ARO 4090 - WEEK 10

- M. MAGGIA

Lecture 16

STABILITY OF TORQUE-FREE MOTION EQUILIBRIA (CONT.)

NONLINEAR SYSTEM

$$\dot{\vec{x}} = f(\vec{x})$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dfrac{I_{yy} - I_{zz}}{I_{xx}} qr \\ \dfrac{I_{zz} - I_{xx}}{I_{yy}} rp \\ \dfrac{I_{xx} - I_{yy}}{I_{zz}} pq \end{bmatrix}$$

$\xrightarrow{\text{LINEARIZATION}}$

LINEARIZED SYSTEM ABOUT THE EQUILIBRIUM $\vec{x}_0 = [p_0, q_0, r_0]^T$

$$\dot{\Delta\vec{x}} = J(\vec{x}_0)\,\Delta\vec{x}$$

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{I_{yy} - I_{zz}}{I_{xx}} r_0 & \dfrac{I_{yy} - I_{zz}}{I_{xx}} q_0 \\ \dfrac{I_{zz} - I_{xx}}{I_{yy}} r_0 & 0 & \dfrac{I_{zz} - I_{xx}}{I_{yy}} p_0 \\ \dfrac{I_{xx} - I_{yy}}{I_{zz}} q_0 & \dfrac{I_{xx} - I_{yy}}{I_{zz}} p_0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

NOTATION:

$\Delta p(t) = p(t) - p_0 := \varepsilon_x(t)$ (p-component error)

$\Delta q(t) = q(t) - q_0 := \varepsilon_y(t)$ (q-component error)

$\Delta r(t) = r(t) - r_0 := \varepsilon_z(t)$ (r-component error)

$\Delta\vec{x}(t) = \vec{x}(t) - \vec{x}_0 := \vec{\varepsilon}(t)$ (error vector

$$\frac{I_{zz} - I_{yy}}{I_{xx}} := \sigma_x$$

$$\frac{I_{xx} - I_{zz}}{I_{yy}} := \sigma_y$$

$$\frac{I_{yy} - I_{xx}}{I_{zz}} := \sigma_z$$

RE-WRITING THE LINEARIZED SYSTEM:

$$\dot{\varepsilon}_x = -\sigma_x r_0 \varepsilon_y - \sigma_x q_0 \varepsilon_z$$

$$\dot{\varepsilon}_y = -\sigma_y r_0 \varepsilon_x - \sigma_y p_0 \varepsilon_z$$

$$\dot{\varepsilon}_z = -\sigma_z q_0 \varepsilon_x - \sigma_z p_0 \varepsilon_y$$

★ STABILITY OF AN EQUILIBRIUM POINT $\vec{x}_0 = [p_0, q_0, r_0]^T$

* STABILITY OF AN EQUILIBRIUM POINT $\vec{x}_0 = [p_0, q_0, r_0]^T$

ASYMPTOTIC STABILITY : FOR $t \to \infty$ $\quad \|\vec{\varepsilon}(t)\| \to 0$

NEUTRAL STABILITY : $\forall t, \exists \delta$ s.t. $\|\vec{\varepsilon}(t)\| < \delta$

INSTABILITY : FOR $t \to \infty$, $\nexists \delta$ s.t. $\|\vec{\varepsilon}(t)\| < \delta$

* SOLUTION OF LINEARIZED SYSTEM: LAPLACE TRANSFORM ($\mathcal{L}$)

$$\dot{\varepsilon}_x(t) = -\sigma_x r_0 \varepsilon_y(t) - \sigma_x q_0 \varepsilon_z(t)$$
$$sE_x(s) - \varepsilon_x(0) = -\sigma_x r_0 E_y(s) - \sigma_x q_0 E_z(s) \quad \mathcal{L}$$

$$\dot{\varepsilon}_y(t) = -\sigma_y r_0 \varepsilon_x(t) - \sigma_y p_0 \varepsilon_z(t)$$
$$sE_y(s) - \varepsilon_y(0) = -\sigma_y r_0 E_x(s) - \sigma_y p_0 E_z(s) \quad \mathcal{L}$$

$$\dot{\varepsilon}_z(t) = -\sigma_z q_0 \varepsilon_x(t) - \sigma_z p_0 \varepsilon_y(t)$$
$$sE_z(s) - \varepsilon_z(0) = -\sigma_z q_0 E_x(s) - \sigma_z p_0 E_y(s) \quad \mathcal{L}$$

$$sE_x + \sigma_x r_0 E_y + \sigma_x q_0 E_z = \varepsilon_x(0)$$
$$sE_y + \sigma_y r_0 E_x + \sigma_y p_0 E_z = \varepsilon_y(0)$$
$$sE_z + \sigma_z q_0 E_x + \sigma_z p_0 E_y = \varepsilon_z(0)$$

$$\underbrace{\boxed{\begin{bmatrix} s & \sigma_x r_0 & \sigma_x q_0 \\ \sigma_y r_0 & s & \sigma_y p_0 \\ \sigma_z q_0 & \sigma_z p_0 & s \end{bmatrix}}}_{A} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x(0) \\ \varepsilon_y(0) \\ \varepsilon_z(0) \end{bmatrix}$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} s^2 - p_0^2\sigma_y\sigma_z & -s\cdot r_0\sigma_x + p_0 q_0 \sigma_x\sigma_z & -s q_0\sigma_x + p_0 r_0 \sigma_x \sigma_y \\ -s r_0 \sigma_y + p_0 q_0 \sigma_y \sigma_z & s^2 - q_0^2 \sigma_x\sigma_z & -s p_0 \sigma_y + q_0 r_0 \sigma_x \sigma_y \\ -s q_0 \sigma_z + p_0 r_0 \sigma_y \sigma_z & -s p_0 \sigma_z + q_0 r_0 \sigma_x \sigma_z & s^2 - r_0^2 \sigma_x \sigma_y \end{bmatrix} \begin{bmatrix} \varepsilon_x(0) \\ \varepsilon_y(0) \\ \varepsilon_z(0) \end{bmatrix}$$

$$\det(A) = \boxed{s^3 - s\left(p_0^2\sigma_y\sigma_z + q_0^2\sigma_x\sigma_z + r_0^2\sigma_x\sigma_y\right) + 2 p_0 q_0 r_0 \sigma_x\sigma_y\sigma_z}$$

↳ CHARACTERISTIC POLYNOMIAL

★ THE ZEROS OF THE CHARACTERISTIC POLYNOMIAL ARE KNOWN AS THE POLES OF THE SYSTEM AND THEY COINCIDE WITH THE EIGENVALUES OF THE JACOBIAN MATRIX (see lecture 16). THUS, WE CAN DRAW CONCLUSIONS ABOUT THE STABILITY OF AN EQUILIBRIUM BY LOOKING AT THE POLES OF THE LINEARIZED SYSTEM. ★

↳ SINCE THE CH. POLYNOMIAL HAS ORDER 3, THE POLES ARE 3.

Examples

- FOR THE EQUILIBRIUM $[p_0, q_0, r_0]^T = [0, 0, n]^T$ (same as Example 1, Lec. 16)

$$\boxed{\begin{aligned} \dot{\varepsilon}_x &= -\sigma_x n\, \varepsilon_y \\ \dot{\varepsilon}_y &= -\sigma_y n\, \varepsilon_x \end{aligned}}$$ X-Y DYNAMICS

DECOUPLED FROM

$$\boxed{\dot{\varepsilon}_z = 0}$$ Z DYNAMICS

$$\det(A) = s^3 - s n^2 \sigma_x \sigma_y$$

FINDING POLES: $s^3 - s n^2 \sigma_x\sigma_y = 0$

$$s(s^2 - n^2\sigma_x\sigma_y) = 0$$

$$s_1 = 0 \qquad (= \lambda_1)$$

$$s_{2,3} = \pm |n| \sqrt{\sigma_x \sigma_y} \qquad (= \lambda_{2,3})$$

- $\boxed{\text{CASE 1:}}$ $\sigma_x \sigma_y > 0$

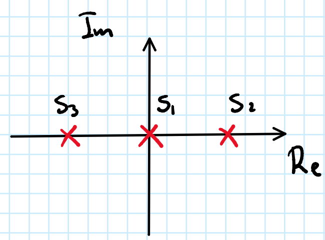
$$\Downarrow \text{ IMPLIES}$$

$$I_{xx} < I_{zz} < I_{yy} \quad \text{OR} \quad I_{yy} < I_{zz} < I_{xx}$$

$\hat{b}_3$ IS AXIS OF INTERMEDIATE INERTIA

$$s_1 = 0$$

$$s_{2,3} = \pm |n| \sqrt{|\sigma_x||\sigma_y|} \in \mathbb{R}$$



UNSTABLE SYSTEM

- $\boxed{\text{CASE 2:}}$ $\sigma_x = 0 \ \vee \ \sigma_y = 0$ $\quad (\vee : \text{"VEL"} := \text{AND/OR})$

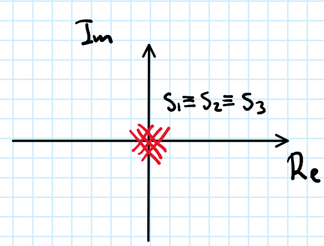
$$\Downarrow \text{ IMPLIES}$$

$$I_{zz} = I_{yy} \ \vee \ I_{zz} = I_{xx}$$

$$s_1 = 0$$

$$s_2 = 0$$

$$s_3 = 0$$



UNSTABLE SYSTEM

- $\boxed{\text{CASE 3:}}$ $\sigma_x \sigma_y < 0$

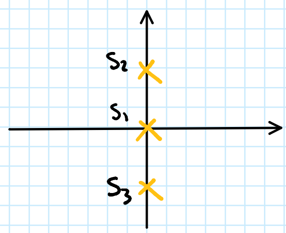
$\Downarrow$ IMPLIES

$$\begin{cases} I_{zz} < I_{xx} \\ I_{zz} < I_{yy} \end{cases} \quad \vee \quad \begin{cases} I_{zz} > I_{xx} \\ I_{zz} > I_{yy} \end{cases}$$

$\hat{b}_3$ IS <u>MINOR</u> AXIS OF INERTIA $\qquad$ $\hat{b}_3$ IS <u>MAJOR</u> AXIS OF INERTIA

$$S_1 = 0$$

$$S_{2,3} = \pm i|n|\sqrt{|\sigma_x|\cdot|\sigma_y|}$$



NEUTRAL STABILITY

LET'S GO BACK TO THE LINEARIZED SYSTEM IN THE LAPLACE DOMAIN (S):

$$\begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix} = \frac{1}{s(s^2 - n^2\sigma_x\sigma_y)} \begin{bmatrix} s^2 & -s\cdot n\cdot\sigma_x & 0 \\ -s n \sigma_y & s^2 & 0 \\ 0 & 0 & s^2 - n^2\sigma_x\sigma_y \end{bmatrix} \begin{bmatrix} \mathcal{E}_x(0) \\ \mathcal{E}_y(0) \\ \mathcal{E}_z(0) \end{bmatrix}$$

DECOUPLED DYNAMICS

$$\mathcal{E}_x(s) = \frac{1}{\cancel{s}(s^2 - n^2\sigma_x\sigma_y)}\left(s^{\cancel{2}}\mathcal{E}_x(0) - \cancel{s}n\sigma_x\mathcal{E}_y(0)\right) = \frac{s\mathcal{E}_x(0) - n\sigma_x\mathcal{E}_y(0)}{s^2 - n^2\sigma_x\sigma_y}$$

$$\mathcal{E}_y(s) = \frac{1}{\cancel{s}(s^2 - n^2\sigma_x\sigma_y)}\left(-\cancel{s}n\sigma_y\mathcal{E}_x(0) + s^{\cancel{2}}\mathcal{E}_y(0)\right) = \frac{s\mathcal{E}_y(0) - n\sigma_y\mathcal{E}_x(0)}{s^2 - n^2\sigma_x\sigma_y}$$

$$\mathcal{E}_z(s) = \frac{1}{s\cancel{(s^2 - n^2\sigma_x\sigma_y)}}\cancel{(s^2 - n^2\sigma_x\sigma_y)}\,\mathcal{E}_z(0) = \frac{\mathcal{E}_z(0)}{s}$$

$\boxed{\text{CASE 1}}$ $\quad \sigma_x \sigma_y > 0 \Rightarrow -n^2 \sigma_x \sigma_y < 0$

$$E_x(s) = \frac{s\,\mathcal{E}_x(0)}{s^2 - n^2\sigma_x\sigma_y} - \frac{n\sigma_x\,\mathcal{E}_y(0)}{s^2 - n^2\sigma_x\sigma_y}$$

FROM TABLES : $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \underbrace{\cosh}_{\text{HYPERBOLIC COSINE}}(at)$; $\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \underbrace{\sinh}_{\text{HYPERBOLIC SINE}}(at)$

$a = \sqrt{n^2\sigma_x\sigma_y}$

$$E_x(s) = \mathcal{E}_x(0)\cdot\frac{s}{s^2-\left(\sqrt{n^2\sigma_x\sigma_y}\right)^2} - \frac{\mathcal{E}_y(0)\cdot n\cdot\sigma_x}{\sqrt{n^2\sigma_x\sigma_y}}\cdot\frac{\sqrt{n^2\sigma_x\sigma_y}}{s^2-\left(\sqrt{n^2\sigma_x\sigma_y}\right)^2}$$

$$= \mathcal{E}_x(0)\frac{s}{s^2-\left(\sqrt{n^2\sigma_x\sigma_y}\right)^2} - \mathcal{E}_y(0)\,\frac{\text{sign}(n\cdot\sigma_x)\sqrt{\cancel{n^2}\sigma_x^{\cancel{2}}}}{\sqrt{\cancel{n^2}\cancel{\sigma_x}\sigma_y}}\,\frac{\sqrt{n^2\sigma_x\sigma_y}}{s^2-\left(\sqrt{n^2\sigma_x\sigma_y}\right)^2}$$

$$\mathcal{E}_x(0)\frac{s}{s^2-\left(\sqrt{n^2\sigma_x\sigma_y}\right)^2} - \mathcal{E}_y(0)\,\text{sign}(n\cdot\sigma_x)\sqrt{\frac{\sigma_x}{\sigma_y}}\,\frac{\sqrt{n^2\sigma_x\sigma_y}}{s^2-\left(\sqrt{n^2\sigma_x\sigma_y}\right)^2}$$

INVERSE LAPLACE TRANSFORM $\mathcal{L}^{-1}$

NECESSARY TO GO FROM LAPLACE DOMAIN (S) TO TIME DOMAIN (t)

$$\mathcal{E}_x(t) = \mathcal{E}_x(0)\cosh\left(\sqrt{n^2\sigma_x\sigma_y}\cdot t\right) - \mathcal{E}_y(0)\,\text{sign}(n\sigma_x)\sqrt{\frac{\sigma_x}{\sigma_y}}\,\sinh\left(\sqrt{n^2\sigma_x\sigma_y}\cdot t\right)$$

$$= \frac{\mathcal{E}_x(0)}{2}\left[e^{\sqrt{n^2\sigma_x\sigma_y}t} + e^{-\sqrt{n^2\sigma_x\sigma_y}t}\right] - \frac{\mathcal{E}_y(0)}{2}\,\text{sign}(n\sigma_x)\sqrt{\frac{\sigma_x}{\sigma_y}}\left[e^{\sqrt{n^2\sigma_x\sigma_y}t} - e^{-\sqrt{n^2\sigma_x\sigma_y}t}\right]$$

$$\boxed{\mathcal{E}_x(t) = \frac{1}{2}\left(\mathcal{E}_x(0) - \mathcal{E}_y(0)\,\text{sign}(n\sigma_x)\sqrt{\frac{\sigma_x}{\sigma_y}}\right)\underbrace{e^{\sqrt{n^2\sigma_x\sigma_y}t}}_{\text{FOR } t\to\infty\text{, IT TENDS TO }\infty} + \frac{1}{2}\left(\mathcal{E}_x(0) + \mathcal{E}_y(0)\,\text{sign}(n\sigma_x)\sqrt{\frac{\sigma_x}{\sigma_y}}\right)\underbrace{e^{-\sqrt{n^2\sigma_x\sigma_y}t}}_{\text{FOR } t\to\infty\text{, IT TENDS TO }0}}$$

EXP. DIVERGING

(TERM RESPONSIBLE FOR INSTABILITY)

IN A SIMILAR FASHION, WE CAN SHOW THAT FROM :

$$\varepsilon_y(s) = \frac{s\,\varepsilon_y(0)}{s^2 - n^2\sigma_x\sigma_y} - \frac{n\sigma_y\,\varepsilon_x(0)}{s^2 - n^2\sigma_x\sigma_y}$$

$\mathcal{L}^{-1}$

$$\boxed{\varepsilon_y(t) = \frac{1}{2}\left(\varepsilon_y(0) - \varepsilon_x(0)\,\mathrm{sign}(n\sigma_y)\sqrt{\frac{\sigma_y}{\sigma_x}}\right) e^{\sqrt{n^2\sigma_x\sigma_y}\,t} + \frac{1}{2}\left(\varepsilon_y(0) + \varepsilon_x(0)\,\mathrm{sign}(n\sigma_y)\sqrt{\frac{\sigma_y}{\sigma_x}}\right) e^{-\sqrt{n^2\sigma_x\sigma_y}\,t}}$$

FOR $t \to \infty$, IT TENDS TO ∞

FOR $t \to \infty$, IT TENDS TO 0

$$\varepsilon_z(s) = \frac{\varepsilon_z(0)}{s}$$

$$\boxed{\varepsilon_z(t) = \varepsilon_z(0) = \text{CONST.}}$$

BOUNDED

$\boxed{\text{CASE 2}}$ (LET'S CONSIDER $\sigma_x = 0$)

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} = \frac{1}{s^3} \begin{bmatrix} s^2 & 0 & 0 \\ -s\,n\,\sigma_y & s^2 & 0 \\ 0 & 0 & s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x(0) \\ \varepsilon_y(0) \\ \varepsilon_z(0) \end{bmatrix}$$

- $$\varepsilon_x(s) = \frac{\cancel{s^2}}{s^{\cancel{3}}}\,\varepsilon_x(0)$$

  $\mathcal{L}^{-1}$

  $$\boxed{\varepsilon_x(t) = \varepsilon_x(0) = \text{CONST.}}$$

  BOUNDED

- $$\varepsilon_y(s) = \frac{1}{s^{\cancel{3}\,2}}(-\cancel{s}\,n\,\sigma_y)\,\varepsilon_x(0) + \frac{\cancel{s^2}}{s^{\cancel{3}}}\,\varepsilon_y(0)$$

  $$\varepsilon_y(s) = -\frac{n\sigma_y}{s^2}\,\varepsilon_x(0) + \frac{1}{s}\,\varepsilon_y(0)$$

  $\mathcal{L}^{-1}$

  $$\boxed{\varepsilon_y(t) = -n\sigma\cdot\varepsilon_x(0)\cdot t + \varepsilon_y(0)}$$

  DIVERGING (LINEARLY)

GROWS TO ∞ !

- $$E_z(s) = \frac{\varepsilon_z(s)}{s}$$
  $\mathcal{L}^{-1}$
  $$\boxed{\varepsilon_z(t) = \varepsilon_z(0)}$$ BOUNDED

$\boxed{\text{CASE 3}}$ $\quad \sigma_x \sigma_y < 0 \Rightarrow -n^2 \sigma_x \sigma_y = b^2 > 0$

$$E_x(s) = \frac{s\,\varepsilon_x(0)}{s^2 - n^2\sigma_x\sigma_y} - \frac{n\sigma_x\,\varepsilon_y(0)}{s^2 - n^2\sigma_x\sigma_y}$$

FROM TABLES : $\mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos(bt)$; $\mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right] = \sin(bt)$

$b = \sqrt{-n^2\sigma_x\sigma_y}$

$$E_x(s) = \varepsilon_x(0)\cdot\frac{s}{s^2+\left(\sqrt{-n^2\sigma_x\sigma_y}\right)^2} - \frac{\varepsilon_y(0)\cdot n\cdot\sigma_x}{\sqrt{-n^2\sigma_x\sigma_y}}\cdot\frac{\sqrt{-n^2\sigma_x\sigma_y}}{s^2+\left(\sqrt{-n^2\sigma_x\sigma_y}\right)^2}$$

$$= \varepsilon_x(0)\cdot\frac{s}{s^2+\left(\sqrt{-n^2\sigma_x\sigma_y}\right)^2} - \frac{\varepsilon_y(0)\,\text{sign}(n\cdot\sigma_x)\sqrt{\cancel{n^2}\sigma_x^{\cancel{2}}}}{\sqrt{-\cancel{n^2}\cancel{\sigma_x}\sigma_y}}\cdot\frac{\sqrt{-n^2\sigma_x\sigma_y}}{s^2+\left(\sqrt{-n^2\sigma_x\sigma_y}\right)^2}$$

$\mathcal{L}^{-1}$

$$\boxed{\varepsilon_x(t) = \varepsilon_x(0)\cos\left(\sqrt{-n^2\sigma_x\sigma_y}\cdot t\right) - \varepsilon_y(0)\,\text{sign}(n\sigma_x)\cdot\sqrt{-\frac{\sigma_x}{\sigma_y}}\,\sin\left(\sqrt{-n^2\sigma_x\sigma_y}\cdot t\right)}$$ BOUNDED

$$E_y(s) = \frac{s\,\varepsilon_y(0)}{s^2 - n^2\sigma_x\sigma_y} - \frac{n\sigma_y\,\varepsilon_x(0)}{s^2 - n^2\sigma_x\sigma_y}$$
$\mathcal{L}^{-1}$
$\varepsilon_y(t) =$ (YOU SOLVE IT!)

$$E_z(s) = \frac{\varepsilon_z(0)}{s}$$
$\mathcal{L}^{-1}$
$$\boxed{\varepsilon_z(t) = \varepsilon_z(0) = \text{CONST.}}$$ BOUNDED