Stat 244, Aut 2015: HW2

due: Tuesday 10.10.17

1. Suppose that errors in HW solutions obey the following model. There is a 20% chance that there is an error in the posted solutions. If an error is present, then the number of students who send emails about the error, follows a Poisson(3) distribution. If no emails were received, what is the probability that there was no error?

   Solution:

$$P(\text{no error} \mid \text{no emails}) = \frac{P(\text{no error and no emails})}{P(\text{no emails})}$$

$$= \frac{P(\text{no error}) \cdot P(\text{no emails} \mid \text{no error})}{P(\text{no error}) \cdot P(\text{no emails} \mid \text{no error}) + P(\text{error}) \cdot P(\text{no emails} \mid \text{error})} = \frac{0.8 \cdot 1}{0.8 \cdot 1 + 0.2 \cdot \frac{3^0 e^{-3}}{0!}} = 0.9878.$$

2. Calculate $P(X \text{ is odd})$ in each setting below. Show your calculation or explain your answer for each part.

   (a) $X \sim \text{Geometric}(0.7)$

   Solution:

$$P(X \text{ is odd}) = P(X=1)+P(X=3)+P(X=5)+\cdots = 0.7\cdot(0.3)^0+0.7\cdot(0.3)^2+0.7\cdot(0.3)^5+\dots$$

$$= \frac{0.7}{1-(0.3)^2} = 0.77$$

   (b) $X \sim \text{Binomial}(101, 0.5)$

   Solution: By symmetry, the probability is 0.5, because $P(X = 0) = P(X = 101)$, and $P(X = 1) = P(X = 100)$, and $P(X = 2) = P(X = 99)$, etc.

   (c) First let $Y \sim N(0, 1)$ (a standard normal random variable), and then let X be the answer you get when you round Y to the nearest integer. Your answer does not need to be simplified to a number—you can write it with summations/fractions/integrals etc as needed. However, your final answer cannot have any probability type notation in it, e.g. it cannot include terms like $p_X(3)$ or $f_Y(1)$ or $P(X \le 1)$ etc.

   Solution: $P(X \text{ is odd})$ is equal to $P(Y \text{ is in the range } (i-0.5, i+0.5) \text{ for some odd integer } i)$. (Note that we do not need to worry about the case that e.g. $Y = 4.5$ exactly, since Y is a continuous random variable so there is zero probability at any exact value.) Then

$$P(X \text{ is odd}) = \sum_{\text{odd integers } i} \int_{y=i-0.5}^{i+0.5} f_Y(y)\, dy = \sum_{\text{odd integers } i} \int_{y=i-0.5}^{i+0.5} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy.$$

3. One person draws cards from a full deck (all 52 cards), while a second person draws cards from a red deck (only the 26 red cards). As usual, cards are drawn at random and without replacement. In each problem below, do X and Y have the same distribution?

(a) X is the indicator variable for the event that the first card drawn from the full deck is a face card i.e. an Ace, King, Queen, or Jack (that is, $X = 1$ if the first card drawn from the full deck is a face card, and $X = 0$ if not). Y is the indicator variable for the event that the first card drawn from the red deck is a face card.

Solution: We have $P(X = 1) = P(\text{first card from the full deck is a face card}) = \frac{16}{52} = \frac{4}{13}$, and $P(X = 0) = 1 - P(X = 1) = \frac{9}{13}$ since 0 and 1 are the only possible values for X. Similarly, $P(Y = 1) = P(\text{first card from the red deck is a face card}) = \frac{8}{26} = \frac{4}{13}$, and $P(Y = 0) = 1 - P(Y = 1) = \frac{9}{13}$ since 0 and 1 are the only possible values for Y. Therefore, X and Y have the same set of possible values, and the same probability for each value, and so they have the same distribution.

(b) X is the number of face cards, when you draw 10 cards from the full deck. Y is the number of face cards, when you draw 10 cards from the red deck.

Solution: They do not have the same distribution—in fact, they do not even have the same set of possible values. For example, $P(X = 10) > 0$ since it's possible (although unlikely) that the 10 cards drawn from the full deck are all face cards. On the other hand, $P(Y = 10) = 0$ since there are only 8 face cards in the red deck.

4. Suppose that $X \sim N(0, 1)$. Calculate the following two density functions. For each one, be sure to specify the support i.e. the range of possible values of the random variable.

(a) Calculate the density of $Y = X^3$.

Solution: We can write $Y = g(X)$ where $g(x) = x^3$. Here g is differentiable and strictly monotonic with $g^{-1}(y) = y^{1/3}$ and so $\left|\frac{d}{dy} g^{-1}(y)\right| = \left|\frac{1}{3} y^{-2/3}\right|$, and so

$$f_Y(y) = f_X(y^{1/3}) \cdot \left|\frac{1}{3} y^{-2/3}\right| = \frac{1}{\sqrt{2\pi}} \cdot e^{-y^{2/3}/2} \cdot \frac{1}{3|y|^{2/3}}.$$

The random variable Y is supported on the entire real line $\mathbb{R}$.

(b) Calculate the density of $Z = |X|$.

Solution: In this case, we have $Z = g(X)$ where $g(x) = |x|$. This is not a strictly monotonic function and is not invertible, so we can't apply the rule for that special case. However, we know that for any $0 < a < b$, we have

$$P(a < Z < b) = P(a < X < b \text{ or } -a > X > -b) = 2P(a < X < b) ,$$

where the last step is true because X has a distribution that's symmetric around zero. So we have

$$f_Z(x) = 2f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \text{ for all } x \geq 0$$

with Z supported on $[0, \infty)$. Equivalently we could instead write

$$f_Z(x) = \begin{cases} 2f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \text{ for all } x \geq 0, \\ 0 \text{ for all } x < 0. \end{cases}$$

5. Let the random variable T be the time until some event occurs (e.g. time until an atom decays, time until next rainfall, etc). Suppose it's a continuous random variable supported on $[0, \infty)$. The hazard rate for T is defined as

$$h(t) = \frac{f(t)}{1 - F(t)},$$

where f and F are the density and CDF for the distribution of T. On an intuitive level, this is the chance that the event will occur in the very near future, given that it has not yet occurred. Hazard rate is a function of time since it can rise or fall as time goes on.

(a) Calculate the hazard rate $h(t)$ if $T \sim \text{Exponential}(\lambda)$.

Solution:

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda e^{-\lambda t}}{1 - (1 - e^{-\lambda t})} = \lambda.$$

Note that this is constant with respect to time.

(b) Now suppose that T follows a Weibull distribution with shape k and scale α, which is defined by the CDF $F(t) = 1 - e^{-(t/\alpha)^k}$. Calculate the density $f(t)$, and the hazard rate function $h(t)$, for this distribution.

Solution:

$$f(t) = F'(t) = -e^{-(t/\alpha)^k} \cdot \frac{-kt^{k-1}}{\alpha^k}$$

and so

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{e^{-(t/\alpha)^k} \cdot \frac{kt^{k-1}}{\alpha^k}}{1 - (1 - e^{-(t/\alpha)^k})} = \frac{kt^{k-1}}{\alpha^k}\ .$$

(c) For the Weibull distribution, for which values of k and α is $h(t)$ decreasing over time, increasing over time, or constant over time?

Solution: The hazard rate is increasing over time for $k > 1$, constant for $k = 1$, and decreasing over time for $k < 1$, regardless of the value of α.

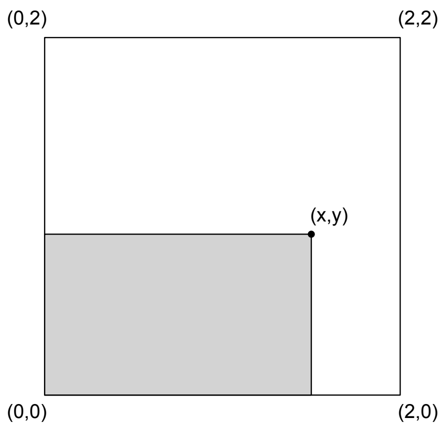
(d) Next, suppose that you buy a new watch. When you purchase it, you put in a new battery. Let T be defined as the time from putting in the new battery, until the battery runs out. Do you think the hazard rate function for T should be decreasing over time, increasing over time, or constant over time (or some other shape)? Explain. (There may be multiple plausible answers.)

Solution: It should probably be increasing over time. An old battery has a higher chance of running out during the next time interval, compared to a newer battery. However if some batteries might be defective, then a U shape could also be plausible.

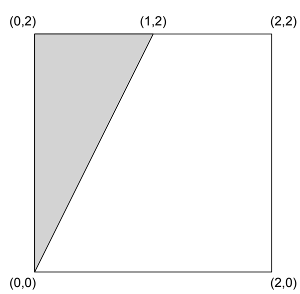
(e) Finally, a common shape for the hazard function is a "U" shape, where the hazard rate is high initially, goes down to a low rate for a long time, and then rises again later on. Give a real-life example for what T could measure that would likely have this type of hazard rate function, and explain.

Solution: Some examples: mortality of any organism (high mortality in infancy, high mortality in old age, low mortality in between); time to the first time your computer crashes (high chance at the beginning when many programs are being run for the first time, high chance when the computer is old, low chance in the middle when everything is running smoothly).

6. Suppose that you have a 2-foot-by-2-foot square, and you choose a point on the square, call it (x, y), uniformly at random from the square. Then you draw the rectangle spanning $(0, 0)$ on the bottom left and (x, y) on the top right.

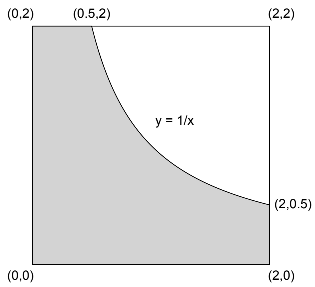


(a) What is $P(A)$, where A is the event that the rectangle is at least twice as tall as it is wide? (Hint: draw a picture of this event.)



Solution:
This figure shows the region that (x, y) would have to land in, in order for the event to occur. Since the point is chosen uniformly at random, we just need to look at the fraction of the total area, which is $1/4$ because the area of the shaded triangle is 1 while the total area is 4.

(b) What is $P(B)$, where B is the event that the area of this rectangle is ≤ 1? (Hint: draw a picture of this event.)



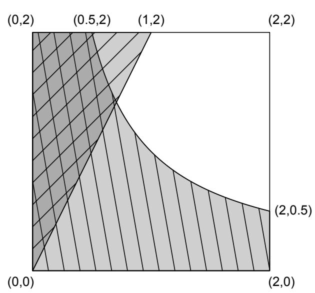
Solution:
This figure shows the region that (x, y) would have to land in, in order for the event to occur. The shaded region is split into two parts to make computation easier. Since the point is chosen uniformly at random, we just need to look at the fraction of the total area. The area of the shaded region is

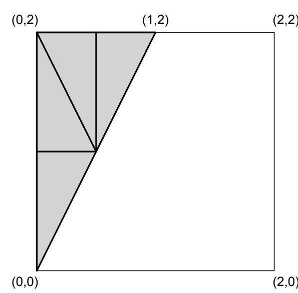
$$2 \cdot 0.5 + \int_{x=0.5}^{2} \frac{1}{x}\, dx = 1 + [\log(x)]_{x=0.5}^{2} = 1 + \log(2) - \log(0.5) = 2.386$$

So the probability is

$$\frac{2.386}{4} = 0.5965$$

(c) Are the events A and B independent? No need to do any complicated integrals for this answer—you should be able to explain it without a lot of calculation.





Solution:

In order for A and B to be independent, we would need $P(B \mid A) = P(B) = 0.5965$. However, we can see that the part of region A (the triangle) which is also in region B (the curved region) is $> 75\%$ (compare the two figures above—the one on the right splits the region corresponding to event A, into four equal parts; three of these parts are contained entirely inside the overlapping region, which corresponds to $A \cap B$). This means that $P(B \mid A) > 0.75$. So they cannot be independent.