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Exercise 11

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Question 1: Trapezoidal rule and Implicit time-stepping

Consider the Initial value problem (IVP)

$$\dot{y} = f(y), \quad y(0) = y_0$$

and the trapezoidal method

$$\tilde{y}_{j+1} = \tilde{y}_j + \frac{h}{2}\left(f(\tilde{y}_j) + f(\tilde{y}_{j+1})\right); \quad \tilde{y}_0 = y(0)$$

To calculate the value $\tilde{y}_{j+1}$, select the starting value of $\tilde{y}^0_{j+1} = \tilde{y}_j + hf(\tilde{y}_j)$ for the fixed-point iteration

$$\tilde{y}^{k+1}_{j+1} = \tilde{y}_j + \frac{h}{2}\left(f(\tilde{y}_j) + f(\tilde{y}^k_{j+1})\right), \quad k = 0, 1, 2....$$

Show that the sequence $\tilde{y}^k_{j+1}$ converges for $k \to \infty$ to the fixed point $\tilde{y}_{j+1}$, if h is small enough and $f(y)$ does not vary much in the chosen interval.

Question 2: Fixed-point and Implicit schemes

Given the differential equation of the damped harmonic oscillator

$$\ddot{x} + 0.5\, x + x = 0$$

with the initial conditions $x(0) = 1,\ \dot{x}(0) = 0$

a) Convert the second order differential equation to a system of first order Differential equations.

b) Approximate $x(h)$ using trapezoidal method for $h = 0$.

Question 3: Butcher tableau

Given the Butcher Tableau of the ϑ-procedure

$$\begin{array}{c|c} \vartheta & \vartheta \\ \hline & 1 \end{array}$$

a) for which ϑ is the procedure explicit or implicit?

b) determine the order of error of the procedure depending on the ϑ.

c) sketch the stability area for the methods with $\vartheta = 0, \frac{1}{2}, 1$.

Question 4: Programming Task

a) Implement the 3-step Adams-Bashforth procedure

$$\tilde{x}^{n+1} = \tilde{x}^n + \frac{h}{12}\left(23f^n - 16f^{n-1} + 5f^{n-2}\right)$$

Assume that the 3 starting values x^0, x^1, x^2 are known.

b) apply your program to

$$\begin{aligned} \dot{x}_1 &= bx_1 - cx_1x_2 \\ \dot{x}_2 &= -dx_2 + cx_1x_2 \end{aligned}$$

for $b = 1, d = 10, c = 1$, initial conditions $x_1(0) = \frac{1}{2}, x_2(0) = 1$ and $t_f = 10$. The starting values are, for $h = \frac{1}{100}$,

t	$x_1(t)$	$x_2(t)$
0.0	0.50000000000000	1.00000000000000
0.01	0.50023020652423	0.90937363770619
0.02	0.50089337004375	0.82696413439848

The above system of ODEs represents the predator-prey model, where x_1 is the prey and x_2 is the predator concentrations.

c) plot the two populations x_1, x_2 as a function of time t and also the trajectory $x_1(t), x_2(t)$ in the (x_1, x_2) plane (phase plane).