\vec{r}\,\vec{r}^T = [\vec{r}]_\times^2 + (\vec{r}^T\vec{r}) I_d$

$$= -3\frac{\mu}{R^5}[\vec{R}]_\times\left[\boxed{\left([\vec{r}]_\times^2 \, dm\right)} + \boxed{\left(\vec{r}^T\vec{r}\, dm\right)} I_d\right]\vec{R}$$

$$= -3\frac{\mu}{R_c^5}[\vec{R}_c]_\times \Bigg[ \underbrace{\boxed{\int_{S/C} [\vec{r}]_\times^2 \, dm}}_{-I^c} + \underbrace{\boxed{\int_{S/C} \vec{r}^T\vec{r} \, dm}}_{\text{SCALAR, SO WE CAN MOVE IT TO THE LEFT}} \cdot I_d \Bigg] \vec{R}_c$$

$$= 3\frac{\mu}{R_c^5}[\vec{R}_c]_\times I^c \vec{R}_c - 3\frac{\mu}{R_c^5}\int_{S/C} \vec{r}^T\vec{r} \, dm \cdot I_d \underbrace{[\vec{R}_c]_\times \vec{R}_c}_{\vec{R}_c \wedge \vec{R}_c = 0}$$

$$\boxed{\vec{M}_g^c = 3\frac{\mu}{R_c^5}[\vec{R}_c]_\times \cdot I^c \cdot \vec{R}_c}$$




- BODY-FIXED RF ($\mathcal{B}$-RF)

  $\mathcal{B} = \{C, \hat{b}_1, \hat{b}_2, \hat{b}_3\}$

  (MOVES & ROTATES WITH S/C)

- ORBITAL RF ($\mathcal{O}$-RF)

  $\mathcal{O} = \{C, \hat{o}_1, \hat{o}_2, \hat{o}_3\}$

  (MOVES WITH S/C, BUT HAS ALWAYS Z-AXIS POINTING TOWARDS EARTH.)

- ECI RF ($\mathcal{I}$-RF)

  $\mathcal{I} = \{O, \hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3\}$

  (EARTH-CENTERED INERTIAL, FIXED WITH EARTH. DOESN'T MOVE OR ROTATE)

S/C'S EQUATIONS OF MOTION :

EXT. TORQUE IS THE GRAVITY GRADIENT

EULER'S EQUATIONS OF MOTION :

$$I^c \dot{\vec{\omega}}^{B/I} + [\vec{\omega}^{B/I}]_\times I^c \vec{\omega}^{B/I} = \overbrace{\vec{M}^c_g}^{\text{EXT. TORQUE IS THE GRAVITY GRADIENT}}$$

$$I^c \dot{\vec{\omega}}^{B/I} + [\vec{\omega}^{B/I}]_\times I^c \vec{\omega}^{B/I} = 3\frac{\mu}{R_c^5}[\vec{R}_c]_\times I^c \vec{R}_c$$

RECALL THAT ALL VECTORS / INERTIA MATRIX MUST BE WRITTEN IN THE SAME RF COORDINATES

WE NORMALLY CHOOSE $\mathcal{B}$-RF COORDINATES.

$$\boxed{I^c_{\mathcal{B}} \dot{\vec{\omega}}^{B/I}_{\mathcal{B}} + [\vec{\omega}^{B/I}_{\mathcal{B}}]_\times I^c_{\mathcal{B}} \vec{\omega}^{B/I}_{\mathcal{B}} = 3\frac{\mu}{R_c^5}[\vec{R}_{c_{\mathcal{B}}}]_\times I^c_{\mathcal{B}} \vec{R}_{c_{\mathcal{B}}}}$$

NOTICE THAT :

$$\vec{R}_{c_{\mathcal{O}}} = R_c \hat{o}_3$$

SO WE'LL NEED A ROTATION MATRIX TO GO FROM $\mathcal{O}$-RF TO $\mathcal{B}$-RF COORDINATES : $R_{\mathcal{B}\mathcal{O}}$.

THUS

$$\vec{R}_{c_{\mathcal{B}}} = \boxed{R_{\mathcal{B}\mathcal{O}}}\ \boxed{\vec{R}_{c_{\mathcal{O}}}}$$

$$= \begin{bmatrix} c\theta\, c\psi & c\theta\, s\psi & -s\theta \\ s\phi\, s\theta\, c\psi - c\phi\, s\psi & s\phi\, s\theta\, s\psi + c\phi\, c\psi & s\phi\, c\theta \\ c\phi\, s\theta\, c\psi + s\phi\, s\psi & c\phi\, s\theta\, s\psi - s\phi\, c\psi & c\phi\, c\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -R_c \end{bmatrix}$$

(3-2-1 ROTATION SEQUENCE FROM $\mathcal{O}$-RF TO $\mathcal{B}$-RF COORDS) (${\vec{R}_c}$ IN $\mathcal{O}$-RF COORDS.)

$$= \begin{bmatrix} s\theta \cdot R_c \\ -s\phi\, c\theta\, R_c \\ -c\phi\, c\theta\, R_c \end{bmatrix}$$

$$I^c_{\mathcal{B}} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$ (INERTIA MATRIX W.R.T. C IN $\mathcal{B}$-COORDS)

$$\vec{\omega}^{B/I}_{\mathcal{B}} = \vec{\omega}^{B/O}_{\mathcal{B}} + \vec{\omega}^{O/I}_{\mathcal{B}}$$

$$\vec{\omega}_B^{B/I} = \vec{\omega}_B^{B/O} + \omega_B^{O/I}$$

$$= \begin{bmatrix} p \\ q \\ r \end{bmatrix} + R_{BO}\,\vec{\omega}_O^{O/I}$$

$\omega_0$ = CONST. IF CIRCULAR ORBIT
$\omega_0 \neq$ CONST. IF ELLIPTICAL ORBIT

FROM LEC. 9/10

$$= \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta\, s\phi & c\phi & 0 \\ c\theta\, c\phi & -s\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} c\theta\, c\psi & c\theta\, s\psi & -s\theta \\ s\phi\, s\theta\, c\psi - c\phi\, s\psi & s\phi\, s\theta\, s\psi + c\phi\, c\psi & s\phi\, c\theta \\ c\phi\, s\theta\, c\psi + s\phi\, s\psi & c\phi\, s\theta\, s\psi - s\phi\, c\psi & c\phi\, c\theta \end{bmatrix} \begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -s\theta\,\dot{\psi} + \dot{\phi} - c\theta\, s\psi\, \omega_0 \\ c\theta\, s\phi\, \dot{\psi} + c\phi\,\dot{\theta} - (s\phi\, s\theta\, s\psi + c\phi\, c\psi)\,\omega_0 \\ c\theta c\phi\,\dot{\psi} - s\phi\,\dot{\theta} - (c\phi\, s\theta\, s\psi - s\phi\, c\psi)\,\omega_0 \end{bmatrix}$$

ASSUMPTIONS :

(i) WE WANT $\mathcal{B}$-RF TO BE AS CLOSE AS POSSIBLE TO $\mathcal{O}$-RF, WHICH MEANS

$$\phi \simeq \theta \simeq \psi \simeq 0 \qquad \text{(SMALL ANGLES)}$$
$$\dot{\phi} \simeq \dot{\theta} \simeq \dot{\psi} \simeq 0 \qquad \text{(SMALL ANGULAR VARIATIONS)}$$

LINEARIZING ABOUT $\phi_0 = \theta_0 = \psi_0 = \dot{\phi}_0 = \dot{\theta}_0 = \dot{\psi}_0 = 0$

$$\begin{cases} s\psi \simeq \psi \\ c\psi \simeq 1 \end{cases} \& \begin{cases} s\theta \simeq \theta \\ c\theta \simeq 1 \end{cases} \& \begin{cases} s\phi \simeq \phi \\ c\phi \simeq 1 \end{cases}$$

EVERYTHING IN RADIANS!

$$\downarrow$$

$$|s\phi\, s\theta\, s\psi| \simeq |\phi \cdot \theta \cdot \psi| \lll 1 \simeq |c\phi\, c\psi|$$
$$|c\phi\, s\theta\, s\psi| \simeq |\theta \cdot \psi| \ll |\phi| \simeq |s\phi\, c\psi|$$

THUS :

$$\vec{\omega}_B^{B/I} \simeq \begin{bmatrix} -\theta\,\dot{\psi} + \dot{\phi} - \psi\,\omega_0 \\ \phi\,\dot{\psi} + \dot{\theta} - \omega_0 \\ \dot{\psi} - \phi\,\dot{\theta} + \phi\,\omega_0 \end{bmatrix}$$

AND FURTHERMORE :

$$\begin{cases} |\theta\,\dot{\psi}| \ll \dot{\phi} \\ |\theta\,\dot{\psi}| \ll |\psi\,\omega_0| \end{cases} \& \begin{cases} |\phi\,\dot{\psi}| \ll |\dot{\theta}| \\ |\phi\,\dot{\psi}| \ll |\omega_0| \end{cases} \& \begin{cases} |\phi\,\dot{\theta}| \ll \dot{\psi} \\ |\phi\,\dot{\theta}| \ll |\phi\,\omega_0| \end{cases}$$

FINALLY:

$$\boxed{\vec{\omega}_B^{B/X} \simeq \begin{bmatrix} \dot\phi - \psi\omega_0 \\ \dot\theta - \omega_0 \\ \dot\psi + \phi\omega_0 \end{bmatrix}}$$

(ii) S/C TRAVELS ON A CIRCULAR ORBIT

$$\omega_0 = \text{CONSTANT}$$

1ST TERM

$$I_B^c \dot{\vec{\omega}}_B^{B/X} \simeq \begin{bmatrix} (\ddot\phi - \dot\psi\omega_0) I_{xx} \\ \ddot\theta I_{yy} \\ (\ddot\psi + \dot\phi\omega_0) I_{zz} \end{bmatrix}$$

2nd TERM

$$[\vec{\omega}_B^{B/X}]_\times I_B^c \vec{\omega}_B^{B/X} \simeq \begin{bmatrix} 0 & -\dot\psi - \phi\omega_0 & \dot\theta - \omega_0 \\ \dot\psi + \phi\omega_0 & 0 & -\dot\phi + \psi\omega_0 \\ -\dot\theta + \omega_0 & \dot\phi - \psi\omega_0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot\phi - \psi\omega_0 \\ \dot\theta - \omega_0 \\ \dot\psi + \phi\omega_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (-\dot\psi - \phi\omega_0) I_{yy} & (\dot\theta - \omega_0) I_{zz} \\ (\dot\psi + \phi\omega_0) I_{xx} & 0 & (-\dot\phi + \psi\omega_0) I_{zz} \\ (-\dot\theta + \omega_0) I_{xx} & (\dot\phi - \psi\omega_0) I_{yy} & 0 \end{bmatrix} \begin{bmatrix} \dot\phi - \psi\omega_0 \\ \dot\theta - \omega_0 \\ \dot\psi + \phi\omega_0 \end{bmatrix}$$

$$= \begin{bmatrix} (\dot\theta - \omega_0)(\dot\psi + \phi\omega_0)(I_{zz} - I_{yy}) \\ (\dot\psi + \phi\omega_0)(\dot\phi - \psi\omega_0)(I_{xx} - I_{zz}) \\ (\dot\theta - \omega_0)(\dot\phi - \psi\omega_0)(I_{yy} - I_{xx}) \end{bmatrix}$$

$$= \begin{bmatrix} (\cancelto{0}{\dot\theta\dot\psi} - \omega_0\dot\psi + \cancelto{0}{\dot\theta\phi}\omega_0 - \phi\omega_0^2)(I_{zz} - I_{yy}) \\ (\cancelto{0}{\dot\psi\dot\phi} + \cancelto{0}{\dot\phi\phi}\omega_0 - \cancelto{0}{\dot\psi\psi}\omega_0 - \cancelto{0}{\phi\psi}\omega_0^2)(I_{xx} - I_{zz}) \\ (\cancelto{0}{\dot\theta\dot\phi} - \omega_0\dot\phi - \cancelto{0}{\dot\theta\psi}\omega_0 + \psi\omega_0^2)(I_{yy} - I_{xx}) \end{bmatrix}$$

(SIMPLIFICATION DUE TO ASSUMPTION #i)

$$\simeq \begin{bmatrix} \omega_0 (\dot\psi + \phi\omega_0)(I_{yy} - I_{zz}) \\ 0 \\ -\omega_0 (\dot\phi - \psi\omega_0)(I_{yy} - I_{xx}) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -\omega_o(\dot{\phi} - \psi\omega_o)(I_{yy} - I_{xx}) \end{bmatrix}$$

3rd TERM

$$\frac{3\mu}{R_c^5}\left[\vec{R}_{c_B}\right]_\times \cdot I_B^c \cdot \vec{R}_{c_B} = \frac{3\mu}{R_c^5}\begin{bmatrix} 0 & c\phi c\theta R_c & -s\phi c\theta R_c \\ -c\phi c\theta R_c & 0 & -s\theta R_c \\ s\phi c\theta R_c & s\theta R_c & 0 \end{bmatrix}\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} s\theta \cdot R_c \\ -s\phi c\theta R_c \\ -c\phi c\theta R_c \end{bmatrix}$$

$$c\phi c\theta \simeq 1 \qquad s\phi c\theta \simeq \phi \qquad s\theta \simeq \theta$$

$$= \frac{3\mu}{R_c^5}\begin{bmatrix} 0 & R_c & -\phi R_c \\ -R_c & 0 & -\theta R_c \\ \phi R_c & \theta R_c & 0 \end{bmatrix}\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} \theta R_c \\ -\phi R_c \\ -R_c \end{bmatrix}$$

$$= \frac{3\mu}{R_c^3}\begin{bmatrix} 0 & I_{yy} & -\phi I_{zz} \\ -I_{xx} & 0 & -\theta I_{zz} \\ \phi I_{xx} & \theta I_{yy} & 0 \end{bmatrix}\begin{bmatrix} \theta \\ -\phi \\ -1 \end{bmatrix}$$

$$= \frac{3\mu}{R_c^3}\begin{bmatrix} \phi(I_{zz} - I_{yy}) \\ \theta(I_{zz} - I_{xx}) \\ \cancelto{0}{\phi\theta}(I_{xx} - I_{yy}) \end{bmatrix}$$

$$\simeq 3\omega_o^2\begin{bmatrix} \phi(I_{zz} - I_{yy}) \\ \theta(I_{zz} - I_{xx}) \\ 0 \end{bmatrix}$$

FOR CIRCULAR ORBITS:

$$v_o = \sqrt{\frac{\mu}{R_c}}$$

$$\omega_o = \frac{v_o}{R_c} = \sqrt{\frac{\mu}{R_c^3}}$$

$$\omega_o^2 = \frac{\mu}{R_c^3}$$

PUTTING EVERYTHING TOGETHER, THE LINEARIZED E.O.M. OF A S/C SUBJECTED TO GG ARE:

$$(\ddot{\phi} - \dot{\psi}\omega_o)I_{xx} + \omega_o(\dot{\psi} + \phi\omega_o)(I_{yy} - I_{zz}) - 3\omega_o^2\phi(I_{zz} - I_{yy}) = 0$$

$$\ddot{\theta} I_{yy} - 3\omega_o^2\theta(I_{zz} - I_{xx}) = 0$$

$$(\ddot{\psi} + \dot{\phi}\omega_o)I_{zz} - \omega_o(\dot{\phi} - \psi\omega_o)(I_{yy} - I_{xx}) = 0$$

EULER EOM FOR GG TORQUE AROUND $\vec{0}$ EQUILIBRIUM

$$I_{xx}\ddot{\phi} - (I_{xx} - I_{yy} + I_{zz})\omega_o\dot{\psi} + 4\omega_o^2\phi(I_{yy} - I_{zz}) = 0$$ ROLL EQUATION

$$\boxed{\begin{aligned} I_{xx}\ddot{\phi} - (I_{xx}-I_{yy}+I_{zz})\omega_0\dot{\psi} + 4\omega_0^2\phi(I_{yy}-I_{zz}) &= 0 \quad \text{ROLL EQUATION} \\ I_{yy}\ddot{\theta} + 3\omega_0^2\theta(I_{xx}-I_{zz}) &= 0 \quad \text{PITCH EQUATION} \\ I_{zz}\ddot{\psi} + (I_{xx}-I_{yy}+I_{zz})\omega_0\dot{\phi} + \omega_0^2\psi(I_{yy}-I_{xx}) &= 0 \quad \text{YAW EQUATION} \end{aligned}}$$

⚠ NOTICE THAT THE PITCH DYNAMICS ARE UNCOUPLED FROM ROLL-YAW DYNAMICS SO WE CAN ANALYZE THEIR STABILITY SEPARATELY.

## PITCH STABILITY

APPLYING LAPLACE TRANSFORM $\left(\theta(t) \xrightarrow{\mathcal{L}} \Theta(s)\right)$

$$I_{yy}\ddot{\theta} + 3\omega_0^2\theta(I_{xx}-I_{zz}) = 0$$

DIVIDING BY $I_{yy}$ $\left(\sigma_y = \frac{I_{xx}-I_{zz}}{I_{yy}}\right)$

$$\ddot{\theta} + 3\omega_0^2\theta\,\sigma_y = 0$$

$\mathcal{L}$

$$s^2\Theta(s) - \dot{\theta}(0) - s\,\theta(0) + 3\omega_0^2\,\Theta(s)\,\sigma_y = 0$$

$$\Theta(s)\left(s^2 + 3\omega_0^2\sigma_y\right) = \dot{\theta}(0) + s\,\theta(0)$$

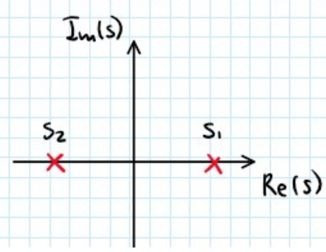
$$\Theta(s) = \frac{1}{\underbrace{s^2 + 3\omega_0^2\sigma_y}_{\text{CH. POLYNOMIAL}}}\left(\dot{\theta}(0) + s\,\theta(0)\right)$$

POLES : $s$ s.t. $s^2 + 3\omega_0^2\sigma_y = 0$

$$s_{1,2} = \pm\sqrt{-3\omega_0^2\sigma_y}$$

CASE I : $\sigma_y < 0 \Rightarrow I_{zz} > I_{xx}$

$$s_{1,2} = \pm\sqrt{3\omega_0^2|\sigma_y|} \in \mathbb{R}$$



CASE I : $\sigma_y < 0 \Rightarrow I_{zz} > I_{xx}$

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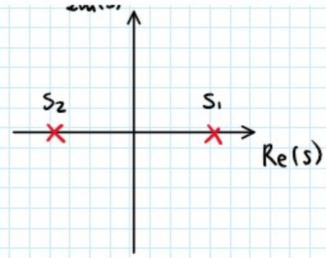
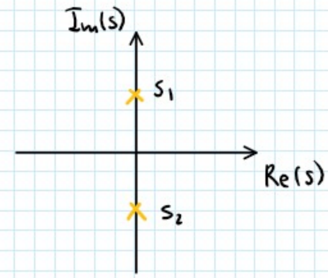


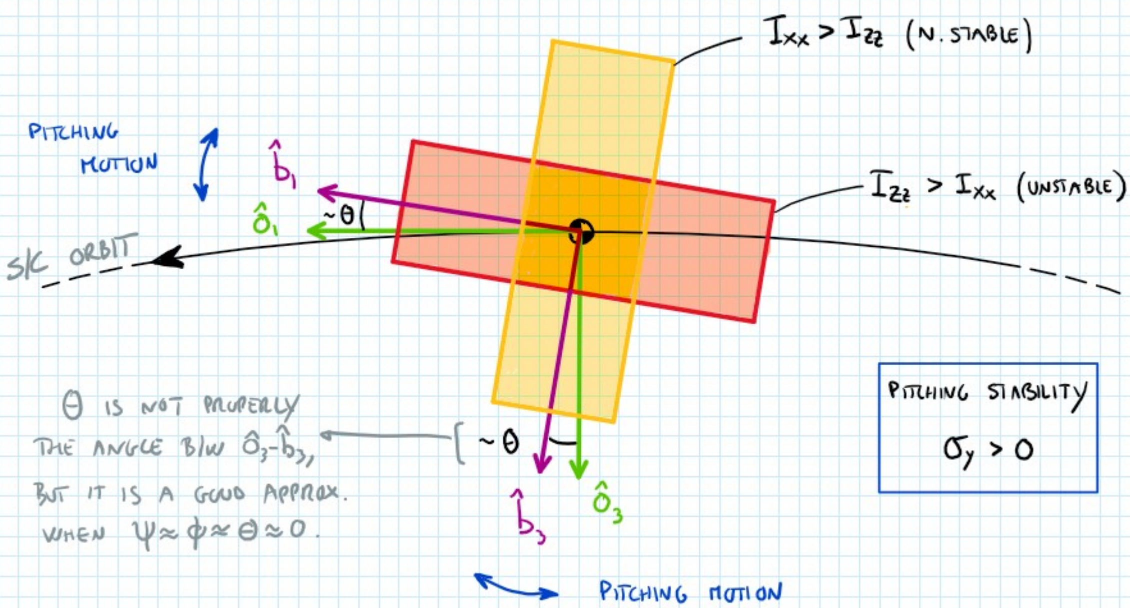

UNSTABLE PITCHING MOTION

CASE II : $\sigma_y > 0 \rightarrow I_{xx} > I_{zz}$

$$S_{1,2} = \pm i\sqrt{3\omega_0^2\sigma_y} \in \mathbb{I}m$$




N. STABLE PITCHING MOTION




ROLL-YAW STABILITY

LAPLACE TRANSFORM $\phi(t) \xrightarrow{\mathcal{L}} \Phi(s)$ & $\psi(t) \xrightarrow{\mathcal{L}} \Psi(s)$

$$\begin{cases} I_{xx}\ddot{\phi} - (I_{xx} - I_{yy} + I_{zz})\omega_0\dot{\psi} + 4\omega_0^2\phi(I_{yy} - I_{zz}) = 0 & \text{(DIVIDING BY } I_{xx}) \\ I_{zz}\ddot{\psi} + (I_{xx} - I_{yy} + I_{zz})\omega_0\dot{\phi} + \omega_0^2\psi(I_{yy} - I_{xx}) = 0 & \text{(DIVIDING BY } I_{zz}) \end{cases}$$

$$\begin{cases} \ddot{\phi} - (1+\sigma_x)\omega_0\dot{\psi} - 4\omega_0^2\phi\sigma_x = 0 \\ \ddot{\psi} + (1-\sigma_z)\omega_0\dot{\phi} + \omega_0^2\psi\sigma_z = 0 \end{cases}$$

$$\left(\sigma_x = \frac{I_{zz} - I_{yy}}{I_{xx}} \quad \sigma_z = \frac{I_{yy} - I_{xx}}{I_{zz}}\right)$$

$$\ddot{\psi} + (1-\sigma_z)\omega_0\dot{\phi} + \omega_0^2\psi\sigma_z = 0 \qquad \left(\sigma_z = \frac{I_{yy}-I_{xx}}{I_{zz}}\right)$$

$\mathcal{L}$

$$\begin{cases} s^2\Phi(s) - \dot{\phi}(0) - s\phi(0) - (1+\sigma_x)\omega_0 s\Psi(s) + (1+\sigma_x)\omega_0\psi(0) - 4\omega_0^2\sigma_x\Phi(s) = 0 \\ s^2\Psi(s) - \dot{\psi}(0) - s\psi(0) + (1-\sigma_z)\omega_0 s\Phi(s) - (1-\sigma_z)\omega_0\phi(0) + \omega_0^2\sigma_z\Psi(s) = 0 \end{cases}$$

$$\underbrace{\boxed{\begin{bmatrix} s^2 - 4\omega_0^2\sigma_x & -(1+\sigma_x)\omega_0 s \\ (1-\sigma_z)\omega_0 s & s^2 + \omega_0^2\sigma_z \end{bmatrix}}}_{A} \begin{bmatrix} \Phi(s) \\ \Psi(s) \end{bmatrix} = \begin{bmatrix} \dot{\phi}(0) + s\phi(0) - (1+\sigma_x)\omega_0\psi(0) \\ \dot{\psi}(0) + s\psi(0) + (1-\sigma_z)\omega_0\phi(0) \end{bmatrix}$$

$$\begin{bmatrix} \Phi(s) \\ \Psi(s) \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} s^2 + \omega_0^2\sigma_z & (1+\sigma_x)\omega_0 s \\ -(1-\sigma_z)\omega_0 s & s^2 - 4\omega_0^2\sigma_x \end{bmatrix} \begin{bmatrix} \dot{\phi}(0) + s\phi(0) - (1+\sigma_x)\omega_0\psi(0) \\ \dot{\psi}(0) + s\psi(0) + (1-\sigma_z)\omega_0\phi(0) \end{bmatrix}$$

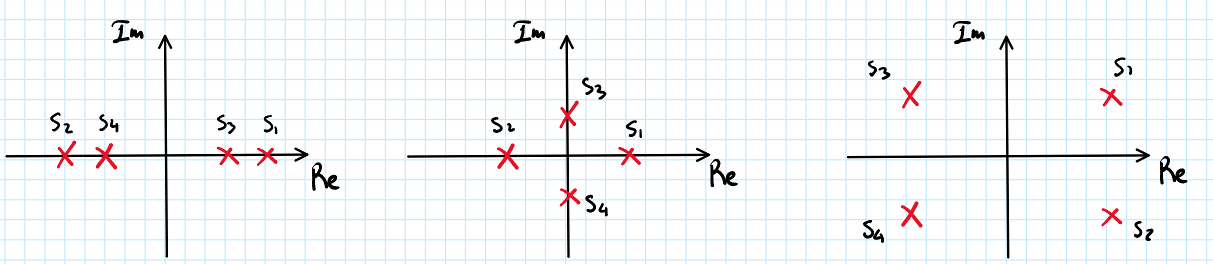
POLES : $s$ s.t. $\det(A) = 0$

$$\det(A) = s^4 + \underbrace{\boxed{(1 - 3\sigma_x - \sigma_x\sigma_z)\omega_0^2}}_{b} s^2 \underbrace{\boxed{-4\sigma_x\sigma_z\omega_0^4}}_{c} = 0$$

$$s^4 + bs^2 + c = 0 \qquad \text{BI-QUADRATIC EQUATION}$$

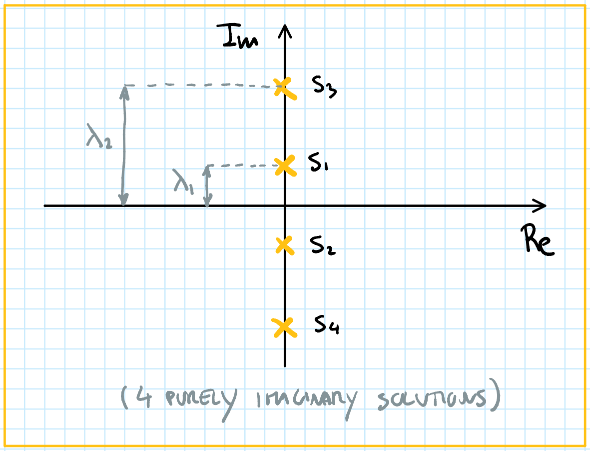
POSSIBLE SOLUTIONS (POLES) OF BI-QUADRATIC EQUATION (SYMMETRY ABOUT ORIGIN & Re-AXIS)

UNSTABLE CONFIGS.




(4 REAL SOLUTIONS) (2 REAL + 2 IMAG. SOLUTIONS) (2 PAIRS OF COMPLEX CONJ. SOLUTIONS)

N. STABLE CONFIG.




TO OBTAIN THE NEUTRALLY STABLE CONFIGURATION:

$$S_{1,2} = \pm i \lambda_1 \Rightarrow (S_{1,2})^2 = X_1 = -\lambda_1^2$$
$$S_{3,4} = \pm i \lambda_2 \Rightarrow (S_{3,4})^2 = X_2 = -\lambda_2^2$$

WHERE $X_1, X_2$ ARE THE REAL, NEGATIVE SOLUTIONS TO

$$x^2 + bx + c = 0$$

THUS, THE CONDITIONS MUST BE:

- $$b^2 - 4c > 0$$
- - $$X_1 = -\frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4c} < 0$$

$$X_2 = -\frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4c} < 0$$ (LESS STRINGENT THAN $X_1 < 0$; IN FACT $X_1 > X_2$)

- $b^2 - 4c > 0$

$$(1 - 3\sigma_x - \sigma_x\sigma_z)^2 \cancel{\omega_0^4} + 16\,\sigma_x\sigma_z \cancel{\omega_0^4} > 0$$
$$(1 - 3\sigma_x - \sigma_x\sigma_z)^2 > -16\,\sigma_x\sigma_z$$
$$-(1 - 3\sigma_x - \sigma_x\sigma_z)^2 < 16\,\sigma_x\sigma_z \Rightarrow \sigma_x\sigma_z > -\frac{1}{16}(1 - 3\sigma_x - \sigma_x\sigma_z)^2$$

- - $$-\frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4c} < 0$$

$$-b + \underbrace{\sqrt{b^2 - 4c}}_{>0} < 0 \Rightarrow b > 0$$

$> 0$

$$\sqrt{b^2 - 4c} < b$$

$$\cancel{b^2} - 4c < \cancel{b^2} \Rightarrow c > 0$$

$$b > 0 : \quad (1 - 3\sigma_x - \sigma_x \sigma_z)\cancel{\omega_0^2} > 0$$

$$\sigma_x \sigma_z + 3\sigma_x - 1 < 0 \quad \Rightarrow \quad \sigma_x \sigma_z < 1 - 3\sigma_x$$

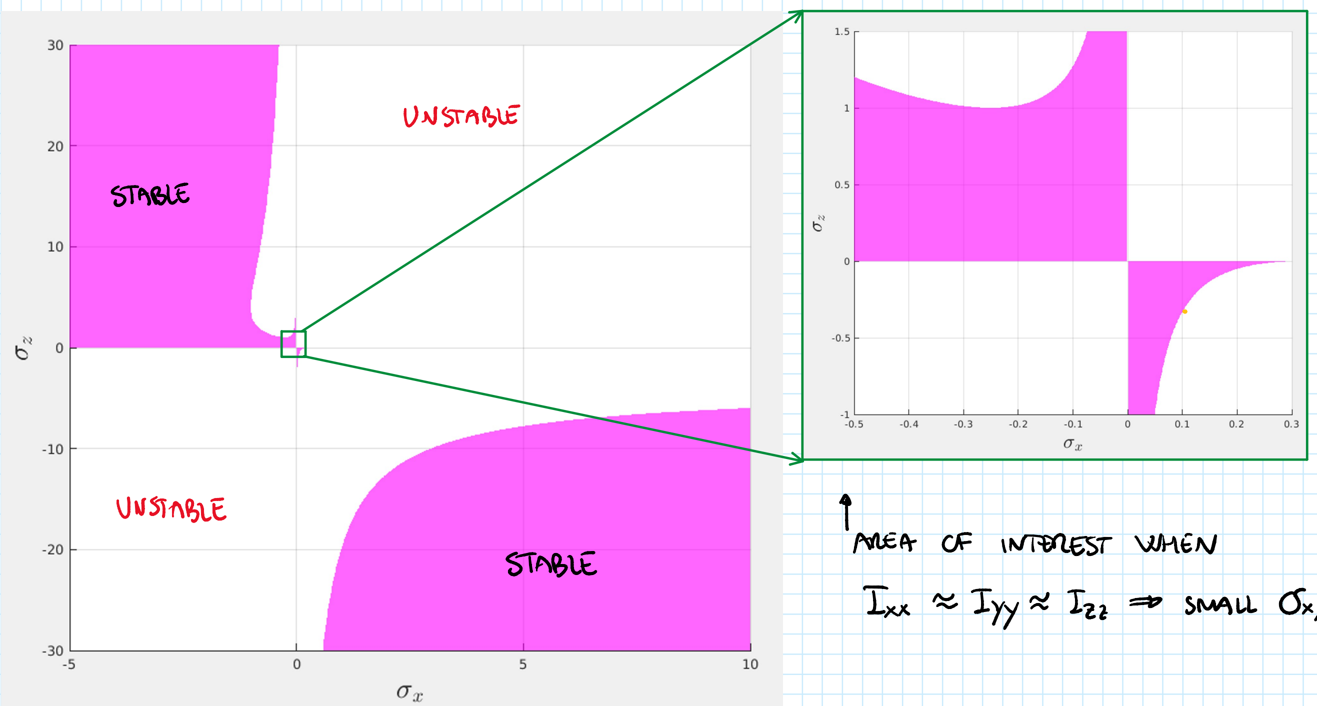
$$c > 0 : \quad -4\sigma_x \sigma_z > 0 \quad \Rightarrow \quad \sigma_x \sigma_z < 0$$

ROLLING & YAWING STABILITY

$$\sigma_x \sigma_z > -\frac{1}{16}(1 - 3\sigma_x - \sigma_x \sigma_z)^2$$

$$\sigma_x \sigma_z < 1 - 3\sigma_x$$

$$\sigma_x \sigma_z < 0$$




AREA OF INTEREST WHEN

$I_{xx} \approx I_{yy} \approx I_{zz} \Rightarrow$ SMALL $\sigma_x, \sigma_y, \sigma_z$

e.g. FOR THE EXAMPLE OF LECTURE 19

$$I_{xx} = 420 \ kg \cdot m^2$$

$I_{xx} = 420 \ Kg \cdot m^2$

$I_{yy} = 300 \ Kg \cdot m^2$

$I_{zz} = 350 \ Kg \cdot m^2$

$$\sigma_x = \frac{I_{zz} - I_{yy}}{I_{xx}} = \frac{350 - 300}{420} = 0.119$$

$$\sigma_y = \frac{I_{xx} - I_{zz}}{I_{yy}} = \frac{420 - 350}{300} = 0.233$$

$$\sigma_z = \frac{I_{yy} - I_{xx}}{I_{zz}} = \frac{300 - 420}{350} = -0.343$$

$$\sigma_x \sigma_z = 0.119 \cdot (-0.343) = -0.041$$

$$-\frac{1}{16}(1 - 3\sigma_x - \sigma_x\sigma_z)^2 = -\frac{1}{16}(1 - 3 \cdot 0.119 + 0.041)^2 = -0.029$$

$$1 - 3\sigma_x = 1 - 3 \cdot 0.119 = 0.643$$

$$\boxed{\sigma_y > 0 \quad ✓}$$

PITCHING STABLE

$$\boxed{\begin{array}{ll} \sigma_x\sigma_z > -\frac{1}{16}(1 - 3\sigma_x - \sigma_x\sigma_z)^2 & ✗ \\ \sigma_x\sigma_z < 1 - 3\sigma_x & ✓ \\ \sigma_x\sigma_z < 0 & ✓ \end{array}}$$

YAW-ROLL UNSTABLE

SIDE QUESTION : DOES THE S/C ATTITUDE AFFECT ITS ORBIT ?

• THE GRAVITATIONAL FORCE ACTING ON AN INFINITESIMAL MASS $dm$ IS GIVEN BY

$$d\vec{F}_g \simeq -\mu \frac{\vec{R}_c + \vec{r}}{R_c^3} \left( 1 - \frac{3 \vec{R}_c^T \vec{r}}{R_c^2} \right) dm$$

INTEGRATING OVER S/C VOLUME

$$\vec{F}_g = \int_{S/C} d\vec{F}_g = \underbrace{\boxed{-\frac{\mu m}{R_c^3}\vec{R}_c}}_{\vec{F}_{g\,2B}} + \underbrace{\boxed{-\frac{3\mu}{R_c^5}\left( I^c + \frac{1}{2}\left( tr(I^c) - 5\hat{R}_c^T I^c \hat{R}_c \right) I^c \right)\vec{R}_c}}_{\Delta\vec{F}_g}$$

$$\hookrightarrow \vec{F}_g = \int_{S/C} d\vec{F}_g = \underbrace{\boxed{-\frac{\mu m}{R_c^3} R_c}}_{\text{2-BODY DYNAMICS GRAV. FORCE.}} + \underbrace{\boxed{-\frac{3\mu}{R_c^5}\left(I^c + \frac{1}{2}\left(tr(I^c) - 5\,\hat{R}_c^{T} I^c \hat{R}_c\right) I^c\right)\vec{R}_c}}_{\text{ADDITIONAL TERMS DUE TO ATTITUDE OF S/C}}$$

FOR NORMAL-SIZED S/C IN LEO THESE TERMS ARE IN THE ORDER OF:

$$\|\vec{F}_{g_{2B}}\| \sim 10^4\,N \quad \gg \quad \|\Delta\vec{F}_g\| \sim 10^{-13}\,N$$

SIDE ANSWER : NO (OTHER DISTURBANCES WILL AFFECT THE ORBIT MUCH MORE THAN THE ATTITUDE).