Prof. Dr. I. F. Sbalzarini
TU Dresden, 01187 Dresden, Germany

Exercise 4

Release: 16.11.2020
Due: 23.11.2020

Question 1: Conjugate gradient (CG)

Given a matrix $A(n \times n)$ and the cost function $F(u)$ defined as

$$F(x) = \frac{1}{2}x^T Ax - x^T b + c$$

a) Show that $\nabla F(x) = Ax - b$

b) For what property of the matrix A is the solution to the problem $\nabla F(x) = 0$ unique ?

c) Given

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \quad , \quad c = 0$$

Compute and plot the quadratic form $F(x)$ with $x_1 \in (-4, 6)$ and $x_2 \in (4, -6)$ as both surface and the contour plots.

d) In the previous problem, check if the solution to $\nabla F(x) = 0$ sits at the global minimum in the plot. Also comment on the positive definiteness of A.

Question 2: Convergence analysis of CG method

Given the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

a) Evaluate the number of iterations required for CG algorithm to reduce the absolute initial error $\|e^0\|_A$ (in the A-norms) by a factor of 10^{-10}.

Question 3: Programming task

a) Implement the CG method in MATLAB/python.

b) Test your code for the solution of linear system $Ax = b$ given by,

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 4 & 1 & 3 \\ 0 & 1 & 10 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

c) Compare the solution from your code with MATLAB command `pcg(A,b,tol,maxit)` with `tol` $= 10^{-7}$ and `maxit` $= 1000$. Evaluate the number of iterations and the CPU time required to reach the tolerance `tol`. See if this is consistent with the analysis of the problem 2 a).

d) Repeat b) with the preconditioning Matrix $C = HH^T$, where H is the incomplete Cholesky Decomposition of A. Compare your solution from your code with MATLAB command `pcg(A,b,tol,maxit,M1,M2)`, here `M1` $= H$ and `M2` $= H^T$.
(Hint : Use `ichol(A)` for incomplete Cholesky Decomposition in MATLAB)

e) Given the sparse matrix A ($n \times n$) and the load vector b ($n \times 1$), use your CG algorithm to find the solution of the linear system $Ax = b$ for $n \in (100, 500, 1000, 10000)$ and $h = \frac{1}{n}$.

$$A = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & \dots & \dots & \dots & \vdots \\ \vdots & 1 & -2 & 1 & \dots & \dots & \vdots \\ \vdots & \vdots & 1 & -2 & 1 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 & -2 & 1 \\ 0 & 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 2h^2 \\ 2h^2 \\ 2h^2 \\ \vdots \\ \vdots \\ \vdots \\ 2h^2 \end{pmatrix}$$

f) For the above problem, plot the number of iterations (k), CPU time taken (t) and the condition number (κ) versus the size (n) of the matrix A.