ARO 4090 - WEEK 12

- M. MAGGIA

Lecture 19

Example: SINGLE-SPIN vs. DUAL-SPIN S/C

LET'S CONSIDER A SPACECRAFT ROTATING ABOUT ITS $\hat{b}_3$-AXIS (I.E. BODY-FIXED Z-AXIS) AT NOMINAL ANGULAR SPEED $n = 60$ rpm. ASSUMING THAT THE MOTION IS TORQUE-FREE AND KNOWING THAT

$$I_{xx} = 420 \ \mathrm{Kg \cdot m^2}$$
$$I_{yy} = 300 \ \mathrm{Kg \cdot m^2}$$
$$I_{zz} = 350 \ \mathrm{Kg \cdot m^2}$$

(a) IS THE S/C STABLE IN ITS ROTATIONAL MOTION?

(b) IF WE ADD A FLYWHEEL ($I_w = 10 \ \mathrm{Kg \cdot m^2}$) ROTATING ABOUT THE $\hat{b}_3$-AXIS, FIND THE RANGE OF THE FLYWHEEL ANGULAR SPEEDS THAT STABILIZE THE S/C'S ROTATIONAL MOTION.

$$\left[n = 60 \ \text{rpm} = 60 \ \frac{\cancel{\text{rotations}}}{\cancel{\text{min}}} \cdot 2\pi \ \frac{\text{rad}}{\cancel{\text{rotation}}} \cdot \frac{1}{60} \ \frac{\cancel{\text{min}}}{\text{s}} = 2\pi \ \frac{\text{rad}}{\text{s}} \right]$$

(a) SINGLE-SPIN STABILITY

FROM Lectures 16 & 17

- N. STABLE IF $\sigma_x \sigma_y < 0$

  UNSTABLE IF $\sigma_x \sigma_y \geq 0$

$$\left.\begin{aligned} \sigma_x &= \frac{I_{zz} - I_{yy}}{I_{xx}} = \frac{350 - 300}{420} = 0.119 \\ \sigma_y &= \frac{I_{xx} - I_{zz}}{I_{yy}} = \frac{420 - 350}{300} = 0.233 \end{aligned}\right\} \ \sigma_x \cdot \sigma_y = 0.0278 > 0$$

$$\sigma_y = \frac{I_{xx} - I_{zz}}{I_{yy}} = \frac{420 - 350}{300} = 0.233$$

- THE S/C MOTION IS UNSTABLE. IN FACT, I_{zz} IS THE AXIS OF INTERMEDIATE INERTIA.

- POLES OF LINEARIZED SYSTEM

$$s_1 = 0$$

$$s_{2,3} = \pm |n| \sqrt{\sigma_x \sigma_y} = \pm 2\pi \sqrt{0.0278} = \pm 1.047$$

(b) DUAL-SPIN STABILITY

FROM Lecture 14, SINCE $I_{xx} \neq I_{yy}$ WE ARE IN CASE I, THUS:

$$\Omega_1 = \frac{n}{I_w}\left(\frac{I_{xx}+I_{yy}}{2} - I_{zz}\right) - \frac{|n(I_{xx} - I_{yy})|}{2 I_w}$$

$$= \frac{60}{10}\left(\frac{420+300}{2} - 350\right) - \frac{|60(420-300)|}{2 \cdot 10}$$

$$= 60 - 360 = -300 \text{ RPM}$$

$$\Omega_2 = \frac{n}{I_w}\left(\frac{I_{xx}+I_{yy}}{2} - I_{zz}\right) + \frac{|n(I_{xx} - I_{yy})|}{2 I_w}$$

$$= \frac{60}{10}\left(\frac{420+300}{2} - 350\right) + \frac{|60(420-300)|}{2 \cdot 10}$$

$$= 60 + 360 = 420 \text{ RPM}$$

$\boxed{\Omega < -300 \text{ RPM} \quad \text{OR} \quad \Omega > 420 \text{ RPM}}$ FOR (NEUTRAL) STABILITY

- e.g. IF $\Omega = 600$ RPM ($= 20\pi$ rad/s), THE POLES ARE

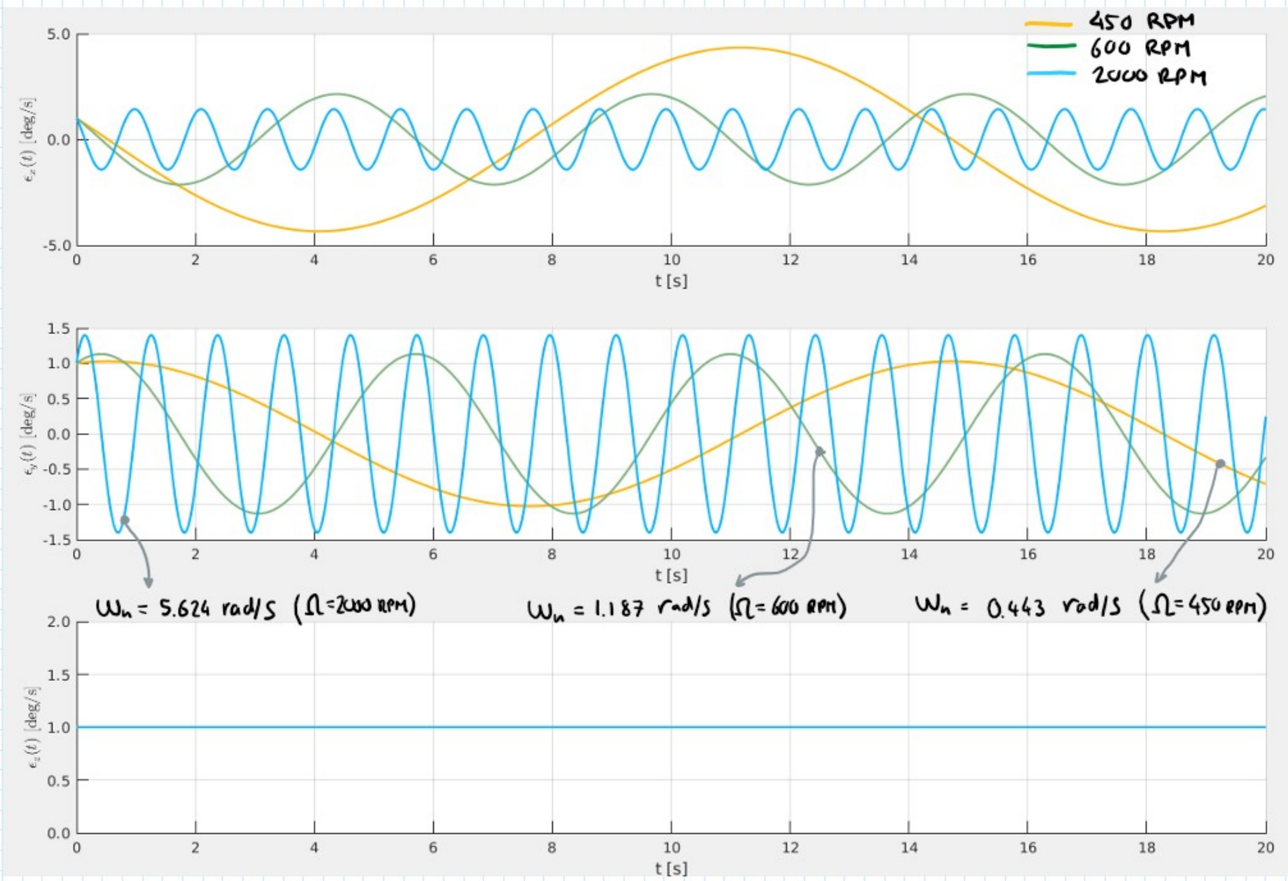
$$S_1 = 0$$

$$S_{2,3} = \pm\sqrt{n^2\sigma_x\sigma_y - I_w\Omega\left(\frac{\sigma_x n}{I_{yy}} - \frac{\sigma_y n}{I_{xx}} + \frac{I_w\Omega}{I_{xx}I_{yy}}\right)}$$

$$= \pm\sqrt{4\pi^2\cdot 0.0278 - 10\cdot 20\pi\left(\frac{0.119\cdot 2\pi}{300} - \frac{0.233\cdot 2\pi}{420} + \frac{10\cdot 2\pi}{420\cdot 300}\right)}$$

$$= \pm\, i\cdot 1.187$$

SOLUTIONS OF THE LINEARIZED SYSTEM WITH $\varepsilon_x(0) = \varepsilon_y(0) = \varepsilon_z(0) = 1$ deg/s AND FOR 3 DIFFERENT VALUES OF Ω (450, 600, 2000 RPM)



NOTICE THAT :

- THE ERROR ε_z REMAINS CONSTANT IN TIME AND THUS IT'S BOUNDED.
- THE ERRORS ε_x, ε_y ARE SINUSOIDAL FUNCTIONS (THUS BOUNDED) WHOSE AMPLITUDE AND FREQUENCY CHANGE AS Ω CHANGES

• THE ERRORS $\varepsilon_x, \varepsilon_y$ ARE SINUSOIDAL FUNCTIONS (THUS BOUNDED) WHOSE AMPLITUDE AND FREQUENCY CHANGE AS Ω CHANGES

$$\varepsilon_x(t) = \boxed{A_x(\Omega, \varepsilon_x(0), \varepsilon_y(0))} \cdot \cos\left(\boxed{\omega_n(\Omega)}\, t + \boxed{\phi_x(\Omega, \varepsilon_x(0), \varepsilon_y(0))}\right)$$

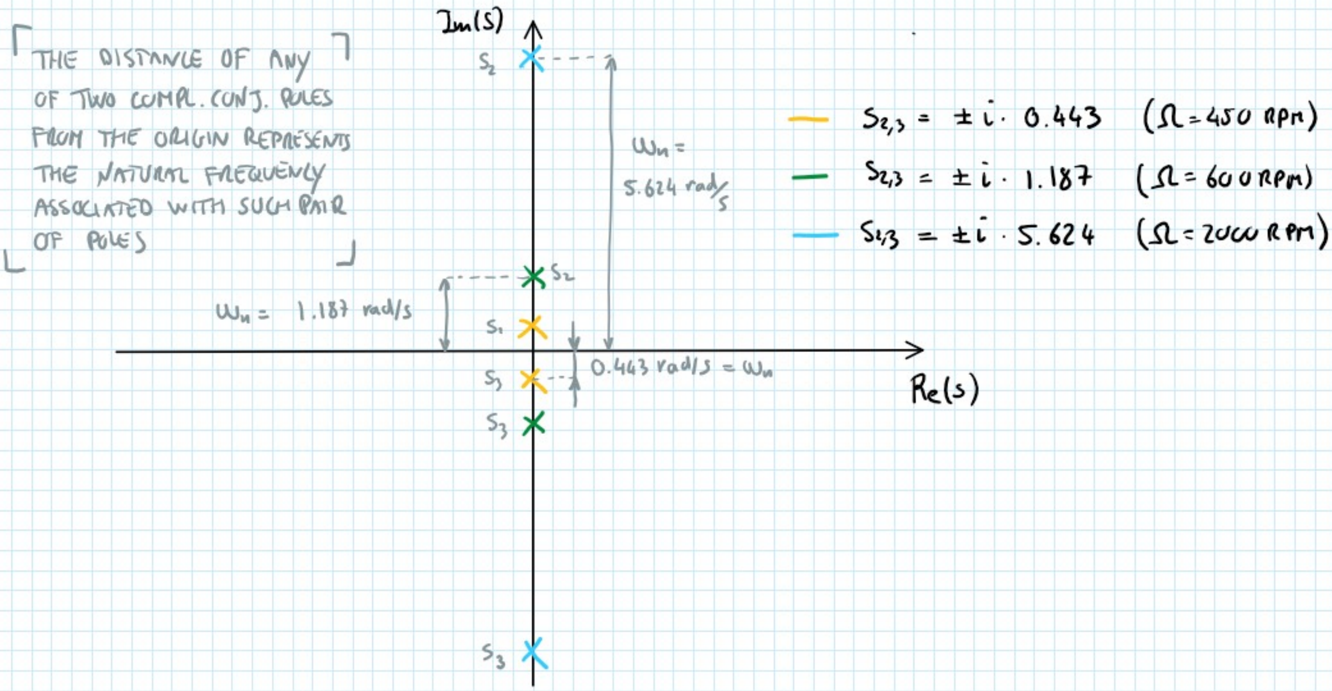
$$\varepsilon_y(t) = \boxed{A_y(\Omega, \varepsilon_x(0), \varepsilon_y(0))} \cos\left(\boxed{\omega_n(\Omega)}\, t + \boxed{\phi_y(\Omega, \varepsilon_x(0), \varepsilon_y(0))}\right)$$

AMPLITUDES OF OSCILLATIONS
(DIFFERENT B/W ε_x AND ε_y)

(NATURAL) FREQUENCY OF OSCILLATIONS
(SAME FOR BOTH ε_x AND ε_y)

IT CAN BE SHOWN THAT, IF THE SYSTEM IS (MARGINALLY) STABLE:

$$\omega_n(\Omega) = |s_2| = |s_3| = \sqrt{n^2 \sigma_x \sigma_y - I_w \Omega \left(\frac{\sigma_x n}{I_{yy}} - \frac{\sigma_y n}{I_{xx}} + \frac{I_w \Omega}{I_{xx} I_{yy}}\right)}$$



★ SO FAR WE'VE CONSIDERED THE FLYWHEEL SPINNING AT A NOMINAL, CONSTANT ANGULAR SPEED Ω.

$$\dot{\Omega} = 0 \Rightarrow \Omega = \text{const.} \in [\Omega_{MIN}, \Omega_{MAX}]$$

HOWEVER, GENERALLY WE CAN HAVE $\dot{\Omega} \neq 0$

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- WHEN THE WHEEL GETS UP TO SPEED $0 \longrightarrow \Omega_{NOMINAL}$ (TRANSIENT PHASE)
- WE CAN EXPLOIT $\dot{\Omega} \neq 0$ TO ACTIVELY CONTROL THE S/C ATTITUDE IF THE EXTERNAL TORQUE ABOUT THE WHEEL'S AXIS (i.e., M_z) IS NOT NULL.

LET'S SEE HOW THE EQUATIONS OF MOTION ACCOUNTING FOR $\dot{\Omega} \neq 0$. FROM lecture 19, THE S/C ANGULAR MOMENTUM VECTOR IS:

$$\vec{H}^{c}_{S/C_{\mathcal{B}}} = \overbrace{\boxed{\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}}}^{\text{BODY}} + \overbrace{\boxed{\begin{bmatrix} I_{t_w} + m_w z_w^2 & 0 & 0 \\ 0 & I_{t_w} + m_w z_w^2 & 0 \\ 0 & 0 & I_{a_w} \end{bmatrix} \begin{bmatrix} p \\ q \\ r + \Omega \end{bmatrix}}}^{\text{WHEEL}}$$

$$\vec{M}^{c}_{\mathcal{B}} = \frac{{}^{\mathcal{I}}d\vec{H}^{c}_{S/C_{\mathcal{B}}}}{dt} = \dot{\vec{H}}^{c}_{S/C_{\mathcal{B}}} + \omega^{\mathcal{B}/\mathcal{I}}_{\mathcal{B}} \wedge \vec{H}^{c}_{S/C_{\mathcal{B}}}$$

$$= \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} I_{t_w} + m_w z_w^2 & 0 & 0 \\ 0 & I_{t_w} + m_w z_w^2 & 0 \\ 0 & 0 & I_{a_w} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} + \boxed{\dot{\Omega}} \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \left(\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \begin{bmatrix} I_{t_w} + m_w z_w^2 & 0 & 0 \\ 0 & I_{t_w} + m_w z_w^2 & 0 \\ 0 & 0 & I_{a_w} \end{bmatrix} \begin{bmatrix} p \\ q \\ r + \Omega \end{bmatrix} \right) = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

$$\begin{cases} M_x = (I_{xx} + I_{tw} + m_w z_w^2)\dot{p} + (I_{zz} - I_{yy}) qr + (I_{aw} - I_{tw} - m_w z_w^2) q (r + \Omega) \\ M_y = (I_{yy} + I_{tw} + m_w z_w^2)\dot{q} + (I_{xx} - I_{zz}) rp + (I_{tw} + m_w z_w^2 - I_{aw}) p (r + \Omega) \\ M_z = (I_{zz} + I_{aw})\dot{r} + (I_{yy} - I_{xx}) pq + I_{aw}\dot{\Omega} \end{cases}$$

see Lecture 18 for simplific. details

$$\begin{cases} M_x \approx I_{xx}\dot{p} + (I_{zz} - I_{yy}) qr + I_w q \Omega \\ M_y \approx I_{yy}\dot{q} + (I_{xx} - I_{zz}) rp - I_w p \Omega \\ M_z \approx I_{zz}\dot{r} + (I_{yy} - I_{xx}) pq + I_w \dot{\Omega} \end{cases}$$

THE WHEEL IS SPED UP BY APPLYING A TORQUE ABOUT ITS AXIS. LET'S ASSUME THAT SUCH TORQUE IS CONSTANT, WHICH RESULTS IN A CONSTANT WHEEL ANGULAR ACCELERATION

$$\dot{\Omega} = C = \text{const.}$$

⚠ WE'VE JUST INTRODUCED A 4^{th} ODE! IN FACT, THE STATE VECTOR HAS BEEN AUGMENTED TO ACCOUNT FOR THE NEW STATE VARIABLE $\Omega = \Omega(t)$.

	NO WHEEL or WHEEL SPINNING @ $\Omega = \text{const.}$ ($\dot{\Omega} = 0$)	WHEEL SPINNING AT $\Omega \neq \text{const.}$ ($\dot{\Omega} = \text{const.}$)
STATE	$\vec{X} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$	$\vec{X}^* = \begin{bmatrix} p \\ q \\ r \\ \boxed{\Omega} \end{bmatrix}$
EQS. OF MOTION	$\begin{cases} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr + I_w q \Omega = M_x \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})rp - I_w p \Omega = M_y \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq = M_z \end{cases}$	$\begin{cases} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr + I_w q \Omega = M_x \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})rp - I_w p \Omega = M_y \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + \boxed{I_w C} = M_z \\ \boxed{\dot{\Omega} = C} \end{cases}$

★ SINCE $\dot{\Omega} = C = \text{const}$, THE WHEEL'S ANGULAR SPEED INCREASES LINEARLY AND THE TIME IT TAKES TO REACH ITS NOMINAL SPEED Ω_{NOM} IS REFERRED TO AS SPIN-UP TIME (t_{SU})

$$\dot{\Omega} = C$$

$$\frac{d\Omega}{dt} = C$$

$$\int_{\Omega_0}^{\Omega_{NOM}} d\Omega = \int_0^{t_{SU}} C\, dt$$

ANGULAR SPEED OF WHEEL @ $t = 0$

$\Omega_0 = 0$ IF WHEEL STARTS AT REST.

$$\Omega_{NOM} - \Omega_0 = C\, t_{SU}$$

$$\boxed{t_{SU} = \frac{\Omega_{NOM} - \Omega_0}{C}}$$

• LET'S STUDY THE CASE FOR $M_x = M_y = M_z = 0$ (NO EXTERNAL TORQUE)

* THE TORQUE NEEDED TO SPIN UP THE WHEEL IS NOT CONSIDERED EXTERNAL AS IT COMES FROM ENERGY EXCHANGE WITHIN THE S/C

$$\begin{cases} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr + I_w q\Omega = 0 \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})rp - I_w p\Omega = 0 \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + I_w c = 0 \\ \dot{\Omega} = c \end{cases}$$

$$\begin{cases} \dot{p} = -\sigma_x qr - \frac{I_w}{I_{xx}} q\Omega \\ \dot{q} = -\sigma_y rp + \frac{I_w}{I_{yy}} p\Omega \\ \dot{r} = -\sigma_z pq - \frac{I_w}{I_{zz}} c \\ \dot{\Omega} = c \end{cases}$$

RECALL:

$\frac{I_{zz} - I_{yy}}{I_{xx}} := \sigma_x$

$\frac{I_{xx} - I_{zz}}{I_{yy}} := \sigma_y$

$\frac{I_{yy} - I_{xx}}{I_{zz}} := \sigma_z$

IS $\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$ AN EQUILIBRIUM FOR THE SYSTEM (LIKE FOR SINGLE-SPIN & DUAL-SPIN W/ Ω=CONST)?

$0 = 0$ ✓

$0 = 0$ ✓

$0 = -\frac{I_w}{I_{zz}} c$ ✗

$\Rightarrow$ WHILE THE WHEEL IS SPINNING UP, THE BODY ANGULAR SPEED ABOUT ITS Z-AXIS ($\hat{b}_3$-AXIS) CHANGES AND CANNOT BE MAINTAINED AT NOMINAL SPEED n.

THIS IS CAUSED BY AN ANGULAR MOMENTUM EXCHANGE B/W WHEEL AND S/C BODY. AS THE WHEEL SPINS UP, THE BODY SLOWS DOWN.

SIMILAR SET UP AS PREVIOUS EXAMPLE : $\varepsilon_x(0) = \varepsilon_y(0) = \varepsilon_z(0) = 1$ deg/s

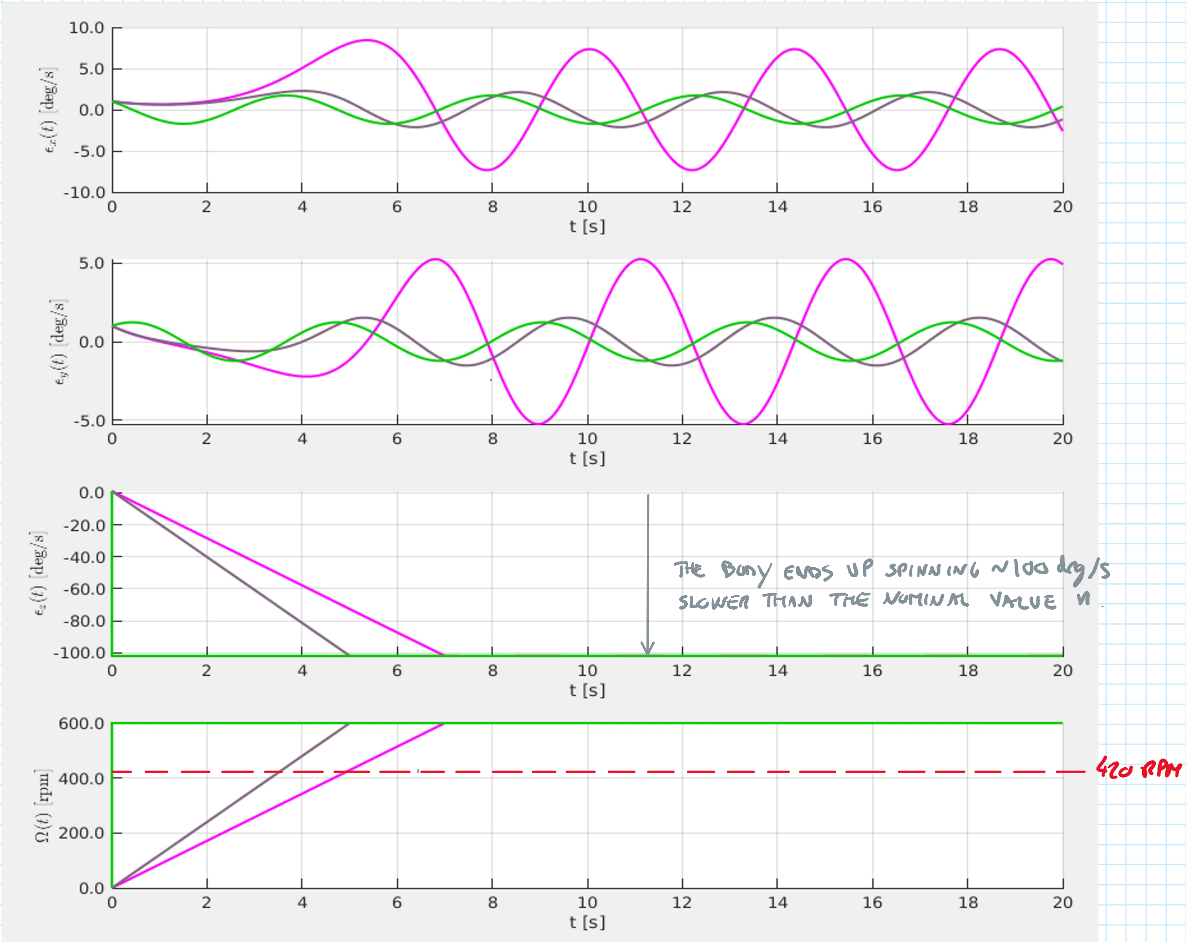
$I_{xx} = 420$ kg·m², $I_{yy} = 300$ kg·m², $I_{zz} = 350$ kg·m²

$n = 60$ RPM $(= r(0))$

$I_w = 10$ kg·m²

$\Omega_{NOM} = 600$ RPM & $\Omega_0 = 0$ RPM

SEVERAL SPIN-UP TIMES HAVE BEEN CHOSEN $t_{su} = [10^{-9}, 5, 7]$ s



COMMENTS :

i. AS t_{su} INCREASES, THE SYSTEM SPENDS MORE TIME IN ITS UNSTABLE CONFIGURATION (ACCORDING TO THE PREVIOUS ANALYSIS $\Omega > 420$ RPM FOR STABILITY), THEREFORE THE OSCILLATIONS IN $\varepsilon_x, \varepsilon_y$ GET LARGER AND LARGER AND THEN STABILIZE ONCE THE WHEEL'S SPEED INCREASES PASS 420 RPM.

ii. IF $t_{su} \to 0$, THE WHEEL REACHES ITS NOMINAL SPEED INSTANTANEOUSLY AND THE OSCILLATIONS IN $\varepsilon_x, \varepsilon_y$ COINCIDE WITH WHAT SEEN FOR THE Ω = CONST. CASE

iii. UNLIKE THE Ω = CONST. CASE, IF WE NEED TO SPEED UP THE WHEEL (NO MATTER HOW FAST WE DO IT), WE'LL END UP WITH AN ERROR $\varepsilon_z(t) \neq \varepsilon_z(0)$.
IN FACT, AS THE WHEEL SPEEDS UP, IT TRADES ANG. MOMENTUM WITH THE BODY OF THE S/C, WHICH ENDS UP SLOWING DOWN.

WHAT IF WE WANTED TO KEEP THE S/C BODY SPINNING @ $n = 66$ RPM?

$\Downarrow$

WE NEED TO APPLY EXTERNAL TORQUE ABOUT Z-AXIS

$\Downarrow$

WE NEED TO APPLY EXTERNAL TORQUE ABOUT Z-AXIS TO BALANCE THE WHEEL'S ANG. ACCELERATION.

$$I_{xx}\dot{p} + (I_{zz}-I_{yy})qr + I_w q\Omega = 0$$
$$I_{yy}\dot{q} + (I_{xx}-I_{zz})rp - I_w p\Omega = 0$$
$$I_{zz}\dot{r} + (I_{yy}-I_{xx})pq + I_w C = M_z$$
$$\dot{\Omega} = C$$

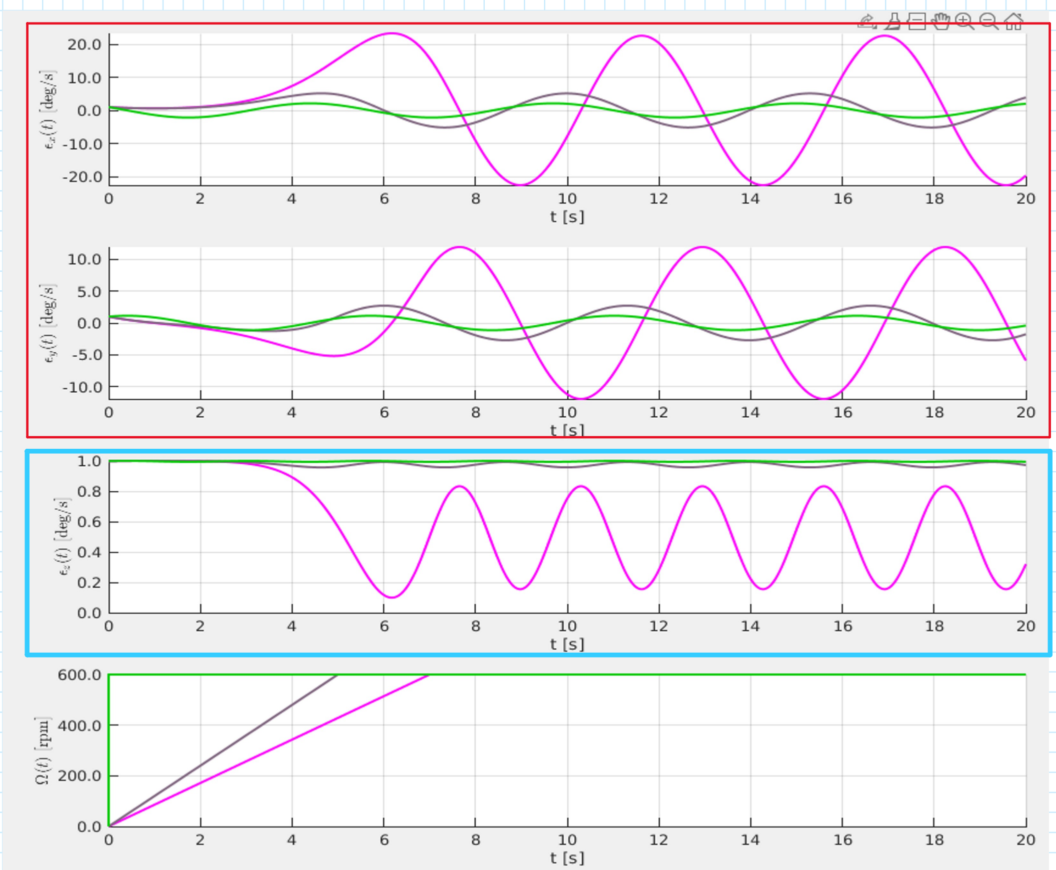
CHOOSING $M_z = I_w \cdot C$ LEADS TO :

$$\begin{cases} \dot{p} = -\sigma_x qr - \dfrac{I_w}{I_{xx}} q\Omega \\ \dot{q} = -\sigma_y rp + \dfrac{I_w}{I_{yy}} p\Omega \\ \dot{r} = -\sigma_z pq \\ \dot{\Omega} = C \end{cases} \qquad (1)$$

WE CAN CLEARLY SEE THAT $\vec{X}_0 = [0,0,n]^T$ ONCE AGAIN LEADS TO $\dot{p} = \dot{q} = \dot{r} = 0$

PASSIVE STABILIZATION

ACTIVE STABILIZ.



PASSIVE STABILITY (NEUTRAL OR ASYMPTOTIC IF NUTATION DAMPERS ARE USED)

BY ACTIVELY CONTROLLING THE Z-AXIS DYNAMICS WE ARE ABLE TO KEEP THE S/C SPINNING ABOUT ITS Z-AXIS CLOSE TO NOMINAL SPEED

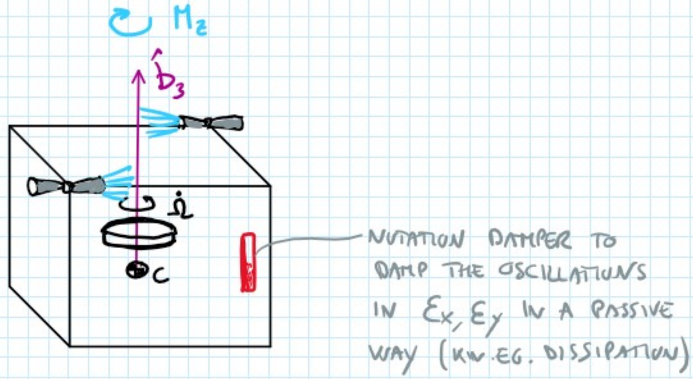
$\epsilon_z(t) \simeq \epsilon_z(0)$

NOTICE THAT, AS $t_{su} \nearrow$, NONLINEARITIES KICK IN. THAT'S WHY $\epsilon_z(t)$ IS NOT PROPERLY EQUAL

THAT'S WHY $\varepsilon_z(t)$ IS NOT PROPERLY EQUAL TO ITS INITIAL VALUE (AS HAPPENS FOR THE LINEARIZED DYNAMICS)

★ WHAT IS THE SOURCE OF THE EXTERNAL TORQUE M_z?

↳ WE CAN USE ATTITUDE CONTROL SYSTEM (ACS) THRUSTERS MOUNTED OUTSIDE THE S/C, FOR THE DURATION OF t_{SU}



↳ TO PASSIVELY DAMP THE OSCILLATIONS IN $\varepsilon_x, \varepsilon_y$ (AND THUS DRIVE THE NUTATION ANGLE TO 0) NUTATION DAMPERS CAN BE INSTALLED. THEY USUALLY CONTAIN VISCOUS FLUIDS THAT SLOSHES AND THUS DISSIPATES KINETIC ENERGY. THIS IS VERY USEFUL AND NECESSARY IF WE NEED TO POINT INSTRUMENTS IN PRECISE DIRECTIONS (e.g. JAMES WEBB / HUBBLE TELESCOPES)

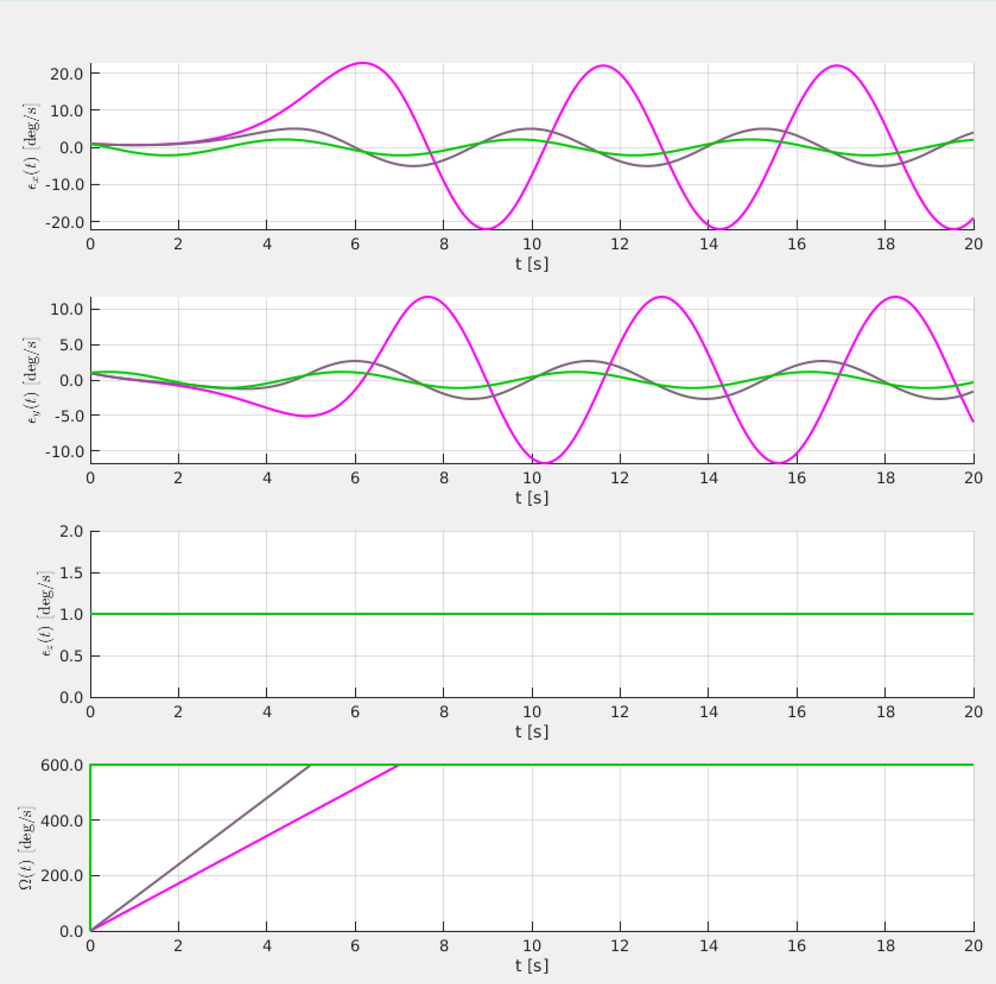
↳ ⚠ THE PREVIOUS RESULTS (i.e. $\dot{\Omega} \neq 0$, $M_z \neq 0$) HAVE BEEN OBTAINED BY INTEGRATING THE NONLINEAR DYNAMICS. ONE COULD LINEARIZE THE NONLINEAR SYSTEM ABOUT THE TRAJECTORY (NOT EQUILIBRIUM, AS ONE OF THE STATES, Ω, IS NOT CONSTANT)

$$\vec{X}^*_{TRAJ} = \begin{bmatrix} 0 \\ 0 \\ n \\ C\cdot t \end{bmatrix} \qquad (\text{VALID FOR } t \leq t_{SU})$$

SUBSTITUTING IN NONLINEAR DYNAMICS (1)

$0 = 0$ ✓
$0 = 0$ ✓
$0 = 0$ ✓
$C = C$ ✓

RESULTS OF LINEARIZED SYSTEM:



$\epsilon_x(t)$ [deg/s]
t [s]
$\epsilon_y(t)$ [deg/s]
t [s]
$\epsilon_z(t)$ [deg/s]
t [s]
$\Omega(t)$ [deg/s]
t [s]