ARO 4090 - WEEK 10

- M. MAGGIA

Lecture 15

STATIC STABILITY OF EQUILIBRIA OF TORQUE-FREE MOTION

RECALL THE EOM'S FOR A TORQUE-FREE MOTION OF A RIGID BODY WITH:

$$I_B^c = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \qquad (I_{xx} \neq 0, I_{yy} \neq 0, I_{zz} \neq 0)$$

$$\boxed{\begin{aligned} \dot{p} &= \frac{I_{yy} - I_{zz}}{I_{xx}} qr \\ \dot{q} &= \frac{I_{zz} - I_{xx}}{I_{yy}} rp \\ \dot{r} &= \frac{I_{xx} - I_{yy}}{I_{zz}} pq \end{aligned}}$$

EQUILIBRIA:

CONDITIONS FOR WHICH THE STATE VARIABLES REMAIN CONSTANT

$$\begin{cases} p = \text{CONST.} \\ q = \text{CONST.} \\ r = \text{CONST.} \end{cases} \Leftrightarrow \begin{cases} \dot{p} = 0 \\ \dot{q} = 0 \\ \dot{r} = 0 \end{cases}$$

WHAT MAKES $\dot{p} = \dot{q} = \dot{r} = 0$?

I) • $I_{xx} = I_{yy} = I_{zz}$ — ALL MOMENTS OF INERTIA ARE THE SAME. VERY PECULIAR CASE e.g. HOM. SPHERE / CUBE, NOT CONSIDERED IN FOLLOWING ANALYSIS

OR

II) • $p = q = r = 0$ — THE RIGID BODY IS NOT ROTATING. TRIVIAL CASE, NOT CONSIDERED.

OR

III a) • $p = q = 0, \ r \neq 0$

OR

III b) • $p = r = 0, \ q \neq 0$ — THE RIGID BODY ROTATES ABOUT ONE OF ITS BODY AXES

OR

III c) • $q = r = 0, \ p \neq 0$

OR

IV a) • $\begin{cases} I_{yy} = I_{zz} \\ p = 0 \end{cases}$

$(I \quad I$

TWO OUT THREE OF THE MOMENTS OF INERTIA ARE THE SAME

$$\left\{ p = 0 \right.$$

IV b) • $\begin{cases} I_{xx} = I_{zz} \\ q = 0 \end{cases}$

IV c) • $\begin{cases} I_{xx} = I_{yy} \\ r = 0 \end{cases}$

TWO OUT THREE OF THE MOMENTS OF INERTIA ARE THE SAME AND ONE SPECIFIC COMPONENT OF THE ANGULAR VELOCITY IS ZERO.

STABILITY OF EQUILIBRIA : IF THE SYSTEM IS AT ONE OF THE EQUILIBRIUM POINTS AND IT GETS PERTURBED FROM IT, 3 THINGS CAN HAPPEN

1) THE SYSTEM COMES BACK TO THE EQUILIBRIUM POINT (ASYMPTOTIC STABILITY)

e.g. PENDULUM PERTURBED FROM ITS DOWNWARD VERTICAL EQUILIBRIUM (DVE), WHEN FRICTION IS CONSIDERED. THE PENDULUM EVENTUALLY GOES BACK TO THE DVE.

2) THE SYSTEM STAYS INDEFINITELY WITHIN A BOUNDED NEIGHBORHOOD OF THE EQUILIBRIUM POINT (NEUTRAL STABILITY)

e.g. PENDULUM PERTURBED FROM ITS DOWNWARD VERTICAL EQUILIBRIUM (DVE), WHEN FRICTION IS NEGLECTED THE PENDULUM KEEPS OSCILLATING INDEFINITELY ABOUT ITS DVE, WITHOUT EVER REACHING IT.

3) THE SYSTEM DRIFTS AWAY FROM THE EQUILIBRIUM POINT (INSTABILITY)

e.g. PENDULUM PERTURBED FROM ITS UPWARD VERTICAL EQUILIBRIUM (UVE), WHEN FRICTION IS CONSIDERED OR NEGLECTED. THE PENDULUM WILL DRIFT AWAY FROM ITS UVE WITHOUT COMING BACK TO IT.

TO STUDY THE STATIC STABILITY CHARACTERISTICS OF AN EQUILIBRIUM POINT WE NEED TO

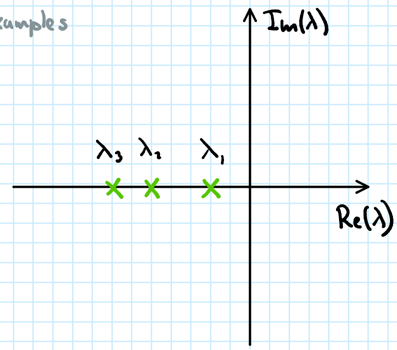
1) LINEARIZE THE SYSTEM ABOUT AN EQUILIBRIUM POINT

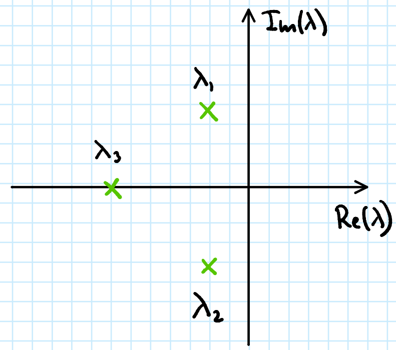
2) COMPUTE THE EIGENVALUES OF THE (JACOBIAN) MATRIX ASSOCIATED WITH THE LINEARIZED SYSTEM: $\lambda_1, \lambda_2, \lambda_3$

ASYMPTOTIC STABILITY

IF $\begin{cases} Re(\lambda_1) < 0 \\ Re(\lambda_2) < 0 \\ Re(\lambda_3) < 0 \end{cases}$ $\Longleftrightarrow$ ALL E.VALUES LIE IN THE LEFT-HAND SIDE OF THE COMPLEX PLANE

Examples

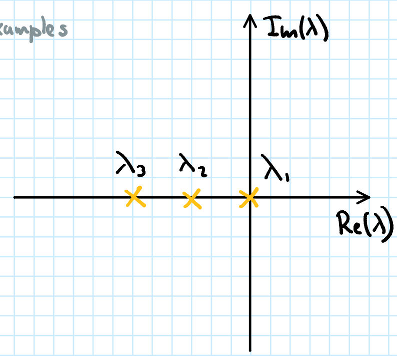


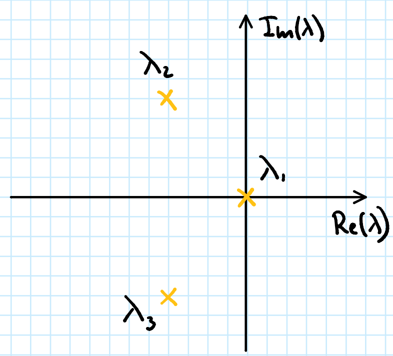


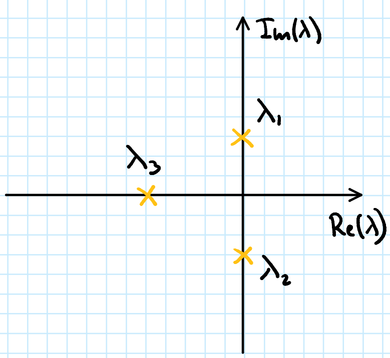
NEUTRAL STABILITY

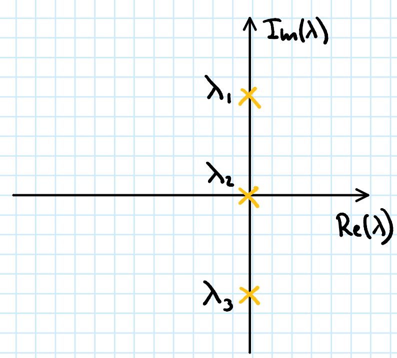
IF $\begin{cases} Re(\lambda_1) \leq 0 \\ Re(\lambda_2) \leq 0 \\ Re(\lambda_3) \leq 0 \\ \text{NO REPEATED ZERO EIGENVALUE} \end{cases}$ $\Longleftrightarrow$ ALL E.VALUES LIE ON THE LEFT-HAND SIDE OF THE COMPLEX PLANE, OR ON THE IMAGINARY AXIS (AT MOST 1 AT THE ORIGIN!)

Examples







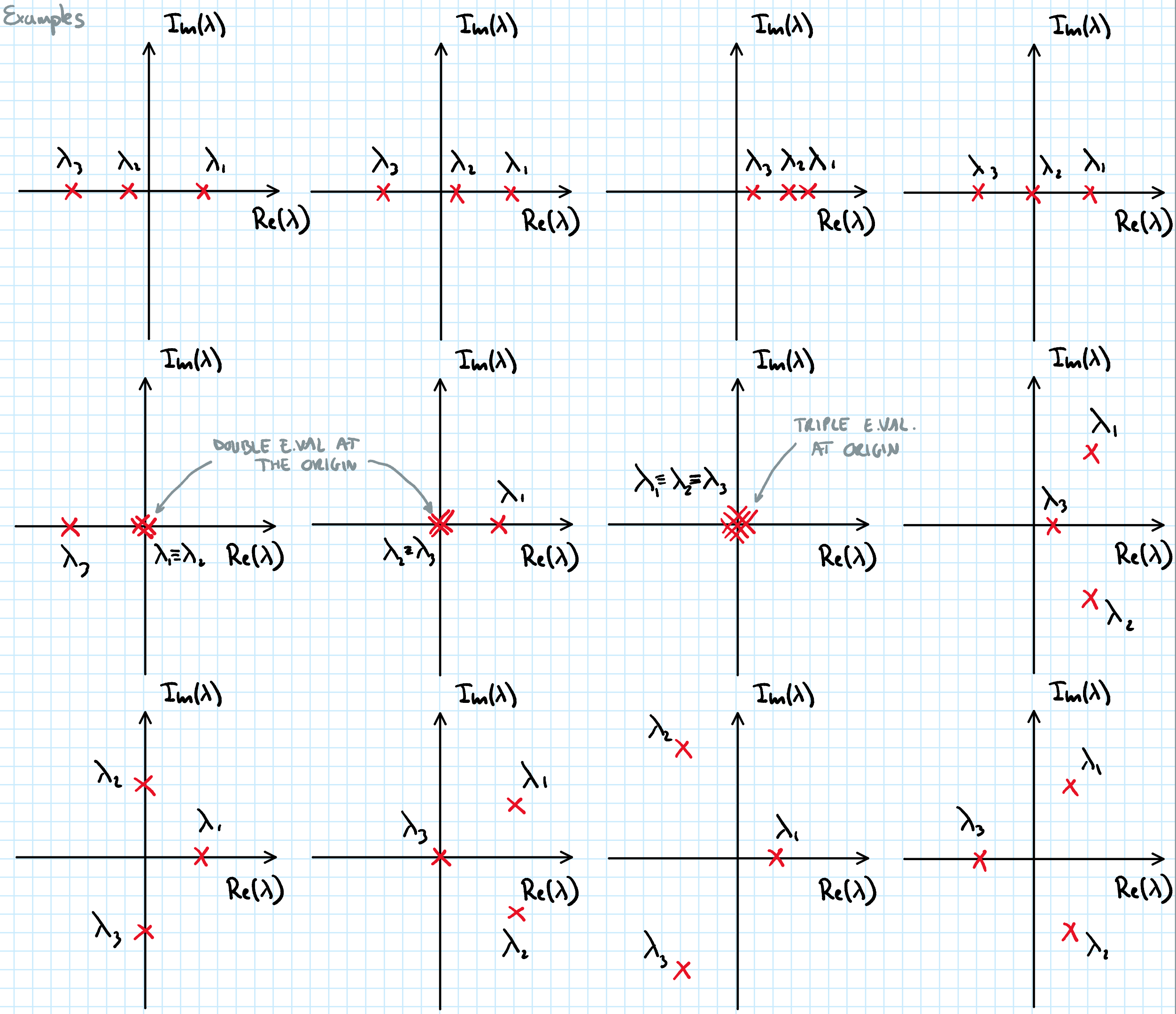


INSTABILITY

IF AT LEAST ONE $Re(\lambda_i) > 0 \quad i = 1,2,3$
OR
REPEATED ZERO EIGENVALUES

$\Longleftrightarrow$ THERE IS AT LEAST ONE E.VALUE IN THE RIGHT-HAND SIDE OF THE COMPLEX PLANE AND/OR TWO OR MORE E.VALUES AT THE ORIGIN.

Examples



NONLINEAR SYSTEM TO LINEARIZE:

$$\dot{p} = \frac{I_{yy} - I_{zz}}{I_{xx}} qr$$

$$\dot{q} = \frac{I_{zz} - I_{xx}}{I_{yy}} rp$$

$$\dot{q} = \frac{I_{zz} - I_{xx}}{I_{yy}} rp$$

$$\dot{r} = \frac{I_{xx} - I_{yy}}{I_{zz}} pq$$

CALLING $\vec{x} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ THE STATE VECTOR AND ITS DERIVATIVE $\dot{\vec{x}} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}$, WE CAN REWRITE IT AS:

NONLINEAR SYSTEM: $$\boxed{\dot{\vec{x}} = f(\vec{x})} \qquad (1)$$

USING TAYLOR'S SERIES EXPANSION OF (1), ABOUT AN EQUILIBRIUM POINT (STATE) $\vec{x}_0$:

$$\boxed{\dot{\vec{x}} = \dot{\vec{x}}_0 + \left.\frac{\partial f(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_0} \cdot (\vec{x} - \vec{x}_0)} + \text{HIGHER ORDER TERMS}$$

SERIES TRUNCATED AT LINEAR TERM

CALLING $\vec{x} - \vec{x}_0 = \Delta\vec{x}$ AND $\dot{\vec{x}} - \dot{\vec{x}}_0 = \Delta\dot{\vec{x}}$:

$$\Delta\dot{\vec{x}} = \left.\frac{\partial f(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_0} \cdot \Delta\vec{x}$$

JACOBIAN MATRIX EVALUATED AT $\vec{x} = \vec{x}_0$

LINEARIZED SYSTEM: $$\boxed{\Delta\dot{\vec{x}} = J(\vec{x}_0) \cdot \Delta\vec{x}} \qquad (2)$$

$$J(\vec{x}_0) = \left.\frac{\partial f(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_0} = \left[\left(\frac{\partial f(\vec{x})}{\partial p}\right), \left(\frac{\partial f(\vec{x})}{\partial q}\right), \left(\frac{\partial f(\vec{x})}{\partial r}\right) \right]_{\vec{x}=\vec{x}_0}$$

$$= \begin{bmatrix} 0 & \frac{I_{yy}-I_{zz}}{I_{xx}} r_0 & \frac{I_{yy}-I_{zz}}{I_{xx}} q_0 \\ \frac{I_{zz}-I_{xx}}{I_{yy}} r_0 & 0 & \frac{I_{zz}-I_{xx}}{I_{yy}} p_0 \\ \frac{I_{xx}-I_{yy}}{I_{zz}} q_0 & \frac{I_{xx}-I_{yy}}{I_{zz}} p_0 & 0 \end{bmatrix}$$

JACOBIAN MATRIX THAT NEEDS TO BE EVALUATED @ AN EQUILIBRIUM POINT $\vec{x}_0 = [p_0, q_0, r_0]^T$

Example # 1: LET'S CHOOSE THE EQUILIBRIUM POINT FOR CASE III a):

$$\vec{x}_0 = \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$$

(RB IS ROTATING ABOUT ITS $\hat{b}_3$ AXIS)

THE JACOBIAN BECOMES:

$$J(\vec{x}_0) = \begin{bmatrix} 0 & \frac{I_{yy}-I_{zz}}{I_{xx}} n & 0 \\ \frac{I_{zz}-I_{xx}}{I_{yy}} n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

WHOSE EIGENVALUES ARE:

$$\lambda_1 = 0$$

$$\lambda_{2,3} = \pm |n| \sqrt{\boxed{\frac{(I_{zz}-I_{xx})(I_{yy}-I_{zz})}{I_{xx} I_{yy}}}}$$

$$\lambda_{2,3} = \pm|n|\sqrt{\boxed{\frac{(I_{zz}-I_{xx})(I_{yy}-I_{zz})}{I_{xx}\, I_{yy}}}} \;\rightarrow\; ☆$$

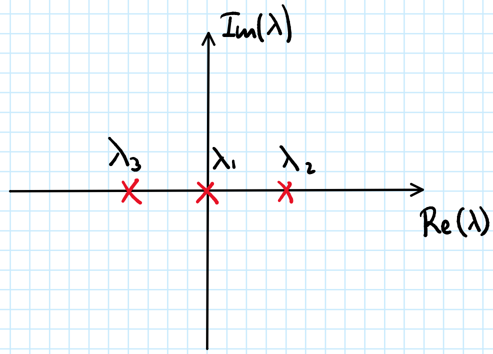
LET'S TAKE A LOOK AT THE SIGN OF ☆

• IF $I_{xx} < I_{zz} < I_{yy}$ OR $I_{yy} < I_{zz} < I_{xx}$ $\Rightarrow ☆ > 0 \Rightarrow \underbrace{\lambda_{2,3} \in \mathbb{R}}_{\text{REAL NUMBERS}} \Rightarrow \begin{cases} Re(\lambda_1) = \lambda_1 = 0 \\ \underline{Re(\lambda_2) = \lambda_2 > 0} \\ Re(\lambda_3) = \lambda_3 < 0 \end{cases}$ UNSTABLE SYSTEM

$\Downarrow$

$\hat{b}_3$ IS THE AXIS OF INTERMEDIATE INERTIA

$\lambda_1 = 0$
$\lambda_2 \neq 0$
$\lambda_3 \neq 0$



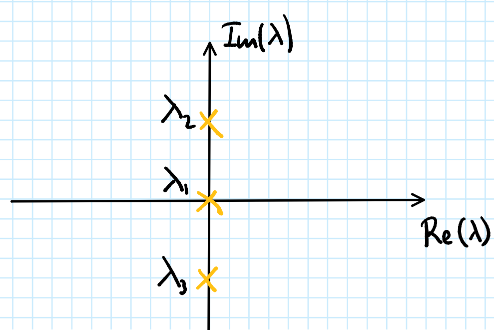
$$\lambda_1 = 0 \;, \quad \lambda_{2,3} = \pm|n|\sqrt{\frac{|I_{zz}-I_{xx}|\cdot|I_{yy}-I_{zz}|}{I_{xx}\, I_{yy}}}$$

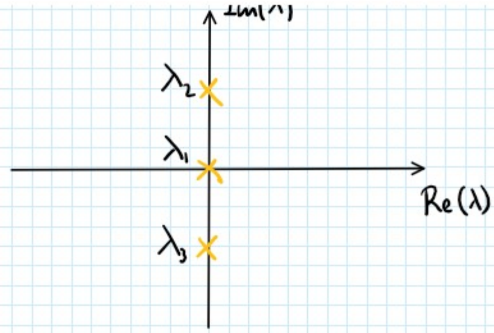
• IF $\begin{cases} I_{zz} > I_{yy} \\ I_{zz} > I_{xx} \end{cases}$ OR $\begin{cases} I_{zz} < I_{xx} \\ I_{zz} < I_{yy} \end{cases}$ $\Rightarrow ☆ < 0 \Rightarrow \underbrace{\lambda_{2,3} \in \mathbb{I}}_{\text{PURELY IMAGINARY NUMBERS}} \Rightarrow \begin{cases} \overbrace{Re(\lambda_1) = \lambda_1 = \underline{0}}^{\text{MULTIPLICITY} = 1 \ \checkmark} \\ Re(\lambda_2) = Re(\lambda_3) \underline{= 0} \end{cases}$ NEUTRAL STABILITY

$\lambda_1 = 0$
$\lambda_2 \neq 0$
$\lambda_3 \neq 0$

$\Downarrow$

$\hat{b}_3$ IS THE AXIS OF MAXIMUM OR MINIMUM INERTIA.





$$\lambda_1 = 0 \quad , \quad \lambda_{2,3} = \pm\, i|n|\sqrt{\frac{|I_{zz}-I_{xx}|\cdot|I_{yy}-I_{zz}|}{I_{xx}\, I_{yy}}}$$

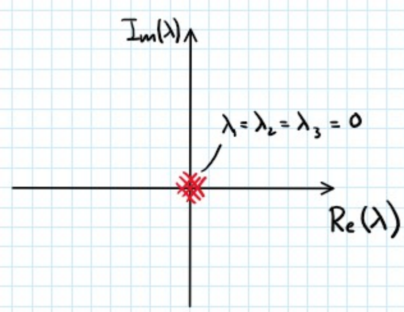
Example #2: LET'S CHOOSE THE EQUILIBRIUM POINT FOR CASE IVc)

$$\vec{X}_0 = \begin{bmatrix} p_0 \\ q_0 \\ 0 \end{bmatrix} \quad \& \quad \begin{cases} I_{xx} = I_{yy} = I_t \\ I_{zz} = I_a \end{cases}$$

THE JACOBIAN BECOMES:

$$J(\vec{X}_0) = \begin{bmatrix} 0 & 0 & \frac{I_t - I_a}{I_t} q_0 \\ 0 & 0 & \frac{I_a - I_t}{I_t} p_0 \\ 0 & 0 & 0 \end{bmatrix}$$

WHOSE EIGENVALUES ARE $\underline{\lambda_1 = \lambda_2 = \lambda_3 = 0}$ $\Big\}$ MULTIPLICITY $= 3 > 1$ (3 EVALS @ ORIGIN) UNSTABLE SYSTEM



MATLAB DEMOS.