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Solution 7

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Question 1: Interpolation

Below are the function values of a function f at nodes x_i, $i = 0, 1, 2, 3$

x_i	1	2	4	8
f_i	0	-2	-1	2

a) Determine the Lagrange interpolation polynomial for the above data points and evaluate the polynomial at $x = 3$.

Solution: Lagrange's interpolation follows :

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \quad \text{for i = 0,1,2,..n}$$

$$\text{The polynomial } P_n(x) = \sum_{i=0}^{n} l_i(x) f_i$$

For $n = 3$ and $x = 3$, we have

$$\begin{aligned}
l_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}; \quad l_0(3) = -\frac{5}{21} \\
l_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}; \quad l_1(3) = \frac{5}{6} \\
l_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}; \quad l_2(3) = \frac{5}{12} \\
l_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}; \quad l_3(3) = -\frac{1}{84} \\
P_3(3) &= l_0(3)f_0 + l_1(3)f_1 + l_2(3)f_2 + l_3(3)f_3 = -\frac{59}{28}
\end{aligned}$$

b) Evaluate the interpolation polynomial at $x = 3$ using the Barycentric formula.

Solution: Barycentric formula for the polynomial $P_n(x)$,

$$P_n(x) = \frac{\sum_{j=0}^{n} \frac{w_j}{(x-x_j)} f_j}{\sum_{j=0}^{n} \frac{w_j}{(x-x_j)}}, \quad \text{where } w_j = \frac{1}{\prod_{j\neq k}(x_j - x_k)}$$

For the above problem for $n = 3$ and $x = 3$, we get,

$$\begin{aligned}
w_0 &= \frac{1}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = -\frac{1}{21} \\
w_1 &= \frac{1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{1}{12} \\
w_2 &= \frac{1}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = -\frac{1}{24} \\
w_3 &= \frac{1}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{1}{168}
\end{aligned}$$

Substituting the values of w_j in the formula for Barycentric interpolation $P_n(x) = -\frac{59}{28}$

c) Evaluate the interpolation polynomial at $x = 5$ using the algorithm for Aitken-Neville interpolation.

Solution :

$$\begin{array}{c|c|c|c|c}
x_0 & P_0 = 0, & & & \\
x_1 & P_1 = -2, & P_{01}(5) = -8 & & \\
x_2 & P_2 = -1, & P_{12}(5) = -\frac{1}{2} & P_{012}(5) = 2 & \\
x_3 & P_3 = 2, & P_{23}(5) = -\frac{1}{4} & P_{123}(5) = -\frac{3}{8} & P_{0123}(5) = \frac{9}{14}
\end{array}$$

where, $P_{0,1\ldots,k,k+1}(x) = \frac{1}{(x_{k+1} - x_0)} \begin{vmatrix} x - x_0 & P_{0,1\ldots,k}(x) \\ x - x_{k+1} & P_{1,2,\ldots,k+1}(x) \end{vmatrix}$

Question 2: Lagrange Interpolation

The following table of values is given by the function $f : x \mapsto y = f(x)$

x_i	1.9	2.3	3.2	4.0
$y_i = f(x_i)$	-3.0	-1.0	2.0	4.0

Find the approximate root $x^* \in [0, 3]$ of the function $f(x)$, i.e. $f(x^*) = 0$ using the following procedure : use the y_i points as the reference points and x_i as reference values to construct the Lagrange polynomial $P_n(y)$. Evaluate the polynomial $P_n(y = 0)$ to obtain x^*.

Solution : Taking x_i as the ordinate and y_i as the abscissa, the Lagrange formulation looks like,

$$l_i(y) = \prod_{j=0, j\neq i}^{n} \frac{y - y_j}{y_i - y_j} \quad \text{for i = 0,1,2,..n}$$

$$\text{The polynomial } P_n(y) = \sum_{i=0}^{n} l_i(y) x_i$$

We can then compute for the approximate root $\implies x^* \approx P_n(y = 0)$

$$l_0(0) = -\frac{4}{35}, \quad l_1(0) = \frac{4}{5}, \ \ l_2(0) = -\frac{2}{5}, \ \ l_3(0) = -\frac{3}{35}$$

$$x^* \approx P_3(0) = l_0(0)x_0 + l_1(0)x_1 + l_2(0)x_2 + l_3(0)x_3 = 2.56$$

Question 3: Spline Interpolation

Set-up a periodic spline interpolator through the data points

x_i	0	1/2	1	3/2	2
f_i	0	1	0	-1	0

Evaluate them at $x = 1/4$ For the mesh $M := \{t_0 < t_1 < \cdots < t_n\}$, Cubic spline derived from Hermite basis polynomials looks like,

$$\begin{aligned} Q_i(t) = {} & f_i \cdot \left(1 - 3t^2 + 2t^3\right) \\ & + f_{i+1} \cdot \left(3t^2 - 2t^3\right) \\ & + h_i f_i^{'} \cdot \left(t - 2t^2 + t^3\right) \\ & + h_i f_{i+1}^{'} \cdot \left(-t^2 + t^3\right) \end{aligned}$$

where $h_i = x_{i+1} - x_i$ and $t = (x - x_i)/h_i$

$$\begin{pmatrix} b_1 & a_1 & b_2 & 0 & \dots & & & \dots & 0 \\ 0 & b_2 & a_2 & b_3 & \dots & & & & \\ & & \ddots & \ddots & \ddots & & & \dots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & \\ 0 & & & & \dots & & b_{n-2} & a_{n-2} & b_{n-1} \end{pmatrix} \begin{pmatrix} f_1^{'} \\ f_2^{'} \\ \vdots \\ \vdots \\ \vdots \\ f_n^{'} \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{n-1} \end{pmatrix}$$

where $a_i = \frac{2}{h_i} + \frac{2}{h_{i+1}}, b_i = \frac{1}{h_i}, c_i = \frac{f_{i+1} - f_i}{h_i^2}, d_{i+1} = 3(c_i + c_{i+1}),\ \ i = 1, 2.....n$

This linear system of equations is underdetermined ((N - 2) equations for N unknowns). We need to add conditions at the boundary nodes in order to render the system solvable. From Boor's "not a knot" condition: $P_1 = P_2, P_1^{'''} = P_2^{'''}$ and $P_{n-2} = P_{n-1}, P_{n-2}^{'''} = P_{n-1}^{'''}$, we get 2 more equations:

$$\begin{pmatrix} b_1 & a_1 & b_2 & 0 & \dots & & & \dots & 0 \\ 0 & b_2 & a_2 & b_3 & \dots & & & & \\ & & \ddots & \ddots & \ddots & & & \dots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & \\ & & & & \dots & b_{n-2} & a_{n-2} & b_{n-1} & \\ b_1^{'} & a_1^{'} & 0 & \dots & & & \dots & & 0 \\ 0 & & & \dots & & & & a_{n-2}^{'} & b_{n-1}^{'} \end{pmatrix} \begin{pmatrix} f_1^{'} \\ f_2^{'} \\ \vdots \\ \vdots \\ \vdots \\ f_n^{'} \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_2^{'} \\ d_{n-1}^{'} \end{pmatrix}$$

where:

$a_1^{'} = \frac{1}{h_1} + \frac{1}{h_2}, b_1^{'} = \frac{1}{h_1}, d_2^{'} = 2c_1 + \frac{h_1}{h_1 + h_2}(c_1 + c_2),$

$a_{n-2}^{'} = \frac{1}{h_{n-2}} + \frac{1}{h_{n-1}}, b_{n-1}^{'} = \frac{1}{h_{n-1}}, d_n^{'} = 2c_{n-1} + \frac{h_{n-1}}{h_{n-1} + h_{n-2}}(c_{n-1} + c_{n-2}),\ \ i = 1, 2.....n$

Now we have N equations for N unknowns.

Periodic cubic spline interpolation requires $f_1^{'} = f_n^{'}$. This reduces the number of unknowns to (N-1) and the above linear system turns to

$$\begin{pmatrix} a_1 & b_2 & 0 & \dots & & & \dots & b_1 \\ b_2 & a_2 & b_3 & \dots & & & & \\ 0 & \ddots & \ddots & \ddots & & & \dots & \vdots \\ & & \ddots & \ddots & \ddots & & & \\ & & & \dots & & b_{n-2} & a_{n-2} & b_{n-1} \\ 0 & & & & \dots & & a_{n-2}^{'} & b_{n-1}^{'} \end{pmatrix} \begin{pmatrix} f_2^{'} \\ f_3^{'} \\ \vdots \\ \vdots \\ \vdots \\ f_n^{'} \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{n-1} \\ d_{n-1}^{'} \end{pmatrix}$$

For the given problem:

- Compute $f_i^{'}$, for $i = 1, 2, 3, 4, 5$ by solving the linear system.
- For $x_i < x < x_{i+1}$, compute the value of the spline at $x = 1/4 \implies h_1 = 1/2,\ t = 1/2$

$$\begin{aligned} Q_1(t) &= f_1 \cdot \left(1 - 3t^2 + 2t^3\right) \\ &= f_2 \cdot \left(3t^2 - 2t^3\right) \\ &= h_1 f_1^{'} \cdot \left(t - 2t^2 + t^3\right) \\ &= h_1 f_2^{'} \cdot \left(-t^2 + t^3\right) & = 0.875 \end{aligned}$$

Question 4: Programming task

a) Write a program to evaluate the Spline interpolation function.

b) Apply the program to the data $(x_j = -5 + 2(j-1), f(x_j))\ j = 1, 2, ..6$ for $f(x) = 1/(1 + x^2)$. Evaluate the spline function for the x values $-4, -2, 0, 2, 4$

c) solve b) again using the MATLAB function spline.