CL exercise for
Tutorial 4

Introduction

Objectives

In this tutorial, you will:

- learn more about *sequents* and *combining predicates*;
- derive de Morgan's second law;
- do proofs in *sequent calculus*.

Tasks

Exercises 1 and 2 are mandatory. Exercises 3 is optional. Exercise 4 is for your own interest only.

Submit

a file called `cl-tutorial-4` with your answers (image or pdf).

Deadline

16:00 Tuesday 18 October

Reminder

Good Scholarly Practice

Please remember the good scholarly practice requirements of the University regarding work for credit.

You can find guidance at the School page

https://web.inf.ed.ac.uk/infweb/admin/policies/academic-misconduct.

This also has links to the relevant University pages. Please do not publish solutions to these exercises on the internet or elsewhere, to avoid others copying your solutions.

Exercise 1 –mandatory—marked–

Read Chapter 14 (*Sequent Calculus*) of the textbook.

Derive the second of de Morgan's laws

$$\neg(a \wedge b) = \neg a \vee \neg b$$

using a similar argument to the one presented in the textbook for the first law on page 122.

Solution to Exercise 1

We start with the following proofs:

$$\dfrac{\dfrac{\dfrac{\neg a \models c}{\neg c \models a}\,\neg R, \neg L \qquad \dfrac{\neg b \models c}{\neg c \models b}\,\neg R, \neg L}{\neg c \models a \wedge b}\,\wedge R}{\neg(a \wedge b) \models c}\,\neg R, \neg L$$

and

$$\dfrac{\neg a \models c \qquad \neg b \models c}{\neg a \vee \neg b \models c}\,\vee L$$

which gives $\dfrac{\neg(a \wedge b) \models c}{\neg a \vee \neg b \models c}$.

Now take c to be each of the terms in turn, to get the desired result. (Don't insist on them spelling out this bit. If they do, it should look like:

$$\dfrac{\dfrac{}{\neg a \vee \neg b \models \neg a \vee \neg b}\,I}{\neg(a \wedge b) \models \neg a \vee \neg b}\,\text{above rule} \qquad \dfrac{\dfrac{}{\neg(a \wedge b) \models \neg(a \wedge b)}\,I}{\neg a \vee \neg b \models \neg(a \wedge b)}\,\text{above rule}$$

Exercise 2 –mandatory—marked–

Write a proof which reduces the conclusion

$$(x \vee y) \wedge (x \vee z) \models x \vee (y \wedge z)$$

to premises that can't be reduced further.

Is it universally valid? If not, give a counterexample.

Solution to Exercise 2

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y, x \vee z \models x, y}\,I \qquad \dfrac{}{x, x \vee z \models x, y}\,I}{x \vee y, x \vee z \models x, y}\,\vee L \qquad \dfrac{\dfrac{\dfrac{}{y, x \models x, z}\,I \qquad \dfrac{}{y, z \models x, z}\,I}{y, x \vee z \models x, z}\,\vee L \qquad \dfrac{}{x, x \vee z \models x, z}\,I}{x \vee y, x \vee z \models x, z}\,\vee L}{x \vee y, x \vee z \models x, y \wedge z}\,\wedge R}{(x \vee y) \wedge (x \vee z) \models x \vee (y \wedge z)}\,\wedge L, \vee R$$

(If students abbreviate things, that's fine, as long as you can see what they mean. Equally, I've telescoped rules here – students probably won't, but it's fine if they do.)

Because the conclusion has been shown to follow from the empty set of premises, it is universally valid.

Exercise 3 –optional—marked–

Write a proof which reduces the conclusion

$$\models (x \wedge y) \vee (\neg(x \vee z) \vee (\neg y \vee z))$$

to premises that can't be reduced further.

Expressions φ like $(x \wedge y) \vee (\neg(x \vee z) \vee (\neg y \vee z))$ used in the antecedents and succedents of sequents are called:

- *tautologies* when $\models \varphi$ is valid (the antecedent is empty);
- *contradictions* when $\varphi \models$ is valid (the succedent is empty);

Is $(x \wedge y) \vee (\neg(x \vee z) \vee (\neg y \vee z))$ a tautology, a contradiction, or neither?

Solution to Exercise 3

In somewhat abbreviated form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y, x \models x, z}\,I \qquad \dfrac{}{y, x \models y, z}\,I}{y, x \models x \wedge y, z}\,\wedge R \qquad \dfrac{}{y, z \models x \wedge y, z}\,I}{y, x \vee z \models x \wedge y, z}\,\vee L}{\models x \wedge y, \neg(x \vee z), \neg y, z}\,\neg R}{\models (x \wedge y) \vee (\neg(x \vee z) \vee (\neg y \vee z))}\,\vee R \times 3$$

Because the conclusion follows from the empty set, it is a tautology.

Exercise 4 –optional—not marked–

Do not submit a solution for this exercise. Discuss in tutorials if you wish!

Write proofs which reduce the conclusions

$$\neg a \wedge \neg b \models \neg(a \wedge b)$$

and

$$\neg(a \wedge b) \models \neg a \wedge \neg b$$

to premises that can't be reduced further.

Is one or both universally valid?

- If not, give a counterexample.
- If so, explain how that shows that $\neg a \wedge \neg b = \neg(a \wedge b)$.

Solution to Exercise 4

The first sequent is universally valid, with the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a, b \vDash b, a}\ I}{\neg a, \neg b, a, b \vDash}\ \neg L \times 2}{\neg a \wedge \neg b, a \wedge b \vDash}\ \wedge L \times 2}{\neg a \wedge \neg b \vDash \neg(a \wedge b)}\ \neg R$$

Trying to prove the second sequent results in

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vDash a}\ I}{\vDash a, \neg a}\ \neg R \quad \dfrac{b \vDash a}{\vDash a, \neg b}\ \neg R}{\vDash a, \neg a \wedge \neg b}\ \wedge R \quad \dfrac{\dfrac{a \vDash b}{\vDash b, \neg a}\ \neg R \quad \dfrac{\dfrac{}{b \vDash b}\ I}{\vDash b, \neg b}\ \neg R}{\vDash b, \neg a \wedge \neg b}\ \wedge R}{\vDash a \wedge b, \neg a \wedge \neg b}\ \wedge R}{\neg(a \wedge b) \vDash \neg a \wedge \neg b}\ \neg L$$

or similar (this is automatically generated, not necesarily the shortest!).

The proof shows that the conclusion follows from the two premises $a \models b$ and $b \models a$, meaning that it is true whenever both of those sequents are true. One counterexample is a universe containing a thing x for which $a(x)$ is true and $b(x)$ is false. Another counterexample is a universe containing a thing x for which $b(x)$ is true and $a(x)$ is false.

If both conclusions were universally valid, then $\neg a \wedge \neg b = \neg(a \wedge b)$ would hold since $\models$ corresponds to set inclusion and we would have shown that each side of the equation is a subset of the other.