Stat 244, Aut 2017: HW3

due: Tuesday 10.17.17

1. In class, we had the following example: $X \sim \text{Poisson}(100)$ is the number of photons emitted by an X-ray beam, and then Y is the number of photons that successfully pass through an object (i.e. the person being imaged). In that example, the distribution of Y, if we know how many photons X were sent into the object, is given by $Y \mid X \sim \text{Binomial}(X, 0.4)$, i.e. given that X many photons are sent into the object, each one has a 40% chance of making it through and passing out the other side.

   Calculate the probability mass function for (X, Y), that is, $p(k, \ell) = P(X = k, Y = \ell)$ (as a function of k and ℓ).

2. (a) Let X and Y be random variables with $X \sim \text{Exponential}(\lambda_1)$ and $Y \sim \text{Exponential}(\lambda_2)$. Suppose that X and Y are independent. Let $Z = \max\{X, Y\}$. Calculate the CDF of Z.

   (b) Let $X \sim \text{Exponential}(1)$ and let $Y \sim \text{Bernoulli}(0.5)$. Let $Z = X + Y$. Calculate the CDF of Z.

3. You have n random number generators, where the ith one draws a number uniformly at random from the interval $[0, t_i]$. Here the t_i's are arbitrary positive integers. Let X_i be the number drawn by the ith random number generator.

   (a) What is the expected value of the sum, $S = X_1 + \cdots + X_n$? (Your answer will be in terms of $t_1, \ldots, t_n$.)

   (b) What is the expected value of Y, which counts how many of the X_i's are ≤ 1? (Your answer will be in terms of $t_1, \ldots, t_n$.)

4. Let $X \sim \text{Exponential}(\lambda)$ and let t be a constant with $0 < t < \lambda$.

   (a) What is $\mathbb{E}\left[e^{tX}\right]$?

   (b) Use the Markov inequality to prove a bound on $P(e^{tX} \geq a)$ (here $a > 0$ is any positive number, while we assume $0 < t < \lambda$ as before).

   (c) Now reformulate this into a bound on $P(X \geq b)$ (here $b > 0$ is any positive number, and again $0 < t < \lambda$). How does this compare to the true value of the probability $P(X \geq b)$?

5. (a) Suppose you flip a fair coin 10 times. Let X be the total number of times that you see the sequence HT. What is $P(X = 0)$?

   (b) What is $E(X)$? (Hint: think of X as a sum.)

   (c) Suppose you draw cards one at a time from a standard deck, without replacement, and record their colors. Let Y be the total number of times that you see the sequence RB (i.e. a red card followed by a black card). What is $P(Y = 0)$?

   (d) What is $E(Y)$? (Hint: think of Y as a sum.)

6. Suppose that (X, Y) is a point chosen uniformly at random from the triangular region formed by connecting the points $(1, 0)$, $(0, 1)$ and $(0, -1)$.

   (a) Calculate $P(X > 0.1 \mid Y > 0.1)$.

   (b) What is the CDF of the variable X?