Stat 244, Aut 2017: HW1

due: Tuesday 10.3.17

1. Three cards are drawn randomly from a standard 52 card deck.

   (a) What is the probability that all three cards are from the same suit (i.e. all hearts, or all diamonds, or all clubs, or all spades)?

   Solution: The number of ways to choose all three from one specific suit is $\binom{13}{3}$ since there are 13 cards of this suit. So, the probability is $\frac{4\cdot\binom{13}{3}}{\binom{52}{3}} = 0.052$.

   (b) What is the probability that the three cards come from three different suits?

   Solution: The probability is $\frac{52}{52} \cdot \frac{39}{51} \cdot \frac{26}{50} = 0.398$, since after the first card is drawn there are 39 remaining cards which are *not* the same suit, and after the second card is drawn there are 26 remaining which are not from *either* of the two suits already seen.

2. You roll two dice, one red and one blue. Let R and B be the numbers that you see on the red one and the blue one. What is the probability that $R < B$? (Your answer should be short!)

   Solution: By symmetry, we have $P(R < B) = P(B > R)$. We also know that $P(R = B) = 1/6$, and $P(R < B) + P(R = B) + P(R > B) = 1$. So, $P(R < B) = \frac{5/6}{2} = \frac{5}{12}$.

3. You have two coins: one fair (50% chance of Heads) and one biased (60% chance of Heads). However, you pick up one coin at random and don't know which one it is.

   (a) Suppose you flip the coin two times, and get Heads both times. Conditional on this outcome, what is the probability that this is the fair coin?

   Solution: Let F be the event that you picked the fair coin and H be the event that you get two Heads. Then $P(F \cap H) = P(F) \cdot P(H \mid F) = 0.5 \cdot 0.5^2 = 0.125$ and $P(F^c \cap H) = P(F^c) \cdot P(H \mid F^c) = 0.5 \cdot 0.6^2 = 0.18$. So,

   $$P(F \mid H) = \frac{P(F \cap H)}{P(F \cap H) + P(F^c \cap H)} = \frac{0.125}{0.125 + 0.18} = 0.41.$$

   (b) After flipping the coin twice and seeing two Heads, what is the probability that the next toss is also Heads?

   Solution: The probability is 0.5 if it's the fair coin, and 0.6 if it's the biased coin. So, using the answer from the above,

   $$0.41 \cdot 0.5 + (1 - 0.41) \cdot 0.6 = 0.56.$$

4. Let A and C be events, with the following known facts:

$$P(A) = 0.4, P(A \mid C) = 0.1 \ .$$

Explain why it's not possible to have $P(C) = 0.8$. More specifically, can you prove either an upper bound or a lower bound on $P(C)$?

Solution: If $P(C) = 0.8$, then we would have $P(A \cap C) = P(A \mid C) \cdot P(C) = 0.1 \cdot 0.8 = 0.08$. This means that $P(A \cap C^c) = P(A) - P(A \cap C) = 0.4 - 0.08$. But this is already more than $P(C^c) = 0.2$, which is not possible. More generally,

$$\begin{aligned} 0.4 = P(A) = P(A \cap C) + P(A \cap C^c) \leq P(A \cap C) + P(C^c) \\ = P(C) \cdot P(A \mid C) + (1 - P(C)) = 0.1 \cdot P(C) + 1 - P(C) = 1 - 0.9 \cdot P(C), \end{aligned}$$

which proves that $P(C) \leq \frac{0.6}{0.9} = \frac{2}{3}$.

5. A diagnostic test for a disease returns one of two answers: Positive (has the disease), negative (does not have the disease). Here are the probabilities:

- For a patient with the disease, 95% positive / 5% negative
- For a patient that does not have the disease, 10% positive / 90% negative

(a) Suppose that 10% of the population has the disease. If everyone is tested, what proportion of the test results will be positive?

Solution: Counting first the patients with the disease who get a positive result, and then the ones without the disease who get a positive result, we get $0.1 \cdot 0.95 + 0.9 \cdot 0.1 = 0.185$

(b) For a patient who gets an Positive result, what is the probability of having the disease?

Solution: This is $P(\text{has disease}|\text{positive result})$. We have $P(\text{has disease and gets positive result}) = 0.1 \cdot 0.95 = 0.095$, so the probability is $\frac{0.095}{0.185} = 0.513$.

(c) Now, let d be the prevalence of the disease in the population (for example if 10% of the population has the disease, then $d = 0.1$).
First, everyone in the population is screened for the disease. For each person that got a positive result, they are brought back for a second screening. It turns out that 25% of the return patients, test positive the second time. (You can assume the test results are independent—for example, if I have the disease and get a Positive on the first test, the second time I take the test it's a fresh "roll of the dice" with the same chances of getting any of the two possible results.)
What is d? You can assume that it's a large population so all the probabilities work out exactly as expected (e.g. if you flip n fair coins then assume exactly $0.5 \cdot n$ of them are Heads, etc).

Solution: The information we are given is that $P(\text{2nd test Positive} \mid \text{1st test positive}) = 0.25$. Writing it out,

$$0.25 = P(\text{2nd test Positive} \mid \text{1st test positive}) = \frac{P(\text{1st and 2nd tests both positive})}{P(\text{1st test positive})} = \frac{d \cdot 0.95^2 + (1-d) \cdot 0.1^2}{d \cdot 0.95 + (1-d) \cdot 0.1}.$$

Simplifying,

$$0.25 = \frac{0.8925d + 0.01}{0.85d + 0.1}.$$

Solving for d, we get $d = 0.022$.

6. A magician plays the following game. First, a volunteer from the audience gets to choose any two pieces of paper, where the nine papers are labeled 1 through 9. The volunteer does not have to choose the numbers randomly—he can choose them however he likes.

   The volunteer then shuffles the two numbers and randomly place one number in each hand. Finally, the magician gets to see the number in the volunteer's left hand, and has to guess which number is larger, i.e. whether the number he sees is the larger number or the smaller number of the two.

   The magician claims that, while he can't answer correctly 100% of the time, his guesses are better than random (i.e. greater than 50% chance of guessing correctly), regardless of which two numbers the volunteer decides to use. You should assume that everyone is being honest, i.e. the volunteer really does shuffle the two numbers randomly, the magician doesn't peek to see which numbers are left in the hat, etc.

   (a) Here is one possible strategy for the magician. If he sees a 1, 2, 3, or 4, then he will guess that the other number is larger. If he sees a 6, 7, 8, or 9, he will guess that the number he sees is larger. And if he sees a 5, since that's exactly in the middle, he will flip a coin to randomly guess one way or the other.

   Calculate his chances of guessing correctly, in each of the following scenarios: the volunteer chooses 2 and 3; the volunteer chooses 3 and 6; the volunteer chooses 5 and 7.

   Solution: For 2 and 3, half the time the revealed number is 2 and the guess is correct. Half the time the revealed number is 3 and the guess is incorrect. So, chance = 50%. For 3 and 6 the chance is 100%. For 5 and 7, half the time the revealed number is 5 and then the guess has a 50% chance of being correct based on the coin flip. Otherwise, 7 is revealed and then the guess is correct. So the total chance is 75%.

   (b) Develop a strategy for the magician to use, so that for *any* choice of two numbers, his chances of guessing correctly are better than random.

   Solution: There is a range of possible solutions. In general, if the number revealed is i, then with probability p_i declare it to be the larger number, and with probability $1 - p_i$ declare it to be the smaller number. As long as

$$0 \leq p_1 < p_2 < \cdots < p_9 \leq 1 \ ,$$

   the method will work. Here is why. Let i and j be the two numbers chosen, with $i < j$.

   - There is a $1/2$ probability that the revealed number is i. In this case, with probability p_i our guess is wrong and with probability $1 - p_i$ our guess is right.
   - There is a $1/2$ probability that the revealed number is j. In this case, with probability p_j our guess is right and with probability $1 - p_j$ our guess is wrong.

   So our total probability of guessing correctly is $\frac{1}{2} \cdot (1 - p_i) + \frac{1}{2} \cdot p_j > \frac{1}{2}$, since we chose $p_j > p_i$.