Stat 244, Aut 2017: HW1

due: Tuesday 10.3.17

1. Three cards are drawn randomly from a standard 52 card deck.
   (a) What is the probability that all three cards are from the same suit (i.e. all hearts, or all diamonds, or all clubs, or all spades)?
   (b) What is the probability that the three cards come from three different suits?
2. You roll two dice, one red and one blue. Let R and B be the numbers that you see on the red one and the blue one. What is the probability that $R < B$? (Your answer should be short!)
3. You have two coins: one fair (50% chance of Heads) and one biased (60% chance of Heads). However, you pick up one coin at random and don't know which one it is.
   (a) Suppose you flip the coin two times, and get Heads both times. Conditional on this outcome, what is the probability that this is the fair coin?
   (b) After flipping the coin twice and seeing two Heads, what is the probability that the next toss is also Heads?
4. Let A and C be events, with the following known facts:

$$P(A) = 0.4, P(A \mid C) = 0.1 \ .$$

   Explain why it's not possible to have $P(C) = 0.8$. More specifically, can you prove either an upper bound or a lower bound on $P(C)$?
5. A diagnostic test for a disease returns one of two answers: Positive (has the disease), negative (does not have the disease). Here are the probabilities:
   - For a patient with the disease, 95% positive / 5% negative
   - For a patient that does not have the disease, 10% positive / 90% negative

   (a) Suppose that 10% of the population has the disease. If everyone is tested, what proportion of the test results will be positive?
   (b) For a patient who gets an Positive result, what is the probability of having the disease?
   (c) Now, let d be the prevalence of the disease in the population (for example if 10% of the population has the disease, then $d = 0.1$).
   First, everyone in the population is screened for the disease. For each person that got a positive result, they are brought back for a second screening. It turns out that 25% of the return patients, test positive the second time. (You can assume the test results are independent—for example, if I have the disease and get a Positive on the first test, the second time I take the test it's a fresh "roll of the dice" with the same chances of getting any of the two possible results.)
   What is d? You can assume that it's a large population so all the probabilities work out exactly as expected (e.g. if you flip n fair coins then assume exactly $0.5 \cdot n$ of them are Heads, etc).
6. A magician plays the following game. First, a volunteer from the audience gets to choose any two pieces of paper, where the nine papers are labeled 1 through 9. The volunteer does not have to choose the numbers randomly—he can choose them however he likes.

   The volunteer then shuffles the two numbers and randomly place one number in each hand. Finally, the magician gets to see the number in the volunteer's left hand, and has to guess which number is larger, i.e. whether the number he sees is the larger number or the smaller number of the two.

   The magician claims that, while he can't answer correctly 100% of the time, his guesses are better than random (i.e. greater than 50% chance of guessing correctly), regardless of which two numbers the volunteer decides to use. You should assume that everyone is being honest, i.e. the volunteer really does shuffle the two numbers randomly, the magician doesn't peek to see which numbers are left in the hat, etc.

(a) Here is one possible strategy for the magician. If he sees a 1, 2, 3, or 4, then he will guess that the other number is larger. If he sees a 6, 7, 8, or 9, he will guess that the number he sees is larger. And if he sees a 5, since that's exactly in the middle, he will flip a coin to randomly guess one way or the other.

Calculate his chances of guessing correctly, in each of the following scenarios: the volunteer chooses 2 and 3; the volunteer chooses 3 and 6; the volunteer chooses 5 and 7.

(b) Develop a strategy for the magician to use, so that for *any* choice of two numbers, his chances of guessing correctly are better than random.