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Solution 14

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Question 1: Von Neumann-stability analysis

$$u_t = \alpha u_{xx}$$

a) Show, using the Von Neumann-stability analysis, that the Crank-Nicolson method applied to the heat equation with central finite differences in space, is unconditionally stable

Solution:
The numerical discretization using Crank-Nicolson method for time-integration and central-difference for space derivatives looks like

$$u_i^{n+1} = \Delta t\, u_i^n + \frac{\alpha \Delta t}{2\Delta x^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) + \frac{\alpha \Delta t}{2\Delta x^2}\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right)$$

For linear-problems, we can show that the error at position i and time n, evolves according to

$$e_i^{n+1} = \Delta t\, e_i^n + \frac{\alpha \Delta t}{2\Delta x^2}\left(e_{i-1}^n - 2e_i^n + e_{i+1}^n\right) + \frac{\alpha \Delta t}{2\Delta x^2}\left(e_{i-1}^{n+1} - 2e_i^{n+1} + e_{i+1}^{n+1}\right)$$

Expanding error as Fourier series $e(x) = \sum_{m=1}^{M} e^{at} e^{ik_m x}$, substituting for the above form for single k yields

$$e^{at} = 1 + \frac{\alpha \Delta t}{2\Delta x^2}\left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x}\right) + \frac{\alpha \Delta t e^{at}}{2\Delta x^2}\left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x}\right)$$

$$e^{at} = \frac{1 + \frac{\alpha \Delta t}{2\Delta x^2}\left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x}\right)}{\left(1 - \frac{\alpha \Delta t}{2\Delta x^2}\left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x}\right)\right)} \quad (*)$$

Using Euler-formula $e^{ix} = \cos(\mathrm{x}) + \mathrm{i}\sin(\mathrm{x})$, we can show that

$$\sin(\frac{k_m \Delta x}{2}) = \frac{e^{ik_m \Delta x/2} - e^{-ik_m \Delta x/2}}{2i} \implies \sin^2(\frac{k_m \Delta x}{2}) = -\frac{e^{ik_m \Delta x/2} + e^{-ik_m \Delta x/2} - 2}{4}$$

Substituting in(*),

$$|e^{at}| = |\frac{e^{n+1}}{e^n}| = |\frac{1 - \frac{2\alpha \Delta t}{\Delta x^2}\sin^2(\frac{k_m \Delta x}{2})}{1 + \frac{2\alpha \Delta t}{\Delta x^2}\sin^2(\frac{k_m \Delta x}{2})}| \leq 1$$

So, the amplification factor is always bounded by 1. So, Crank-Nicolson with central difference for space-derivatives is unconditionally stable.

b) In a similar way, show that the Leap-frog method applied to above equation is unconditionally unstable.

Solution:
The discretized numerical solution with leap-frog time-stepping looks like

$$u_i^{n+1} = u_i^{n-1} + 2r\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right)$$

where $r = \frac{\alpha \Delta t}{\Delta x^2}$.
The corresponding error evolution equation is

$$e_i^{n+1} = e_i^{n-1} + 2r\left(e_{i-1}^n - 2e_i^n + e_{i+1}^n\right)$$

Repeating the same analysis as before, we get the following quadratic $g = e^{at}$,

$$g^2 + 8rg\sin^2\left(\frac{k\Delta x}{2}\right) - 1 = 0$$

The roots of the equation are,

$$g_{\pm} = -4r\sin^2\left(\frac{k\Delta x}{2}\right) \pm \sqrt{16r^2\sin^4\left(\frac{k\Delta x}{2}\right) + 1}$$

Consider the cases,
case 1: $r = 1$ and $k\Delta x = \pi, |g_{-1}| > 1$,
case 2: $r > 1$ and $k\Delta x = \pi, |g_{-1}| > 1$
case 3: $r < 1$, stable for some r

Question 2: Hyperbolic equation

Consider the PDE for advection equation

$$u_t + cu_x = 0$$

Show that for the CTCS-method (Leapfrog?) the local truncation error is of the form

$$\text{error} = -\frac{1}{6}\Delta t^2 u_{ttt}|_i^n - \frac{c}{6}\Delta x^2 u_{xxx}|_i^n + \text{H.O.T in } \Delta t \text{ and } \Delta x$$

Solution:

The CTCS discretization for the advection equation,

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \quad (*)$$

Taylor expanding around the solution (x_i, t^n), we obtain,

$$u_i^{n\pm1} = u_i^n \pm \Delta t u_t|_i^n + \frac{\Delta t^2}{2}u_{tt}|_i^n \pm \frac{\Delta t^3}{3!}u_{ttt}|_i^n + \text{H.O.T in } \Delta t$$
$$u_{i\pm1}^n = u_i^n \pm \Delta x u_x|_i^n + \frac{\Delta x^2}{2}u_{xx}|_i^n \pm \frac{\Delta x^3}{3!}u_{xxx}|_i^n + \text{H.O.T in } \Delta x$$

Substituting the above expansions into the difference equation (*), we obtain.

$$\frac{2\Delta t u_t|_i^n + \frac{\Delta t^3}{3!}u_{ttt}|_i^n + \text{H.O.T in } \Delta t}{2\Delta t} + c\frac{2\Delta x u_x|_i^n + \frac{\Delta t^3}{3!}u_{xxx}|_i^n + \text{H.O.T in } \Delta x}{2\Delta x}$$

The PDE satisfies the $(u_t + cu_x)\,|_i^n = 0$, simplifying the above expression to

$$\tau = -\frac{1}{6}\Delta t^2 u_{ttt}|_i^n - \frac{c}{6}\Delta x^2 u_{xxx}|_i^n + \text{H.O.T in } \Delta t \text{ and } \Delta x$$

Question 3: Stability of hyperbolic PDEs

Work out the Von Neumann stability analysis for the wave equation with the CTCS scheme

$$u_{tt} = c^2 u_{xx}$$
$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (CTCS)$$

Solution:
the error propagation equation looks like,

$$e_j^{n+1} - 2e_j^n + e_j^{n-1} = p^2\left(e_{j+1}^n - 2e_j^n + e_{j-1}^n\right) \quad (CTCS)$$

where $p^2 = \frac{c^2\Delta t^2}{\Delta x^2}$ Repeating similar analysis as in previous problems,

$$g^2 - 2g + 1 = -4p^2 g \sin^2\left(\frac{k\Delta x}{2}\right)$$
$$g^2 - 2g(1 - 2p^2)\sin^2\left(\frac{k\Delta x}{2}\right) + 1 = 0$$

The roots of the equation are,

$$g_\pm = 1 - 2p^2\sin^2\left(\frac{k\Delta x}{2}\right) \pm \sqrt{4p^2\sin^2\left(\frac{k\Delta x}{2}\right)\left[p^2\sin^2\left(\frac{k\Delta x}{2}\right) - 1\right]}$$

Let us consider three cases,
case 1: $p^2 < 1 : |g_\pm| < 1$,
case 2: $p^2 = 1 : |g_\pm| = 1$, Hence the scheme is stable for $p^2 \leq 1$
case 3: $p^2 > 1$, consider the scenario, $k\Delta x = \pi$,

$$g_\pm = 1 - 2p^2 \pm 2p\sqrt{p^2 - 1}$$

So $q_{-1} < 1 - 2p^2 < -1$, $\forall p^2 > 1$, thus $|g_{-1}| > 1$ at $k\Delta x = \pi$

So the CTCS scheme is only stable for $p^2 < 1$

Question 4: Mass conservation

Show that for the non-linear hyperbolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial [F(u)]}{\partial x} = 0 \quad (*)$$

the following property holds

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx \ \ \forall t \geq 0$$

if we assume that $\lim_{x\to\pm\infty} F(u(x,t)) = 0, \ \ \forall t \geq 0$

Solution

Integrating the equation (*) over the interval $[-\infty, \infty] \times [0, t]$ and using the fundamental theorem of calculus, we obtain

$$\int_0^t \int_{-\infty}^{\infty} (u_t + [F(u)]_x)\, dxdt = \int_{-\infty}^{\infty} [u(x,t) - u(x,0)]dx + \int_0^t [F(u(\infty,t)) - F(u(-\infty,t))]dt = 0$$

Taking the fluxes to be zero at infinity i.e. $\lim_{x\to\pm\infty} F(u(x,t)) = 0, \ \ \forall t \geq 0$, we have

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx, \ \ \forall t \geq 0$$