# Flux Limiters and TVD Criteria

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- **Total Variation Diminishing Criteria**
- Summary of Flux Limiters

# Total-Variation

For a quantity *Q* specified on a numerical grid, the “total variation” of the solution at a given time is given by:

$$TV(Q) = \sum_{i=1}^{N-1} |Q_{i+1} - Q_i|$$

If the system is not to develop nonphysical maxima or minima (indicating oscillations, for example), this must remain constant or decreasing over each time step:

$$\boxed{TV(Q^{n+1}) \leq TV(Q^n)}$$

[We follow the descriptions of *LeVeque*, 2002; *Durran*, 2010; both provide excellent introductions.]

# "Monotonicity" of a Method

Methods which satisfy TVD criteria also preserve monotonicity of solutions. A scheme is **monotone** if:

**Given:** $Q_i^n \geq Q_{i+1}^n$

**Then:** $Q_i^{n+1} \geq Q_{i+1}^{n+1}$

Pure second order methods cannot be monotone; indeed, any linear monotone scheme is at best first-order accurate.

We seek to use **flux limiters** to control the inclusion of higher-order terms, creating schemes with greater than first order accuracy, while preserving TVD

# TVD as a Goal for FVMs

TVD criteria is independent of a scheme’s stability, and is not a sufficient condition for a “satisfactory” solution.

Indeed, useful higher order methods do not satisfy TVD alone, and many TVD low-order methods are too diffusive for practical application. (Hence interest in flux limiters...)

**“High resolution”** TVD methods will be of greatest utility for systems with initially discontinuous or steep solutions, or those apt to produce shocks or steep solutions.

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# Flux Limited Methods

We will focus first on Flux-Limited flux form finite difference methods, using fluxes defined by combinations of low ($F_L$) and high ($F_H$) order flux approximations.

$$\boxed{F_{i-1/2} = F^L_{i-1/2} + \phi_{i-1/2}(F^H_{i-1/2} - F^L_{i-1/2})}$$

Here, the "flux limiter" is given by *phi*.

Flux limiters seek to maintain TVD behavior of the solution while achieving greater than first order accuracy.

# Flux Limited Methods

$$\boxed{F_{i-1/2} = F^L_{i-1/2} + \phi_{i-1/2}(F^H_{i-1/2} - F^L_{i-1/2})}$$

**We can explore a few limiting cases:**

$\phi_{i-1/2} = 0$ *Defaults to low-order method.*

$\phi_{i-1/2} = 1$ *Defaults to high-order method.*

For discussion, we will assume that $F^H$ is defined by a **Lax-Wendroff** flux, $F^L$ is defined by an **Upwind** flux (i.e., our example code online).

# Calculations for Flux Limiter

The flux limiter function will be dependent on the ratio of solution slopes across an interface between two cells:

$$\theta^n_{i+1/2} = \frac{Q^n_i - Q^n_{i-1}}{Q^n_{i+1} - Q^n_i}$$

Note that this expression is valid only for a positive upwind direction (c>0), where if c<0 the ratio of slopes becomes:

$$\theta^n_{i+1/2} = \frac{Q^n_{i+2} - Q^n_{i+1}}{Q^n_{i+1} - Q^n_i}$$

The flux limiter value *phi* is a function of *theta*: $\phi(\theta^n_{i+1/2})$

# Flux Limited Methods

$$\boxed{F_{i-1/2} = F^L_{i-1/2} + \phi_{i-1/2}(F^H_{i-1/2} - F^L_{i-1/2})}$$

**We can explore a few more limiting cases:**

$$\phi_{i-1/2} = \theta$$ *Beam-Warming Method*

$$\phi_{i-1/2} = \frac{1}{2}(1+\theta)$$ *Fromm Method*

Beam-warming takes upwind differences, Fromm takes centered differences (asymmetrically), Lax-Wendroff takes downwind differences (symmetrically).

$$\phi_{i-1/2} = 1$$

*Lax-Wendroff Method*

$$Q_i^{n+1} = Q_i^n - \frac{c\Delta t}{2\Delta x}(Q_{i+1}^n + Q_{i-1}^n) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2 (Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

$$\phi_{i-1/2} = \theta$$

*Beam-Warming Method*

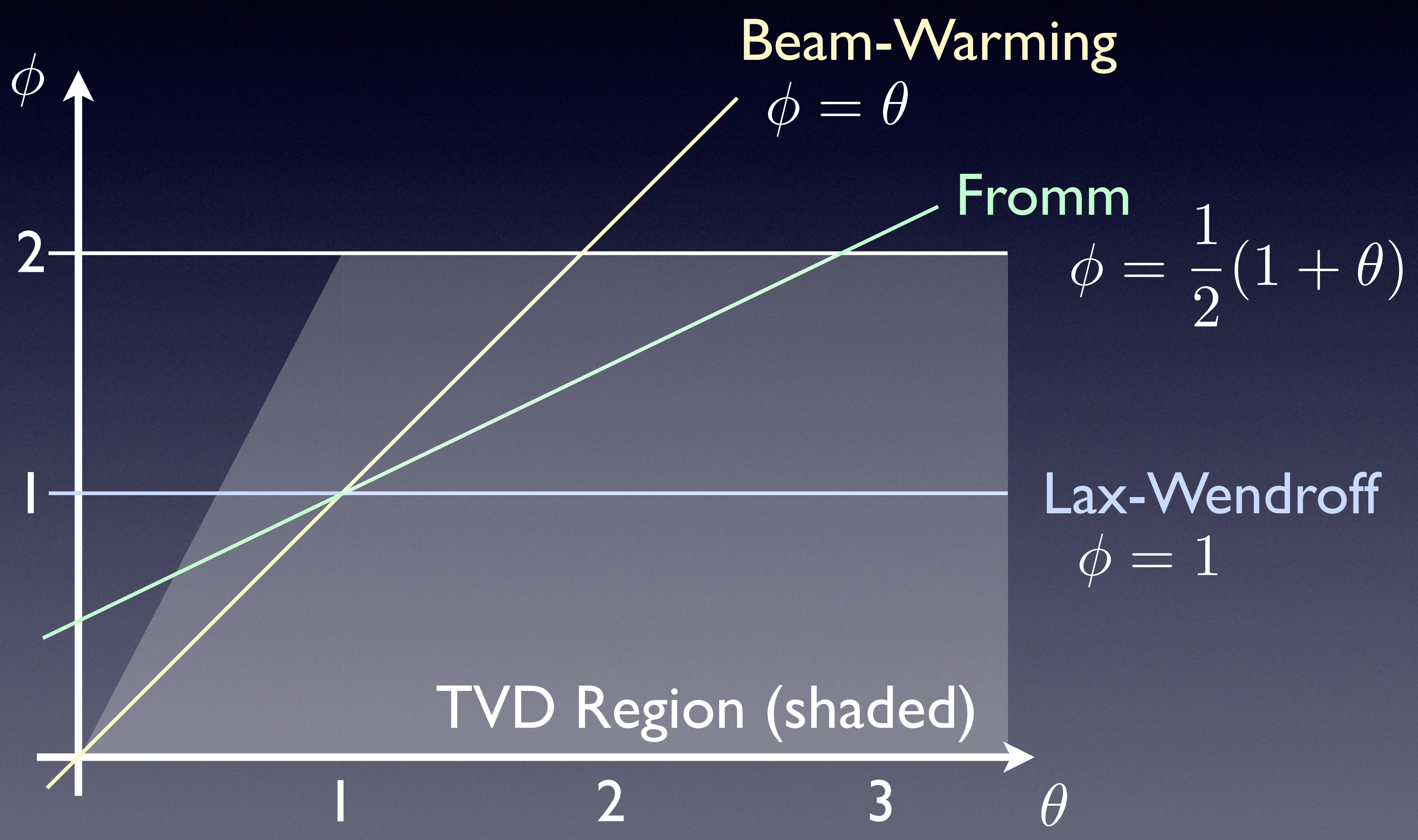
$$Q_i^{n+1} = Q_i^n - \frac{c\Delta t}{2\Delta x}(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2 (Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n)$$

$$\phi_{i-1/2} = \frac{1}{2}(1+\theta)$$

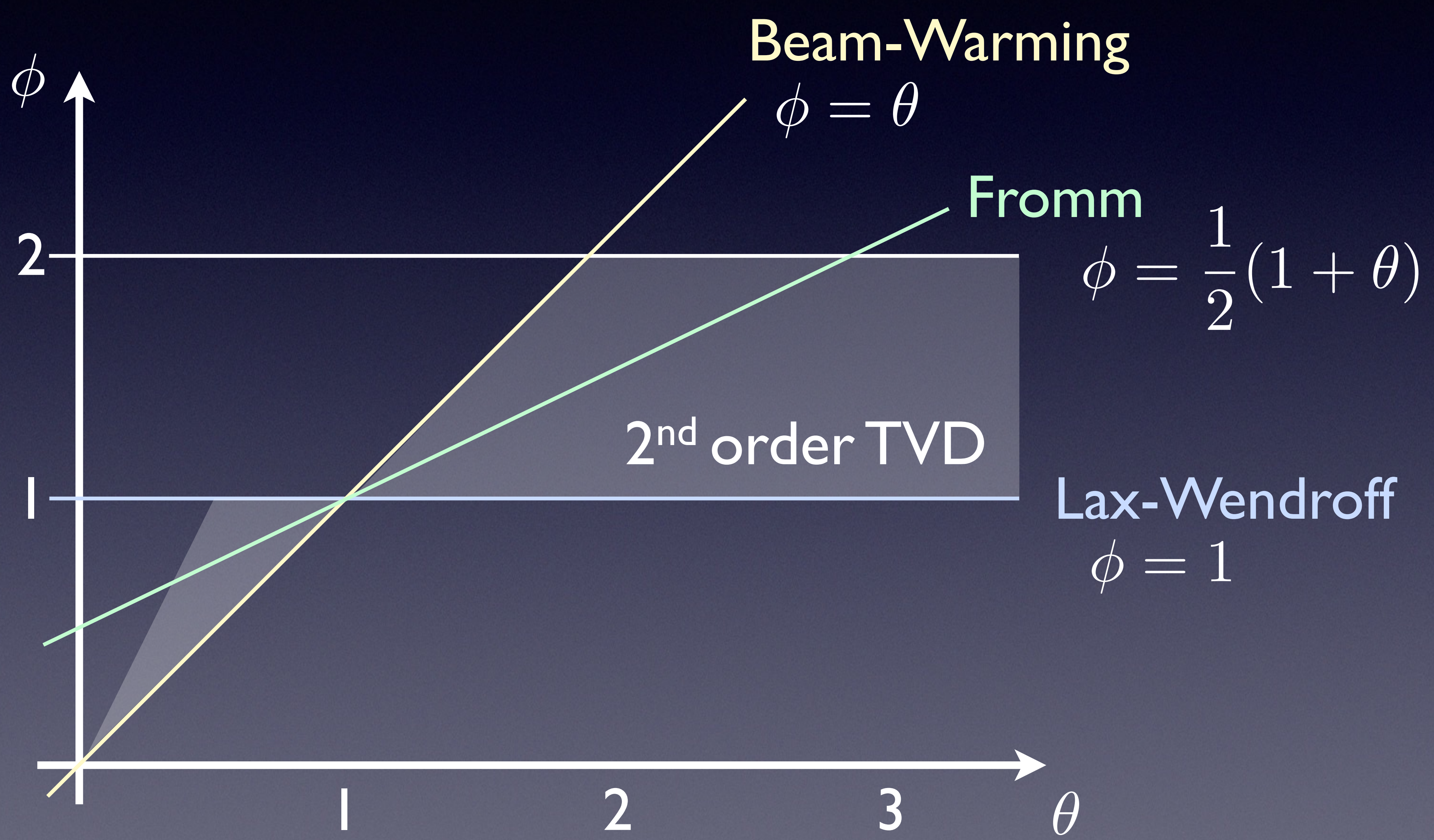
*Fromm Method*

$$Q_i^{n+1} = Q_i^n - \frac{c\Delta t}{4\Delta x}(Q_{i+1}^n + 3Q_i^n - 5Q_{i-1}^n + Q_{i-2}^n) + \frac{1}{4}\left(\frac{c\Delta t}{\Delta x}\right)^2 (Q_{i+1}^n - Q_i^n - Q_{i-1}^n + Q_{i-2}^n)$$

# Plotting Phi vs. Theta

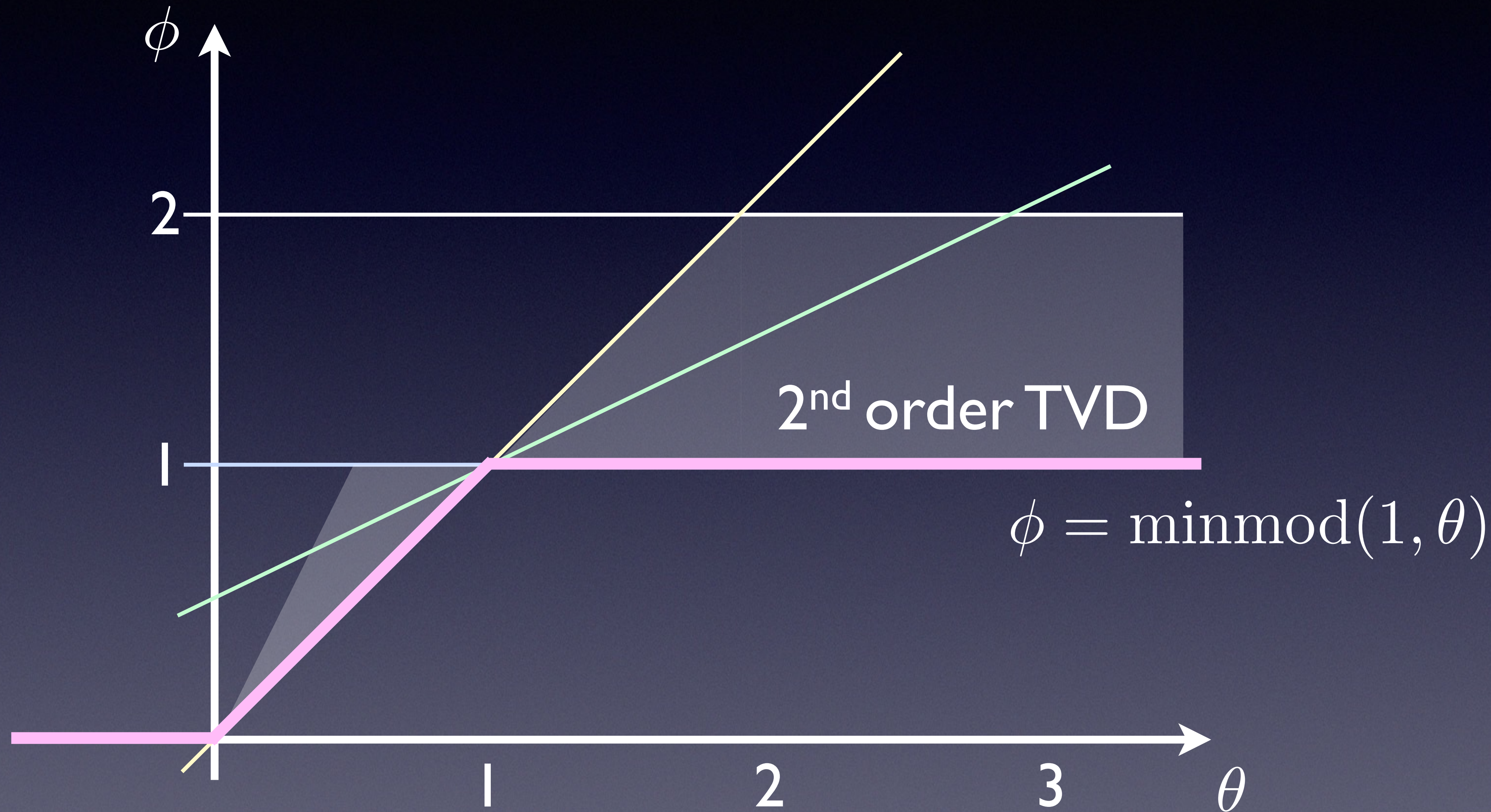


# Plotting Phi vs. Theta



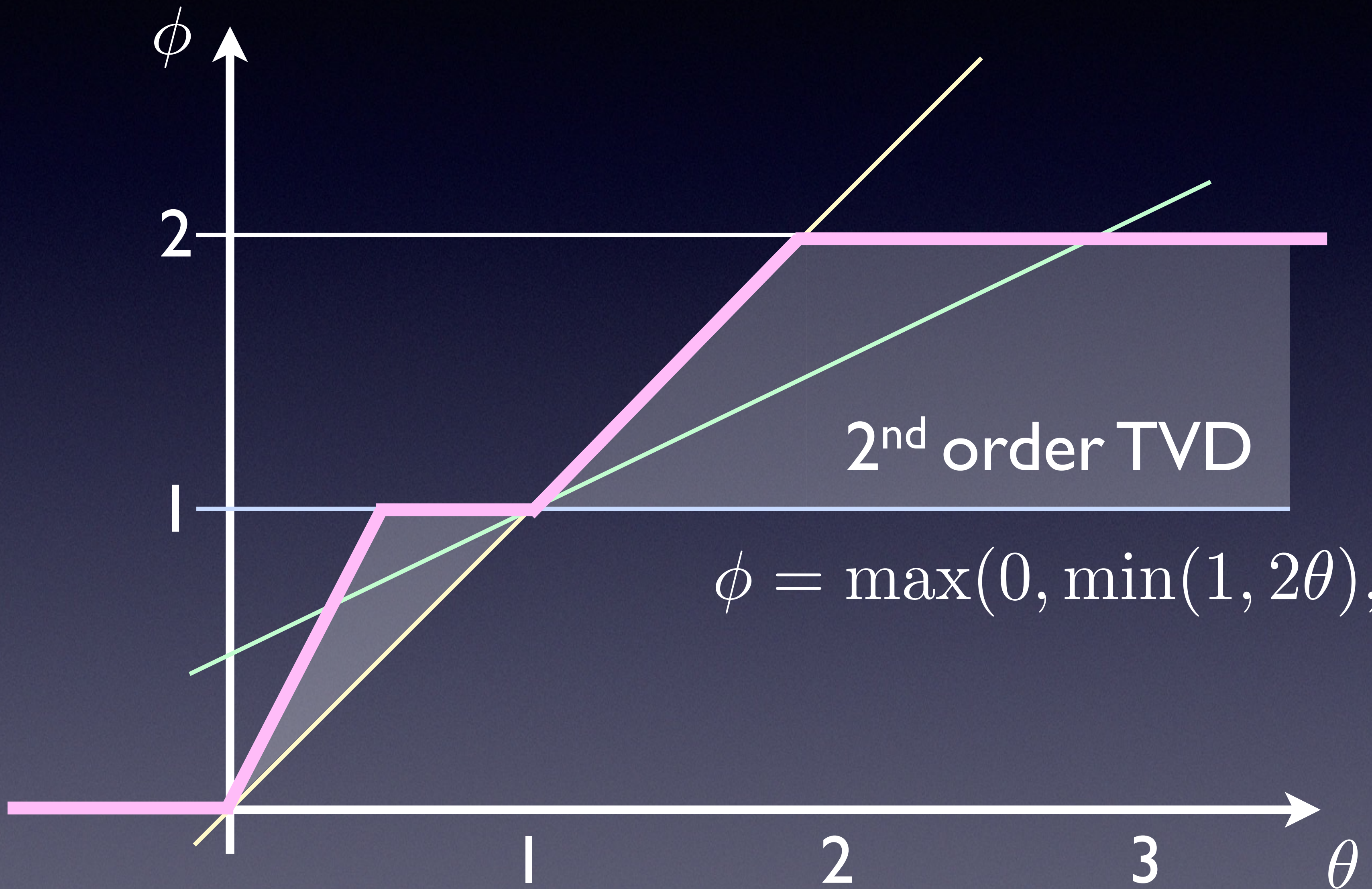

Only the region bounded by these methods is 2nd order!

# Minmod Flux Limiter




Errs on the “safe” side

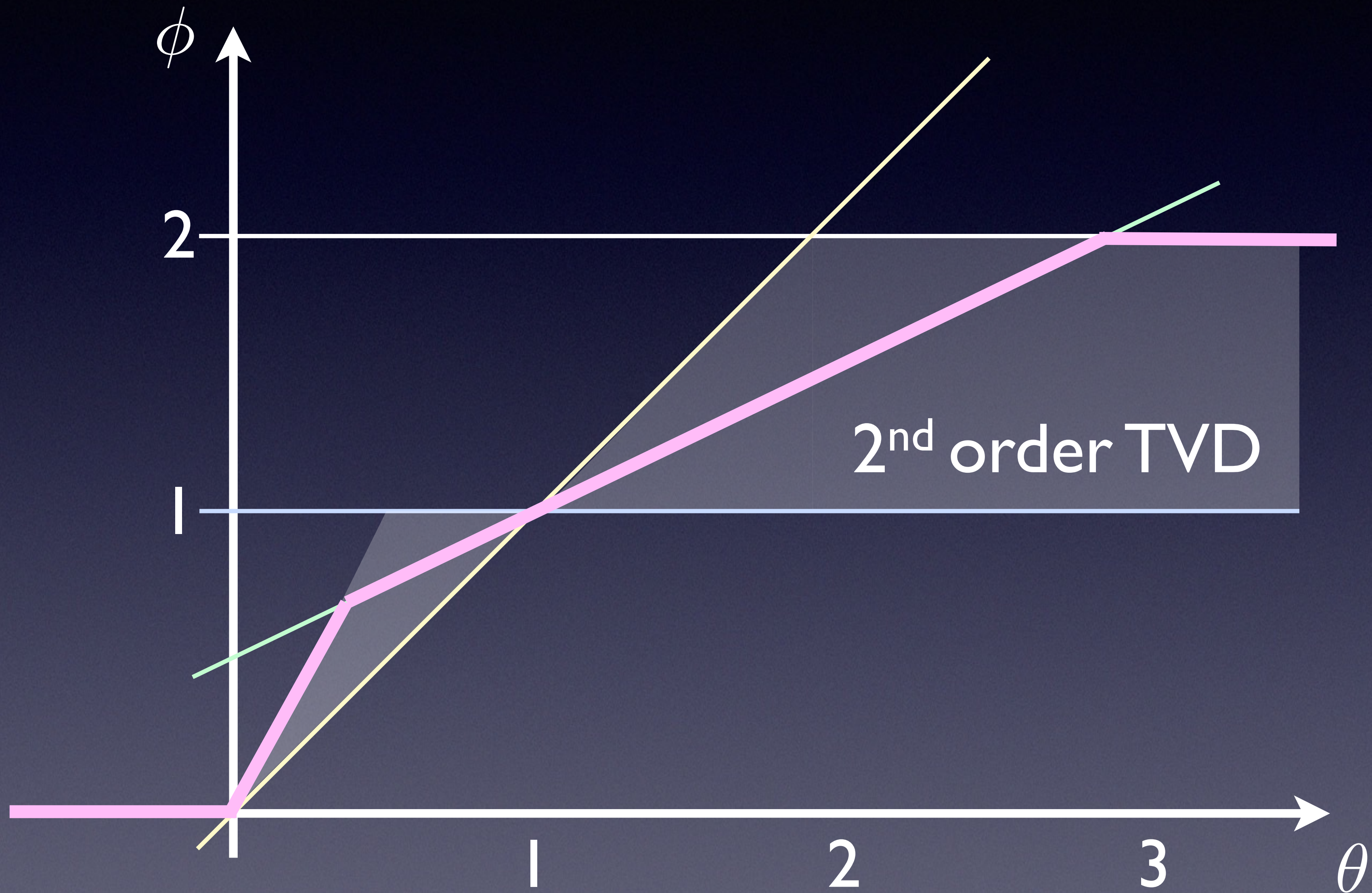
# Superbee Flux Limiter




$$\phi = \max(0, \min(1, 2\theta), \min(2, \theta))$$

Takes the riskier path.

# Monotized Center Flux Limiter




Splits the difference.