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Solution 5

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Question 1: Least-Squares formulation, Normal equation

a) Given three points $P_i = (x_i, y_i),\ i = 1, 2, 3$, wherein

x_i	0	1	2
y_i	5.41	5.17	5.93

Determine a linear function $y = f(x) = ax + b$, so that the sum of the error squares in the y-direction

$$\sum_{i=1}^{3} |f(x_i) - y_i|^2 \qquad \text{is minimized}$$

Solution : We would like to minimize $\sum_{i=1}^{3} (f(x_i) - y_i)^2$

$$\begin{aligned} ax_1 + b - y_1 &= r_1 \\ ax_2 + b - y_2 &= r_2 \\ ax_3 + b - y_3 &= r_3 \end{aligned}$$

We can reduce the problem to linear-system $Ap = c$, and solve the problem for least-squares using Normal Equations i.e. $p = (A^T A)^{-1} A^T c$

$$A = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{pmatrix}; \quad c = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}; \; p = \begin{pmatrix} a \\ b \end{pmatrix}$$

$p = (A^T A)^{-1} A^T c \implies p = (0.26, 5.243)$.
So $f(x) = ax + b$, where $\mathbf{a = 0.26,\ b = 5.243}$

b) Consider the matrix $A\,(3 \times 2)$ and vector $b\,(3 \times 1)$, given by

$$A = \begin{pmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

i) Write the least-square form which leads to the solution of linear system $Ax = b$.

ii) Check if the method of Normal Equations is stable for matrix A and $0 < \epsilon \ll 1$.
Solution : i) $x = (A^T A)^{-1} A^T b$
ii) $(A^T A)$ is singular, so the method Normal equations is unstable.

Question 2: Least-squares, QR decomposition and SVD

Following are the velocity measurements $f(t)$ in ms^{-1} from the pitot-tube of a descending airplane at time t_i, $i = 1, 2, ..., 10$.

t_i	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_i	100	34	17	12	9	6	5	4	4	2

We express the unknown function $f(t) = \sum_{j=1}^{4} \lambda_j \phi_j(t)$ as the linear combination of known functions $\phi_j(t)$, $j = 1, 2, 3, 4$, given by,

$$\phi_1(t) = \frac{1}{t},\ \ \phi_2(t) = \frac{1}{t^2},\ \ \phi_3(t) = e^{-(t-1)},\ \ \phi_4(t) = e^{-2(t-1)}$$

Determine the coefficients λ_j, the linear combination such that

$$\sum_{10}^{i=1} |f(t_i) - f_i|^2 \qquad \text{is minimized}$$

a) using normal equations

b) using the QR decomposition of the matrix A

c) by means of singular value decomposition (SVD)
(MATLAB command `[U, S, V] = svd(A)`)

Solution : Set up the linear system $Ax = f$ for least-square solution, i.e

$t = [0.1\ \ 0.2\ \ 0.3\ \ ....\ \ 1.0]^T$

$f = [100\ \ 34\ \ 17\ \ ....\ \ 12]^T$

$A = [1.0/t\ \ 1.0/t^2\ \ e^{-(t-1)}\ \ e^{-2(t-1)}]$

a) Method of normal equations $x = (A^T A)^{-1} A^T f$,

$\implies \lambda_1 = 4.0591, \lambda_2 = 0.6140, \lambda_3 = -2.5315, \lambda_4 = 0.7058$

b) $A = QR \implies QRx = b \implies Rx = Q^T b$

$\implies \lambda_1 = 4.0591, \lambda_2 = 0.6140, \lambda_3 = -2.5315, \lambda_4 = 0.7058$

c) $A = USV^T \implies x = VS^{-1}U^T b$

$\implies \lambda_1 = 4.0591, \lambda_2 = 0.6140, \lambda_3 = -2.5315, \lambda_4 = 0.7058$

Question 3: SVD decomposition by hand

Given the matrices A and B,

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

a) Find the rank and singular values of the matrix A and matrix B.

b) Find the SVD decomposition of the matrix A and B. Also comment on the uniqueness of the decomposition.

Solution :
a)Rank(A) = 2

$$A^T A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \quad ; \quad \lambda_1 = \sqrt{3 + \sqrt(3)},\ \lambda_2 = \sqrt{3 - \sqrt(3)},\ \lambda_3 = 0$$

Rank(B) = 2

$$B^T B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad ; \quad \lambda_1 = \sqrt{2},\ \lambda_2 = \sqrt{2}$$

Note λ are the singular values (i.e., square root of the Eigen values of the product with transpose).
b) Refer to the following tutorial:
`http://web.mit.edu/be.400/www/SVD/Singular_Value_Decomposition.htm`
Only the singular value matrix S is always unique for the SVD.