

A Graph Theory Based Approach to MDAO Problem Formulation

David Pate, *Dr. Brian German [†]Justin Gray,[‡]

blah blah blah

Nomenclature

AAO	All-At-
MDAO	Multidisciplinary Design Analysis and Optimization
FPF	Fundamental Problem Formulation

I. Introduction

The number of analysis tools required in multidisciplinary design optimization (MDO) studies is growing in parallel with the increasing scope of typical MDO problems. An example of this growth in scope can be observed in the historical evolution of the disciplines involved in MDO problems in aircraft design. Multidisciplinary optimization emerged as a separate field from structural optimization through the need to introduce formal techniques for managing the coupling of aerodynamic loads and structural deformations, through the linking of aerodynamic vortex lattice or panel methods with structural finite element models [[REF]]. Subsequently, flight performance and life cycle economics tools were integrated into MDO analysis workflows for conceptual and preliminary design studies [[REF]]. Currently, MDO problems for aircraft design also often include tools for aircraft noise and emissions. In sum, it is now commonplace for 5-10 analysis tools to be employed for typical aircraft design optimization studies [[REF]].

The number of analysis tools is expected to grow in the future as the scope of MDO problems continues to evolve, and as computing is increasingly commoditized. This expansion of scope will be driven, in part, by consideration of additional disciplines. Current trends on the horizon for aircraft MDO studies include incorporation of manufacturing analyses [[REF]], subsystem performance [[REF]], and fleet-level aggregate models of emissions, noise, or economics [[REF]]. A recent survey by Shan and Wang covered a range of different techniques proposed to solve High Dimensional, Expensive, Black-box (HEB) problems. The large amount of relevant research they identified demonstrates the growing interest large scale MDO problems that arise from multiple disciplines.¹

As the size and complexity of engineering systems grow, the time and expense for setting up analysis models grow with them. Multidisciplinary Design Analysis and Optimization (MDAO) frameworks such as OpenMDAO² and ModelCenter have enabled a new level of analysis tool integration and paved the way for models with more analyses and increasing numbers of interdisciplinary couplings. That new capability has created a new challenge. Even for models with 10's of analyses there could be hundreds or thousands of variables that need to be linked together. Commonly different analyses will provide competing values for the same physical quantity, and these conflicts need to be resolved somehow. To make matters worse, the competing values may not even be in the same format. These occurrences are particularly acute when analysis tools have differing fidelities. For example, an abstracted aerodynamic analysis such as an empirical drag buildup model may return only integrated drag, whereas a CFD tool may return pressure and shear stress distributions across the entire surface grid. If an analysis downstream of the aerodynamics tool needs only integrated drag as an input, then the designer has a free choice of which of the two possible aerodynamics

*Georgia Tech

[†]Georgia Tech...

[‡]Aerospace Engineer, MDAO Branch, Mail Stop 5-11, AIAA Member

analysis tools to select to provide the drag estimate (presuming that drag can be computed from surface distributions by a simple integration algorithm). On the other hand, if a downstream analysis needs a pressure distribution in order to compute pitching moment, for example, then any feasible MDO problem formulation must include CFD or similar analysis in the data flow, regardless of whether the empirical drag buildup model is also included.

With all the added complexity from larger models it is not hard to imagine that the task of combining all the analyses into a consistent system model capable of solving a relevant engineering design problem could easily become much more costly than creating the discipline analyses themselves. Just as for a large system the couplings between the disciplines begin to dominate the design space, the couplings between the analyses begin to dominate the job of setting up the model. It is this problem of determining sets of analysis tools and their interconnectivities to form realistic multidisciplinary problems that is the subject of this paper. We are motivated by the following notional but real problem of organizing an MDO study for a complex system:

A new system is being designed for which there is little or no historical precedent. The system is complex, as measured by the number of disciplines and/or components involved in the analysis. A general optimization problem statement has been formulated based on system-level objectives and constraints; however, it is unclear which engineering analysis tools should be interconnected in order to solve the optimization problem. A team of disciplinary and/or component design engineers has been formed in which each engineer has expertise in a particular analysis tool or component model. The engineers meet to discuss the approach to interconnecting their tools to achieve the required system-level MDO model.

Our goal is to develop formalism for expressing analysis interconnectivity and for determining feasible data flows to assist an engineering team conducting this task. Because the problem deals with interconnectivity, we base our approach on the representations and techniques of graph theory. The proposed graph syntax is similar to the REMS system proposed by Alexandrov and Lewis.³ The approach taken here could be viewed as an extension of the concepts proposed in REMS with the goals of providing greater insight into design problem structure. The approach begins by constructing the *maximal connectivity graph (MCG)* describing all possible interconnections between the analysis tools proposed by the engineers. Graph operations are then conducted to reduce the MCG down to a *fundamental problem graphs (FPG)* that describes the minimum set of analysis tools needed to solve the specified system-level design problem. The FPG does not predispose any particular solution procedure for the problem; any relevant MDO solution architecture, e.g. MDF, IDF, CO, BLISS, could be selected *post facto* to implement the system-level optimization by constructing a *problem solution graph (PSG)* and the corresponding problem solution approach. The concept of the FPG and the identification of feasible FPGs are the main contributions of the paper. For completeness we include node and edge types for building a PSG, but the details of how a given problem can be decomposed are beyond the scope of this work.

The paper is organized as follows. First, we describe the differences between a fundamental problem formulation based only on the design problem that the engineers desire to solve and on the available analysis tools, and a specific problem formulation, which additionally presumes a specific solution to the problem. Next, we survey the literature in applying graph theoretic and formal language approaches to multidisciplinary design problem formulation. We then discuss our graph syntax and representation of MDO problems and describe how different structures within a graph map to important features of an MDAO problem. Next we present a procedure for determining possible FPG from a given MCG. Finally, we present the graph based formulation of commercial aircraft design problem and discuss additional applications enabled by our approach.

II. Specific vs Fundamental Problem Formulation

The Fundamental Problem Formulation (FPF) for any given problem will be constant regardless of which MDAO framework, optimization algorithm, iterative solver, or solution strategy is used to solve the problem. However, it is possible to formulate a problem with an assumed execution ordering that implies a solution strategy. For example, consider the Sellar test problem:⁷

$$\begin{aligned}
&\text{given } y_1 = D_1(x_1, y_2, z_1, z_2) \\
&\quad y_2^t = D_2(y_1, z_1, z_2) \\
&\text{min. } F(x_1, y_1, y_2, z_2) \\
&\text{w.r.t. } x_1, y_1, y_2, z_1, z_2 \\
&\text{s.t. } G(y_2, y_2^t) = 0
\end{aligned} \tag{1}$$

D_1 and D_2 represent analysis tools, and F and G are the objective and constraint functions respectively. Equation 2 makes an inherent assumption about the solution strategy for the problem. Analysis D_1 outputs y_1 , which is an input to D_2 . Hence, D_1 should be run before D_2 with y_2 being iterated on to convergence with y_2^t . However, a slightly different formulation is equally valid and still represents the exact same problem:

$$\begin{aligned}
&\text{given } y_2 = D_2(y_1, z_1, z_2) \\
&\quad y_1^t = D_1(x_1, y_2, z_1, z_2) \\
&\text{min. } F(x_1, y_1, y_2, z_2) \\
&\text{w.r.t. } x_1, y_1, y_2, z_1, z_2 \\
&\text{s.t. } G(y_1, y_1^t) = 0
\end{aligned} \tag{2}$$

Equation ?? differs only slightly from Eq. 2. D_1 is now dependent on the output of D_2 , and y_1 will be iterated on to convergence with y_1^t . Now the problem can be solved by running D_2 first and then D_1 . Since the formulations in Eqs. 2 and ?? both describe the same problem then neither can be the FPF. They are both specific versions of a more fundamental description of the problem that is common between them. We present the fundamental formulation as follows:

$$\begin{aligned}
&\text{given } D_1.y_1 = D_1(x_1, y_2, z_1, z_2) \\
&\quad D_2.y_2 = D_2(y_1, z_1, z_2) \\
&\text{min. } F(x_1, y_1, y_2, z_2) \\
&\text{w.r.t. } x_1, y_1, y_2, z_1, z_2 \\
&\text{s.t. } G_1(D_1.y_1, D_2.y_1) = 0 \\
&\quad G_2(D_1.y_2, D_2.y_2) = 0
\end{aligned} \tag{3}$$

The formulation in Eq. 3 differs from both Eq. 2 and Eq. ?? because it has two constraints which both must be met. The presence of both of these constraints fully decouples the problem so that either D_1 or D_2 could be run first or both could be run simultaneously. Alexandrov and Lewis, in their work with REMS, assert that fully decoupling the analyses is key to achieving reconfigurability in the problem formulation.³ The key feature they identify to allow this decoupling is the creation of separate variables for each analysis so that a variable is identified by its analysis and it's name. For example, in Eqn. 2 there is only one y_1 variable vs Eqn. 3 where there are two ($D_1.y_1$ vs $D_2.y_1$).

III. Existing Graph-Based Syntax

As shown above, the mathematical language for specifying problem formulations is very general and can be used both for fundamental and specific problem formulations. Tedford and Martins used the above mathematical syntax to specify the FPF for a set of test problems and also to describe specific formulations for solving them with a number of optimization architectures.⁴ Their work demonstrates clearly how multiple specific problem formulations can all relate back to a common FPG. The challenge with using this traditional mathematical syntax is that it is not easily manipulated or analyzed. A number of graph-based methods have been used successfully to translate the mathematical syntax into a more useful computational form.

Steward's Design Structure Matrix (DSM) is a square adjacency matrix which captures the relationship between analysis tools where off diagonal elements of the matrix indicate coupling.⁵ Since a DSM describes

a square adjacency matrix, it can be represented in an equivalent directed graph where nodes represent analysis tools and edges represent information dependence between those tools. The ordering of elements in a DSM can be used to indicate execution order. For more complex problems, choosing the proper order to run analysis tools is a non-trivial task. Rogers et. al developed DeMAID to manipulate a DSM to find an ordering for analysis tools that reduces the cost of solving highly coupled systems.^{6,7} This re-ordering is done through row operations on the DSM matrix and yields multiple specific problem formulations which all solve the same FPF. In other words, manipulation of the DSM does not fundamentally alter the problem formulation, which makes DSM an excellent foundation for specifying the FPF itself.

Despite its attractive properties, a DSM by itself is insufficient to describe complete problem formulations. Traditional DSM only captures information about data dependency between analyses. Objective and constraint information is missing from the description of the problem. An alternative matrix-based syntax, called a Functional Dependency Table (FDT), was proposed by Wagner and Papalambros. FDT represents the relationship between functions, including objectives and constraints, and specific variables that affect them.⁸ Similar to DSM, FDT also describes an adjacency matrix of a graph. Unlike the DSM graph, however, the graph is undirected and nodes can represent analysis tools, objectives, or constraints. Edges between nodes represent a dependence on the same variable. Michelena and Papalambros made use of the FDT to solve a graph partitioning problem that yielded more efficient optimization problem decompositions.⁹ While FDT succeeds at capturing the information about objectives and constraints, it can not capture the coupled data dependency that DSM captures. For instance, we know from the FPF in Eq. 3 that the objective, f , is dependent on the output, m , of analysis A . You could not determine that from the FDT in Fig. 1 alone. This missing information means that, while FDT is very useful for partitioning problems, it is also not sufficient to contain a complete problem formulation.

	x	y	y^t	z	z^t	m
A	1	1				
B	1			1		
f						1
$g1$				1	1	
$g2$		1	1			

Figure 1: Functional Dependency Table (FDT) for Eq. 3

Alexandrov and Lewis introduced a graph based syntax called REMS which incorporates objectives and constraints into a graph, effectively combining FDT and DSM.³ REMS retains the square adjacency matrix form from DSM, but by adding in the new elements it partially combined a traditional DSM with an FDT. This allows REMS to represent data dependency between multiple analysis tools as well as between analysis tools and objective/constraint functions. One significant conclusion from REMS is the need for variables that are local to discipline analyses. This allows all information exchange to be stated as equality constraints, which become edges in the graph. Additionally, REMS address the need to be able to transition between multiple solution strategies while keeping a constant graph representation of the fundamental problem formulation. REMS only provides syntax for variables and functions, but does not provide for inclusion of solvers or optimizers in the graph.

Lamb and Martins also included objectives and constraint functions as nodes in an Extended DSM (XDSM) in order to capture a more complete description of solution strategies for MDAO problems.¹⁰ Unlike REMS, XDSM also includes nodes for solvers and optimizers to enable complete definition of MDAO architectures. Martins uses XDSM to describe 13 different optimization architectures in a survey paper that provides a novel classification of the different techniques.¹¹ With the additional information included in an XDSM, Lu and Martins applied both ordering and partitioning algorithms on an MDAO test problem named the Scalable Problem.¹²

Although XDSM captures part of the functional aspects of FDT, it leaves out the variables themselves. As a result, all variable information is aggregated so that for the problem from Eq. 3 you can say that A depends on B or vice versa, but you can't identify which individual variables are interacting in the dependency cycle. Without the detailed variable information, you can't construct the compatibility constraints necessary to implement the problem. Additionally, XDSM requires the use of solver and optimizer blocks to represent

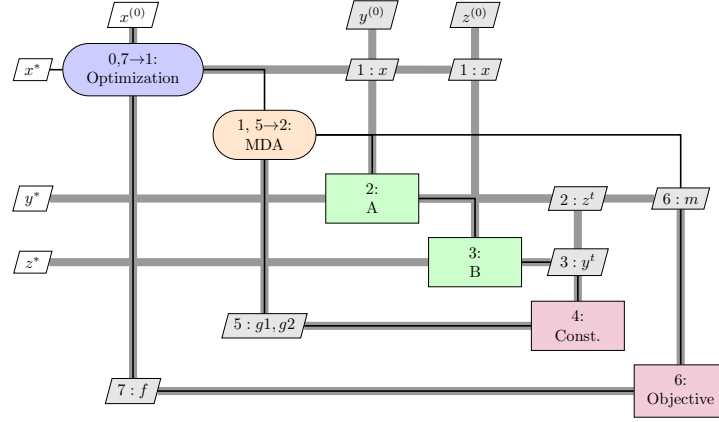


Figure 2: XDSM for Eq. 2, with a gauss-siedel iteration and MDF solution architecture.

the relationship between design variables and objectives/constraints. By introducing solver or optimizer blocks XDSM automatically provides some kind of solution strategy. The XDSM for Eq. 2 is given in Figure 2. This diagram is shown with an assumed gauss-siedel iteration scheme and a MDF solution architecture. Hence XDSM is too specific for use with a fundamental problem formulation.

Of all the methods, REMS comes the closest to fully describing a problem formulation, but lacks the ability to represent optimizers and solvers in more specific problem formulations. While XDSM does include optimizers and constraints, it relies on their presence and can't be used for a more fundamental problem statement. We propose a new graph syntax that combines the key features of REMS and XDSM and provides new language to enable more complete description of MDAO problems.

IV. Requirements for New Graph Syntax

The ultimate goal of the new graph-based syntax presented here is to be able to fully describe the general structure of an MDAO problem independently of any solution information, while still being able to accomodate the more specific case when a solution strategy is applied. In order to achieve that goal the graph needs to accomodate a number of constructs of MDAO problems:

- Analysis tools and connections between them
- Design variables, objectives, and constraints
- Local and global properties
- Coupling between analyses
- Multi-fidelity analyses

Beyond those basic constructs, there are also three phases of a design process that all need to be representable with the new syntax. Firstly there is the initial problem definition phase where the specific analysis tools and design goals are identified. At the end of this phase, a single formal problem formulation is selected specifying design variables, constraints, objectives, analysis tools, etc. Lastly some specific procedure for solving the problem is selected, for example picking an MDAO optimization architecture. Using the proposed graph syntax, each of these phases can be represented with the following three graphs:

- Maximal Connectivity Graph (MCG)
- Fundamental Problem (FPG)
- Problem Solution Graph (SPG)

The *maximal connectivity graph* represents the first phase with all analysis codes being considered and all possible connections between them also present. The second graph is the *fundamental problem formulation graph*, which is the smallest possible graph that still fully defines a given problem formulation. Finally, a *specific problem formulation* may be represented by including additional edges and nodes to represent the solution strategy being employed to solve the problem.

The relationship between these three graphs is depicted in Figs. 3(a) and 3(b). The tree diagram demonstrates the fact that it is generally possible to obtain multiple FPGs from a single maximal connectivity graph. This may correspond to different down-selections of analysis codes, different connections between them, or both. Each down-selection shrinks the number of possible FPGs that could be reached until only one is remaining. Then, from a single FPG, different PSGs may be obtained by implementing different solution strategies. If you consider the size of a graph to be the sum of all of its edges and nodes then the hourglass shape in Fig. 3(b) illustrates how the MCG gets reduced to a single FPG, then multiple possible PSGs exist to solve the problem. In other words the FPG is obtained from the MCG by removing nodes and edges, and the PSG is obtained from the FPG by adding nodes and edges.

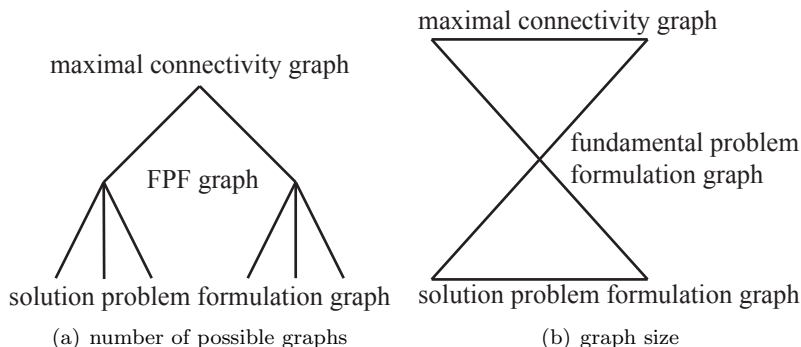


Figure 3: The relationship between the MCG, FPG, and PSG.

V. Syntax Definition

In the rest of this section we present the necessary graph theory fundamentals to construct any of the different graphs discussed above. The notation used in this work is adapted from Diestel.¹³ A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq V \times V$, which means that the elements of E are 2-element subsets of V . The set V contains the *vertices* or *nodes* and the set E contains the *edges*. For a *directed graph*^{*} (or *digraph*) we construct E as a set of ordered pairs instead of a set of sets. Each ordered pair represents an edge starting at the node indicated by the first entry and directed to the node indicated by the second entry. Edge $e = (x, y)$ may be referred to simply as xy . For node $v \in V$ the edges directed out are given by $E(v)$ and the edges directed into v are given by $E^{-1}(v)$; $E(E(v))$ is denoted as $E^2(v)$, and likewise for additional levels. If E is not a one-to-one mapping, $E(v)$ may be the empty set, a single node, or a set of nodes. As

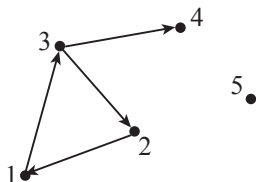


Figure 4: Example directed graph.

an example, for the directed graph shown in Fig. 4 we have

$$V = \{1, 2, 3, 4, 5\},$$

$$E = \{(1, 2), (3, 2), (1, 3), (3, 4)\}.$$

Let I be a nonempty set of counting numbers such that for each $i \in I$ there is a corresponding set A_i .

The set of sets $\mathcal{A} = \{A_i \mid i \in I\}$ is called an indexed family of sets with index i and indexing set I .¹⁴ The union over this family of sets can be described in a few different ways:

$$\bigcup_{i \in I} A_i = \bigcup_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for some } A \in \mathcal{A}\}. \quad (4)$$

Lastly, the cardinality of a set B is the number of elements in B and is denoted as $|B|$.

A. Node and Edge Types

We now define different types of nodes and edges with specific properties to compose graphs used to represent an MDAO problem. There are three node types:

variable: represents scalar or array data

model: responsible for mapping inputs to outputs

driver: control logic capable of managing iteration

In addition to the three node types, there are two edge types:

connection edge: Exchange of information between two nodes. These edges can either be fixed (can not be removed from graph) or free (can be removed).

implicit edge: Implicit passing of information from a driver node to a variable node. A single variable node can have many incoming implicit edges. These edges are free and can be added or removed from the graph.

[[Insert legend for node and edge types]]

In Alexandrov and Lewis’s REMS syntax only two node types were present, variable and function, and only one edge type.³ We’ve chosen to rename the “function” node type to “model” to be more consistent with modern MDAO frameworks. This work adds two more node types and one more edge type to the syntax to allow descriptive graphs for all three phases of the design process.

B. Rules for Nodes and Edges

There are specific rules for the usage of these nodes and edges. The driver node and the implicit edge are only allowed to be present in PSGs. All other node and edge types can be present in any of the three graphs, subject to these restrictions:

1. A model node can only have one edge directed to or from another model node.
2. A model node can only have fixed connection edges directed in or out.
3. A model node must have at least one edge directed in and at least one edge directed out.
4. If a variable node has an outgoing edge to a model node then it may not have any other outgoing edges.

C. In-degree and Out-degree

The *in-degree* of a node is the number of connection edges directed in and is denoted as $\deg^-(v)$, and the *out-degree* is the number of connection edges directed out and it is denoted as $\deg^+(v)$. The degree of a given node is only a function of the connection edges attached to it. The number of implicit edges is not relevant since any number of drivers could be involved, at different parts of an solution process. For example in a sequential optimization where a global optimizer is first applied and a gradient based one is applied second, design variables would have incoming implicit edges from both optimizers but this would not affect the in-degree of those variable nodes.

We also define the *upper in-degree limit*

$$\deg_u^-(v) : V \rightarrow \mathbb{N} \quad (5)$$

and the *lower in-degree limit* as

$$\deg_l^-(v) : V \rightarrow \mathbb{N}. \quad (6)$$

These user-specified limits govern the number of connection edges that may be directed into a variable node for a valid graph. Consider a variable node v with $\deg_u^-(v) = \deg_l^-(v) = 1$. In this case, v must have exactly one incoming explicit edge or the graph is invalid. Two possibilities exist for violating these limit conditions:

hole: The number of incoming edges is less than the lower indegree limit:

$$\deg^-(v) < \deg_l^-(v) \quad (7)$$

collision: The number of incoming edges is greater than $\deg_u^-(v)$.

$$\deg^-(v) > \deg_u^-(v) \quad (8)$$

These violations will be managed in order to seek a working problem formulation.

VI. Graph Representation of MDAO Constructs

In this section we will make use of the Sellar problem, described in Eqn. 3. The graph representation of the Sellar problem, using the new syntax, is presented in Fig. 5. Throughout the rest of this section, different sub-graphs or slight modifications of the full graph will be used to demonstrate the important MDAO constructs within this graph syntax.

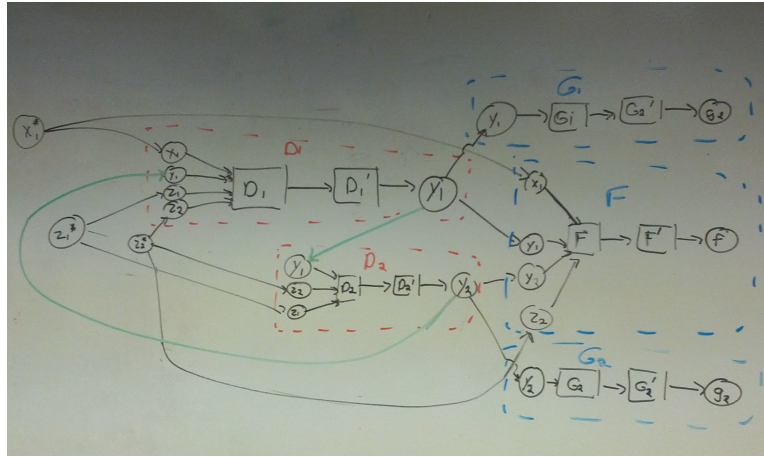


Figure 5: Graph of the Sellar problem formulation

A. Analysis Blocks and Connections

Analysis tools take in a number input variables and then perform some work to calculate the values for their respective outputs. In an MDAO graph, this process is represented by a group of nodes and edges called an *analysis block*. The Sellar problem has two analyses, D_1 and D_2 , so the graph will have two analysis blocks in it. Fig. 6 shows the analysis block for D_2 with some important annotations. Within an analysis block each variable node represents a single input or output and is connected to a single model node via a fixed edge. Note that in Fig. 6, the analysis block contains two model nodes, with a single edge connecting them. This edge represents the necessary calculations to map given inputs to proper output values. If constructing a weighted graph, the weight calculation edge provides the opportunity to represent computational cost. However, it is also fine to skip this edge and have both incoming and outgoing edges to variables through a single model node for a given analysis block. Although analysis blocks are comprised of multiple nodes and edges, since all the edges within them are fixed, they become fixed sub-graphs within a graph.

Via the analysis block structure, we can also distinguish between input and output variable nodes, as shown in Fig. 6. Inputs are any variable nodes that have an outgoing edge into a model node. Conversely,

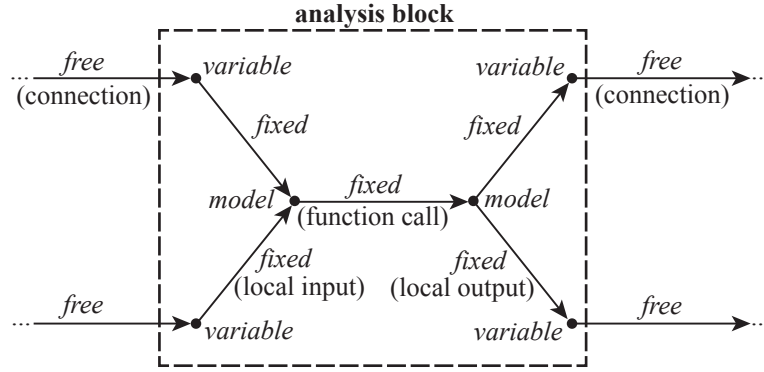


Figure 6: Sellar problem D_2 analysis block The each node type and edge type is labeled in italics and annotated parenthetically.

outputs are any variable nodes that have an incoming edge from a model node. Note that by this definition, inputs and outputs can only exist when variable nodes poses an edge joining them to a model node. These are labeled out as local inputs and local outputs. The variables are local because they are explicitly tied to the analysis block and have no relevance without it.

MDAO problems require that information be passed between sets of analyses. When information from the output of one analysis block is passed, or connected, to the input of another analysis block, a new connection edge is added connecting the two variable nodes involved in the exchange. These connection edges are free, unlike the edges within an analysis block, they could be added or removed depending on the needs of a given design problem. In other words, free connection edges are created by linking the output of one tool to the input of another one. Note that it is allowed for a single output to have outgoing connection edges to multiple downstream inputs.

B. Design Variables

For the Sellar problem, there are three design variables, x_1^* , z_1^* , and z_2^* . In Fig. 5, these are the only three holes (nodes without out incomming edges) in the graph. Based on this example it's tempting to simply call all holes in the graph design variables. However, in the general case this is not completely correct. For any given problem, the set of holes only identifies all the potential design variables. However, some variable holes might not make good design variables. For instance, if your aircraft mission analysis code requires a drag input which ends up being a hole then it's likely that some new analysis tool is needed to predict drag before the design process can continue.

[NOTE: Re-work this one to match D_2 from Sellar]

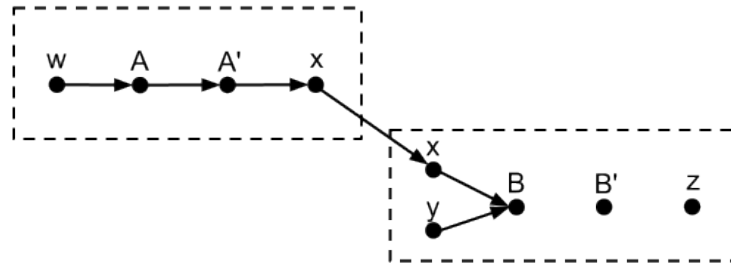


Figure 7: Notional graph with two potential design variables

More formally stated, given a MCG with the $deg_l^-(v) = 1$ for all nodes, there will be a set of holes. Each hole represents a node that could potentially become a design variable. It is up to the designer to examine that variable and decide if it is appropriate to use it as a design variable. If a variable should be a design variable, then set it's $deg_l^-(v) = 0$. Otherwise you leave $deg_l^-(v) = 1$ and find some tool to fill the hole in

graph.

So a design variable is defined as any variable node with $\deg_l^-(v) = 0$. This, by definition, means that design variables are not holes in the graph, since they have not violated their lower in-degree limit. Regardless, when working with a MCG or FPG, only connection edges are allowed, so a design variable in either graph will be a variable with no incoming edges. For a valid PSG any design variable node would have at least one, but possibly multiple, incoming implicit edges from one or more driver nodes.

C. Objectives, and Constraints

Like design variables, objectives and constraints are also constructs that need to be identified when setting up a given design problem. In the case of objective functions single output values could be selected, but commonly multiple values are assembled together via simple algebraic expressions to form a composite objective function. (e.g. The objective function from Eqn. 3 is $x_1^2 + z_2 + y_1 + e^{-y_2}$). Constraints are always given in the form of either an inequality or an equality. (e.g. a constraint from Eqn. 3 is $\frac{y_1}{3.16} - 1 \geq 0$.) By convention, constraints are usually given such that some expression will be less than or equal to zero, so any positive value would violate the constraint. For both objective and constraints, these simple expressions take some inputs and map them to an output value of significance to the overall design problem.

These operations, though very simple, are effectively the same task performed by an analysis block. A constraint or objective function can therefore be represented as another analysis block in the graph, with its own input and output variable nodes. Although fundamentally no different than an analysis block, for clarity and convenience it is useful to distinguish between regular analysis blocks and those that arise from the addition of objectives or constraints to the graph. So we define an expression block, as the collection of variable and model nodes related to a given objective or constraint function. Fig. 8 shows the objective and constraint blocks from the Sellar problem in isolation from the rest of the problem. Note how each block has its own input variables, which in the full graph would be linked via connection edges. Treating objectives and constraints in this way introduces extra variables to the problem formulation, but provides an important benefit for describing local and global properties of MDAO problems.

[NOTE: Show just the G1, G2, and F blocks from Sellar Graph]

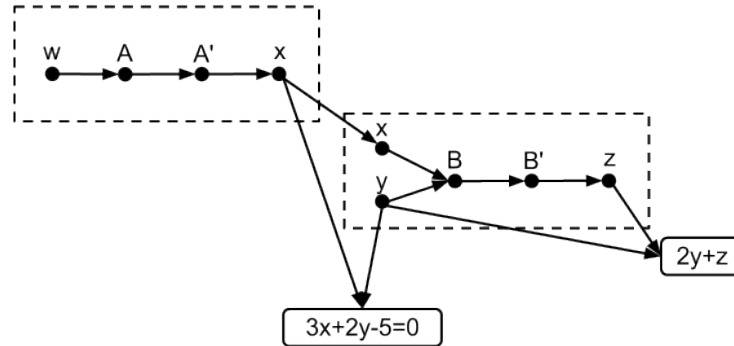


Figure 8: Notional graph with and objective and constraint nodes

D. Local vs Global Properties

A simple definition of a global node is any node that involves information from more than one analysis block. Likewise, a local node is any node that involves information from only a single analysis block. As stated before, variable nodes with fixed edges directed to model nodes (as part of an analysis block) are inherently local and similarly model nodes are also local. Global variables are a part of many MDAO problems and must be representable in the graph. Following our simple definition any variable node that had multiple incoming or outgoing edges, would be global. In Fig. 5 the variable nodes for x_1^* , z_1^* , and z_2^* have multiple outgoing edges and would be considered global.

Note that the variable x_1 from the Sellar problem is usually referred to as a local variable, which is contradicted by the given graph. This discrepancy arises from the explicit treatment of objectives and

constraints as separate expression blocks. Normally in the Sellar problem x_1 is considered a local variable because it only directly affects analysis block D_1 . By forcing the expansion of F into an expression block with its own inputs, then F gets its own copy of the x_1 variable. This necessitates the creation of the global x_1^* node in the graph with two outgoing edges to link the two x_1 inputs nodes in the different blocks. Interestingly, using this definition, where x_1^* is a global variable fits nicely within the structure of Braun's Collaborative Optimization (CO) architecture.[?] The rules of setting up a problem with CO require that each discipline be given an independent local copy of all global variables, while any local variables are retained uniquely within their respective disciplines. But when a local design variable shows up explicitly in the objective function or global constraint then a global variable is created, and the discipline again gets an independent local copy of it. Given our definition of global vs local variable nodes, no such exception for local variables is necessary. When you include a given variable in an expression node, it becomes global by definition and should be treated as such in CO.

Analysis and expression blocks can also have a locality defined for the set of nodes and edges within the block. Their locality is determined by the incoming and outgoing edges from the block as a whole. Fig. 5 has expression blocks for each constraint, G_1 and G_2 . Both constraints are local to their respective analysis blocks. But the expression block for F has incoming edges from D_1 and D_2 , so it would be considered global.

[[insert graph here]]

Figure 9: A simple graph with local and global nodes

Simply stating that any node or block with incoming or outgoing edges from more than one other node or block is global works glosses over a more subtle aspect of some MDAO problems. The locality of any node in a graph is a relative property. For instance, a single node might have outgoing edges to two separate analysis blocks, A and B , but none to a third, C . Then you could say that this node was global with respect to A and B , but local to the group A, B with respect to C . This situation produces a natural hierarchy in a graph, where you can form increasingly smaller groups as you segment the problem into finer localities. When solving a problem using a hierarchical decomposition approach, there are various techniques for partitioning that are effective in different scenarios.^{15, ?, ?, 16, 17}

E. Coupling Between Analyses

Coupling exists in a design problem when a set of two or more analyses each depend on the others' outputs. Consider the notional problem from section II where two analyses, A and B are coupled through their respective variables. The graph of this is shown in Fig. 10. In the graph $A.z$ has an outgoing connection edge to $B.z$, which refers to the coupling constraint g_1 from Eqn. 3. Similarly a connection exists between $B.y$ and $A.y$ referring to the coupling constraint. These two connections form a cycle which represents the coupling between them. This cycle means that blocks A and B need to be iterated on until the two equality constraints from Eqn. 3.

[INSERT GRAPH HERE]

Figure 10: Graph of a notional problem with simple coupling

It is important to note that the cycle in the graph in Fig. 10 does not contain any driver nodes. That means that no solver, optimizer, or other iterative construct is involved in the coupling. This absence is intentional, as the coupling is an inherent part of the problem. This corresponds directly to Eqn. 3, where each equality constraint is one edge in the graph. Transitioning to either Eqn. 2 or Eqn. ?? means selecting one of the nodes as the starting point for the iteration, indicating the associated analysis block should be executed first.

Larger problems can contain more complex cycles in their FPG, indicating more complex coupling between analyses. A cycle can contain more than just two analysis blocks. Multiple independent cycles can exist, indicating independent coupling relationships. Cycles can also overlap, where the same analysis blocks are involved in multiple different coupling cycles. All of these situations arise naturally as the size of problems grows, and managing all of this coupling becomes difficult. From the perspective of the graph, the problem reduces down to an ordering challenge. For each cycle in the graph, you need to pick an analysis block to start

the iteration from. Rogers DEMAID tool uses a genetic algorithm to find an ordering that minimizes the overall coupling of the system by separating independent cycles in the graph.^{6,7}

F. Multi-fidelity Problems

A multi-fidelity problem can be characterized by two or more different analyses each calculating the same data. Mapping this type of multi-fidelity situation onto a graph will yield two analysis blocks, each with their own output variable nodes, having outgoing edges to a third analysis block that needs the data as input. Figure 11 shows a notional problem that exhibits this structure, where the key feature is that variable node y in analysis block C has two incoming edges.

[INSERT GRAPH HERE]

Figure 11: Graph of a notional multi-fidelity problem

When an input variable node has more than one incoming edge, designers must make some kind of decision about how to manage the flow of information in their problem. In Eqn. 8 we defined a collision as the situation where there are more incoming connection edges than the $\deg_u^-(y)$. So designers can indicate how to handle conflicting edges in a problem graph by setting the $\deg_u^-(y)$ for a given node. In more concrete terms, setting this limit equates to defining the part of the graph in question as single or multi-fidelity.

When $\deg_u^-(y) = 1$, then any conflicting edges in the graph will cause a collision and a choice between the colliding analysis blocks must be made. This results in a single fidelity design problem. On the other hand if $\deg_u^-(y) = 2$, then two different analysis blocks can co-exist with a given graph without causing a collision. The design problem is then a multi-fidelity problem.

Multi-fidelity problems are always characterized by graphs with variable nodes that have an upper-degree-limit greater than one. These problems require special techniques for resolving the conflicting edges by introducing some mechanism to manage when each of the different fidelity analyses are run.^{18,19,20}

VII. Building MCG and FPG

A. Maximal Connectivity Graph

To construct the maximal connectivity graph, we assume that a set of codes, global inputs, and objectives and constraints (collectively called global outputs) are provided. The codes are represented by analysis block graphs ((ABGs?)) $A_i = (V_{A_i}, E_{A_i})$, $i \in I$, where m is the number of codes and $I = \{1, 2, \dots, m\}$. The global inputs are represented as a set of variable nodes V_{in} , and the global outputs are represented by a set of variable nodes V_{out} . We assume that V_{in} , V_{out} , and each A_i are given, and that any potential connection between variables is given in the form of connection edges in the set C_M . Then we may construct the maximal connectivity graph $M = (V_M, E_M)$ as

$$V_M = V_{\text{in}} \cup V_{\text{out}} \cup \left(\bigcup_{i \in I} V_{A_i} \right),$$

$$E_M = C_M \cup \left(\bigcup_{i \in I} E_{A_i} \right),$$

The MCG M is uniquely determined by the given set of analysis blocks, the required outputs, and the given global inputs. In the cases where the set of global inputs is not known a priori, the process of obtaining the FPG will reveal the required inputs, as discussed subsequently.

B. Fundamental Problem Formulation Graph

We now define the fundamental problem formulation graph $F = (V_F, E_F)$ as a directed graph meeting the following conditions, which are explained subsequently,

- (1) $I_F \subset I$ such that the remaining conditions hold
- (2) $\forall i \in I_F, \exists v \in V_{A_i}$ with $t_{\text{node}}(E^{-1}(v)) = \text{'model'}$ and $\deg^+(v) > 0$

(3) $V_{in,F} = \{v \in V_{in} \mid \deg^+(v) > 0\}$ ((maybe this should be 1 instead of 0 to match the previous definition of global input))

$$(4) V_F = V_{in,F} \cup V_{out} \cup \left(\bigcup_{i \in I_F} V_{A_i} \right)$$

$$(5) E_F = C_F \cup \left(\bigcup_{i \in I_F} E_{A_i} \right), C_F \subset C_M$$

$$(6) \forall v \in V_F \text{ with } t_{node}(v) = \text{'variable'}, \deg_l^-(v) \leq \deg^-(v) \leq \deg_u^-(v)$$

The set I_F is an index set containing the indices of the analysis blocks in F and it may only include the analysis blocks A_i that meet requirements (2) and (6). Requirement (2) stipulates that each analysis blocks in F must have at least one local output that is being used. Requirement (3) stipulates that only the global inputs that are being used should be included. Requirements (4) and (5) provide the construction of the sets of nodes and edges, respectively. Lastly, requirement (6) stipulates that the number of edges directed into the variable nodes must be within the lower and upper in-degree limits; if $\deg^-(v) < \deg_l^-(v)$ the node is called a *hole*, and if $\deg^-(v) > \deg_u^-(v)$ the node is called a *collision*.

The set C_F is a set containing connection edges representing connections. Since C_M is the set of all potential connections, we must have $C_F \subset C_M$. While no other conditions are explicitly stated for C_F , requirement (6) is actually a requirement on both I_F and C_F .

C. Obtaining the Fundamental Problem Graph

In general, there may be multiple different graphs that satisfy the FPG conditions, though there may be none at all. Here, we describe a process for obtaining an FPG by starting with the MCG and disconnecting connection edges until the FPG conditions are met. Then the problem is reduced to deciding which connection edges to remove.

Then, mathematically, the FPG starts with $C_{F,0} = C_M$, $E_{F,0} = E_M$, $V_{F,0} = V_M$. The nodes where $\deg^-(v) < \deg_l^-(v)$ are considered first.

1. The first step is to detect holes and disconnect the connection edges following them. These connection edges are removed because they represent variables which cannot be determined because the analysis function does not have adequate inputs. The set of variable nodes which are holes is created as

$$H = \{v \in V_M \mid t_{node}(v) = \text{'variable'} \text{ and } \deg^-(v) < \deg_l^-(v)\}, \quad (9)$$

which is the set of variable nodes with fewer incoming edges than are allowed by the lower in-degree limit. Then the updated set of edges is created by removing the first set of connection edges following the hole:

$$C_{F,1} = C_{F,0} \setminus \{(x, y) \in E_M \mid x = E_M^3(v) \text{ for } v \in H\}, \quad (10)$$

and this step is demonstrated by Fig. 12. Because removing these edges can create new holes, this step must be repeated until no more holes are found.

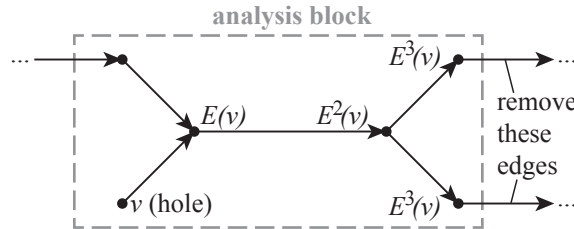


Figure 12: Example variable node indicating a hole.

2. The second step is to detect collisions and to then disconnect enough connection edges so that the collisions are resolved. The set of variable nodes which contain collisions is created as

$$S_{\text{nodes}} = \{v \in V_M \mid t_{\text{node}}(v) = \text{'variable'} \text{ and } \deg^-(v) > \deg_u^-(v)\}. \quad (11)$$

For each collision node we can construct a set contained the edges directed in. The set containing all of these sets is constructed as

$$S_{\text{edges}} = \{\{(x, y) \in E_M\} \mid y \in S_{\text{nodes}}\} \quad (12)$$

Let $J = \{1, 2, \dots, |S_{\text{edges}}|\}$ be an indexing set for S_{edges} such that each $S_{\text{edges},j}$ corresponds to a set in S_{edges} for $j \in J$. An example collision is shown in Fig. 13 to indicate the definition of $S_{\text{edges},j}$. We can assume that J also indexes S_{nodes} because there is a one-to-one correspondence between the elements in S_{nodes} and the elements in S_{edges} (which are sets). Then we may construct sets of edges as

$$B_j = \{e_k \in S_{\text{edges},j} \mid k \in \{1, 2, \dots, K\} \text{ with } K \leq \deg_u^-(v_j)\}, \quad j \in J, \quad (13)$$

which means that each set B_j is constructed from the set $S_{\text{edges},j}$ by taking only as many edges as are allowed by the upper in-degree limit of v_j . The construction of each B_j corresponds to making a decision on which edges to include and which edges not to include. Then let

$$C_{F,2} = \{e \in C_{F,1} \mid e \in B_j \text{ for some } j \in J\}, \quad (14)$$

$$E_{F,2} = E_M \setminus (C_M \setminus C_{F,2}) \quad (15)$$

and

$$F_2 = (V_M, E_{F,2}) \quad (16)$$

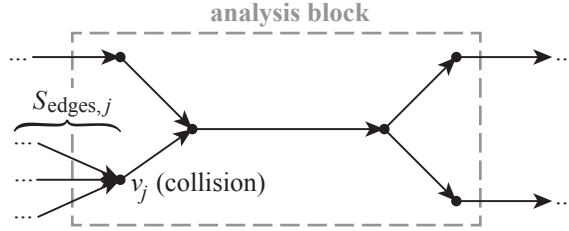


Figure 13: Example variable node indicating a collision.

3. The third and final step is to remove any unused nodes and edges. The indexing set for only the analysis blocks with at least one used local output may be constructed as

$$I_F = \{i \in I \mid \exists v \in V_{A_i} \text{ such that } t_{\text{node}}(E_{F,2}^{-1}(v)) = \text{'model'} \text{ and } \deg^+(v) > 0\}, \quad (17)$$

where the degree is calculated with respect to F_2 . This set excludes both the analysis blocks that were unused in the original MCG and those that became unused in step 1 or step 2.

The set of used global inputs is constructed as

$$V_{\text{in},F} = \left\{v \in V_{\text{in}} \mid |\{(x, y) \in E_{F,2}(v) \mid y \in V_{A_i} \text{ for } i \in I_F\}| > 1\right\}, \quad (18)$$

which requires that at least two edges directed out of the nodes in $V_{\text{in},F}$ must be to the analysis blocks indexed by I_F .

Finally, the set of nodes and edges describing a fundamental problem formulation is given by

$$V_F = V_{\text{in},F} \cup V_{\text{out}} \cup \left(\bigcup_{i \in I_F} V_{A_i}\right), \quad (19)$$

$$C_F = \{(x, y) \in C_{F,2} \mid x, y \in V_F\}, \quad (20)$$

$$E_F = C_F \cup \left(\bigcup_{i \in I_F} E_{A_i}\right), \quad (21)$$

$$F = (V_F, E_F). \quad (22)$$

VIII. Example Problem

This section presents an example problem to demonstrate the process of obtaining the FPG and the advantages of doing so. The task is the conceptual sizing of a single-aisle subsonic transport using an MDAO framework with objectives for performance and total weight for a required mission. The full set of analysis codes available for use is given in Table 1. Each of these analysis codes contribute individual

Table 1: Analysis codes for subsonic transport sizing

analysis code	description
VSP	parametric geometry
PDCYL	wing and fuselage weight estimation
NPSS	engine sizing and performance
VORLAX	aerodynamics using the vortex lattice method
PMARC	aerodynamics using a low-order panel method
WATE	engine weight estimation
FLOPSa	mission performance, engine sizing, and weight estimation
FLOPSb	mission performance only

disciplinary analysis but they are not mutually exclusive. For example, VORLAX and PMARC are both aerodynamics codes that predict inviscid drag but with different levels of fidelity. The Flight Optimization System (FLOPS) is included twice to represent two different uses. FLOPSa denotes FLOPS being executed for both mission performance and engine analysis, while FLOPSb indicates that FLOPS is executed for only mission performance analysis. This sort of representation is useful for “supercodes” that are capable of being used in different ways and with different sets of inputs and outputs.

In this example, the MCG is formed using a consistent variable naming convention and then connecting every variable with the same name, though this is not the only way to form an MCG. Table 2 presents the full list of variables in the leftmost column and then indicates whether the variable is an input or an output of each analysis code. Some of the variables, such as geometry and performance, represent many different variables which have been bundled together due to their similarity. This bundling is not necessary and does not limit the generality of this example, it is done to simplify the presentation.

Table 2: Analysis code input and output description

variable	analysis code							
	VSP	PDCYL	NPSS	VORLAX	PMARC	WATE	FLOPSa	FLOPSb
geometry	in	in		in	in	in	in	in
number of engines			in				in	in
mission							in	in
fuselage weight		out					in	in
wing weight		out					in	in
engine weight						out	out	in
wetted area	out						in	in
inviscid drag				out	out			in
drag			in				out	out
engine performance			out			in	out	in
performance							out	out
total weight		in					out	out

The MCG M is formed in four steps:

1. An analysis block is created for each analysis code using the information in each column of Table 2
2. Each analysis block is formed by creating variable nodes for each input and directing them into a single model node, which is directed into another model node, which is then directed into variable nodes corresponding to each output. A sample analysis block is shown for analysis code FLOPSb in Fig. 14.

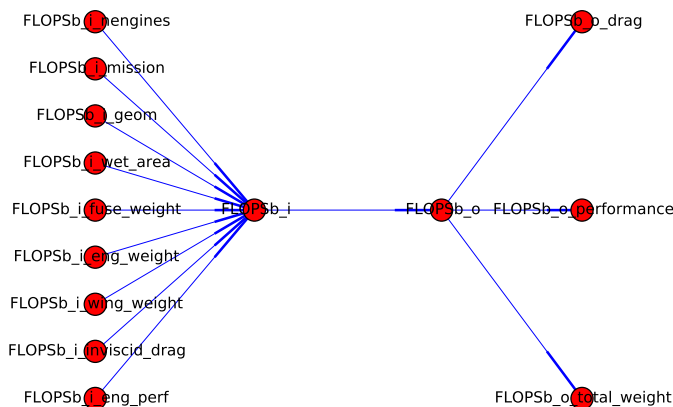


Figure 14: Sample analysis block for analysis code FLOPSb using Table 2

2. Variable ((change)) nodes are created to represent the objectives for performance and total weight.
3. Variable nodes are created to represent the geometry variable and mission variable as global inputs.
4. Explicit edges are created from each variable node to every other variable node representing variables with the same name. The direction is determined by whether the variable node has an edge directed into or out of a model node (local input or local output).

The resulting MCG is shown in Fig. ?? . Emerging cycles are shown in yellow and indicate potential cycles that may exist when an FPG is formed.

A. Obtaining an FPG

The process of obtaining an FPG from the given MCG, but it is expected to include the user because of required changes to the MCG or make decisions

((I should show a figure of the MCG with edges and nodes colored as to when they were removed))
 ((mention python))

As presented in Sec.(), the first step to obtaining an FPG is to detect holes. To start, we set the lower indegree limit for every node to be one to find any potential holes. In this case, it turns out that the variable nodes representing the number of engines is a hole for NPSS, FLOPSa, and FLOPSb. This implies than an FPG cannot be obtained from the given MCG, as shown in Fig. 16. In this figure, the green nodes indicate analysis blocks with a hole, and the cyan nodes indicate analysis blocks that became unused after the invalid analyses were removed ((connections were removed, color edges too)).

Therefore, we may now create another variable node to represent the number of engines as a global input and also create explicit edges to supply this new input. Now this step is repeated with the new MCG and no holes are found.

While Sec.() indicated the process for obtaining an FPG, it did not specify how to decide which edges to keep when resolving a collision, that freedom is left to the specific implementation. The benefit of this approach is that standard graph theory algorithms can be used to make these decisions. The first

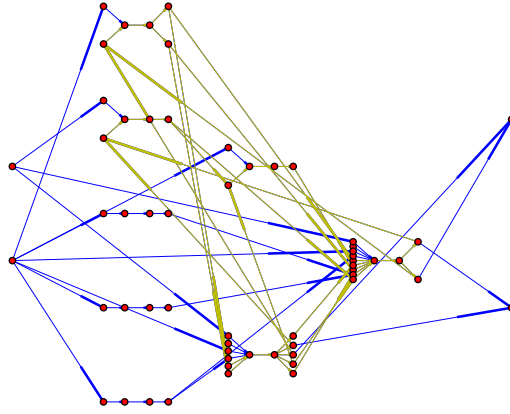


Figure 15: Maximal connectivity graph for the subsonic transport example problem

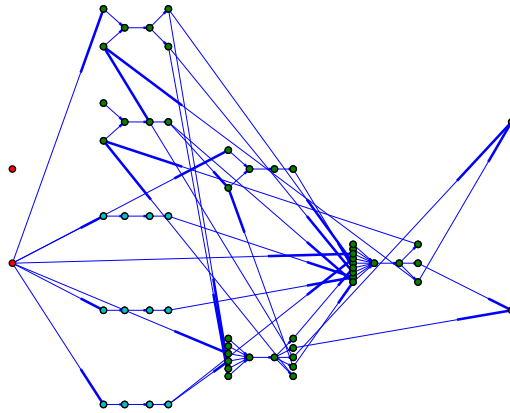


Figure 16: The MCG with one global variable missing. The resulting holes are shown in green and unused analyses shown in cyan.

example of this is to obtain an FPG with the fewest possible number of cycles, which are found using the implementation of Johnson's algorithm²¹ in the Python package NetworkX. The FPG with the fewest cycles is shown in Fig. 17, which indicates one cycle shown in yellow. A similar alternative would be to minimize the number of analysis blocks involved in cycles, counting multiplicity.

Alternatively, it may be desirable to resolving collisions by including the higher fidelity codes, such as NPSS in this case. One method to ensure that the edges directed from certain analysis blocks are the ones chosen to resolve the conflicts is to use a ranking system. For this case, consider the rankings for each analysis code given in Table 3. Step two in Sec. (()) (the resolution of collisions) is now conducted by selecting the edges directed from the analysis blocks with the highest rank for each collision. The resulting FPG is shown in Fig. 18, and this graph has four cycles, which are enumerated in Table 4.

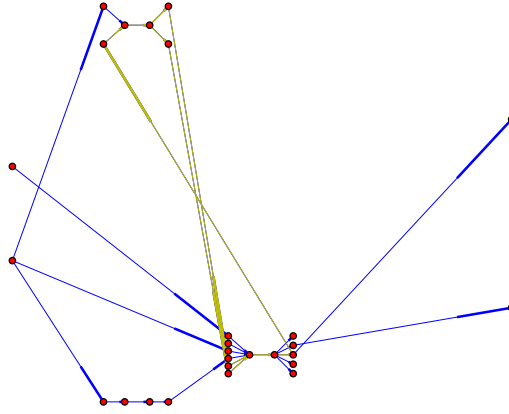


Figure 17: FPG with the fewest number of cycles.

Table 3: Example ranking of importance

analysis code	rank
VSP	5
PDCYL	5
NPSS	4
PMARC	4
FLOPSb	4
VORLAX	3
WATE	3
FLOPSa	2

Finally, now consider the case where it is desired to have the inviscid drag input into FLOPSb be a multi-fidelity input, meaning that it is sought to use multiple analysis codes to calculate the same variable. This is done simply by setting the upper indegree limit for this node as two and then repeating the (automated) process. In this case, both VORLAX and PMARC are retained, resulting in an FPG that omits only FLOPSa.

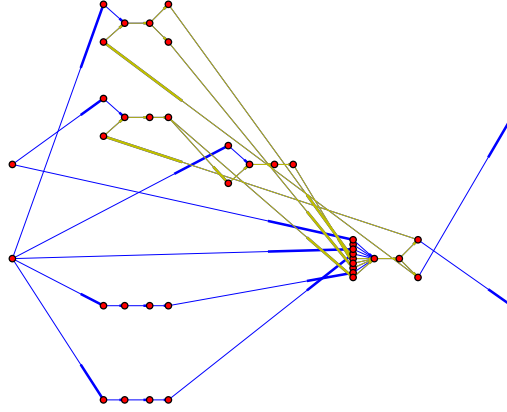


Figure 18: FPG obtained using the ranking system.

Table 4: Cycles for the FPG obtained using ranking

output/input	analysis code	output/input	analysis code	output/input	analysis code	output/input
engine weight	FLOPSb	drag	NPSS	engine performance	WATE	engine weight
engine performance	FLOPSb	drag	NPSS	engine performance		
fuselage weight	FLOPSb	total weight	PDCYL	fuselage weight		
wing weight	FLOPSb	total weight	PDCYL	wing weight		

IX. Applications

X. Conclusions

References

¹Shan, S. and Wang, G., "Survey of modeling and optimization strategies to solve high-dimensional design problems with computationally-expensive black-box functions," *Structural and Multidisciplinary Optimization*, Vol. 41, No. 2, 2010, pp. 219–241.

²Gray, J., Moore, K. T., Hearn, T. A., and Naylor, B. A., "A Standard Platform for Testing and Comparison of MDAO Architectures," *8th AIAA Multidisciplinary Design Optimization Specialist Conference (MDO)*, Honolulu, Hawaii, 2012, pp. 1–26.

³Alexandrov, N. and Lewis, R., "Reconfigurability in MDO Problem Synthesis, Part 1 and Part 2," Tech. rep., Papers AIAA-2004-4307 and AIAA-2004-4308, 2004.

⁴Tedford, N. P. and Martins, J. R. R. a., "Benchmarking multidisciplinary design optimization algorithms," *Optimization and Engineering*, Vol. 11, No. 1, March 2009, pp. 159–183.

⁵Steward, D. V., "The Design Structure System: A Method for Managing the Design of Complex Systems," *IEEE Transactions on Engineering Management*, Vol. EM-28, No. 3, 1981, pp. 71–74.

⁶Rogers, J., McCulley, C., and Bloebaum, C., *Integrating a genetic algorithm into a knowledge-based system for ordering complex design processes*, NASA Technical Memorandum, Hampton Virginia, 1996.

⁷Rogers, J., "DeMAID/GA-an enhanced design managers aid for intelligent decomposition (genetic algorithms)," *the Proceedings of the 6th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, AIAA-96-4157-CP, 1996, pp. 1497–1504.

⁸Wagner, T. C. and Papalambros, P. Y., "A general framework for decomposition analysis in optimal design," *Advances in Design Automation*, Vol. 2, 1993, pp. 315–325.

⁹Michelena, N. F. and Papalambros, P. Y., "A Hypergraph Framework for Optimal Model-Based Decomposition of Design Problems," *Mechanical Engineering*, Vol. 8, No. 2, 1997, pp. 173–196.

- ¹⁰Lambe, A. B. and Martins, J. R. R. A., “Extensions to the Design Structure Matrix for the Description of Multidisciplinary Design, Analysis, and Optimization Processes,” *Structural and Multidisciplinary Optimization*, 2012.
- ¹¹Martins, J. R. R. A. and Lambe, A. B., “Multidisciplinary Design Optimization: Survey of Architectures,” *AIAA Journal*, 2012, pp. 1–46.
- ¹²Lu, Z. and Martins, J., “Graph Partitioning-Based Coordination Methods for Large-Scale Multidisciplinary Design Optimization Problems,” *ISSMO Multidisciplinary Analysis Optimization Conference*, No. September, Indianapolis, Indiana, 2012, pp. 1–13.
- ¹³Diestel, R., *Graph Theory*, Springer, 2010.
- ¹⁴Smith, D., Eggen, M., and Andre, R. S., *A Transition to Advanced Mathematics*, Brooks/Cole Pub Co, 2006.
- ¹⁵Krishnamachari, R. S. and Papalambros, P. Y., “Optimal hierarchical decomposition synthesis using integer programming,” *J MECH DES, TRANS ASME*, Vol. 119, No. 4, 1997, pp. 440–447.
- ¹⁶Allison, J. T., Kokkolaras, M., and Papalambros, P. Y., “Optimal partitioning and coordination decisions in decomposition-based design optimization,” *Journal of Mechanical Design*, Vol. 131, 2009, pp. 081008.
- ¹⁷Michelena, N. F. and Papalambros, P. Y., “A hypergraph framework for optimal model-based decomposition of design problems,” *Computational Optimization and Applications*, Vol. 8, No. 2, 1997, pp. 173–196.
- ¹⁸March, A. and Willcox, K., “Provably Convergent Multifidelity Optimization Algorithm Not Requiring High-Fidelity Derivatives,” *AIAA journal*, Vol. 50, No. 5, 2012, pp. 1079–1089.
- ¹⁹Alexandrov, N. M., Lewis, R. M., Gumbert, C. R., Green, L. L., and Newman, P. A., “Approximation and model management in aerodynamic optimization with variable-fidelity models,” *Journal of Aircraft*, Vol. 38, No. 6, 2001, pp. 1093–1101.
- ²⁰Huang, D., Allen, T. T., Notz, W. I., and Miller, R. A., “Sequential kriging optimization using multiple-fidelity evaluations,” *Structural and Multidisciplinary Optimization*, Vol. 32, No. 5, 2006, pp. 369–382.
- ²¹Johnson, D. B., “Finding all the elementary circuits of a directed graph,” *SIAM Journal on Computing*, Vol. 4, No. 1, March 1975, pp. 77–84.