MODALITIES AND INTENSIONAL LANGUAGES*

The subject of this paper is the foundations of modal logic. By foundations, we generally mean the underlying assumptions, the underpinnings. There is a normative sense in which it has been claimed that modal logic is without foundation. Professor Quine, in *Word and Object,* suggests that it was conceived in sin: the sin of confusing use and mention. The original transgressors were Russell and Whitehead. Lewis followed suit and constructed a logic in which an operator corresponding to 'necessarily' operates on sentences whereas 'is necessary' ought to be viewed as a predicate of sentences. As Professor Quine reconstructs the history of the enterprise,[1] the operational use of modalities promised only one advantage: the possibility of quantifying into modal contexts. This several of us[2] were enticed into doing. But the evils of the sentential calculus were found out in the functional calculus, and with it – to quote again from *Word and Object* – 'the varied sorrows of modality transpose'.

I do not intend to claim that modal logic is wholly without sorrows, but only that they are not those which Professor Quine describes. I do claim that modal logic is worthy of defense, for it is useful in connection with many interesting and important questions such as the analysis of causation, entailment, obligation and belief statements, to name only a few.

If we insist on equating formal logic with strongly extensional functional calculi then Strawson[3] is correct in saying that 'the analytical equipment (of the formal logician) is inadequate for the dissection of most ordinary types of empirical statement.'

INTENSIONAL LANGUAGES

I will begin with the notion of an intensional language. I will make a further distinction between those which are explicitly and implicitly

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intensional. Our notion of intensionality does not divide languages into mutually exclusive classes but rather orders them loosely as strongly or weakly intensional. A language is explicitly intensional to the degree to which it does not equate the identity relation with some weaker form of equivalence. We will assume that every language must have some constant objects of reference (things), ways of classifying and ordering them, ways of making statements, and ways of separating true statements from false ones. We will not go into the question as to how we come to regard some elements of experience as things, but one criterion for sorting out the elements of experience which we regard as things is that they may enter into the identity relation. In a formalized language, those symbols which name things will be those for which it is meaningful to assert that I holds between them, where 'I' names the identity relation.

Ordinarily, and in the familiar constructions of formal systems, the identity relation must be held appropriate for individuals. If 'x' and 'y' are individual names then

$$xIy \tag{1}$$

is a sentence, and if they are individual variables, then (1) is a sentential function. Whether a language confers thinghood on attributes, classes, propositions is not so much a matter of whether variables appropriate to them can be quantified upon (and we will return to this later), but rather whether (1) is meaningful where 'x' and 'y' may take as values names of attributes, classes, propositions. We note in passing that the meaningfulness of (1) with respect to attributes and classes is more frequently allowed than the meaningfulness of (1) in connection with propositions.

Returning now to the notion of explicit intensionality, if identity is appropriate to propositions, attributes, classes, as well as individuals, then any weakening of the identity relation with respect to any of these entities may be thought of as an extensionalizing of the language. By a weakening of the identity relation is meant equating it with some weaker equivalence relation.

On the level of individuals, one or perhaps two equivalence relations are customarily present: identity and indiscernibility. This does not preclude the introduction of others such as similarity or congruence, but the strongest of these is identity. Where identity is defined rather than taken as primitive, it is customary to define it in terms of indiscernibility, one form of which is

(2) $$x \text{ Ind } y =_{df} (\varphi)(\varphi x \text{ eq } \varphi y)$$

In a system of material implication (*Sm*), eq is taken as $\equiv$. In modal systems, eq may be taken as $\equiv\!\!\equiv$. In more strongly intensional systems eq may be taken as the strongest equivalence relation appropriate to such expressions as 'φx'. In separating (1) and (2) I should like to suggest the possibility that to equate (1) and (2) may already be an explicit weakening of the identity relation, and consequently an extensionalizing principle. This was first suggested to me by a paper of Ramsey.[4] Though I now regard his particular argument in support of the distinction as unconvincing, I am reluctant to reject the possibility. I suppose that at bottom my appeal is to ordinary language, since although it is obviously absurd to talk of two things being the same thing, it seems not quite so absurd to talk of two things being indiscernible from one another. In equating (1) and (2) we are saying that to be distinct is to be discernibly distinct in the sense of there being one property not common to both. Perhaps it is unnecessary to mention that if we confine things to objects with spatio-temporal histories, then it makes no sense to distinguish (1) and (2). And indeed, in my extensions of modal logic, I have chosen to define identity in terms of (2). However the possibility of such a distinction ought to be mentioned before it is obliterated. Except for the weakening of (1) by equating it with (2), extensionality principles are absent on the level of individuals.

Proceeding now to functional calculi with theory of types, an extensionality principle is of the form

(3) $$x \text{ eq } y \rightarrow xIy\,.$$

The arrow may be one of the implication relations present in the system or some metalinguistic conditional. eq is one of the equivalence relations appropriate to x and y, but not identity. Within the system of material implication, 'x' and 'y' may be taken as symbols for classes, eq as class equality (in the sense of having the same members); or 'x' and 'y' may be taken as symbols for propositions and eq as the triple bar. In extended modal systems eq may be taken as the quadruple bar where 'x' and 'y' are symbols for propositions. If the extended modal system has symbols for classes, eq may be taken as 'having the same members' or alternatively, 'necessarily having the same members', which can be expressed within such a language. If we wish to distinguish classes from attributes in such a system (although I regard this as the perpetuation of a confusion), eq may be taken as 'necessarily applies to the same thing', which is directly expressible within the system. In a language which permits epistemic contexts such as belief contexts, an even stronger equivalence relation would have to be present, than either material or strict equivalence. Taking that stronger relation as eq, (3) would still be an extensionalizing principle in such a strongly intensional language.

I should now like to turn to the notion of implicit extensionality, which is bound up with the kinds of substitution theorems available in a language. Confining ourselves for the sake of simplicity of exposition to a sentential calculus, one form of the substitution theorem is

(4) $$x \text{ eq}_1\, y \rightarrow z \text{ eq}_2\, w$$

where x, y, z, w are well-formed, w is the result of replacing one or more occurrences of x by y in z, and '$\rightarrow$' symbolizes implication or a metalinguistic conditional. In the system of material implication (*Sm* or *QSm*), (4) is provable where eq_1 and eq_2 are both taken as material equivalence for appropriate values of x, y, z, w. That is

(5) $$(x \equiv y) \supset (z \equiv w)\,.$$

Now (5) is clearly false if we are going to allow contexts involving belief, logical necessity, physical necessity and so on. We are familiar with the examples. If 'x' is taken as 'John is a featherless biped', and 'y' as 'John is a rational animal', then (5) fails. Our choice is to reject (5) as it stands, or to reject all contexts in which it fails. If the latter choice is made, the language is implicitly extensional since it cannot countenance predicates or contexts which might be permissible in a more strongly intensional language. Professor Quine's solution is the latter. All such contexts are dumped indiscriminately onto a shelf labelled 'referential opacity' or more precisely 'contexts which confer referential opacity', and are disposed of. But the contents of that shelf are of enormous interest to some of us and we would like to examine them in a systematic and formal manner. For this we need a language which is appropriately intensional.

In the modal calculus, since there are two kinds of equivalence which may hold between 'x' and 'y', (4) represents four possible substitution theorems, some of which are provable. We will return to this shortly. Similarly, if we are going to permit epistemic contexts, the modal analogue of (4) will fail in those contexts and a more appropriate one will have to supplement it.

IDENTITY AND SUBSTITUTION IN QUANTIFIED MODAL LOGIC

In the light of the previous remarks I would like to turn specifically to the criticisms raised against extended modal systems in connection with identity and substitution. In particular, I will refer to my [5] extension of Lewis'[6] $S4$ which consisted of introducing quantification in the usual manner and the addition of the axiom [7]

(6) $$\Diamond(\exists x)A \prec (\exists x)\Diamond A\,.$$

I will call this system $QS4$. $QS4$ does not have an explicit axiom of extensionality, although it does have an implicit extensionalizing principle in the form of the substitution theorem. It would appear that for many uses to which modal calculi may be put, $S5$ is to be preferred[8]. In an extended $S4$, Prior[9] has shown that (6) is a theorem. My subsequent remarks, unless otherwise indicated, apply equally to $QS5$. In $QS4$ (1) is defined in terms of (2). (2), and consequently (1), admit of alternatives where 'eq' may be taken as material or strict equivalence: 'I_m' and 'I' respectively. The following are theorems of $QS4$:

(7) $$(xI_m y) \equiv (xIy)$$ and

(8) $$(xIy) \equiv \Box(xIy)$$

where '$\Box$' is the modal symbol for logical necessity. In (7) 'I_m' and 'I' are strictly equivalent. Within such a modal language, they are therefore indistinguishable by virtue of the substitution theorem. Contingent identities are disallowed by (8).

(9) $$(xIy) \cdot \Diamond \sim (xIy)$$

is a contradiction.

Professor Quine[10] finds these results offensive, for he sees (8) as 'purifying the universe.' Concrete entities are said to be banished and replaced by pallid concepts. The argument is familiar:

(10) The evening star eq the morning star

is said to express a 'true identity', yet they are not validly intersubstitutable in

(11) It is necessary that the evening star is the evening star.

The rebuttals are also familiar. Rather than tedious repetition, I will try to restate them more persuasively. This is difficult, for I have never appreciated the force of the original argument. In restating the case, I would like to consider the following informal argument:

(12) If p is a tautology, and p eq q, then q is a tautology

where 'eq' names some equivalence relation appropriate to p and q. In S_m if 'eq' is taken as $\equiv$ then a restricted (12) is available where $p \equiv q$ is provable.

One might say informally that with respect to any language, if (12) is said to fail, then we must be using 'tautology' in a very peculiar way, or what is taken as 'eq' is not sufficient equivalence relation appropriate to p and q. Consider the claim that

(13) $$aIb$$

is a true identity. Now if (13) is such a true identity, then a and b are the same thing. It doesn't say that a and b are two things which happen, through some accident, to be one. True, we are using two different names for that same thing, but we must be careful about use and mention. If, then, (13) is true, it must say the same thing as

(14) $$aIa\,.$$

But (14) is surely a tautology, and so (13) must surely be a tautology as well. This is precisely the import of my theorem (8). We would therefore expect, indeed it would be a consequence of the truth of (13), that 'a' is replaceable by 'b' in any context except those which are about the names 'a' and 'b'.

Now suppose we come upon a statement like

(15) Scott is the author of *Waverley*

and we have a decision to make. This decision cannot be made in a formal vacuum, but must depend to a considerable extent on some informal consideration as to what it is we are trying to say in (10) and (15). If we decide that 'the evening star' and 'the morning star' are names for the same thing, and that 'Scott' and 'the author of *Waverley*' are names for the same thing, then they must be intersubstitutable in every context. In fact it often happens, in a growing, changing language, that a descriptive phrase comes to be used as a proper name – an identifying tag – and the descriptive meaning is lost or ignored. Sometimes we use certain devices such as capitalization and dropping the definite article, to indicate the change in use. 'The evening star' becomes 'Evening Star', 'the morning star' becomes 'Morning Star', and they may come to be used as names for the same thing. Singular descriptions such as 'the little corporal', 'the Prince of Denmark', 'the sage of Concord', or 'the great dissenter', are as we know often used as alternative proper names of Napoleon, Hamlet, Thoreau and Oliver Wendell Holmes. One might even devise a criterion as to when a descriptive phrase is being used as a proper name. Suppose through some astronomical cataclysm, Venus was no longer the first star of the evening. If we continued to call it alternatively 'Evening Star' or 'the evening star' then this would be a measure of the conversion of the descriptive phrase into a proper name. If, however, we would then regard (10) as false, this would indicate that 'the evening star' was not used as an alternative proper name of Venus. We might mention in passing that although the conversion of descriptions into proper names appears to be asymmetric, we do find proper names used in singular descriptions of something other than the thing named, as in the statement 'Mao Tse-tung is the Stalin of China,' where one intends to assert a similarity between the entities named.

That any language must countenance some entities as things would appear to be a precondition for language. But this is not to say that experience is given to us as a collection of things, for it would appear that there are cultural variations and accompanying linguistic variations as to what sorts of entities are so singled out. It would also appear to be a precondition of language that the singling out of an entity as a thing is accompanied by many – and perhaps an indefinite or infinite number – of unique descriptions, for otherwise how would it be singled out? But to give a thing a proper name is different from giving a unique description. For suppose we took an inventory of all the entities countenanced as things by some particular culture through its own language, with its own set of names and equatable singular descriptions, and suppose that

number were finite (this assumption is for the sake of simplifying the exposition). And suppose we randomized as many whole numbers as we needed for a one-to-one correspondence, and thereby tagged each thing. This identifying tag is a proper name of the thing. In taking our inventory we discovered that many of the entities countenanced as things by that language-culture complex already had proper names, although in many cases a singular description may have been used. This tag, a proper name, has no meaning. It simply tags. It is not strongly equatable with any of the singular descriptions of the thing, although singular descriptions may be equatable (in a weaker sense) with each other where

(16) $\quad \text{Desc}_1 \text{ eq } \text{Desc}_2$

means that Desc_1 and Desc_2 describe the same thing. But here too, what we are asserting would depend on our choice of 'eq'. The principle of indiscernibility may be thought of as equating a proper name of a thing with the totality of its descriptions.

Perhaps I should mention that I am not unaware of the awful simplicity of the tagging procedure I described above. The assumption of finitude; and even if this were not assumed, then the assumption of denumerability of the class of things. Also, the assumption that all things countenanced by the language-culture complex are named or described. But my point is only to distinguish tagging from describing, proper names from descriptions. You may describe Venus as the evening star and I may describe Venus as the morning star, and we may both be surprised that as an empirical fact, the same thing is being described. But it is not an empirical fact that

(17) $\quad \text{Venus } I \text{ Venus}$

and if '*a*' is another proper name for Venus

(18) $\quad \text{Venus } I\ a\,.$

Nor is it extraordinary, that we often convert one of the descriptions of a thing into a proper name. Perhaps we ought to be more consistent in our use of upper-case letters, but this is a question of reforming ordinary language. It ought not to be an insurmountable problem for logicians. What I have been arguing in the past several minutes is, that to say of an identity (in the strongest sense of the word) that it is true, it must be tautologically true or analytically true. The controversial (8) of QS4 no more banishes concrete entities from the universe than (12) banishes from the universe red-blooded propositions.

Let us return now to (10) and (15). If they express a true identity, then 'Scott' ought to be anywhere intersubstitutable for 'the author of *Waverley*' and similarly for 'the morning star' and 'the evening star'. If they are not so universally intersubstitutable – that is, if our decision is that they are not simply proper names for the same thing; that they express an equivalence which is possibly false, e.g., someone else might have written *Waverley,* the star first seen in the evening might have been different from the star first seen in the morning – then they are not identities. One solution is Russell's, whose analysis provides a translation of (10) and (15) such that the truth of (10) and (15) does not commit us to the logical truth of (10) and (15), and certainly not to taking the 'eq' of (10) as identity, except on the explicit assumption of an extensionalizing axiom. Other and related solutions are in terms of membership in a non-empty unit class, or applicability of a unit attribute. But whatever the choice of a solution, it will have to be one which permits intersubstitutability, or some analogue of intersubstitutability for the members of the pairs: 'Scott' and 'the author of *Waverley*', and 'the evening star' and 'the morning star', which is short of being universal. In a language which is implicitly strongly extensional; that is where all contexts in which such substitutions fail are simply eschewed, then of course there is no harm in equating identity with weaker forms of equivalence. But why restrict ourselves in this way when, in a more intensional language, we can still make all the substitutions permissible to this weaker form of equivalence, yet admit contexts in which such a substitutivity is not permitted. To show this, I would like to turn to the instances of (4) which are provable [11] in QS4. I will again confine my remarks, for the purpose of exposition, to S4, although it is the generalizations for QS4 which are actually proved. An unrestricted

(19) $$x \equiv y \rightarrow z \equiv w$$

is clearly not provable whether '$\rightarrow$' is taken as material implication, strict implication or a metalinguistic conditional. It would involve us in a contradiction, if our interpreted system allowed statements such as

(20) $$(x \equiv y) \cdot \sim \Box(x \equiv y)$$

as it must if it is not to reduce itself to the system of material implication. Indeed, the underlying assumption about equivalence which is implicit in the whole 'evening star morning star' controversy is that there are equivalences (misleadingly called 'true identities') which are contingently true. Let x and y of (19) be taken as some p and q which satisfies (20). Let z be $\Box(p \equiv p)$ and w be $\Box(p \equiv q)$. Then (19) is

(21) $$(p \equiv q) \rightarrow (\Box(p \equiv p) \equiv \Box(p \equiv q)) .$$

From (20), simplification, modus ponens and $\Box(p \equiv p)$, which is a theorem of S4, we can deduce $\Box(p \equiv q)$. The latter and simplification of (20) and conjunction leads to the contradiction

(22) $$\Box(p \equiv q) \cdot \sim \Box(p \equiv q) .$$

A restricted form of (19) is provable. It is provable if z does not contain any modal operators. And this is exactly every context allowed in *Sm*, without at the same time banishing modal contexts. Indeed a slightly stronger (19) is provable. It is provable if x does not fall within the scope of a modal operator in z.

Where in (4), eq_1 and eq_2 are both taken as strict equivalence, the substitution theorem

(23) $$(x \equiv y) \rightarrow (z \equiv w)$$

is provable without restriction, and also where eq_1 is taken as strict equivalence and eq_2 is taken as material equivalence as in

(24) $$(x \equiv y) \rightarrow (z \equiv w) .$$

But (23) is also an extensionalizing principle, for it fails in epistemic contexts such as contexts involving 'knows that' or 'believes that'. For consider the statement

(25) When Professor Quine reviewed the paper on identity in QS4, he knew that $\vdash aI_m b \equiv aI_m b$.

and

(26) When Professor Quine reviewed the paper on identity in QS4 he knew that $\vdash aIb \equiv aI_m b$.

Although (25) is true, (26) is false for (7) holds in QS4. But rather than repeat the old mistakes by abandoning epistemic contexts to the shelf labelled 'referential opacity' after having rescued modal contexts as the most intensional permissible contexts to which such a language is appropriate, we need only conclude that (23) confines us to limits of applicability of such modal systems. If it should turn out that statements involving 'knows that' and 'believes that' permit of formal analysis, then such an analysis would have to be embedded in a language with a stronger equivalence relation than strict equivalence. Carnap's intensional isomorphism, Lewis' analytical comparability, and perhaps Anderson and Belnap's mutual entailment are attempts in that direction. But they too would be short of identity, for there are surely contexts in which substitutions allowed by such stronger equivalences, would convert a truth into a falsehood.

It is my opinion [12] that the identity relation need not be introduced for anything other than the entities we countenance as things such as individuals. Increasingly strong substitution theorems give the force of universal substitutivity without explicit axioms of extensionality. We can talk of equivalence between propositions, classes, attributes, without thereby conferring on them thinghood by equating such equivalences with the identity relation. QS4 has no explicit extensionality axiom. Instead we have (23), the restricted (19), and their analogues for attributes (classes).

The discussion of identity and substitution in QS4 would be incomplete without touching on the other familiar example:

(27) 9 eq the number of planets

is said to be a true identity for whcih substitution fails in

(28) $\Box(9 > 7)$

for it leads to the falsehood

(29) $\Box$(the number of planets > 7) .

Since the argument holds (27) to be contingent ($\sim \Box$ (9 eq the number of planets)), 'eq' of (27) is the appropriate analogue of material equivalence and consequently the step from (28) to (29) is not valid for the reason that the substitution would have to be made in the scope of the square. It was shown above that (19) is not an unrestricted theorem in QS4.

On the other hand, since in QS4

(30) $(5 + 4) =_s 9$

where '$=_s$' is the appropriate analogue for attributes (classes) of strict equivalence, '5 + 4' may replace '9' in (28) in accordance with (23). If, however, the square were dropped from (28) as it validly can for

(30a) $\Box p \prec p$

is provable, then by the restricted (19), the very same substitution available to *Sm* is available here.

THE INTERPRETATION OF QUANTIFICATION

The second prominent area of criticism of quantified modal logic involves interpretation of the operations of quantification when combined with modalities. It appears to me that at least some of the problems stem from an absence of an adequate, unequivocal, colloquial translation of the operations of quantification. It is often not quantification but our choice of reading and implicit interpretive consequences of such a reading which leads to difficulties. Such difficulties are not confined to modal systems. The most common reading of existential quantification is

(31) There is (exists) at least one (some) thing (person) which (who)...

Strawson,[13] for example, does not even admit of significant alternatives, for he says of (31): '...we might think it strange that the whole of modern formal logic after it leaves the propositional logic and before it crosses the boundary into the analysis of mathematical concepts, should be confined to the elaboration of sets of rules giving the logical interrelations of formulae which, however complex, begin with these few rather strained and awkward phrases.' Indeed, taking (31) at face value, Strawson gets into a muddle about tense ((31) is in the present tense), and the ambiguities of the word 'exist'. What we would like to have and do not have, is a direct, unequivocal colloquial reading of

(32) $(\exists x)\ \varphi x$

which gives us the force of either of the following:

(33) Some substitution instance of φx is true

or

There is at least one value of x for which φx is true.

I am not suggesting that (33) provides translations of (32), but only that what is wanted is a translation with the force of (32).
As seen from (33), quantification has primarily to do with truth and falsity, and open sentences. Reading in accordance with (31) may entangle us unnecessarily in ontological perplexities. For if quantification has to do with things and if variables for attributes or classes can be quantified upon, then in accordance with (31) they are things. If we still want to distinguish the identifying from the classifying function of language, then we are involved in a classification of different kinds of things and the accompanying platonic involvements. The solution is not to banish quantification on variables other than individual variables, but only not to be taken in by (31). We do in fact have some colloquial counterparts of (33). The non-temporal 'sometimes' or 'in some cases' or 'in at least one case', which have greater ontological neutrality than (31).
Some of the arguments involving modalities and quantification are closely connected with questions of substitution and identity. At the risk of boredom I will go through one again. In QS4 the following definitions are introduced:[14]

(34) $(\varphi =_m \Psi) =_{df} (x)(\varphi x \equiv \Psi x)$

(35) $(\varphi =_s \Psi) =_{df} \Box(\varphi =_m \Psi)$

Since the equality in (10) is contingent, (10) may be written as

(36) (the evening star $=_m$ the morning star) .

It is also the case that

(37) $\Diamond \sim$ (the evening star $=_m$ the morning star) .

One way of writing (11) is as

(38) $\Box$(the evening star $=_m$ the evening star) .

By existential generalization on (38), it follows that

(39) $(\exists\, \varphi)\, \Box\, (\varphi =_m$ the evening star) .

In the words of (31), (39) becomes

(40) There is a thing such that it is necessary that it is equal to the evening star .

The stubborn unlaid ghost rises again. Which thing, the evening star which by (36) is equal to the morning star? But such a substitution would lead to the falsehood

(41) $\Box$(the evening star $=_m$ the morning star) .

The argument may be repeated for (27) through (29).
In QS4 the solution is clear. For since (37) holds, and since in (39) 'φ' occurs within the scope of a square, then we cannot go from (39) to (41). On the other hand the step from (38) to (39) (existential instantiation) is entirely valid. For surely there is a value of φ for which

$$\Box(\varphi = \text{the evening star})$$

is true. In particular, the case where 'φ' is replaced by 'the evening star'.
There is also the specific problem of interpreting quantification in (6), which is a postulate of QS4. Read in accordance with (31) as

(42) If it is logically possible that there is something which φ's, then there is something such that it is logically possible that it φ's ,

it is admittedly odd. The antecedent seems to be about what is logically possible and the consequent about what there is. How can one go from possibility to existence? Read in accordance with (33) we have the clumsy but not so paradoxical

(43) If it is logically possible that φx for some value of x, then there is some value of x such that it is logically possible that φx .

Although the emphasis has now been shifted from things to statements, and the ontological consequences of (42) are absent, it is still indirect and awkward. It would appear that questions such as the acceptability or non-acceptability of (6) are best solved in terms of some semantical construction. This will be returned to in conclusion, but first some minor matters.
A defense of modal logic would be incomplete without touching on criticisms of modalities which stem from confusion about what is or isn't provable in such systems. One example is that of Rosenbloom[15] who seized on the fact that a strong deduction theorem is not available in

QS4, as a reason for discarding strict implication as in any way relevant to the deducibility relation. He failed to note [16] that a weaker and perhaps more appropriate deduction theorem is available. Indeed, Anderson and Belnap,[17] in their attempt to formalize entailment without modalities, reject the strong form of the deduction theorem as 'counter-intuitive for entailment'.

Another example occurs in *Word and Object* [18] which can be summarized as follows:

(44) Modalities yield talk of a difference between necessary and contingent attributes.

(45) Mathematicians may be said to be necessarily rational and not necessarily two-legged.

(46) Cyclists are necessarily two-legged and not necessarily rational.

(47) a is a mathematician and a cyclist.

(48) Is this concrete individual necessarily rational or contingently two-legged or vice versa?

(49) 'Talking referentially of the object with no special bias toward a background grouping of mathematicians as against cyclists... there is no semblance of sense in rating some of his attributes as necessary and others as contingent.'

Professor Quine says that (44) through (47) are supposed to 'evoke the appropriate sense of bewilderment' and they surely do. For I know of no interpreted modal system which countenances necessary attributes in the manner suggested. Translating (45) through (47) we have

(50) $(x)(Mx \prec Rx) \equiv (x)\Box(Mx \supset Rx) \equiv (x) \sim \Diamond(Mx \cdot \sim Rx)$

which is conjoined in (45) with

(51) $(x) \sim \Box(Mx \supset Tx) \equiv (x)\Diamond \sim (Mx \supset Tx) \equiv (x)\Diamond(Mx \cdot \sim Tx)$.

Also

(52) $(x)(Cx \prec Tx) \equiv (x)\Box(Cx \supset Tx) \equiv (x) \sim \Diamond(Cx \cdot \sim Tx)$

which is conjoined in (46) with

(53) $(x) \sim \Box(Cx \supset Rx) \equiv (x)\Diamond \sim (Cx \supset Rx) \equiv (x)\Diamond(Cx \cdot \sim Rx)$.

And in (48)

(54) $Ma \cdot Ca$

Among the conclusions we can draw from (49) through (53) are

$\Box(Ma \supset Ra), \sim \Diamond(Ma \cdot \sim Ra), \Diamond(Ma \cdot \sim Ta), \sim \Box(Ma \supset Ta),$
$\Box(Ca \supset Ta), \sim \Diamond(Ca \cdot \sim Ta), \Diamond(Ca \cdot \sim Ra), \sim \Box(Ca \cdot \sim Ra),$
$Ta, Ra, Ta \cdot Ra$

But nothing to answer question (48), or to make any sense of (49). It would appear that Professor Quine is assuming

(55) $(p \prec q) \prec (p \prec \Box q)$

is provable in QS4, but it is not, except where $p \equiv \Box r$ for some r. Keeping in mind that we are dealing with logical modalities, none of the attributes (M, R, T, C) in (50) through (54) taken separately, or conjoined, are necessary. It is not that sort of attribute which modal logic, even derivatively, countenances as being necessary. A word is appropriate here about the derivative sense in which we can speak of logically necessary and contingent attributes.

In QS4 abstracts are introduced such that to every function there corresponds an abstract, e.g.

(56) $x\varepsilon\hat{y}A =_{df} B$, where B is the result of substituting every free occurence of y in A by x.

If r is some abstract then we can define

(57) $x\varepsilon \boxdot r =_{df} \Box(x\varepsilon r), \quad \vdash \boxdot r =_{df} (x)(x\varepsilon \boxdot r)$

and

(58) $x\varepsilon \dot{\Diamond} r =_{df} \Diamond(x\varepsilon r), \quad \vdash \dot{\Diamond} r =_{df} (x)(x\varepsilon \dot{\Diamond} r)$

It is clear that among the abstracts to which $\vdash \boxdot$ may validly be affixed, will be those corresponding to tautological functions, e.g., $\hat{y}(yIy)$, $\hat{y}(\varphi x \vee \sim \varphi x)$, etc. It would be appropriate to call these necessary attributes, and the symbol '$\boxdot$' is a derivative way of applying modalities to attributes.

Similarly, all of the attributes of (50) through (54) could in the sense of (58) be called contingent, where '$\diamond$' is the derivative modality for contingency of attributes. However, if (50) is true, then the attribute of being either a mathematician or not rational could appropriately be called necessary, for

$$(59) \qquad (x) \Box (x\varepsilon\hat{y}(Mx \vee \sim Rx)) .$$

SEMANTIC CONSTRUCTIONS

I would like in conclusion to suggest that the polemics of modal logic are perhaps best carried out in terms of some explicit semantical construction. As we have seen in connection with (6) it is awkward at best and at worst has the character of a quibble, not to do so.

Let us reappraise (6) in terms of such a construction.[19] Consider for example a language (L), with truth functional connectives, a modal operator ($\Diamond$), a finite number of individual constants, an infinite number of individual variables, one two-place predicate (R), quantification and the usual criteria for being well-formed. A domain (D) of individuals is then considered which are named by the constants of L. A model of L is defined as a class of ordered couples (possibly empty) of D. The members of a model are exactly those pairs between which R holds. To say therefore that the atomic sentence $R(a_1a_2)$ of L holds or is true in M, is to say that the ordered couple (b_1, b_2) is a member of M, where a_1 and a_2 are the names in L of b_1 and b_2. If a sentence A of L is of the form $\sim B$, A is true in M if and only if B is not true in M. If A is of the form $B_1 \cdot B_2$ then A is true in M if and only if both B_1 and B_2 are true in M. If A is of the form $(\exists x)\, B$, then A is true in M if and only if at least one substitution instance of B is true (holds) in M. If A is $\Diamond\, B$ then A is true in M if and only if B is true in some model M_1.

We see that a true sentence of L is defined relative to a model and a domain of individuals. A logically true sentence is one which would be true in every model. We are now in a position to give a rough proof of (6). Suppose (6) is false in some M. Then

$$\sim (\Diamond (\exists x)\, \varphi x \cdot \sim (\exists x) \Diamond \varphi x)$$

is false in M. Therefore

$$\Diamond (\Diamond (\exists x)\, \varphi x \cdot \sim (\exists x) \Diamond \varphi x)$$

is true in M. So

$$\Diamond (\exists x) \phi x \cdot \sim (\exists x) \Diamond \phi x$$

is true in some M_1. Therefore

$$(60) \qquad \Diamond (\exists x)\, \varphi x$$

and

$$(61) \qquad \sim (\exists x) \Diamond \varphi x$$

are true in M_1. Consequently, from (60)

$$(62) \qquad (\exists x)\, \varphi x$$

is true in some model M_2. Therefore there is a member of D (b) such that

$$(63) \qquad \varphi b$$

is true in M_2. But from (61)

$$(\exists x) \Diamond \varphi x$$

is not true in M_1. Consequently there is no member of D such that

$$(64) \qquad \Diamond \varphi b$$

is true in M_1. So there is no model M_2 such that φb is true in M_2. But this result contradicts (63). Consequently, in such a construction, (6) must be true in every model.

If this is the sort of construction one has in mind then we are persuaded of the plausibility of (6). Indeed, going back to (43), it can be seen that this was the sort of construction which was being assumed. If (6) is to be regarded as offensive in a way other (and here I am borrowing an image from Professor White) than the manner in which we regard eating peas with a fork as offensive, it must be in terms of some semantic construction which ought to be made explicit.[20]

We see, that though the rough outline above corresponds to the Leibnizian distinction between true in a possible world and true in all possible worlds, it is also to be noted that there are no specifically intensional objects. No new entity is spawned in a possible world that isn't already in the domain in terms of which the class of models is defined. In such a model modal operators have to do with truth relative to the model, not with things. On this interpretation,[21] Professor Quine's 'flight from intension' may have been exhilarating, but unnecessary.[22]

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NOTES

1. W. V. Quine, *Word and Object*, 1960, pp. 195–196.
2. a. F. B. Fitch, *Symbolic Logic*, New York, 1952.
   b. R. Carnap, 'Modalities and quantification,' *Journal of Symbolic Logic*, Vol. XI (1946), pp. 33–64.
   c. R. C. Barcan (Marcus), 'A functional calculus of first order based on strict implication,' *Journal of Symbolic Logic*, Vol. XI (1946), pp. 1–16.
   d. R. C. Barcan (Marcus), 'The identity of individuals in a strict functional calculus of first order,' *Journal of Symbolic Logic*, Vol. XII (1947), pp. 12–15.
3. P. F. Strawson, *Introduction to Logical Theory*, London, 1952, p. 216.
4. F. P. Ramsey, *The Foundations of Mathematics*, London and New York, 1931, pp. 30–32.
5. *Op. cit*, notes 2c, 2d.
6. C. I. Lewis and C. H. Langford, *Symbolic Logic*, New York, 1932.
7. See A. N. Prior, *Time and Modality*, Oxford, 1932, for an extended discussion of this axiom.
8. S5 results from adding to S4.

$$p \prec \Box \Diamond p.$$

9. A. N. Prior, 'Modality and Quantification in S5,' *Journal of Symbolic Logic* March, 1956.
10. W. V. Quine, *From a Logical Point of View*, Cambridge, 1953, pp. 152–154.
11. *Op. cit.*, note 2c. Theorem XIX* corresponds to (23). The restricted (19), given the conditions of the restriction, although not actually proved, is clearly provable in the same manner as XIX*.
12. See R. Barcan Marcus, 'Extensionality,' *Mind*, Vol. LXIX, n.s., pp. 55–62 which overlaps to some extent the present paper.
13. *Op. cit.*, note 3.
14. *Op. cit.*, note 2c. Abstracts are introduced and attributes (classes) may be equated with abstracts. Among the obvious features of such a calculus of attributes (classes), is the presence of equivalent, non-identical, empty attributes (classes). If the null attribute (class) is defined in terms of identity, then it wil be intersubstitutible with any abstract on a contradictory function.

15. P. Rosenbloom, *The Elements of Mathematical Logic*, New York, 1950, p. 60.
16. R. Barcan Marcus, 'Strict implication, deducibility, and the deduction theorem,' *Journal of Symbolic Logic*, Vol. XVIII (1953), pp. 234–236.
17. A. R. Anderson and N. D. Belnap, *The Pure Calculus of Entailment*, (preprint).
18. *Op. cit.*, note 1, pp. 199–200.
19. The construction here outlined corresponds to that of R. Carnap, *Meaning and Necessity*, Chicago, 1947. The statement of the construction is in accordance with the method of J. C. C. McKinsey. See also, J. C. C. McKinsey, 'On the syntactical construction of systems of modal logic,' *Journal of Symbolic Logic*, Vol. X (1946), pp. 88–94; 'A new definition of truth,' *Synthese*, Vol. VII (1948/49), pp. 428–433.
20. A criticism of the construction here outlined is the assumption of the countability of members of D. McKinsey points this out in the one chapter I have seen (Chapter I, Vol. II), of a projected (unpublished) two volume study of modal logic, and indicates that his construction will not assume the countability of members of D. Whereas Carnap's construction leads to a system at least as strong as S5, McKinsey's (he claims) will be at least as strong as S4 (without (6) I would assume). I've not seen, nor been able to locate any other parts of this study in which the details are worked out out along with completeness proofs for some of the Lewis systems. See also, J. Myhill, *Logique et analyse*, 1958, pp. 74–83; and S. Kripke, *Journal of Symbolic Logic*, Vol. XXIV (1959), pp. 323–324 (abstract).
22. If one wishes to talk about possible things then of course such a construction is inadequate.
22. This paper was written while the author was under N.S.F. Grant 24335.