Stat 244, Aut 2017: HW5

due: Tuesday Nov 7

1. Let U and V be independent Uniform$[0, 1]$ random variables.
   (a) Calculate $E(U^k)$ where $k \geq 0$ is some fixed constant.
   (b) Calculate $\text{Cov}(UV, U + V)$.
   (c) Calculate $E(V^U)$. (Hint: use the tower law.)

2. In Hyde Park there are six taxi companies. When you call company #1, the amount of time (in minutes) that it takes the cab to arrive is distributed as Exponential(1); if you call company #2, then it's distributed as Exponential(2), $\ldots$, if you call company #6, it's generated as Exponential(6). (Recall that an Exponential(λ) distribution, whose parameter is the rate λ, has mean $1/\lambda$ and variance $1/\lambda^2$.) A statistician rolls a single standard die (sides numbered $1, 2, 3, 4, 5, 6$), and looks at the number rolled (let's call it X). Next, she calls the company chosen by the roll of the die. Let T be the amount of time that she waits for a cab to arrive. You are waiting for the cab with her, but you don't know what was the roll of the die.
   (a) Write down a hierarchical model.
   (b) Calculate $P(T > t)$ where t is a fixed value.
   (c) Calculate $E(T)$.
   (d) Suppose that after t minute the cab has not yet arrived. Given this information, what is the conditional distribution of X?
   (e) As t goes to infinity, describe how the conditional distribution of X changes. (Hint: you may want to calculate explicit probabilities at a few values of t—but this is not necessary.)

3. In this problem we will work with the Beta-Binomial setup from class:

$$\begin{cases} \theta \sim \text{Beta}(\alpha, \beta) \\ X \mid \theta \sim \text{Binomial}(n, \theta) \end{cases}$$

   Recall that $E(\theta) = \frac{\alpha}{\alpha+\beta}$ and $E(\theta^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$, from the mean & variance of the Beta distribution.

   The idea is that we cannot see θ, but we want to estimate it based on observed data given by X. Assume that α, β are known constants. Our goal in this problem is to estimate θ as accurately as possible.
   (a) Suppose that we pick a fixed value t as our "guess" for θ. Calculate the mean squared error (MSE) of this (non-random) estimator,
   $$E((\theta - t)^2),$$
   in terms of t. What value of t will minimize the MSE? What is the MSE at this value of t? How does your answer relate to the mean and/or variance of θ?
   (b) Now let's instead use X/n as our estimator. Let's calculate the MSE,
   $$E((X/n - \theta)^2).$$
   Hint: use the tower law (a.k.a. law of total expectation).
   (c) For what values of n, α, β does the first estimator (just guessing the value t, for the optimal value of t) have lower or higher MSE than the second estimator (given by X/n)?

4. The random variable X is obtained from the following random process. First, you draw from the Uniform$[0, 1]$ distribution. If the value you get is ≤ 0.5, then X is set to equal this value. Otherwise, you draw again (i.e. a second, independent draw from the Uniform$[0, 1]$ distribution), and X is set to this second value. Compute the CDF and the density of X.

5. **Rejection sampling.** Suppose that we follow this procedure:

- Draw random variable $Y \sim \text{Uniform}[0,1]$
- After observing Y, with probability Y we throw it away, otherwise we keep it.

Repeat this procedure many times (independently) to obtain a large set of draws. What is the distribution we're sampling from? Compute its density.

6. **Binomials & normals.** Let $X \sim \text{Binomial}(60, 0.22)$ and let $Y = X/60$ be the proportion of successes in the sample. What is the normal distribution that approximates the distribution of Y? Calculate (approximately) the probability $P(Y \leq 0.25)$ (no need to deal with continuity corrections etc). (To obtain values of $\Phi(x)$, the CDF of the normal distribution, you can use Table 2 in the back of your book or just search online for ?standard normal table?. Or, if you have R, you can use the command `pnorm`.)