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Exercise 10

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Question 1: Heun's procedure

a) Implement the Heun's procedure to

$$y^*_{n+1} = \tilde{y}_n + hf(t_n, \tilde{y}_n)$$
$$\tilde{y}_{n+1} = \tilde{y}_n + \frac{h}{2}\left(f(t_n, \tilde{y}_n) + f(t_{n+1}, y^*_{n+1})\right)$$

and solve the 2-body problem given by,

$$\dot{y}_1 = y_3$$
$$\dot{y}_2 = y_4$$
$$\dot{y}_3 = -\frac{y_1}{(y_1^2 + y_2^2)^{3/2}}$$
$$\dot{y}_4 = -\frac{y_2}{(y_1^2 + y_2^2)^{3/2}}$$

with the initial values $y_1(0) = 0.5$, $y_2(0) = 0$, $y_3(0) = 0$, $y_4(0) = \sqrt{(}3)$. To do this, select $t_0 = 0, t_f = 8$, use the Heun's method with two steps $h = \frac{1}{10}, \frac{1}{100}$ and draw the paths $(\tilde{y}_1(t_n), \tilde{y}_2(t_n))$

b) Compare your results with the results of MATLAB's **ode23**

Question 2: RK4

Consider the Linear Differential Equation system

$$\dot{x} = \begin{pmatrix} -1001 & 999 \\ 999 & -1001 \end{pmatrix} x$$

a) Determine the general solution $x(t)$.

b) Apply the explicit Euler and Runge-Kutta 4 procedure to this system with the following initial conditions and compare the error from the analytical solution in a) for different values of h.
i) $x(0) = (-1, 1)^T$
ii) $x(0) = (1, 1)^T$
iii) $x(0) = (2, 0)^T$

Question 3: Adaptive Step Size

Consider the joint Butcher table of the Dormand-Prince method (DOPRI5),

$$
\begin{array}{c|ccccccc}
0 & & & & & & & \\
\frac{1}{5} & \frac{1}{5} & & & & & & \\
\frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & & \\
\frac{4}{5} & \frac{44}{45} & -\frac{56}{15} & \frac{32}{9} & & & & \\
\frac{8}{9} & \frac{19372}{6561} & -\frac{25360}{2187} & \frac{64448}{6561} & -\frac{212}{729} & & & \\
1 & \frac{9017}{3168} & -\frac{355}{33} & \frac{46732}{5247} & \frac{49}{176} & -\frac{5103}{18656} & & \\
1 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & \\
\hline
 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0 \\
\hline
 & \frac{5179}{57600} & 0 & \frac{7571}{16695} & \frac{393}{640} & -\frac{92097}{339200} & \frac{187}{2100} & \frac{1}{40}
\end{array}
$$

a) Using the table, derive the expressions for computing each stage of the DOPRI5 method.

b) Using the DOPRI5 method, implement algorithm 13: "Dynamic step-size adaptation for one-step methods" from the lecture notes and apply it to the skewed logistic map ODE,

$$f'(x) = 0.01\,(1 - f(x))\,f(x)$$

with the initial condition $f(x = -5) = 0.00669$. Find the number of steps required to reconstruct the logistic function up-to $f(x) = 0.99999$. Also plot a graph of the step size vs step-index.