

# NOTES ON THE BURGERS EQUATION

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## 1 Shock speed

The *Burgers equation*

$$u_t + uu_x = \nu u_{xx} \quad (1)$$

is a successful, though rather simplified, mathematical model of the motion of a viscous compressible gas [Barenblatt,1996], where

- $u$  = the speed of the gas,
- $\nu$  = the kinematic viscosity,
- $x$  = the spatial coordinate,
- $t$  = the time.

If the viscosity  $\nu = 0$ , or neglected, the Burgers equation becomes

$$u_t + uu_x = 0 \quad (2)$$

Equation (2) is easier to study theoretically and numerically. From now on, unless indicated otherwise, we will refer to Eq. (2) as the Burgers equation.

Equation (2) has a solution in the form of the traveling wave

$$u(x, t) = V(x - x_0 - st),$$

where  $V(y)$  is a step function:

$$V(y) = \left\{ \begin{array}{ll} u_L & y < 0 \\ u_R & y > 0 \end{array} \right\},$$

where  $u_L > u_R$ . This wave is called the *shock wave*.  $s$  is the speed of propagation of the shock wave. It can be obtained from the law of conservation of momentum. Let  $M$  be some large number (Fig. 1). Then the momentum of the gas in the interval  $[-M, M]$  is given by

$$\int_{-M}^M u(x, t) dx.$$

From the Burgers equation the time derivative of the momentum is

$$\frac{d}{dt} \int_{-M}^M u(x, t) dx = \int_{-M}^M -vv_x dx = \frac{u^2}{2} \Big|_{-M}^M = \frac{u_L^2}{2} - \frac{u_R^2}{2}.$$

On the other hand,

$$\int_{-M}^M u(x, t) dx = (M + st)u_L + (M - st)u_R.$$

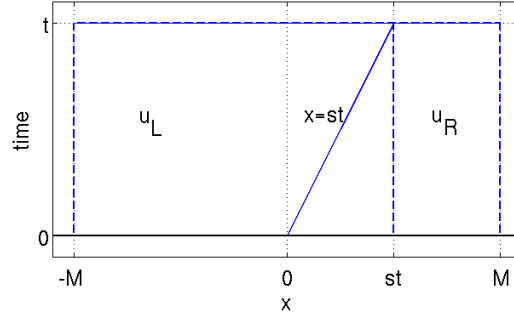


Figure 1: Finding the shock speed using the law of conservation of momentum.

Therefore,

$$\frac{d}{dt} \int_{-M}^M u(x, t) dx = s(u_L - u_R).$$

Hence, the speed of propagation of the wave is

$$s = \left( \frac{u_L^2}{2} - \frac{u_R^2}{2} \right) / (u_L - u_R) = \frac{u_L + u_R}{2}. \quad (3)$$

**Remark** The argument above is valid for a more general equation of the form

$$u_t + [f(u)]_x = 0. \quad (4)$$

Such equations are called *hyperbolic conservation laws*. The shock speed is given by

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{\text{jump in } f(u)}{\text{jump in } u}. \quad (5)$$

This equation is called the *Rankine-Hugoriot condition*.

## 2 Characteristics of the Burgers equation

The characteristics of Eq. (2) are given by

$$\frac{dx}{dt} = u(x, t). \quad (6)$$

Let us show that  $u$  is constant along the characteristics. Let  $(x(t), t)$  be a characteristic. Then

$$\frac{d}{dt} u(x(t), t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} = u_t + uu_x = 0.$$

Therefore, the solution of Eq. (6) is given by

$$x(t) = u(x(0), 0)t + x(0) = u_0(x_0)t + x_0, \quad \text{where } x_0 = x(0), \quad u_0(x) = u(x, 0). \quad (7)$$

Eq. (7) shows that

- the characteristics are straight lines,
- they may intersect,
- they do not necessarily cover the entire  $(x, t)$  space.

This is a new phenomenon in comparison with the linear first order equations  $u_t + a(x, t)u_x = 0$ . For a linear first order equation, there is a unique characteristic passing through every point of the  $(x, t)$  space. Thus, its characteristics never intersect and cover the entire space.

Moreover, even for a smooth initial speed distribution  $u_0(x)$  the solution of the Burgers equation may become discontinuous in a finite time. This happens when the characteristics first intersect, i.e., the wave breaks. Let us find the break time. Consider two characteristics  $x(t) = u_0(x_1)t + x_1$  and  $x(t) = u_0(x_2)t + x_2$ . Then we equate

$$x(t) = u_0(x_1)t + x_1 = u_0(x_2)t + x_2.$$

Then the time at which they intersect is

$$t = -\frac{x_2 - x_1}{u_0(x_2) - u_0(x_1)}.$$

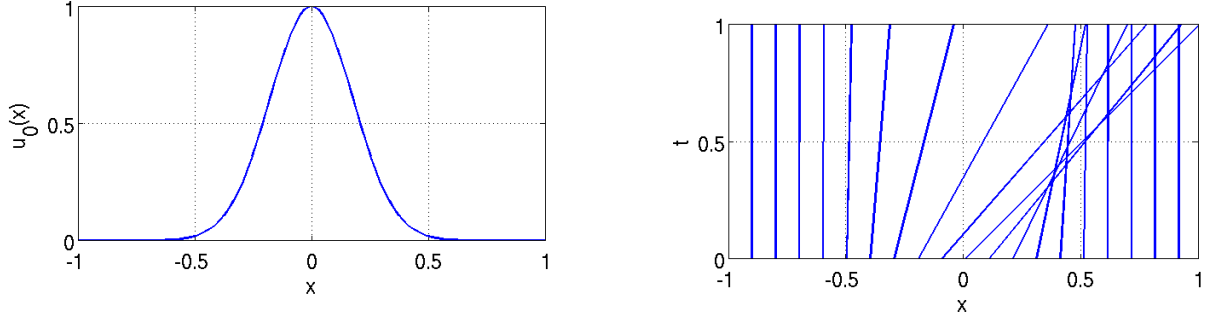
Therefore, the break time is

$$\begin{aligned} T_b &= \min_{x_1, x_2 \in \mathbb{R}} \left( -\frac{x_2 - x_1}{u_0(x_2) - u_0(x_1)} \right) = -\frac{1}{\min_{x_1, x_2 \in \mathbb{R}} \left( -\frac{u_0(x_2) - u_0(x_1)}{x_2 - x_1} \right)} \\ &= -\frac{1}{\min_{x_1, x_2 \in \mathbb{R}} \left( \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} u'_0(x) dx \right)} = -\frac{1}{\min_{x \in \mathbb{R}} u'_0(x)}. \end{aligned}$$

An example is shown in Figures 2 and 3. The initial data  $u_0(x) = \exp(-16x^2)$  and the corresponding characteristics of the Burgers equation are shown in Fig. 2 (a) and (b) respectively. The solution at times  $t = 0.5$  and  $t = 0.8$  obtained by the method of characteristics is shown in Fig. 3 (a) and (c). The numerical solution computed by Godunov's method (see Section 6) is shown in Fig. 3 (b) and (d). The solution obtained by the method of characteristics is triple-valued at some values of  $x$  and non-physical in the sense that it is not the vanishing viscosity solution (see Section 5). The solution computed numerically tends to the vanishing viscosity solution as we refine the mesh.

### 3 Weak solutions

As we have derived in the previous section, a solution to the Burgers equation can become discontinuous even if the initial data are smooth. Then the discontinuity travels with a certain speed,



(a)

(b)

Figure 2: (a) The initial data  $u_0(x) = \exp(-16x^2)$ . (b) The corresponding characteristics of the Burgers equation.

the shock speed  $s$ , given by Eq. (3). In Section 1 we found  $s$  from the law of conservation of momentum. However, at this point we can only call the step function solution considered in Section 1 a solution of the integral rather than the differential Burgers equation. In order to allow discontinuous solutions for differential equation the concept of the weak solutions was introduced. This extension of the concept of solution must satisfy the following requirements:

- a smooth function is a weak solution iff it is a regular solution,
- a discontinuous function can be a weak solution,
- only those discontinuous functions which satisfy the associated integral equation can be weak solutions.

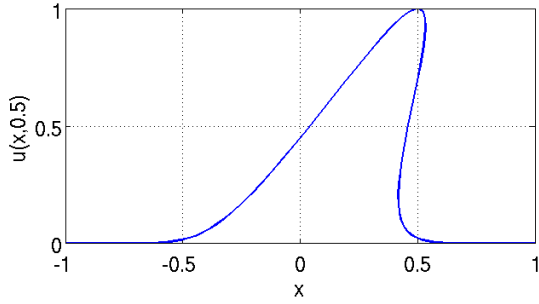
**Motivation.** Let  $\phi(x, t)$  be an infinitely smooth function with a compact support, i.e., it is different from zero only within some compact subset of the space  $(x, t) = \mathbb{R} \times [0, +\infty)$ . Let  $u(x, t)$  be a smooth solution of a hyperbolic conservation law given by Eq. (4). Then

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + [f(u)]_x) \phi dx dt \\ &= \left( \int_{-\infty}^\infty \phi u dx \right) \Big|_0^\infty + \left( \int_0^\infty \phi f(u) dt \right) \Big|_{-\infty}^\infty - \int_0^\infty \int_{-\infty}^\infty \phi_t u + \phi_x f(u) dx dt. \end{aligned} \quad (8)$$

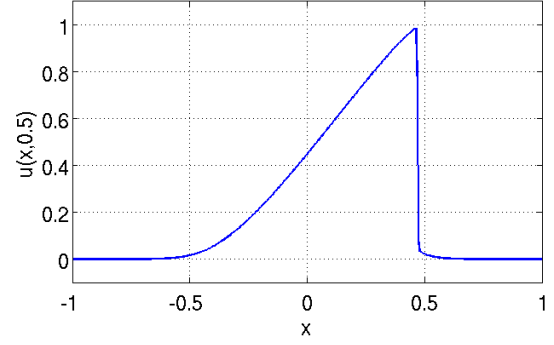
All boundary terms except for the first one evaluated at  $t = 0$  vanish due to the fact that  $\phi(x, t)$  has a compact support. Therefore, here is the definition.

**Definition 1.**  $u(x, t)$  is a weak solution of the conservation law  $u_t + [f(u)]_x = 0$  if for any infinitely differentiable function  $\phi(x, t)$  with a compact support

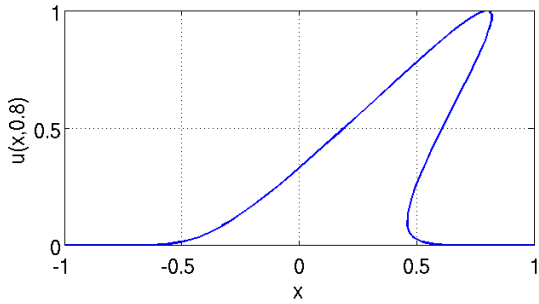
$$\int_0^\infty \int_{-\infty}^\infty \phi_t u + \phi_x f(u) dx dt + \left( \int_{-\infty}^\infty \phi u dx \right) \Big|_{t=0} = 0. \quad (9)$$



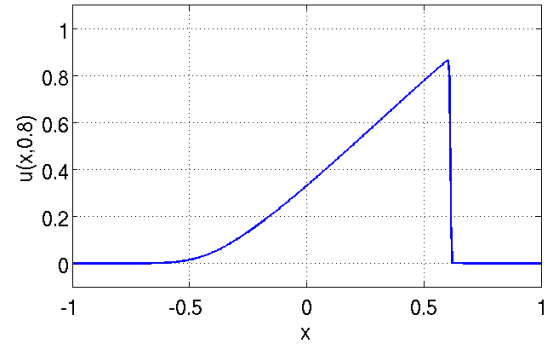
(a)



(b)



(c)



(d)

Figure 3: (a-b) The solution at time  $t = 0.5$  obtained by the method of characteristics (a) and computed numerically (by Godunov's method) (b). (c-d) The solution at time  $t = 0.8$  obtained by the method of characteristics (c) and computed numerically (by Godunov's method) (d).

Such a function  $\phi(x, t)$  is called a *test function*.

## 4 Riemann problem

In this section we consider the following initial value problem for the Burgers equation:

$$u(x, 0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}.$$

This problem is called the *Riemann problem*. We will consider two cases.

**Case 1:**  $u_L > u_R$ . In this case the characteristics cover the entire  $(x, t)$  space but also cross. Hence the construction of the solution using characteristics only is ambiguous. Let us show that in this case there exists a unique weak solution given by

$$u(x, t) = \begin{cases} u_L & x < st \\ u_R & x > st \end{cases}, \quad \text{where } s = \frac{u_L + u_R}{2}. \quad (10)$$

The characteristics for this solution are shown in Fig. 4(a).

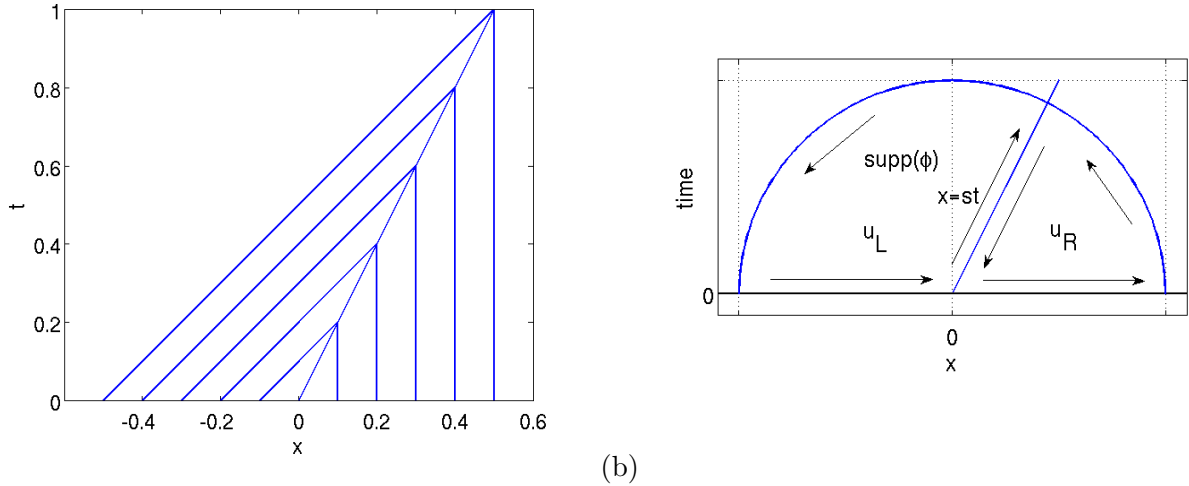


Figure 4: (a) The characteristics for the shock wave. (b) Illustration for the proof that the shock wave is the unique weak solution.

*Proof.* Let  $\phi(x, t)$  be a test function. First suppose that support  $U$  lies entirely in one of the sets  $\{x < st\}$  or  $\{x > st\}$ . Then since  $u(x, t)$  is constant in each of these sets, it satisfies the Burgers equation on the support of  $\phi$ . Then using Eq. (8) we conclude that Eq. (9) holds.

Now suppose that the support  $U$  of  $\phi$  is divided by the line  $x = st$  into two sets  $U_L$  and  $U_R$  (Fig. 4(b)). Then we have

$$\int_0^\infty \int_{-\infty}^\infty \phi_t u + \phi_x \frac{u^2}{2} dx = \int \int_{U_L} \phi_t u + \phi_x \frac{u^2}{2} dx dt + \int \int_{U_R} \phi_t u + \phi_x \frac{u^2}{2} dx dt$$

Applying the Green identity

$$\int \int_D P_x - Q_t dx dt = \int_{\partial D} P dt + Q dx \quad (11)$$

we continue

$$\begin{aligned} &= \int_{\partial U_L} \phi \left( \frac{u_L^2}{2} dt - u_L dx \right) + \int_{\partial U_R} \phi \left( \frac{u_R^2}{2} dt - u_R dx \right) \\ &= \int_{x=st} \phi \left( \frac{u_L^2}{2s} - u_L \right) dx - \int_{x=st} \phi \left( \frac{u_R^2}{2s} - u_R \right) dx - \int_{-\infty}^0 \phi u_L dx - \int_0^\infty \phi u_R dx \\ &= \int_{x=st} \left( \frac{u_L^2 - u_R^2}{2s} - (u_L - u_R) \right) \phi dx - \int_{-\infty}^\infty \phi(x, 0) u(x, 0) dx \end{aligned}$$

The first integral in the last equality is zero for any test function  $\phi$  iff  $s = (u_L + u_R)/2$ . Thus, the solution given by Eq. (10) is the unique weak solution for the Riemann problem in the case  $u_L > u_R$ .  $\square$

In the next section we will show that this solution is the *vanishing viscosity solution*, i.e. the limit of the solutions of Eq. (1) as  $\nu \rightarrow 0$ .

**Case 2:**  $u_L < u_R$ . Then the characteristics do not cross but do not cover the entire space  $(x, t)$ . There are many weak solutions. Two of them are shown in Fig. 5 The one in Fig. (a) is called *rarefaction wave* and given by

$$u(x, t) = \left\{ \begin{array}{ll} u_l, & x < u_l t \\ x/t, & u_l t \leq x \leq u_R t \\ u_R, & x > u_R t \end{array} \right\}. \quad (12)$$

Another solution, shown in Fig. (b) is called the *rarefaction shock*. Despite there are many weak solutions, only one of them is "physical", i.e., the vanishing viscosity solution. It is the rarefaction wave solution. One can reject all of the nonphysical weak solutions by analyzing Eq. (1). However, the analysis of the equation with nonzero viscosity is harder than the analysis of the one with zero viscosity. Then additional simpler-to-verify conditions, so called *entropy conditions* were introduced to eliminate nonphysical weak solutions. There is a number of variations of the entropy condition, We will mention only the simplest one.

**Definition 2.** A discontinuity propagating with speed  $s$  given by Eq. (3) satisfies entropy condition if  $f'(u_L) > s > f'(u_R)$ .

For the Burgers equation this entropy condition reduces to the requirement that if a discontinuity is propagating with speed  $s$  then  $u_L > u_R$ .

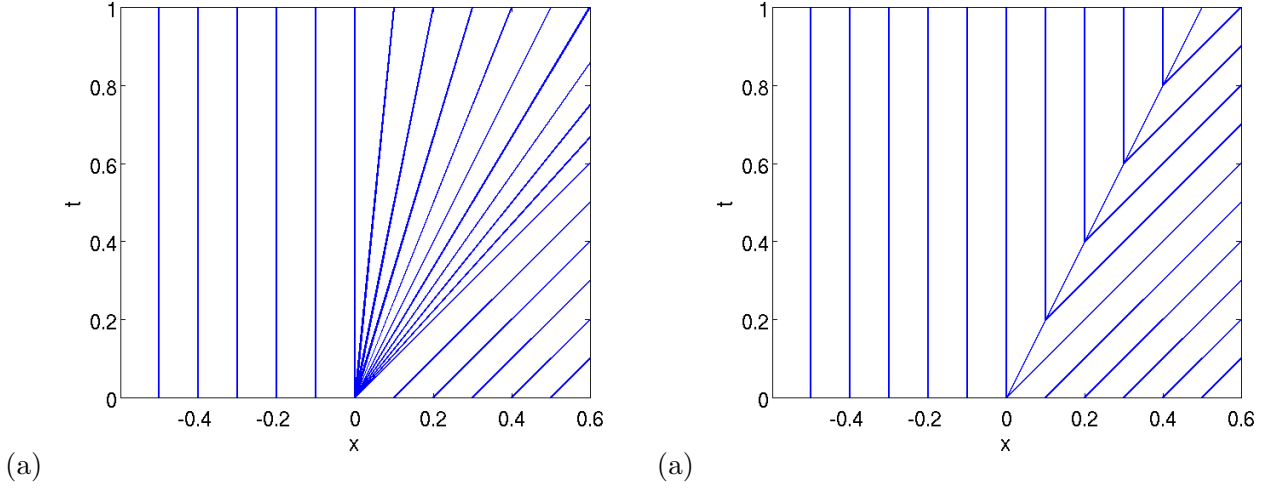


Figure 5: The characteristics for the rarefaction wave (a) and the rarefaction shock (b). Both of these are weak solutions. However, the rarefaction wave is the physical vanishing viscosity solution, while the rarefaction shock is not.

## 5 Solution of the Burgers equation with nonzero viscosity

Now we solve Eq. (1). We will look for a solution of the propagating wave type, i.e.,  $u(x, t) = w(x - st) \equiv w(y)$ . Then  $u_t = -sw'$ ,  $u_x = w'$ , and  $u_{xx} = w''$ . Plugging this into Eq. (1) we obtain

$$\begin{aligned}
 -sw' + ww' &= \nu w'' \\
 -sw' + \left(\frac{w^2}{2}\right)' &= \nu w'' \\
 -sw + \frac{w^2}{2} &= \nu w' + C.
 \end{aligned}$$

We also impose conditions at the  $\pm\infty$ :  $w(-\infty) = u_L$ ,  $w(\infty) = u_R$ , where  $u_L > u_R$ , and  $w'(\pm\infty) = 0$ . Then we have

$$-su_L + \frac{u_L^2}{2} = C = -su_R + \frac{u_R^2}{2}.$$



Therefore,  $s$  must be  $(u_L + u_R)/2$ . Thus, the shock speed is the same as in the case of zero viscosity. Hence  $C = -u_L u_R/2$ . Then we continue.

$$\begin{aligned}\nu w' &= \frac{w^2}{2} - \frac{u_L + u_R}{2}w + \frac{u_L u_R}{2} \\ \frac{dy}{2\nu} &= \frac{dw}{w^2 - (u_L + u_R)w + u_L u_R} \\ \frac{dy}{2\nu} &= \frac{dw}{w^2 - (u_L + u_R)w + \frac{(u_L + u_R)^2}{4} - \frac{(u_L - u_R)^2}{4}} \\ \frac{dy}{2\nu} &= \frac{dw}{\left(w - \frac{u_L + u_R}{2}\right)^2 - \frac{(u_L - u_R)^2}{4}}\end{aligned}$$

Integrating the both parts using

$$\int \frac{dw}{(w - a)^2 - b^2} = \frac{1}{2b} \log \left| \frac{w - a - b}{w - a + b} \right|$$

we obtain

$$\frac{y}{2\nu} + C = \frac{1}{u_L - u_R} \log \left| \frac{w - \frac{u_L + u_R}{2} - \frac{u_L - u_R}{2}}{w - \frac{u_L + u_R}{2} + \frac{u_L - u_R}{2}} \right| = \frac{1}{u_L - u_R} \log \left| \frac{w - u_L}{w - u_R} \right| = \frac{1}{u_L - u_R} \log \frac{u_L - w}{w - u_R}.$$

In the last equality we used the fact that  $u_L > w > u_R$ . Hence,

$$\begin{aligned}\frac{u_L - w}{w - u_R} &= e^{\frac{y(u_L - u_R)}{2\nu} + C} \\ u_L - w &= w e^A - u_R e^A \\ w(e^A + 1) &= (u_L + u_R e^A) \\ w &= \frac{u_L + u_R e^A}{e^A + 1} = u_R + \frac{u_L - u_R}{2} \frac{2}{e^A + 1},\end{aligned}$$

where  $A = y(u_L - u_R)/(2\nu) + C$ . Multiplying and dividing by  $\exp(-A/2)$  and using the identity

$$\frac{2e^{-A/2}}{e^{A/2} + e^{-A/2}} = 1 - \frac{e^{A/2} - e^{-A/2}}{e^{A/2} + e^{-A/2}} = 1 - \tanh \frac{A}{2}$$

we get

$$w(y) = u_R + \frac{u_L - u_R}{2} \tanh \left( \frac{y(u_L - u_R)}{4\nu} + C \right).$$

Hence,

$$u(x, t) = u_R + \frac{u_L - u_R}{2} \tanh \left( \frac{(x - x_0 - st)(u_L - u_R)}{4\nu} \right). \quad (13)$$

The profiles  $w(y)$  for various values of  $\nu$  are shown in Fig. 6. As  $\nu \rightarrow 0$ ,  $u(x, t)$  tends to the step function for every  $t$ , which is the unique weak solution of the Burgers equation given by Eq. (10). Thus, the solution given by Eq. (10) is the vanishing viscosity solution.

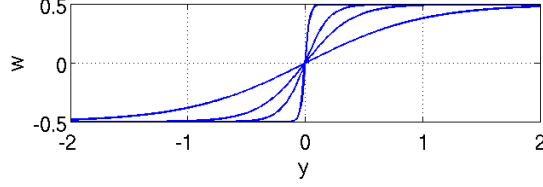


Figure 6: The profiles of the solution of the viscous Burgers equation for  $\nu$  equal to 0.25, 0.1, 0.05, and 0.01 (the steepest one). We see that  $w(y)$  tends to the step function as  $\nu \rightarrow 0$ .

## 6 Numerical solution of the Burgers equation

Numerical solution of the Burgers equation was a challenging problem. Numerical schemes must satisfy certain non-obvious conditions in order to propagate the discontinuity with the right speed [LeVeque,1992]. In this section we consider two methods for solving the Burgers equation, Godunov's and Glimm's methods.

### 6.1 Godunov's method (1959)

The idea of Godunov's method is the following. Let  $U_j^n$  be a numerical solution on the  $n$ -th layer. Then we define a function  $\tilde{u}^n(x, t)$  for  $t_n \leq t \leq t_{n+1}$  as follows. At  $t = t_n$ ,

$$\tilde{u}^n(x, t_n) = U_j^n, \quad x_j - \Delta x/2 < x < x_j + \Delta x/2, \quad j = 2, \dots, n-1.$$

Then  $\tilde{u}(x, t)$  is the solution of the collection of the Riemann problems on the interval  $[t_n, t_{n+1}]$ . If  $\Delta t$  is small enough so that the characteristics starting at the points  $x_j + \Delta x/2$  do not intersect within this interval (i.e., the CFL condition is satisfied), then  $\tilde{u}(x, t_{n+1})$  is determined unambiguously. Then the numerical solution on the next layer,  $U_j^{n+1}$  is defined by averaging  $\tilde{u}(x, t_{n+1})$  over the intervals  $x_j - \Delta x/2 < x < x_j + \Delta x/2$ . This idea reduces to a very simple numerical procedure given by

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} (F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)).$$

The function  $F(U_L, U_R)$  is the numerical flux defined by  $F(U_L, U_R) = \frac{(u^*)^2}{2}$  where  $u^*$  is defined as follows.

If  $U_L \geq U_R$

$$u^* = \begin{cases} U_L, & (U_L + U_R)/2 > 0 \\ U_R, & (U_L + U_R)/2 \leq 0 \end{cases},$$

If  $U_L < U_R$

$$u^* = \begin{cases} U_L, & U_L > 0 \\ U_R, & U_R < 0 \\ 0, & U_L \leq 0 \leq U_R \end{cases}.$$

Godunov's method adds some artificial smearing to the solution.

## 6.2 Glimm's method

Like Godunov's method, Glimm's method is also based on solving a collection of Riemann problems on the interval  $[t_n, t_{n+1}]$ . But instead of averaging procedure for getting the numerical solution on the next layer a random choice procedure is used. Glimm's time step proceeds in two stages

$$\begin{aligned} U_{j+1/2}^{n+1/2} &= \xi(U_j^n, U_{j+1}^n), \\ U_j^{n+1} &= \xi(U_{j-1/2}^{n+1/2}, U_{j+1/2}^{n+1/2}), \end{aligned}$$

where  $\xi(U_L, U_R)$  is a random variable taking values either  $U_L$  or  $U_R$  with probabilities proportional to lengths of the corresponding intervals ( Fig. 7 ):

$$\xi(U_L, U_R) = \begin{cases} U_L, & p = \frac{\Delta x/2 + s\Delta t/2}{\Delta x} \\ U_R, & p = \frac{\Delta x/2 - s\Delta t/2}{\Delta x} \end{cases},$$

where  $s = (U_L + U_R)/2$ . We can rewrite the definition of  $\xi$  as follows.

$$\xi(U_L, U_R) = \begin{cases} U_L, & p = \frac{1}{2}(1 + s\lambda) \\ U_R, & p = \frac{1}{2}(1 - s\lambda) \end{cases},$$

where  $\lambda = \frac{\Delta t}{\Delta x}$ .

## References

- [LeVeque,1992] Randall J. LeVeque, Numerical methods for conservation laws, Basel; Boston; Berlin: Birkhäuser, 1992
- [Barenblatt,1996] G. I. Barenblatt, Scaling, self-similarity, and intermediate asymptotics, Cambridge University press, 1996

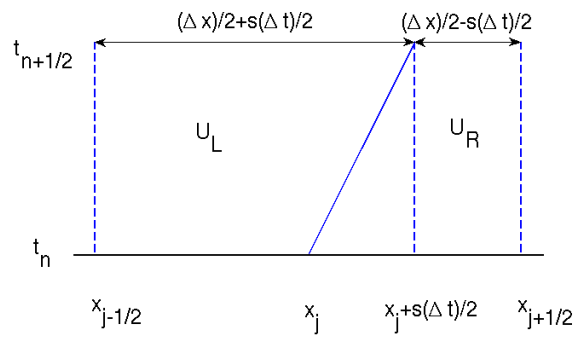


Figure 7: Illustration for the Glimm's random choice method.