Homework #2

Due: **Sunday, February 18, 2024 at 11:59 pm**

Turn in, on Canvas, a well organized and clear-to-read pdf file with your calculations, explanations, plots, as well as a ready-to-run .m files containing the MATLAB codes (when applicable).

Problem 1
Consider the following matrix:

$$R = \begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

(a) Show that R is a rotation matrix.

(b) Calculate its eigenvalues and draw them in the complex plane.

(c) Calculate the direction of the axis of rotation $\hat{\mathbf{a}}$ and the angle of rotation α about $\hat{\mathbf{a}}$.

(d) Using the formula derived at the end of Lecture 6 (i.e., Euler's formula), use α and $\hat{\mathbf{a}}$ to retrieve the matrix R. Comment on your findings.

(e) Find the Euler angles ψ, θ and ϕ, so that $R = R_1(\phi)R_2(\theta)R_3(\psi)$.

Do all calculations by hand. You can use the provided MATLAB code to check on the correctness of your results.

Problem 2
Repeat all steps of Problem 1 for the following matrix:

$$R = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 3
(a) Show that any 3-by-3 coordinate transformation matrix is a rotation matrix.

(b) Explain the difference between rotation matrices and coordinates transformation matrices.

Problem 1

Consider the following matrix:

$$R = \begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

(a) Show that R is a rotation matrix.

(b) Calculate its eigenvalues and draw them in the complex plane.

(c) Calculate the direction of the axis of rotation $\hat{\mathbf{a}}$ and the angle of rotation α about $\hat{\mathbf{a}}$.

(d) Using the formula derived at the end of Lecture 6 (i.e., Euler's formula), use α and $\hat{\mathbf{a}}$ to retrieve the matrix R. Comment on your findings.

(e) Find the Euler angles ψ, θ and ϕ, so that $R = R_1(\phi)R_2(\theta)R_3(\psi)$.

Do all calculations by hand. You can use the provided MATLAB code to check on the correctness of your results.

Solution

(a) Show that R is a rotation matrix.

* Must meet 2 conditions $\Rightarrow \begin{cases} ① & \det(R) = 1 \rightarrow \text{Determinant} \\ ② & RR^T = I \rightarrow \text{orthogonality} \end{cases}$

$$R = \begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

DETERMINANT: $\det(A) \stackrel{?}{=} 1$

$$\det(R) = 0\begin{bmatrix} 0 & 0 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} - 1/2\begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix} + \sqrt{3}/2\begin{bmatrix} -1 & 0 \\ 0 & -\sqrt{3}/2 \end{bmatrix}$$

$$= -1/2(-1/2) + \sqrt{3}/2(\sqrt{3}/2) = 1$$

$$\boxed{\therefore \det(A) \stackrel{\checkmark}{=} 1}$$

ORTHOGONALITY: $RR^T \stackrel{?}{=} I$

$$R = \begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \quad ; \quad R^T = \begin{bmatrix} 0 & -1 & 0 \\ 1/2 & 0 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

$$RR^T = \begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1/2 & 0 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} \overbrace{0 + (1/2)(1/2) + (\sqrt{3}/2)(\sqrt{3}/2)}^{1} & \cancelto{0}{-1(0) + 1/2(0) + \sqrt{3}/2(0)} & \cancelto{0}{0(0) + (1/2)(-\sqrt{3}/2) + (\sqrt{3}/2)(1/2)} \\ \cancelto{0}{-1(0) + (0)(1/2) + 0(\sqrt{3}/2)} & \overbrace{-1(-1) + 0(0) + 0(0)}^{1} & \cancelto{0}{-1(0) + 0(-\sqrt{3}/2) + 0(1/2)} \\ \cancelto{0}{0(0) + (-\sqrt{3}/2)(1/2) + (1/2)(\sqrt{3}/2)} & \cancelto{0}{(0)(0) + (-\sqrt{3}/2)(0) + (1/2)(0)} & \underbrace{0(0) + (-\sqrt{3}/2)(-\sqrt{3}/2) + 1/2(1/2)}_{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\boxed{\therefore RR^T \stackrel{\checkmark}{=} I}$$

(b) Calculate its eigenvalues and draw them in the complex plane.

$$(R - \lambda I) \cdot \vec{v} = \vec{0} \quad \text{with} \quad \vec{v} \neq \vec{0}$$

$\hookrightarrow \det(R - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det\begin{bmatrix} -\lambda & 1/2 & \sqrt{3}/2 \\ -1 & -\lambda & 0 \\ 0 & -\sqrt{3}/2 & 1/2 - \lambda \end{bmatrix} = -\lambda\begin{bmatrix} -\lambda & 0 \\ -\sqrt{3}/2 & 1/2 - \lambda \end{bmatrix} - 1/2\begin{bmatrix} -1 & 0 \\ 0 & 1/2 - \lambda \end{bmatrix} + \sqrt{3}/2\begin{bmatrix} -1 & \lambda \\ 0 & -\sqrt{3}/2 \end{bmatrix} = 0$$

$$= -\lambda[-\lambda(1/2 - \lambda)] - 1/2[-1(1/2 - \lambda)] + \sqrt{3}/2[\sqrt{3}/2] = 0$$

$$= -\lambda[-\lambda/2 + \lambda^2] - 1/2[\lambda - 1/2] + 3/4 = 0$$

$$= -\lambda^3 + \tfrac{1}{2}\lambda^2 - (\lambda/2 - 1/4) + 3/4 = 0$$

$$= -\lambda^3 + \tfrac{1}{2}\lambda^2 - \lambda/2 + 1/4 + 3/4 = 0$$
$$= -\lambda^3 + \lambda^2/2 - \lambda/2 + 1 = 0$$
$$= \lambda^3 - \tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\lambda - 1 = 0$$

- $\lambda_1 = 1 \Rightarrow 1^3 - \tfrac{1}{2} + \tfrac{1}{2} - 1 = 0$ ✓
- $\lambda^3 - \tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\lambda - 1 \iff \lambda^3 + \lambda^2(e-1) + \lambda(f-e) - f$

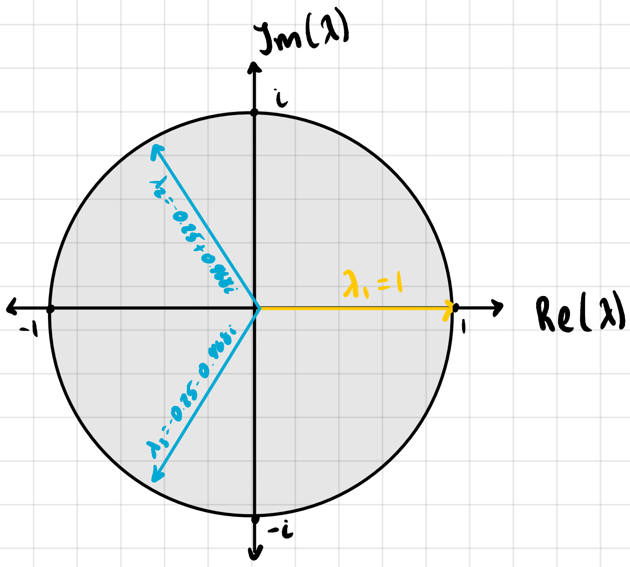
By Comparison

$\lambda^3: 1 = 1$
$\lambda^2: -\tfrac{1}{2} = e - 1 \Rightarrow 1/2$
$\lambda^1: \tfrac{1}{2} = f - e \rightarrow 1/2 = 1/2$
$\lambda^0: -1 = -f \Rightarrow f = 1$

$$(\lambda_1 - 1)(\lambda^2 + e\lambda + f) = 0 \Longrightarrow \lambda^2 + \tfrac{1}{2}\lambda + 1 = 0$$

$$\lambda_{2,3} = \tfrac{1}{2}\left(-\tfrac{1}{2}\right) \pm \tfrac{1}{2}\sqrt{\left(\tfrac{1}{2}\right)^2 - 4(1)(1)}$$
$$= -\tfrac{1}{4} \pm \tfrac{1}{2}\sqrt{\tfrac{1}{4} - 4}$$
$$= -\tfrac{1}{4} \pm \tfrac{1}{2} i \sqrt{\tfrac{15}{4}}$$
$$= -\tfrac{1}{4} \pm \tfrac{i}{4}\sqrt{15} \Rightarrow \lambda_{2,3} = -0.25 \pm i\,0.968$$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -0.25 + i\,0.968 \\ \lambda_3 &= -0.25 - i\,0.968 \end{aligned}}$$



(c) Calculate the direction of the axis of rotation $\hat{a}$ and the angle of rotation α about $\hat{a}$.

1) Find Angle of rotation $\alpha \rightarrow \alpha = \operatorname{acos}\left(\frac{\operatorname{trace}(R) - 1}{2}\right)$; where $R = \begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}$

$$\operatorname{trace}(R) = 0 + 0 + 1/2 = 1/2$$

$$\alpha = \operatorname{acos}\left(\frac{\operatorname{trace}(R) - 1}{2}\right) = \operatorname{acos}\left(\frac{1/2 - 1}{2}\right) = \operatorname{acos}\left(-1/4\right) \approx 104.47°$$

$\alpha = 104.47°$

2) Axis of Rotation $\hat{a}$

$$(R - I)\hat{a} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & 1/2 & \sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} a_x \\ a_y \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1/2 & \sqrt{3}/2 \\ -1 & -1 & 0 \\ 0 & -\sqrt{3}/2 & -1/2 \end{bmatrix}\begin{bmatrix} a_x \\ a_y \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -a_x + \tfrac{1}{2}a_y + \tfrac{\sqrt{3}}{2}a_3 &= 0 \quad (1) \\ -a_x - a_y &= 0 \quad (2) \\ -\sqrt{3}/2\, a_y - 1/2\, a_3 &= 0 \quad (3) \end{aligned}$$

using Eq's 2 & 3

$$\begin{cases} -a_x - a_y = 0 \\ -\sqrt{3}/2\, a_y - 1/2\, a_3 = 0 \\ a_x = -1 \end{cases}$$

$\hookrightarrow -a_x - a_y = 0$

$a_y = -a_x = 1$

$\hookrightarrow a_3 = 2(-\sqrt{3}/2\,(a_y)) = -\sqrt{3}$

$$\Rightarrow \vec{a} = [-1, 1, -\sqrt{3}]^T \Rightarrow \hat{a} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{[-1, 1, -\sqrt{3}]^T}{\sqrt{5}}$$

$$\Rightarrow \boxed{\begin{aligned} \alpha &= 104.47 \\ \hat{a} &= \begin{bmatrix} -1/\sqrt{5} \\ 1/\sqrt{5} \\ -\sqrt{3}/\sqrt{5} \end{bmatrix} \end{aligned}}$$

(d) Using the formula derived at the end of Lecture 6 (i.e., Euler's formula), use α and $\hat{\mathbf{a}}$ to retrieve the matrix R. Comment on your findings.

$$[\hat a]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Using ; $R = \cos\alpha \cdot I + (1-\cos\alpha)\hat a \hat a^T - \sin\alpha [\hat a]_\times$

where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \hat a = \begin{bmatrix} -1/\sqrt5 \\ 1/\sqrt5 \\ -\sqrt3/\sqrt5 \end{bmatrix} ; \quad \hat a^T = [-1/\sqrt5 , 1/\sqrt5 , -\sqrt3/\sqrt5] ; \quad [\hat a]_\times = \begin{bmatrix} 0 & \sqrt3/\sqrt5 & 1/\sqrt5 \\ -\sqrt3/\sqrt5 & 0 & 1/\sqrt5 \\ -1/\sqrt5 & -1/\sqrt5 & 0 \end{bmatrix}$$

$$\hat a \hat a^T = \begin{bmatrix} -1/\sqrt5 \\ 1/\sqrt5 \\ -\sqrt3/\sqrt5 \end{bmatrix} [-1/\sqrt5 , 1/\sqrt5 , -\sqrt3/\sqrt5] = \begin{bmatrix} 1/5 & -1/5 & \sqrt3/5 \\ -1/5 & 1/5 & -\sqrt3/5 \\ \sqrt3/5 & -\sqrt3/5 & 3/5 \end{bmatrix}$$

$$R = \cos(104.47°)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\cos(104.47°))\begin{bmatrix} 1/5 & -1/5 & \sqrt3/5 \\ -1/5 & 1/5 & -\sqrt3/5 \\ \sqrt3/5 & -\sqrt3/5 & 3/5 \end{bmatrix} - \sin(104.47°)\begin{bmatrix} 0 & \sqrt3/\sqrt5 & 1/\sqrt5 \\ -\sqrt3/\sqrt5 & 0 & 1/\sqrt5 \\ -1/\sqrt5 & -1/\sqrt5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0.5 & 0 & -0.86 \\ 0.86 & 0 & 0.5 \end{bmatrix} \times$$

$\Rightarrow$ Must flip $\hat a$ signs to get correct matrix, will use a method in Problem 2 to correct ambiguity

NEW sol.

New $\hat a = \begin{bmatrix} 1/\sqrt5 \\ -1/\sqrt5 \\ \sqrt3/\sqrt5 \end{bmatrix}$

$$\hat a \hat a^T = \begin{bmatrix} 1/\sqrt5 \\ -1/\sqrt5 \\ \sqrt3/\sqrt5 \end{bmatrix} [1/\sqrt5 , -1/\sqrt5 , \sqrt3/\sqrt5] = \begin{bmatrix} 1/5 & -1/5 & \sqrt3/5 \\ -1/5 & 1/5 & -\sqrt3/5 \\ \sqrt3/5 & -\sqrt3/5 & 3/5 \end{bmatrix}$$

$$[\hat a]_\times = \begin{bmatrix} 0 & -\sqrt3/\sqrt5 & -1/\sqrt5 \\ \sqrt3/\sqrt5 & 0 & -1/\sqrt5 \\ 1/\sqrt5 & 1/\sqrt5 & 0 \end{bmatrix}$$

$$R = \cos(104.47°)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\cos(104.47°))\begin{bmatrix} 1/5 & -1/5 & \sqrt3/5 \\ -1/5 & 1/5 & -\sqrt3/5 \\ \sqrt3/5 & -\sqrt3/5 & 3/5 \end{bmatrix} - \sin(104.47°)\begin{bmatrix} 0 & -\sqrt3/\sqrt5 & -1/\sqrt5 \\ \sqrt3/\sqrt5 & 0 & -1/\sqrt5 \\ 1/\sqrt5 & 1/\sqrt5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & \sqrt3/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt3/2 & 1/2 \end{bmatrix} = R \checkmark$$

(e) Find the Euler angles ψ, θ and ϕ, so that $R = R_1(\phi)R_2(\theta)R_3(\psi)$.

$$R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{R_1\phi} \cdot \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{R_2\theta} \cdot \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3\psi}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & \sqrt3/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt3/2 & 1/2 \end{bmatrix}$$

$$\theta = \sin^{-1}(-\sqrt3/2) = 300°$$

$$\phi = \frac{\sin\phi\cancel{\cos\theta}}{\cos\phi\cancel{\cos\theta}} = \text{atan2}(0/(1/2)) = 0°$$

$$\psi = \frac{\sin\psi\cancel{\cos\theta}}{\cos\psi\cancel{\cos\theta}} = \text{atan2}((1/2)/0) = 90°$$

$\Rightarrow$ $\boxed{\psi = 90°, \quad \theta = 300°, \quad \phi = 0°}$

Problem 2
Repeat all steps of Problem 1 for the following matrix:

$$R = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Show that R is a rotation matrix.

(b) Calculate its eigenvalues and draw them in the complex plane.

(c) Calculate the direction of the axis of rotation $\hat{\mathbf{a}}$ and the angle of rotation α about $\hat{\mathbf{a}}$.

(d) Using the formula derived at the end of Lecture 6 (i.e., Euler's formula), use α and $\hat{\mathbf{a}}$ to retrieve the matrix R. Comment on your findings.

(e) Find the Euler angles ψ, θ and ϕ, so that $R = R_1(\phi)R_2(\theta)R_3(\psi)$.

Do all calculations by hand. You can use the provided MATLAB code to check on the correctness of your results.

Solution:

(a) Show that R is a rotation matrix.

* Must meet 2 conditions $\Rightarrow \begin{cases} ① \ \det(R) = 1 \rightarrow \text{Determinant} \\ ② \ RR^T = I \rightarrow \text{Orthogonality} \end{cases}$

$$R = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix}$$

DETERMINANT: $\det(A) \stackrel{?}{=} 1$

$$\det(R) = 1 \begin{vmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{vmatrix}$$

$$= 1\left[\sqrt{2}/2(\sqrt{2}/2) - (-\sqrt{2}/2)(\sqrt{2}/2)\right]$$

$$= 1$$

$$\boxed{\therefore \ \det(A) = 1 \checkmark}$$

ORTHOGONALITY: $RR^T \stackrel{?}{=} I$

$$R = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix} \quad ; \quad R^T = \begin{bmatrix} 0 & 0 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix}$$

$$RR^T = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix} = \begin{bmatrix} 0(0) + \sqrt{2}/2(\sqrt{2}/2) + (-\sqrt{2}/2)(-\sqrt{2}/2) & \cancelto{0}{0(0) + \sqrt{2}/2(\sqrt{2}/2) + -\sqrt{2}/2(\sqrt{2}/2)} & \cancelto{0}{0(1) + \sqrt{2}/2(0)} + -\sqrt{2}/2(0) \\ \cancelto{0}{0(0) + \sqrt{2}/2(\sqrt{2}/2) + (\sqrt{2}/2)(-\sqrt{2}/2)} & 0(0) + \sqrt{2}/2(\sqrt{2}/2) + \sqrt{2}/2(\sqrt{2}/2) & \cancelto{0}{0(1) + \sqrt{2}/2(0) + \sqrt{2}/2(0)} \\ \cancelto{0}{1(0) + 0(\sqrt{2}/2) + 0(-\sqrt{2}/2)} & \cancelto{0}{1(0) + 0(\sqrt{2}/2) + 0(\sqrt{2}/2)} & (1)(1) + 0(0) + 0(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\boxed{\therefore \ RR^T = I \checkmark}$$

(b) Calculate its eigenvalues and draw them in the complex plane.

$(R - \lambda I) \cdot \vec{v} = \vec{0}$ with $\vec{v} \neq \vec{0}$

$\hookrightarrow \det(R - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 - \lambda & \sqrt{2}/2 \\ 1 & 0 & -\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = -\lambda \begin{bmatrix} \sqrt{2}/2 - \lambda & \sqrt{2}/2 \\ 0 & -\lambda \end{bmatrix} - \cancelto{0}{0 \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & -\lambda \end{bmatrix}} + 1 \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 - \lambda & \sqrt{2}/2 \end{bmatrix}$$

$$= -\lambda\left[-\lambda\left(\tfrac{\sqrt{2}}{2} - \lambda\right)\right] + 1\left[(\sqrt{2}/2)(\sqrt{2}/2) - \left(-\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2} - \lambda\right)\right]$$

$$= -\lambda\left[\lambda^2 - \tfrac{\sqrt{2}}{2}\lambda\right] + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{\sqrt{2}}{2}\lambda = -\lambda^3 + \tfrac{\sqrt{2}}{2}\lambda^2 - \tfrac{\sqrt{2}}{2}\lambda + 1$$

$$\lambda_1 = 1 \stackrel{?}{=} -(1)^3 + \frac{\sqrt{2}}{2}(1)^2 - \frac{\sqrt{2}}{2}(1) + 1 = 0$$

$$\Rightarrow \quad \lambda_1 = 1$$

$$\lambda^3 - \frac{\sqrt{2}}{2}\lambda^2 + \frac{\sqrt{2}}{2}\lambda - 1 \iff \lambda^3 + \lambda^2(e-1) + \lambda(f-e) - f$$

By comparison:

$\lambda^3 : 1 = 1$

$\lambda^2 : -\frac{\sqrt{2}}{2} = e - 1 \rightarrow e = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$

$\lambda^1 : \frac{\sqrt{2}}{2} = f - e$

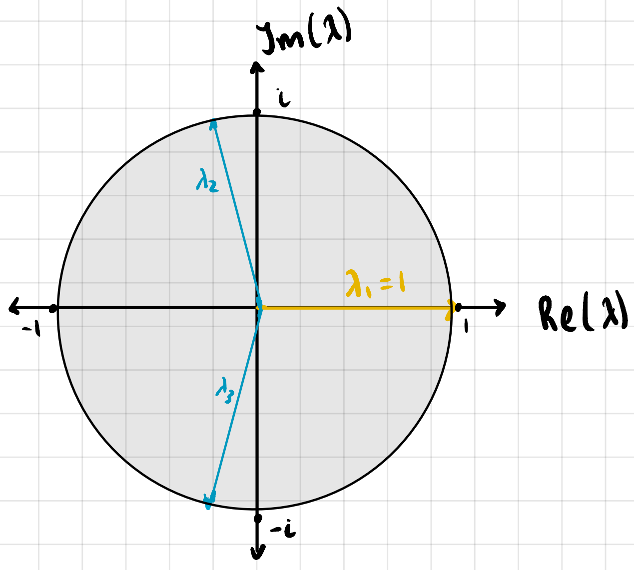
$\lambda^0 : -1 = -f \rightarrow f = 1$

$$\Rightarrow (\lambda - 1)(\lambda^2 + e\lambda + f) \Rightarrow (\lambda - 1)\left(\lambda^2 + \frac{2-\sqrt{2}}{2}\lambda + 1\right)$$

$$\Rightarrow \lambda_{1,2} = -\frac{1}{2}\left(\frac{2-\sqrt{2}}{2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{2-\sqrt{2}}{2}\right)^2 - 4}$$

$$\lambda_{1,2} = -0.146 \pm 0.9892\, i$$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -0.1464 + 0.9892i \\ \lambda_3 &= -0.1464 - 0.9892i \end{aligned}}$$



(c) Calculate the direction of the axis of rotation $\hat{a}$ and the angle of rotation α about $\hat{a}$.

Find Angle of rotation $\alpha \rightarrow \alpha = a\cos\left(\frac{trace(R) - 1}{2}\right)$; where $R = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix}$

$trace(R) = 0 + \sqrt{2}/2 + 0$

$$\alpha = a\cos\left(\frac{trace(R) - 1}{2}\right) = a\cos\left(\frac{\sqrt{2}/2 - 1}{2}\right) = 98.42°$$

$$\boxed{\alpha = 98.42° \qquad \hat{a} = \begin{bmatrix} 0.3574 \\ 0.8629 \\ 0.3574 \end{bmatrix}}$$

Since $0° < \alpha < 180°$

$$\hat{a} = \frac{1}{2\sin\alpha}\begin{bmatrix} r_{23} - r_{32} & r_{31} - r_{13} & r_{12} - r_{21} \end{bmatrix}^T$$

$$\hat{a} = \begin{bmatrix} 0.3574 \\ 0.8629 \\ 0.3574 \end{bmatrix}$$

(d) Using the formula derived at the end of Lecture 6 (i.e., Euler's formula), use α and $\hat{a}$ to retrieve the matrix R. Comment on your findings.

Using ; $R = \cos\alpha \cdot I + (1 - \cos\alpha)\hat{a}\hat{a}^T - \sin\alpha[\hat{a}]_x$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \hat{a} = \begin{bmatrix} 0.3574 \\ 0.8629 \\ 0.3574 \end{bmatrix} ; \quad \hat{a}^T = \begin{bmatrix} 0.3574 & 0.8629 & 0.3574 \end{bmatrix} ; \quad [\hat{a}]_x = \begin{bmatrix} 0 & -0.3574 & 0.8629 \\ 0.3574 & 0 & -0.3574 \\ -0.8629 & 0.3574 & 0 \end{bmatrix}$$

$$R = \cos(98.42)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1 - \cos(98.42))\begin{bmatrix} 0.3574 \\ 0.8629 \\ 0.3574 \end{bmatrix}\begin{bmatrix} 0.3574 & 0.8629 & 0.3574 \end{bmatrix} - \sin(98.42)\begin{bmatrix} 0 & -0.3574 & 0.8629 \\ 0.3574 & 0 & -0.3574 \\ -0.8629 & 0.3574 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix} = R$$

✗ This method is much easier.

(e) Find the Euler angles ψ, θ and ϕ, so that $R = R_1(\phi)R_2(\theta)R_3(\psi)$.

$$R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{R_1\phi} \cdot \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{R_2\theta} \cdot \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3\psi}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$$

$$\phi = \frac{\sin\phi\cos\theta}{\cos\phi\cos\theta} = atan2\left(\frac{\sqrt{2}/2}{0}\right) = 90°$$

$$\psi = \frac{\sin\psi\cancel{\cos\theta}}{\cos\psi\cancel{\cos\theta}} = atan2\left(\frac{\sqrt{2}/2}{0}\right) = 90°$$

$$\Rightarrow \boxed{\begin{aligned} \psi &= 90° \\ \theta &= 45° \\ \phi &= 90° \end{aligned}}$$

Problem 3

(a) Show that any 3-by-3 coordinate transformation matrix is a rotation matrix.

(b) Explain the difference between rotation matrices and coordinates transformation matrices.

A)

Any 3-by-3 coordinate transformation matrix is a rotation matrix mainly because both type of these matrices contain the same matrix to be classified as either a coordinate transformation matrix and a rotation matrix.

B) A rotation matrix only describes rotating without altering size and shape and a coordinate transformation matrix consists of translations.

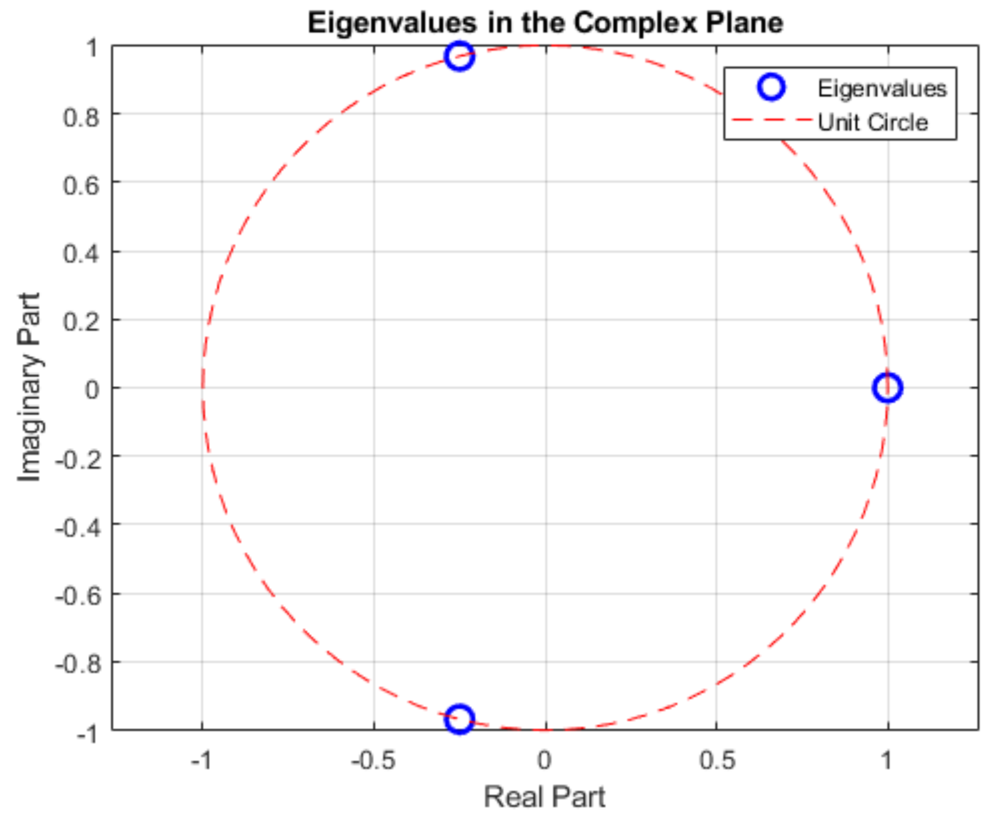
```
% ARO 4090 - Space Vehicle Dyn. & Cntrl. | Dr. Maggia | Justin Millsap |
Homework 5 %
clc; clear; close all;
```

Problem 1

```
clc; clear; close all
R = [0 1/2 sqrt(3)/2 ; -1 0 0 ; 0 -sqrt(3)/2 1/2];

I = [1 0 0 ; 0 1 0 ; 0 0 1];

% ~~~~~~~~~~~~~~ PART A ~~~~~~~~~~~~~~ %

% Show that R is a rotation Matrix
    % Must satisfy det(R) = 1
    % & RR^T = I

condition_1 = det(R);
condition_2 = R*R';

% ~~~~~~~~~~~~~~ PART B ~~~~~~~~~~~~~~ %

% Find eigenvalues
eigenvalues = eig(R)

% Plot the eigenvalues on the complex plane
figure; % Opens a new figure window
plot(real(eigenvalues), imag(eigenvalues), 'bo', 'MarkerSize', 10,
'LineWidth', 2);
xlabel('Real Part');
ylabel('Imaginary Part');
title('Eigenvalues in the Complex Plane');
grid on;
axis equal;
hold on;

% Additionally, plot the unit circle for reference
th = 0:pi/50:2*pi;
xunit = cos(th);
yunit = sin(th);
plot(xunit, yunit, 'r--'); % Unit circle in red dashed line
legend('Eigenvalues', 'Unit Circle');

hold off;

% ~~~~~~~~~~~~~~ PART C ~~~~~~~~~~~~~~ %

% Find the angle of rotation (alpha)

alpha = acosd( (trace(R) - 1) /2 )

% Find axis of rotation (a_hat)
```

```
a = [-1 ; 1 ; -sqrt(3)];

a_hat = -a/norm(a); % Normalize the vector

fprintf('The angle of rotation is alpha = %.2f degrees ',alpha)
disp(' ')
disp('The axis of rotation is a=')
disp(a_hat)




% ~~~~~~~~~~~~~~ PART D ~~~~~~~~~~~~~~ %

% Use alpha and unit vector a to retrive Matrix R

a_x = [ 0 -a_hat(3,1) a_hat(2,1) ; a_hat(3,1) 0 -a_hat(1,1) ; -a_hat(2,1)
a_hat(1,1) 0]

R_prime = cosd(alpha)*I + (1 - cosd(alpha)) * a_hat * a_hat' -
sind(alpha)*a_x

% ~~~~~~~~~~~~~~ PART E ~~~~~~~~~~~~~~ %

% Solve for Eurler Angles by using most opitmal positions to compare between
R_BI & Rotation Sequence Matrix


theta = asind(-R(1,3)) + 360;
phi   = atan2d(R(2,3) , R(3,3));
psi   = atan2d(R(1,2) , R(1,1));

% Check if Eurler angles work
R1 = [ 1 0 0  ; 0 cosd(phi) sind(phi) ; 0 -sind(phi) cosd(phi)];
R2 = [cosd(theta) 0 -sind(theta); 0 1 0; sind(theta) 0 cosd(theta)];
R3 = [ cosd(psi) sind(psi) 0 ; -sind(psi) cosd(psi) 0 ; 0 0 1];


Rot_Seq = R1*R2*R3;          % [3-2-1] Rotation Sequence
disp(' The Eurler Angles for the given R Matrix for a given Rotation
Sequence are')
fprintf('psi = %.2f\n', psi)
fprintf('theta = %.2f\n' , theta)
fprintf('phi = %.2f\n' , phi)


eigenvalues =

  -0.2500 + 0.9682i
  -0.2500 - 0.9682i
   1.0000 + 0.0000i
```

```
alpha =

  104.4775

The angle of rotation is alpha = 104.48 degrees
The axis of rotation is a=
    0.4472
   -0.4472
    0.7746


a_x =

         0   -0.7746   -0.4472
    0.7746         0   -0.4472
    0.4472    0.4472         0


R_prime =

         0    0.5000    0.8660
   -1.0000         0    0.0000
   -0.0000   -0.8660    0.5000

 The Eurler Angles for the given R Matrix for a given Rotation Sequence are
psi = 90.00
theta = 300.00
phi = 0.00
```



Problem 2

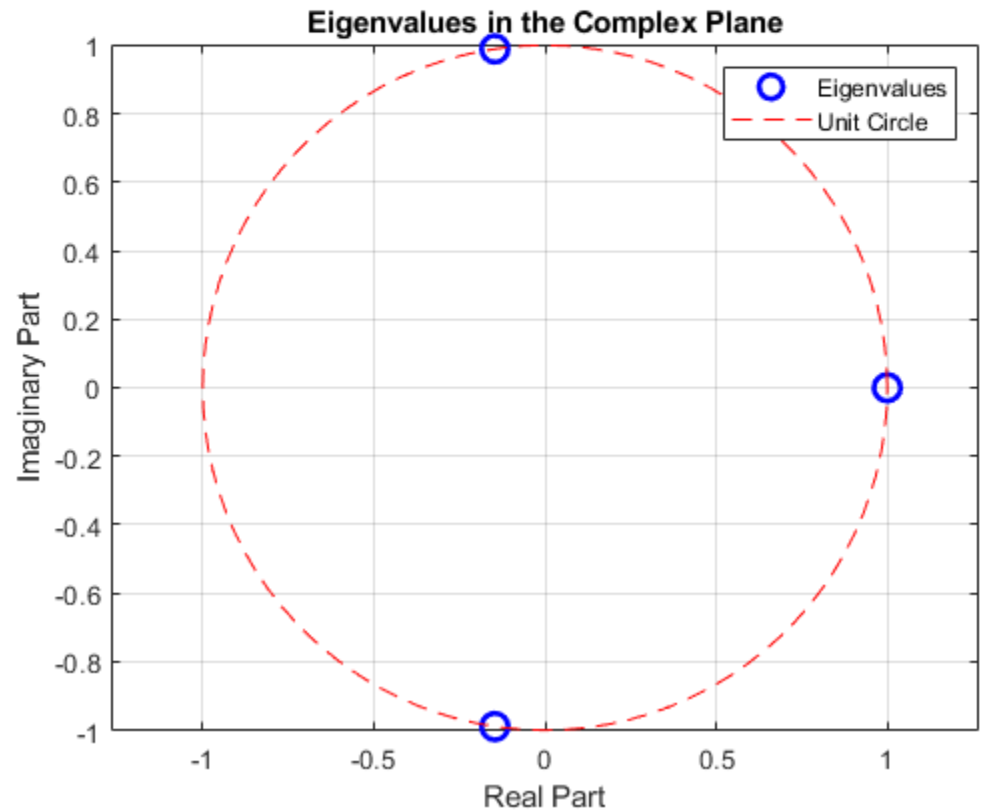
```
clc; clear; close all
R = [0 sqrt(2)/2 -sqrt(2)/2 ; 0 sqrt(2)/2 sqrt(2)/2 ; 1 0 0];

I = [1 0 0 ; 0 1 0 ; 0 0 1];

% ~~~~~~~~~~~~~~ PART A ~~~~~~~~~~~~~~ %

% Show that R is a rotation Matrix
    % Must satisfy det(R) = 1
    % & RR^T = I

condition_1 = det(R)
condition_2 = R*R'

% ~~~~~~~~~~~~~~ PART B ~~~~~~~~~~~~~~ %

% Find eigenvalues
eigenvalues = eig(R)

% Plot the eigenvalues on the complex plane
figure; % Opens a new figure window
plot(real(eigenvalues), imag(eigenvalues), 'bo', 'MarkerSize', 10,
'LineWidth', 2);
```

```
xlabel('Real Part');
ylabel('Imaginary Part');
title('Eigenvalues in the Complex Plane');
grid on;
axis equal;
hold on;

% Additionally, plot the unit circle for reference
th = 0:pi/50:2*pi;
xunit = cos(th);
yunit = sin(th);
plot(xunit, yunit, 'r--'); % Unit circle in red dashed line
legend('Eigenvalues', 'Unit Circle');

hold off;

% ~~~~~~~~~~~~~~ PART C ~~~~~~~~~~~~~~ %

% Find the angle of rotation (alpha)

alpha = acosd( (trace(R) - 1) /2 )

a = [1 ; 0 ; 0]
% Find axis of rotation (a_hat)
if alpha >= 0
    if alpha <= 180
        a = (1/(2*sind(alpha))) * [R(2,3) - R(3,2) ; R(3,1) - R(1,3) ;
R(1,2) - R(2,1)]
    end
end

a_hat = a/norm(a)

 % Normalize the vector

fprintf('The angle of rotation is alpha = %.2f degrees ',alpha)
disp(' ')
disp('The axis of rotation is a=')
disp(a_hat)




% ~~~~~~~~~~~~~~ PART D ~~~~~~~~~~~~~~ %

% Use alpha and unit vector a to retrive Matrix R

a_x = [ 0 -a_hat(3,1) a_hat(2,1) ; a_hat(3,1) 0 -a_hat(1,1) ; -a_hat(2,1)
a_hat(1,1) 0]

R_prime = cosd(alpha)*I + (1 - cosd(alpha)) * a_hat * a_hat' -
sind(alpha)*a_x

% ~~~~~~~~~~~~~~ PART E ~~~~~~~~~~~~~~ %
```

```
% Solve for Eurler Angles by using most opitmal positions to compare between
R_BI & Rotation Sequence Matrix


theta = asind(-R(1,3));
phi   = atan2d(R(2,3) , R(3,3));
psi   = atan2d(R(1,2) , R(1,1));

% Check if Eurler angles work
R1 = [ 1 0 0  ; 0 cosd(phi) sind(phi) ; 0 -sind(phi) cosd(phi)];
R2 = [cosd(theta) 0 -sind(theta); 0 1 0; sind(theta) 0 cosd(theta)];
R3 = [ cosd(psi) sind(psi) 0 ; -sind(psi) cosd(psi) 0 ; 0 0 1];


Rot_Seq = R1*R2*R3;          % [3-2-1] Rotation Sequence
disp(' The Eurler Angles for the given R Matrix for a given Rotation
Sequence are')
fprintf('psi = %.2f\n', psi)
fprintf('theta = %.2f\n' , theta)
fprintf('phi = %.2f\n' , phi)
```

```
condition_1 =

    1.0000


condition_2 =

    1.0000         0         0
         0    1.0000         0
         0         0    1.0000


eigenvalues =

  -0.1464 + 0.9892i
  -0.1464 - 0.9892i
   1.0000 + 0.0000i


alpha =

   98.4211


a =

     1
     0
     0
```

```
a =

    0.3574
    0.8629
    0.3574


a_hat =

    0.3574
    0.8629
    0.3574

The angle of rotation is alpha = 98.42 degrees
The axis of rotation is a=
    0.3574
    0.8629
    0.3574


a_x =

         0   -0.3574    0.8629
    0.3574         0   -0.3574
   -0.8629    0.3574         0


R_prime =

    0.0000    0.7071   -0.7071
         0    0.7071    0.7071
    1.0000    0.0000    0.0000

 The Eurler Angles for the given R Matrix for a given Rotation Sequence are
psi = 90.00
theta = 45.00
phi = 90.00
```



Published with MATLAB® R2023b