Outline

- Scalar nonlinear conservation laws
- Shocks and rarefaction waves
- Entropy conditions
- Finite volume methods
- Approximate Riemann solvers
- Lax-Wendroff Theorem

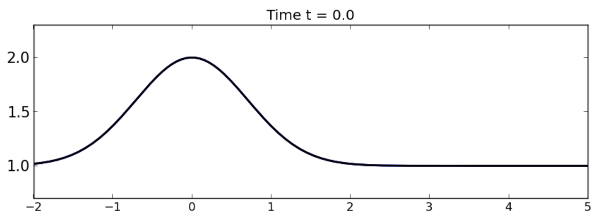
Reading: Chapter 11, 12

Notes:

Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

The solution is constant on characteristics so each value advects at constant speed equal to the value...



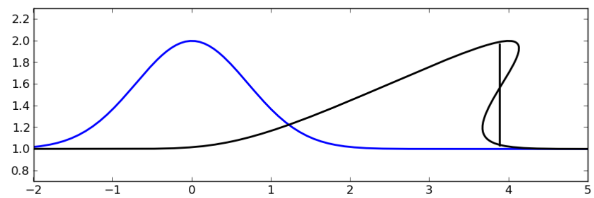
Notes:

Burgers' equation

Equal-area rule:

The area "under" the curve is conserved with time,

We must insert a shock so the two areas cut off are equal.



Notes:

Riemann problem for Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_t + uu_x = 0.$$
$$f(u) = \tfrac{1}{2}u^2, \qquad f'(u) = u.$$

Consider Riemann problem with states u_ℓ and u_r.

For any $u_\ell,\ u_r$, there is a weak solution consisting of this discontinuity propagating at speed given by the Rankine-Hugoniot jump condition:

$$s = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_\ell^2}{u_r - u_\ell} = \frac{1}{2}(u_\ell + u_r).$$

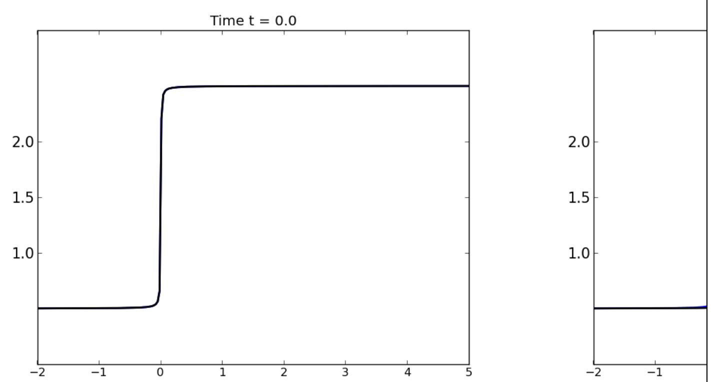
Note: Shock speed is average of characteristic speed on each side.

This might not be the physically correct weak solution!

Notes:

Burgers' equation

The solution is constant on characteristics so each value advects at constant speed equal to the value...



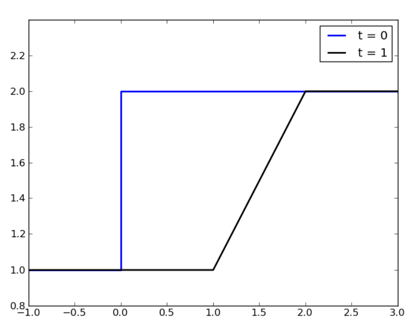
Notes:

Weak solutions to Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \quad u_r = 2$$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

"Physically correct" rarefaction wave solution:



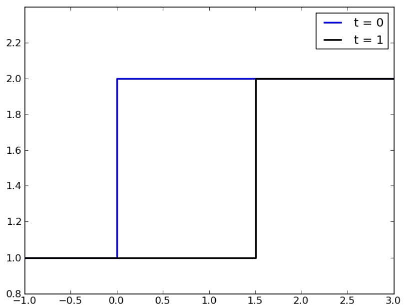
Notes:

Weak solutions to Burgers' equation

$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

Entropy violating weak solution:



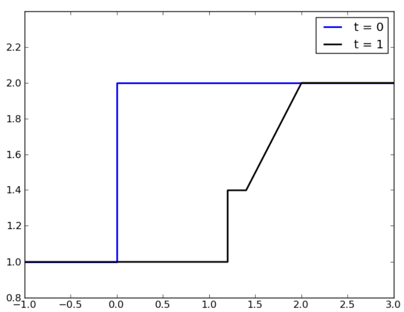
Notes:

Weak solutions to Burgers' equation

$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

Another Entropy violating weak solution:



Notes:

Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

$$q_t + f(q)_x = \epsilon q_{xx}.$$

This selects a unique weak solution:

- Shock if $f'(q_l) > f'(q_r)$,
- Rarefaction if $f'(q_l) < f'(q_r)$.

Lax Entropy Condition:

A discontinuity propagating with speed s in the solution of a convex scalar conservation law is admissible only if $f'(q_\ell) > s > f'(q_r)$, where $s = (f(q_r) - f(q_\ell))/(q_r - q_\ell)$.

Note: This means characteristics must approach shock from both sides as t advances, not move away from shock!

Notes:

Riemann problem for scalar nonlinear problem

$q_t + f(q)_x = 0$ with data

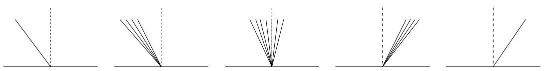
$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Piecewise constant with a single jump discontinuity.

For Burgers' or traffic flow with quadratic flux, the Riemann solution consists of:

- Shock wave if $f'(q_l) > f'(q_r)$,
- Rarefaction wave if $f'(q_l) < f'(q_r)$.

Five possible cases:



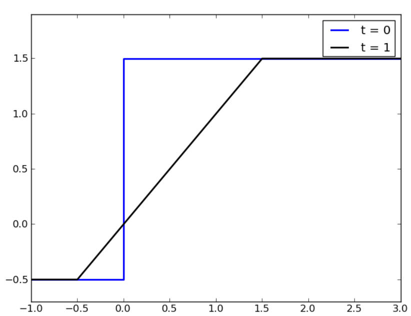
Notes:

Transonic rarefactions

Sonic point: $u_s = 0$ for Burgers' since $f'(0) = 0$.

Consider Riemann problem data $u_\ell = -0.5 < 0 < u_r = 1.5$.

In this case wave should spread in both directions:



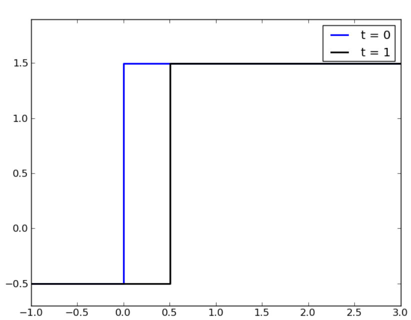
Notes:

Transonic rarefactions

Entropy-violating approximate Riemann solution:

$$s = \frac{1}{2}(u_\ell + u_r) = 0.5.$$

Wave goes only to right, no update to cell average on left.

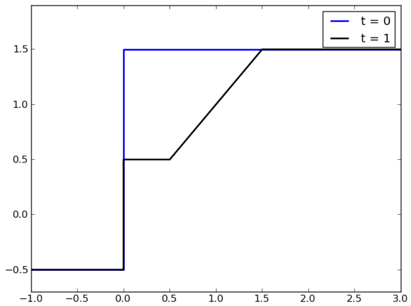


Notes:

Transonic rarefactions

If $u_\ell = -u_r$ then Rankine-Hugoniot speed is 0:

Similar solution will be observed with Godunov's method
if entropy-violating approximate Riemann solver used.

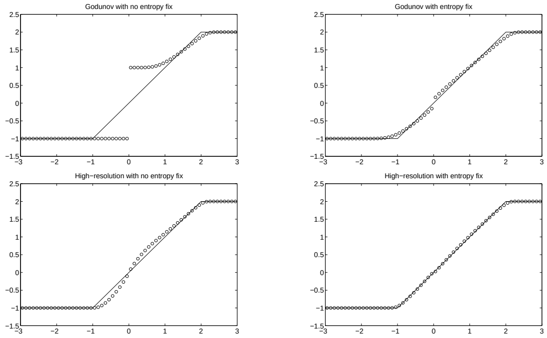


Notes:

Entropy-violating numerical solutions

Riemann problem for Burgers' equation at $t = 1$

with $u_\ell = -1$ and $u_r = 2$:



Notes:

Approximate Riemann solvers

For nonlinear problems, computing the exact solution to each Riemann problem may not be possible, or too expensive.

Often the nonlinear problem $q_t + f(q)_x = 0$ is approximated by

$$q_t + A_{i-1/2} q_x = 0, \qquad q_\ell = Q_{i-1}, \quad q_r = Q_i$$

for some choice of $A_{i-1/2} \approx f'(q)$ based on data $Q_{i-1},\ Q_i$.

Solve linear system for $\alpha_{i-1/2}$: $Q_i - Q_{i-1} = \sum_p \alpha^p_{i-1/2} r^p_{i-1/2}$.

Waves $\mathcal{W}^p_{i-1/2} = \alpha^p_{i-1/2} r^p_{i-1/2}$ propagate with speeds $s^p_{i-1/2}$,

$r^p_{i-1/2}$ are eigenvectors of $A_{i-1/2}$,
$s^p_{i-1/2}$ are eigenvalues of $A_{i-1/2}$.

Notes:

Approximate Riemann solvers

$$q_t + \hat{A}_{i-1/2} q_x = 0, \qquad q_\ell = Q_{i-1}, \quad q_r = Q_i$$

Often $\hat{A}_{i-1/2} = f'(Q_{i-1/2})$ for some choice of $Q_{i-1/2}$.

In general $\hat{A}_{i-1/2} = \hat{A}(q_\ell,\ q_r)$.

Roe conditions for consistency and conservation:

- $\hat{A}(q_\ell, q_r) \to f'(q^*)$ as $q_\ell, q_r \to q^*$,
- $\hat{A}$ diagonalizable with real eigenvalues,
- For conservation in wave-propagation form,

$$\hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}).$$

Notes:

Approximate Riemann solvers

For a scalar problem, we can easily satisfy the Roe condition

$$\hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}).$$

by choosing

$$\hat{A}_{i-1/2} = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}.$$

Then $r^1_{i-1/2} = 1$ and $s^1_{i-1/2} = \hat{A}_{i-1/2}$ (scalar!).

Note: This is the Rankine-Hugoniot shock speed.

$\Longrightarrow$ shock waves are correct,
rarefactions replaced by entropy-violating shocks.

Notes:

Approximate Riemann solver

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

For scalar advection $m = 1$, only one wave.
$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2} = Q_i - Q_{i-1}$ and $s_{i-1/2} = u$,

$$\mathcal{A}^-\Delta Q_{i-1/2} = s^-_{i-1/2}\mathcal{W}_{i-1/2},$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = s^+_{i-1/2}\mathcal{W}_{i-1/2}.$$

For scalar nonlinear: Use same formulas with
$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2}$ and $s_{i-1/2} = \Delta F_{i-1/2}/\Delta Q_{i-1/2}$.

Need to modify these by an entropy fix in the trans-sonic rarefaction case.

Notes:

Entropy fix

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Revert to the formulas

$$\mathcal{A}^-\Delta Q_{i-1/2} = f(q_s) - f(Q_{i-1}) \qquad \text{left-going fluctuation}$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - f(q_s) \qquad \text{right-going fluctuation}$$

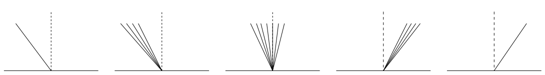
if $f'(Q_{i-1}) < 0 < f'(Q_i)$.

High-resolution method: still define wave $\mathcal{W}$ and speed s by

$$\mathcal{W}_{i-1/2} = Q_i - Q_{i-1},$$
$$s_{i-1/2} = \begin{cases} (f(Q_i) - f(Q_{i-1}))/(Q_i - Q_{i-1}) & \text{if } Q_{i-1} \neq Q_i \\ f'(Q_i) & \text{if } Q_{i-1} = Q_i. \end{cases}$$

Notes:

Godunov flux for scalar problem



The Godunov flux function for the case $f''(q) > 0$ is

$$F_{i-1/2}^n = \begin{cases} f(Q_{i-1}) & \text{if } Q_{i-1} > q_s \text{ and } s > 0 \\ f(Q_i) & \text{if } Q_i < q_s \text{ and } s < 0 \\ f(q_s) & \text{if } Q_{i-1} < q_s < Q_i. \end{cases}$$
$$= \begin{cases} \min\limits_{Q_{i-1} \le q \le Q_i} f(q) & \text{if } Q_{i-1} \le Q_i \\ \max\limits_{Q_i \le q \le Q_{i-1}} f(q) & \text{if } Q_i \le Q_{i-1}, \end{cases}$$

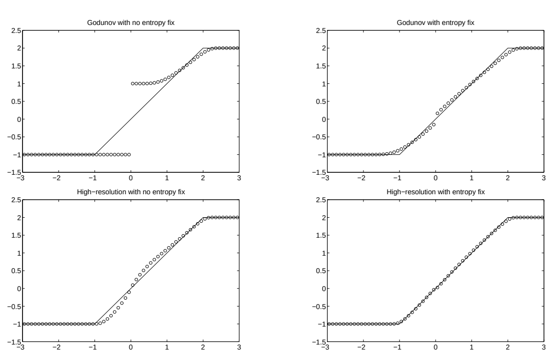
Here $s = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}$ is the Rankine-Hugoniot shock speed.

Notes:

Entropy-violating numerical solutions

Riemann problem for Burgers' equation at $t = 1$

with $u_\ell = -1$ and $u_r = 2$:



Notes:

Entropy (admissibility) conditions

We generally require additional conditions on a weak solution to a conservation law, to pick out the unique solution that is physically relevant.

In gas dynamics: entropy is constant along particle paths for smooth solutions, entropy can only increase as a particle goes through a shock.

Entropy functions: Function of q that "behaves like" physical entropy for the conservation law being studied.

NOTE: Mathematical entropy functions generally chosen to
decrease for admissible solutions,
increase for entropy-violating solutions.

Notes:

Entropy functions

A scalar-valued function $\eta : \mathbb{R}^m \to \mathbb{R}$ is a convex function of q

if the Hessian matrix $\eta''(q)$ with (i,j) element

$$\eta''_{ij}(q) = \frac{\partial^2 \eta}{\partial q^i \partial q^j}$$

is positive definite for all q, i.e., satisfies

$$v^T \eta''(q) v > 0 \qquad \text{for all } \; q, \; v \in \mathbb{R}^m.$$

Scalar case: reduces to $\eta''(q) > 0$.

Notes:

Entropy functions

Entropy function: $\eta : \mathbb{R}^m \to \mathbb{R}$ Entropy flux: $\psi : \mathbb{R}^m \to \mathbb{R}$

chosen so that $\eta(q)$ is convex and:

- $\eta(q)$ is conserved wherever the solution is smooth,

$$\eta(q)_t + \psi(q)_x = 0.$$

- Entropy decreases across an admissible shock wave.

Weak form:

$$\int_{x_1}^{x_2} \eta(q(x,t_2))\,dx \;\le\; \int_{x_1}^{x_2} \eta(q(x,t_1))\,dx + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt$$

with equality where solution is smooth.

Notes:

Entropy functions

How to find η and ψ satisfying this?

$$\eta(q)_t + \psi(q)_x = 0$$

For smooth solutions gives

$$\eta'(q)q_t + \psi'(q)q_x = 0.$$

Since $q_t = -f'(q)q_x$ this is satisfied provided

$$\psi'(q) = \eta'(q)f'(q)$$

Scalar: Can choose any convex $\eta(q)$ and integrate.

Example: Burgers' equation, $f'(u) = u$ and take $\eta(u) = u^2$.

Then $\psi'(u) = 2u^2 \implies$ Entropy function: $\psi(u) = \frac{2}{3}u^3$.

Notes:

Weak solutions and entropy functions

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \qquad \text{and} \qquad \left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions, different shock speeds!

Entropy function: $\eta(u) = u^2$.

A correct Burgers' shock at speed $s = \frac{1}{2}(u_\ell + u_r)$ will have total mass of $\eta(u)$ decreasing.

Notes:

Entropy functions

$$\int_{x_1}^{x_2} \eta(q(x,t_2))\,dx \le \int_{x_1}^{x_2} \eta(q(x,t_1))\,dx + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt$$

comes from considering the vanishing viscosity solution:

$$q_t^\epsilon + f(q^\epsilon)_x = \epsilon q_{xx}^\epsilon$$

Multiply by $\eta'(q^\epsilon)$ to obtain:

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\eta'(q^\epsilon)q_{xx}^\epsilon.$$

Manipulate further to get

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\left(\eta'(q^\epsilon)q_x^\epsilon\right)_x - \epsilon\eta''(q^\epsilon)\,(q_x^\epsilon)^2.$$

Notes:

Entropy functions

Smooth solution to viscous equation satisfies

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\big(\eta'(q^\epsilon)q^\epsilon_x\big)_x - \epsilon\eta''(q^\epsilon)\,(q^\epsilon_x)^2.$$

Integrating over rectangle $[x_1, x_2] \times [t_1, t_2]$ gives

$$\begin{aligned}\int_{x_1}^{x_2} \eta(q^\epsilon(x,t_2))\,dx &= \int_{x_1}^{x_2} \eta(q^\epsilon(x,t_1))\,dx \\ &- \left(\int_{t_1}^{t_2} \psi(q^\epsilon(x_2,t))\,dt - \int_{t_1}^{t_2} \psi(q^\epsilon(x_1,t))\,dt\right) \\ &+ \epsilon \int_{t_1}^{t_2} \left[\eta'(q^\epsilon(x_2,t))\,q^\epsilon_x(x_2,t) - \eta'(q^\epsilon(x_1,t))\,q^\epsilon_x(x_1,t)\right] dt \\ &- \epsilon \int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta''(q^\epsilon)\,(q^\epsilon_x)^2\,dx\,dt.\end{aligned}$$

Let $\epsilon \to 0$ to get result:

Term on third line goes to 0,

Term of fourth line is always ≤ 0.

Notes:

Entropy functions

Weak form of entropy condition:

$$\int_0^\infty \int_{-\infty}^\infty \left[\phi_t \eta(q) + \phi_x \psi(q)\right] dx\,dt + \int_{-\infty}^\infty \phi(x,0)\eta(q(x,0))\,dx \geq 0$$

for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R})$ with $\phi(x,t) \geq 0$ for all $x,\ t$.

Informally we may write

$$\eta(q)_t + \psi(q)_x \leq 0.$$

Notes:

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar q, \bar q) = f(\bar q)$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

Two sequences might converge to different weak solutions.

Also need to satisfy an entropy condition.

Notes:

Sketch of proof of Lax-Wendroff Theorem

Multiply the conservative numerical method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

by Φ_i^n to obtain

$$\Phi_i^n Q_i^{n+1} = \Phi_i^n Q_i^n - \frac{\Delta t}{\Delta x}\Phi_i^n(F_{i+1/2}^n - F_{i-1/2}^n).$$

This is true for all values of i and n on each grid.
Now sum over all i and $n \geq 0$ to obtain

$$\sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty} \Phi_i^n(Q_i^{n+1} - Q_i^n) = -\frac{\Delta t}{\Delta x}\sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty} \Phi_i^n(F_{i+1/2}^n - F_{i-1/2}^n).$$

Use summation by parts to transfer differences to Φ terms.

Notes:

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Sketch of proof of Lax-Wendroff Theorem

Obtain analog of weak form of conservation law:

$$\Delta x \Delta t \left[\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty}\left(\frac{\Phi_i^n - \Phi_i^{n-1}}{\Delta t}\right) Q_i^n \right.$$
$$\left. + \sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty}\left(\frac{\Phi_{i+1}^n - \Phi_i^n}{\Delta x}\right) F_{i-1/2}^n\right] = -\Delta x \sum_{i=-\infty}^{\infty} \Phi_i^0 Q_i^0.$$

Consider on a sequence of grids with $\Delta x, \Delta t \to 0$.

Show that any limiting function must satisfy weak form of conservation law.

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Notes:

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Analog of Lax-Wendroff proof for entropy

Show that the numerical flux function F leads to a
numerical entropy flux Ψ

such that the following discrete entropy inequality holds:

$$\eta(Q_i^{n+1}) \leq \eta(Q_i^n) - \frac{\Delta t}{\Delta x}\left[\Psi_{i+1/2}^n - \Psi_{i-1/2}^n\right].$$

Then multiply by test function Φ_i^n, sum and use summation by parts to get discrete form of integral form of entropy condition.

$\Longrightarrow$ If numerical approximations converge to some function, then the limiting function satisfies the entropy condition.

Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 1, 2011 [FVMHP Sec. 12.11]

Entropy consistency of Godunov's method

For Godunov's method, $F(Q_{i-1}, Q_i) = f(Q^{\Downarrow}_{i-1/2})$
where $Q^{\Downarrow}_{i-1/2}$ is the constant value along $x_{i-1/2}$ in the Riemann solution.

Let $\Psi^n_{i-1/2} = \psi(Q^{\Downarrow}_{i-1/2})$

Discrete entropy inequality follows from Jensen's inequality:

The value of η evaluated at the average value of $\tilde{q}^n$ is less than or equal to the average value of $\eta(\tilde{q}^n)$, i.e.,

$$\eta\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^n(x, t_{n+1})\,dx\right) \le \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta\big(\tilde{q}^n(x, t_{n+1})\big)\,dx.$$

Notes:

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