高数表1

极限 求导

$$a^n-b^n=\frac{a-b}{\sum_{i=0}^{n-1}a^ib^{(n-1)-i}}=\frac{a-b}{a^{n-1}+a^{n-2}b+\cdots+b^{n-1}}$$

$$|\vec{v}|-|\vec{u}|\leq\Big||\vec{v}|-|\vec{u}|\Big|\leq|\vec{v}\pm\vec{u}|\leq|\vec{v}|+|\vec{u}|$$

$$|\alpha|-|\beta|\leq\Big||\alpha|-|\beta|\Big|\leq|\alpha\pm\beta|\leq|\alpha|+|\beta|$$

$$\lim_{x\to x_0^+}f(x)=f(x_0^+)$$

$$\lim_{x\to x_0^-}f(x)=f(x_0^-)$$

$$\lim_{x\to x_0^+}f'(x)=f_+(x_0)$$

$$\lim_{x\to x_0^-}f'(x)=f_-(x_0)$$

$$\mathrm{tg}\,\alpha=\tan\alpha$$

$$\tan^2\alpha=\sec^2\alpha+1$$

$$\cot^2\alpha=\csc^2\alpha+1$$

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\sin\beta\cos\alpha$$

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$$\cot\alpha=\frac{1}{\tan\alpha}$$

$$\sec\alpha=\frac{1}{\cos\alpha}$$

$$\csc\alpha = \frac{1}{\sin\alpha}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2}{\cot\alpha - \tan\alpha}$$

$$a\sin\alpha + b\cos\alpha = \sqrt{a^2+b^2}\sin(\alpha + \arctan\tfrac{b}{a})$$

$$y = \arcsin x = \sin^{-1} x \Rightarrow x = \sin y$$

$$y = \arccos x = \cos^{-1} x \Rightarrow x = \cos y$$

$$y = \arctan x = \tan^{-1} x \Rightarrow x = \tan y$$

$$\sin\alpha + \sin\beta = 2\sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = -\tfrac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

$$\sinh x = \mathrm{sh}\, x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \operatorname{th} x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$$

$$\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$$

$$\operatorname{arctanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\tanh 2x = \frac{2}{\tanh x + \coth x} = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\sinh x \cosh y = \tfrac{1}{2}[\sinh(x+y) + \sinh(x-y)]$$

$$\cosh x \sinh y = \tfrac{1}{2}[\sinh(x+y) - \sinh(x-y)]$$

$$\cosh x \cosh y = \tfrac{1}{2}[\cosh(x+y) + \cosh(x-y)]$$

$$\sinh x \sinh y = \tfrac{1}{2}[\cosh(x+y) - \cosh(x-y)]$$

$$\sinh x + \sinh y = 2 \sinh \tfrac{x+y}{2} \cosh \tfrac{x-y}{2}$$

$$\sinh x - \sinh y = 2 \cosh \tfrac{x+y}{2} \sinh \tfrac{x-y}{2}$$

$$\cosh x + \cosh y = 2 \cosh \tfrac{x+y}{2} \cosh \tfrac{x-y}{2}$$

$\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$

数列极限存在准则：

1. 夹逼定理：

若 $n>k$ 时 $c_n \le a_n \le b_n, \lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} b_n = a$

则 $\lim\limits_{n\to\infty} a_n = a$

2. 单调有解收敛准则：单调有界数列一定收敛

柯西归并原理证明极限不存在：

若存在不完全相同的数列 $\{a_n\},\{b_n\}$，满足 $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$ ，且使得数列 $\{c_n\}$/函数 $f(x)$ 出现

$\lim\limits_{n\to\infty} c_{a_n} \ne \lim\limits_{n\to\infty} c_{b_n}$ / $\lim\limits_{n\to\infty} f(a_n) \ne \lim\limits_{n\to\infty} f(b_n)$

则数列 $\{c_n\}$ / 函数 $f(x)$ 极限不存在

若在相同变化趋势下，存在

$\lim f(x) = A, \lim g(x) = B$

则

$\lim[f(x) \pm g(x)] = A \pm B$

$\lim f(x)\cdot g(x) = A\cdot B$

$\lim\frac{f(x)}{g(x)} = \frac{A}{B}(B\ne 0)$

$\lim cf(x) = cA$

若 $\lim\limits_{u\to u_0} f(u) = A$ 和 $\lim\limits_{x\to x_0}\varphi(x) = u_0$ 存在

则对 $y=f(u), u=\varphi(x)$ 有

$$\lim_{x\to x_0} f[\varphi(x)] = A$$

$$\lim_{\alpha(x)\to 0} \frac{\sin\alpha(x)}{\alpha(x)} = 1$$

> $$\lim_{\alpha(x)\to 0} \frac{\tan\alpha(x)}{\alpha(x)} = 1$$

> $$\lim_{\alpha(x)\to 0} \frac{1-\cos\alpha(x)}{\frac{1}{2}\alpha^2(x)} = 1$$

> $$\lim_{\alpha(x)\to 0} \frac{\tan\alpha(x)-\sin\alpha(x)}{\alpha^3(x)} = \frac{1}{2}$$

$$\lim_{\alpha(x)\to \infty} [1+\frac{1}{\alpha(x)}]^{\alpha(x)} = e$$

> $$\lim_{\alpha(x)\to 0} [1+\alpha(x)]^{\frac{1}{\alpha(x)}} = e$$

> $$\lim_{\alpha(x)\to 0} \frac{\ln[1+\alpha(x)]}{\alpha(x)} = \lim_{\alpha(x)\to 0} \ln[1+\alpha(x)]^{\frac{1}{\alpha(x)}} = \ln e = 1$$

> $$\lim_{\alpha(x)\to 0} \frac{e^{\alpha(x)}-1}{\alpha(x)} = \lim_{\alpha(x)\to 0} \left[\frac{\ln\left(1+[e^{\alpha(x)}-1]\right)}{e^{\alpha(x)}-1}\right]^{-1} = 1^{-1} = 1$$

在同一趋势下

$\lim \frac{\alpha(x)}{\beta(x)} = 0 \Rightarrow \alpha = o(\beta)$

> $\alpha(x)$ 是 $\beta(x)$ 的高阶无穷小

$\lim \frac{\alpha(x)}{\beta(x)} = \infty \Rightarrow \beta = o(\alpha)$

> $\alpha(x)$ 是 $\beta(x)$ 的低阶无穷小

$\lim \frac{\alpha(x)}{\beta(x)} = C(C \neq 0) \Rightarrow \alpha = O(\beta)$

> $\alpha(x)$ 是 $\beta(x)$ 的同阶无穷小

$\lim \frac{\alpha(x)}{\beta(x)} = 1 \Rightarrow \alpha \sim \beta$

> $\alpha(x)$ 是 $\beta(x)$ 的等价无穷小

$\lim \frac{\alpha(x)}{\beta^k(x)} = C(C \neq 0) \Rightarrow \alpha = O(\beta^k)$

> $\alpha(x)$ 是 $\beta(x)$ 的 k 阶无穷小

若在自变量 x 的某一过程中， $\alpha(x)$ 为非零无穷小量

$\alpha(x) \sim \sin\alpha(x) \sim \ln[1+\alpha(x)] \sim [e^{\alpha(x)} - 1]$

> $\alpha(x) \sim \sin\alpha(x) \sim \tan\alpha(x)$

> $\alpha(x) \sim \sin\alpha(x) \sim \arcsin\alpha(x)$

> $\alpha(x) \sim \tan\alpha(x) \sim \arctan\alpha(x)$

> $\alpha(x) \sim [e^{\alpha(x)} - 1] \Rightarrow [a^{\alpha(x)} - 1] \sim \alpha(x)\ln a$

> $\alpha(x) \sim \ln[1+\alpha(x)] \Rightarrow \log_a[1+\alpha(x)] \sim \frac{\alpha(x)}{\ln a}$

$[1-\cos\alpha(x)] \sim \frac{1}{2}\alpha^2(x)$

$[1+\alpha(x)]^\lambda \sim \lambda\alpha(x)$

函数连续定义：

$\lim\limits_{\Delta x \to 0}[f(x_0+\Delta x) - f(x_0)] = 0$

$\lim\limits_{x \to x_0}[f(x) - f(x_0)] = 0$

> $\lim\limits_{x \to x_0} f(x) = f(x_0)$

$f(x_0^+) = f(x_0^-) = f(x_0)$

$f(x_0^+) = f(x_0)$ 即在 $x = x_0$ 处函数右连续

$f(x_0^-) = f(x_0)$ 即在 $x = x_0$ 处函数左连续

对 $\forall x \in (a, b)$, $f(x)$ 连续，即 $f(x)$ 在 (a, b) 上连续

> $f(x) \in C(a, b)$

$f(x) \in C(a, b)$ 且 $f(x)$ 在 $x = a$ 右连续，即 $f(x)$ 在 $[a, b)$ 连续

> $f(x) \in C[a, b)$

同理可表明

> $f(x) \in C(a, b]$

> $f(x) \in C[a, b]$

$C = \bigcup\limits_{a,b \in R} C[a, b]$

最大值最小值定理：

若 $f(x) \in C[a, b]$ 则 $\exists \xi_1, \xi_2 \in [a, b]$ 使得对 $\forall x \in [a, b]$ 都有

$f(\xi_1) \le f(x) \le f(\xi_2)$

零点定理与介质定理：

若 $f(x) \in C[a, b], f(a)f(b) < 0$ 则 $\exists \xi \in (a, b)$ 使得 $f(\xi) = 0$

若 $f(x) \in C[a, b]$ 且 $f(x)$ 有最小值和最大值分别为 m, M 则对 $\forall \mu \in [m, M], \exists \xi \in [a, b]$ 使得 $f(\xi) = \mu$

函数可导定义：

$\lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ 存在

> $\frac{\mathrm{d}f(x)}{\mathrm{d}x}|_{x=x_0} = \frac{\mathrm{d}}{\mathrm{d}x}f(x)|_{x=x_0} = f'(x_0)$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=0$$

$f'_-(x_0)=f'_+(x_0)$

对 $\forall x\in(a,b)$ 都有 $f(x)$ 可导，即 $f(x)$ 在 (a,b) 上可导

> $f(x)\in D(a,b)$

$$D=\bigcup_{a,b\in R}D(a,b)$$

可导一定连续，连续不一定可导

若 $\Delta y=A\Delta x+o(\Delta x)$

则 $\mathrm{d}y=A\Delta x$

由 $f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}=A$

$\therefore \mathrm{d}y=f'(x_0)\Delta x$

$\therefore \Delta y\approx \mathrm{d}y=f'(x_0)\Delta x$

$\therefore y=f(x_0)+\Delta y\approx f(x_0)+f'(x_0)\Delta x$

由对 $y=x$ 上述分析得 $\mathrm{d}x=\Delta x$

对 $\forall x\in R$ 都有 $\mathrm{d}y=f'(x)\mathrm{d}x$

$\mathrm{d}(u+v)=\mathrm{d}u+\mathrm{d}v$

> $$\mathrm{d}(\sum_{i=1}^{n}\alpha_i)=\sum_{i=1}^{n}\mathrm{d}\alpha_i$$

$\mathrm{d}(uv)=u\mathrm{d}v+v\mathrm{d}u$

> $$\mathrm{d}(\prod_{i=1}^{n}\alpha_i)=\sum_{i=1}^{n}\mathrm{d}\alpha_i(\prod_{j\neq i}\alpha_j)$$

$$\mathrm{d}(\frac{1}{v})=-\frac{\mathrm{d}v}{v^2}$$

> $$\mathrm{d}(\frac{u}{v})=\mathrm{d}(u\cdot\frac{1}{v})=\frac{v\mathrm{d}u-u\mathrm{d}v}{v^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}v}\cdots\frac{\mathrm{d}\mu}{\mathrm{d}x}$$

$$\mathrm{d}C=0$$

> $$\mathrm{d}Cu=C\mathrm{d}u$$

$$\mathrm{d}x^\mu=\mu x^{\mu-1}\mathrm{d}x$$

$$\mathrm{d}\sin x=\cos x\mathrm{d}x$$

> $$\mathrm{d}\csc x=\csc x\cot x\mathrm{d}x$$

$$\mathrm{d}\cos x=-\sin x\mathrm{d}x$$

> $$\mathrm{d}\sec x=\sec x\tan x\mathrm{d}x$$

> $$\mathrm{d}\tan x=\sec^2 x\mathrm{d}x$$

> $$\mathrm{d}\cot x=-\csc^2 x\mathrm{d}x$$

$$\mathrm{d}a^x=a^x\ln a\mathrm{d}x$$

> $$\mathrm{d}e^x=e^x\mathrm{d}x$$

$$\mathrm{d}\log_a x=\frac{1}{x\ln a}\mathrm{d}x$$

> $$\mathrm{d}\ln x=\frac{1}{x}\mathrm{d}x$$

> $$\mathrm{d}\ln\ |f(x)|=\frac{f'(x)}{f(x)}\mathrm{d}x$$

$$\mathrm{d}\sinh x=\cosh x\mathrm{d}x$$

$$\mathrm{d}\cosh x=\sinh x\mathrm{d}x$$

> $$\mathrm{d}\tanh x=(1-\tanh^2 x)\mathrm{d}x$$

$$\mathrm{d}\sqrt{a^2-x^2}=-\frac{x}{\sqrt{a^2-x^2}}\mathrm{d}x$$

$\mathrm{d}\sqrt{x^2 \pm a^2} = \frac{x}{\sqrt{x^2 \pm a^2}}\mathrm{d}x$

$\mathrm{d}\ln(x + \sqrt{x^2 \pm a^2}) = \frac{1}{\sqrt{x^2 \pm a^2}}\mathrm{d}x$

> $\mathrm{d}\ \mathrm{arcsinh}\ x = \frac{1}{\sqrt{x^2+1}}\mathrm{d}x$

> $\mathrm{d}\ \mathrm{arccosh}\ x = \frac{1}{\sqrt{x^2-1}}\mathrm{d}x$

$\mathrm{d}\arcsin x = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$\mathrm{d}\arccos x = -\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$\mathrm{d}\arctan x = \frac{1}{1+x^2}\mathrm{d}x$

$\mathrm{d}\ \mathrm{arccot}\ x = -\frac{1}{1+x^2}\mathrm{d}x$

$$\begin{cases} f^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}y = \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}y), n > 1 \\ \frac{\mathrm{d}^0}{\mathrm{d}x^0}y = y \end{cases}$$

$\forall x \in (a,b), f(x)$ 在 (a,b) 上有连续的 n 阶导数，记为

$f \in C^n(a,b)$

> $C^n = \bigcup\limits_{a,b\in R} C^n(a,b)$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^\mu = \begin{cases} \frac{\mu!}{(\mu-n)!}x^{\mu-n}, n \le \mu \\ 0, n > \mu \end{cases}, \mu \ge 0$$

> $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(ax+b)^\mu = \begin{cases} a^n \cdot \frac{\mu!}{(\mu-n)!}(ax+b)^{\mu-n}, n \le \mu \\ 0, n > \mu \end{cases}, \mu \ge 0$$

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\frac{1}{x}) = \frac{(-1)^n n!}{x^{n+1}}$

> $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\frac{1}{ax+b}) = a^n \frac{(-1)^n n!}{(ax+b)^{n+1}}$

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^\mu = (-1)^n \cdot \frac{(-\mu+n-1)!}{(-\mu-1)!}x^{\mu-n}, \mu < 0$

> $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(ab+b)^\mu = (-a)^n \cdot \frac{(-\mu+n-1)!}{(-\mu-1)!}x^{\mu-n}, \mu < 0$

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sin(ax+b) = a^n\sin(ax+b+n\cdot\frac{\pi}{2})$

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\cos(ax+b) = a^n\cos(ax+b+n\cdot\frac{\pi}{2})$

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}a^x = (\ln a)^n \cdot a^x$

$$\begin{aligned}&\frac{\mathrm{d}^n}{\mathrm{d}x^n}\ln(ax+b)\\
=&\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}[\frac{\mathrm{d}}{\mathrm{d}x}\ln(ax+b)]\\
=&\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\frac{1}{ax+b}\times a)\\
=&a^n\frac{(-1)^{n-1}(n-1)!}{(ax+b)^n}\end{aligned}$$

$$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}y}{\frac{\mathrm{d}}{\mathrm{d}t}x} = \frac{\psi'(t)}{\varphi'(t)}$$

$$\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}x^2}y = \frac{\mathrm{d}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}[\frac{\psi'(t)}{\varphi'(t)}]}{\frac{\mathrm{d}}{\mathrm{d}t}x} = \frac{\psi''(t)\varphi'(t)-\varphi''(t)\psi'(t)}{[\varphi'(t)]^3}$$

费马定理：

若函数 $f(x)$ 在点 x_0 的某个邻域 $U(x_0,\delta)$ 内有定义并且在 x_0 处可导，如果对 $\forall x\in U(x_0,\delta)$ 恒有

$f(x)\le f(x_0)$ 或 $f(x)\ge f(x_0)$

则有 $f'(x_0)=0$

罗尔定理、拉格朗日中值定理、柯西中值定理：

罗尔定理：

若 $f(x)\in C[a,b]\bigcap D(a,b), f(a)=f(b)$ 则 $\exists\xi\in(a,b)$ 使得 $f'(\xi)=0$

拉格朗日中值定理：

若 $f(x)\in C[a,b]\bigcap D(a,b)$ 则 $\exists\xi\in(a,b)$ 使得 $f'(\xi)=\frac{f(b)-f(a)}{b-a}$

> $f(b)-f(a)=f'(\xi)(b-a)$

> 有限增量公式：取 $x_0, x_0+\Delta x\in[a,b]$，则在 $x_0, x_0+\Delta x$为端点的区间上，有 $\Delta y=f(x_0+\Delta x)-f(x_0)=f'(x_0+\theta\Delta x)\cdot\Delta x(0<\theta<1)$

柯西中值定理：

若 $f(x), g(x)\in C[a,b]\bigcap D(a,b)$ 且对 $\forall x\in(a,b)$ 都有 $g'(x)\neq 0$ 则 $\exists\xi\in(a,b)$ 使得 $\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}$

泰勒展开：

$$\begin{aligned}f(x)&=f(x_0)+f'(x_0)(x-x_0)+\tfrac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\tfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)\\&=\sum_{i=0}^{n}\frac{f^{(i)}(x_0)}{i!}(x-x_0)^i+R_n(x)\end{aligned}$$

其中拉格朗日余项 $R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \xi\in(x,x_0)$

当 $R_n(x)=o[(x-x_0)^n]$ 时为佩亚诺型余项

麦克劳林展开：

$$\begin{aligned}f(x)&=f(0)+f'(0)x+\tfrac{f''(0)}{2!}x^2+\cdots+\tfrac{f^{(n)}(0)}{n!}x^n+o(x^n)\\&=\sum_{i=0}^{n}\frac{f^{(i)}(0)}{i!}x^i+o(x^n)\end{aligned}$$

$o(x^n)\Rightarrow o(x^{n-m})$ 高阶无穷小可以当低阶无穷小用

$o(x^n)\pm o(x^{n+m})=o(x^n)$

$o(x^n)\cdot o(x^m)=o(x^{n+m})$

$$
\begin{aligned}
e^x &= 1 + x + \tfrac{1}{2!}x^2 + \cdots + \tfrac{1}{n!}x^n + o(x^n) \\
&= \sum_{i=0}^{n} \frac{x^i}{i!} + o(x^n)
\end{aligned}
$$

$$
\begin{aligned}
\sin x &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \cdots + \tfrac{(-1)^{m-1}}{(2m-1)!}x^{2m-1} + o(x^{2m}) \\
&= \sum_{i=0}^{m-1} \frac{(-1)^i x^{2i+1}}{(2i+1)!} + o(x^{2m})
\end{aligned}
$$

$$
\begin{aligned}
\cos x &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \cdots + \tfrac{(-1)^m}{(2m)!}x^{2m} + o(x^{2m}) \\
&= \sum_{i=0}^{m} \frac{(-1)^i x^{2i}}{(2i)!} + o(x^{2m})
\end{aligned}
$$

$$
\begin{aligned}
\ln(1+x) &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots + \tfrac{(-1)^{n-1}}{n!}x^n + o(x^n) \\
&= \sum_{i=1}^{n} \frac{(-1)^{i-1} x^i}{i} + o(x^n)
\end{aligned}
$$

$$
\begin{aligned}
(1+x)^\alpha &= 1 + \alpha x + \tfrac{\alpha(\alpha-1)}{2!}x^2 + \cdots + C_\alpha^n x^n + o(x^n) \\
&= \sum_{i=0}^{n} C_\alpha^i x^i + o(x^n), n \le \alpha
\end{aligned}
$$

洛必达法则：

若 $f(x), g(x)$ 在 x_0 的某去心邻域内可导，且 $g'(x_0) \neq 0$

若满足 $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0, \lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ 存在

则 $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \dfrac{f'(x_0)}{g'(x_0)}$

简记为 $\frac{0}{0}$ 型

洛必达法则的变型

$\frac{\infty}{\infty}=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$

$0\cdot\infty=0\cdot\frac{1}{0}=\frac{0}{0}$

$0^0=e^{0\cdot\ln 0}=e^{0\cdot\infty}$

$1^\infty=e^{\infty\cdot\ln 1}=e^{0\cdot\infty}$

$\infty^0=e^{0\cdot\ln\infty}=e^{0\cdot\infty}$

$\infty\pm\infty=\frac{1}{0}\pm\frac{1}{0}=\frac{1\cdot0\pm0\cdot1}{0\cdot0}=\frac{0}{0}$

（注意：以上的都是记号，不是算术式。使用洛必达法则满足上述条件时即可使用）

$\lim\limits_{x\to\infty/+\infty/-\infty}f(x)=A$ 则称 $y=A$ 为曲线 $f(x)$ 的水平渐近线

$\lim\limits_{x\to x_0/x_0^+/x_0^-}=\infty$ 则称 $x=x_0$ 为曲线 $f(x)$ 的铅直渐近线

$$\begin{cases}a=\lim\limits_{x\to\infty/+\infty/-\infty}\dfrac{f(x)}{x}\\ b=\lim\limits_{x\to\infty/+\infty/-\infty}[f(x)-ax]\end{cases}$$

当且仅当两个极限都存在时，称 $y=ax+b$ 为曲线 $y=f(x)$ 的斜渐近线

曲率 $K=\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}$

$$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}\Rightarrow K=\frac{|\psi''(t)\varphi'(t)-\varphi''(t)\psi'(t)|}{[\varphi'^2(t)+\psi'^2(t)]^{\frac{3}{2}}}$$

曲率半径 $\rho=\frac{1}{K}$