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Solution 11

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Question 1: Trapezoidal rule and Implicit time-stepping

Consider the Initial value problem (IVP)

$$\dot{y} = f(y), \quad y(0) = y_0$$

and the trapezoidal method

$$\tilde{y}_{j+1} = \tilde{y}_j + \frac{h}{2}\left(f(\tilde{y}_j) + f(\tilde{y}_{j+1})\right); \quad \tilde{y}_0 = y(0)$$

To calculate the value $\tilde{y}_{j+1}$, select the starting value of $\tilde{y}^0_{j+1} = \tilde{y}_j + hf(\tilde{y}_j)$ for the fixed-point iteration

$$\tilde{y}^{k+1}_{j+1} = \tilde{y}_j + \frac{h}{2}\left(f(\tilde{y}_j) + f(\tilde{y}^k_{j+1})\right), \quad k = 0, 1, 2....$$

Show that the sequence $\tilde{y}^k_{j+1}$ converges for $k \to \infty$ to the fixed point $\tilde{y}_{j+1}$, if h is small enough and $f(y)$ does not vary much in the chosen interval.

Solution :
According to fixed-point theorem, we need to show that that (i) the interval $I = [y_j - r, y_j + r]$ gets mapped onto itself by g(y) and that g(y) is a contraction map/Lipschitz continuous in this interval.

$$y = g(y) = \tilde{y}_j + \frac{h}{2}\left(f(\tilde{y}_j) + f(y)\right)$$

For the function $g(y)$ to be mapped onto itself, we need in the interval $I = [\tilde{y}_j - r, \tilde{y}_j + r]$,

$$|y - \tilde{y}_j| = \frac{h}{2}\left(f(\tilde{y}_j) + f(y)\right) \leq h \max_{y \in I} f(y) \leq r$$

Lets assume a bound on $f(y)$ i.e. $|f(y)| \leq M_1 \implies M_1 h \leq r \;\; (*)$

For the function $g(y)$ to be lipschitz continous in the interval I

$$\begin{aligned}
|g(y) - g(y')| &= |\tilde{y}_j + \frac{h}{2}\left(f(\tilde{y}_j) + f(y)\right) - \tilde{y}_j + \frac{h}{2}\left(f(\tilde{y}_j) + f(y')\right)| \\
&= \frac{h}{2}|f(y) - f(y')| \\
&= \frac{h}{2}|(y - y')f'(\eta)|
\end{aligned}$$

such that $y \leq \eta \leq y'$. We then can put a bound on the derivative f', i.e. $|f'(\eta)| < M_2$.

$$|g(y) - g(y')| \leq \frac{h}{2} M_2 |(y - y')|$$

So the function $g(y)$ is a contraction map, given $\frac{h}{2} M_2 < 1 \implies h < \frac{2}{M_2}$ $(**)$.

Combining the two bounds $(*)$ and$(**)$, i.e.

$$h < \frac{2}{M_2} \quad , \; r \geq h M_1$$

and for any initial guess $y_0 \in [\tilde{y}_j - r, \tilde{y}_j + r]$, we will converge to the fixed-point $\tilde{y}_{(j+1)}$ according to the fixed-point theorem. By assuming constraints on the function (*) and its derivative (**), we showed that the update rule converges to the fixed-point.

Question 2: Fixed-point and Implicit schemes

Given the differential equation of the damped harmonic oscillator

$$\ddot{x} + 0.5\,\dot{x} + x = 0$$

with the initial conditions $x(0) = 1, \; \dot{x}(0) = 0$

a) Convert the second order differential equation to a system of first order Differential equations.

Make the substitution $x_1 = x$ and $x_2 = \dot{x}$, we end with the system of equations

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -0.5 x_2 - x_1 \end{aligned}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 0$

b) Approximate $x(h)$ using trapezoidal method for $h = 0.5$

Solution:

The trapezoidal rule (semi-implicit) reads,

$$\tilde{x}_{j+1} = \tilde{x}_j + \frac{h}{2}\left(f(t_j, \tilde{x}_j) + f(t_{j+1}, \tilde{x}_{j+1})\right)$$

$x(h) = x_1(h) \approx x_1^{(1)}$

$$\begin{pmatrix} \tilde{x_1}^{(1)} \\ \tilde{x_2}^{(1)} \end{pmatrix} = \begin{pmatrix} \tilde{x_1}(0) \\ \tilde{x_2}(0) \end{pmatrix} + \tfrac{h}{2} \begin{pmatrix} f_1(0, , \tilde{x_1}(0), \tilde{x_2}(0)) + f_1(h, \tilde{x_1}^{(1)}, \tilde{x_2}^{(1)}) \\ f_2(0, \tilde{x_1}(0), \tilde{x_2}(0)) + f_2(h, \tilde{x_1}^{(1)}, \tilde{x_2}^{(1)}) \end{pmatrix}$$

making the substitutions for initial conditions and h, we get

$$\begin{aligned} \tilde{x_1}^{(1)} &= 1 + \frac{h}{2}\left(0 + \tilde{x_2}^{(1)}\right) \\ \tilde{x_2}^{(1)} &= 0 + \frac{h}{2}\left(-1 - 0.5\tilde{x_2}^{(1)} - \tilde{x_1}^{(1)}\right) \\ \tilde{x_2}^{(1)} &= -\frac{h}{2} - \frac{h}{4}\tilde{x_2}^{(1)} - \frac{h}{2} - \frac{h^2}{4}\tilde{x_2}^{(1)} \\ \tilde{x_2}^{(1)} &= -\frac{h}{1 + \frac{h}{4} + \frac{h^2}{4}} = -0.42105 \end{aligned}$$

Substituting for $\tilde{x_2}^{(1)}$, we get

$$x(h) = x_1(h) \approx \tilde{x_1}^{(1)} = 1 + \frac{h}{2}\tilde{x_2}^{(1)} = 0.894737$$

Question 3: Butcher tableau

Given the Butcher Tableau of the ϑ-procedure

$$\begin{array}{c|c} \vartheta & \vartheta \\ \hline & 1 \end{array}$$

a) for which ϑ is the procedure explicit or implicit?

Solution:

$$\begin{aligned} k_1 &= f(t_j + \vartheta h, \tilde{x}_j + h\vartheta k_1) \\ \tilde{x}_{j+1} &= \tilde{x}_j + hk_1 \end{aligned}$$

From the above formula, it is clear that for $\vartheta = 0$ is explicit and $\vartheta \neq 0$ is implicit.

b) determine the order of error of the procedure depending on the ϑ.

$$\begin{aligned} f(t + \vartheta h, \tilde{x} + h\vartheta k_1) = f(t,x) + \vartheta h f_t(t,x) + h\vartheta k_1 f_x(t,x) + \frac{\vartheta^2 h^2}{2} f_{tt}(t,x) \\ + \frac{\vartheta^2 h^2 k_1^2}{2} f_{xx}(t,x) + \vartheta^2 h^2 k_1 f_{tx}(t,x) + \mathcal{O}(h^3) \end{aligned}$$

Substituting the above in the update formulate $x(t+h) = x(t) + hk_1$, we get

$$\begin{aligned} x(t+h) = x(t) + hf(t,x) + \vartheta h^2 f_t(t,x) + h^2 \vartheta k_1 f_x(t,x) + \frac{\vartheta^2 h^3}{2} f_{tt}(t,x) \\ + \frac{\vartheta^2 h^3 k_1^2}{2} f_{xx}(t,x) + \vartheta^2 h^3 k_1 f_{tx}(t,x) + \mathcal{O}(h^4) \end{aligned}$$

after grouping, truncating the 3^{rd} order terms and for substituting $k_1 = f(t,x)$, we get

$$\begin{aligned} x(t+h) &= x(t) + hf(t,x(t)) + h^2\vartheta\left[f_t(t,x) + f_x(t,x)f(t,x)\right] + \mathcal{O}(h^3) \quad (*) \\ x(t+h) &= x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t) + \mathcal{O}(h^3) \quad (**) \end{aligned}$$

Comparing $(*)$ and $(**)$, we can infer that for $\vartheta = \frac{1}{2}$, the method is $\mathcal{O}(h^2)$, else $\mathcal{O}(h)$

c) sketch the stability area for the methods with $\vartheta = 0, \frac{1}{2}, 1$.

Let us take the model problem

$$\dot{x} = \lambda x, \quad x(0) = x_0, \quad \lambda \in \mathbb{C}$$

For $\vartheta = 0$, the update rule is

$$\begin{aligned} \tilde{x}_{j+1} &= \tilde{x}_j + hf(t_j, \tilde{x}_j) \\ &= \tilde{x}_j + h\lambda\tilde{x}_j \\ &= \tilde{x}_j(1 + h\lambda) \end{aligned}$$

$\mu = h\lambda \implies R(\mu) = 1 + \mu$. The update is stable if $|R(\mu)| < 1$.
Since $\mu \in \mathcal{C} \implies \mu = u + iv \implies |R(\mu)|^2 = (1+u)^2 + v^2$.
Solving for $|R(\mu)|^2 = (1+u)^2 + v^2 < 1$, we get the stability regions.

For $\vartheta = \frac{1}{2}$, the update rule is

$$\begin{aligned} \tilde{x}_{j+1} &= \tilde{x}_j + hf(t_j + \frac{h}{2}, \frac{\tilde{x}_j + \tilde{x}_{j+1}}{2}) \\ &= \tilde{x}_j + h\lambda\left(\frac{1}{2}\tilde{x}_j + \frac{1}{2}\tilde{x}_{j+1}\right) \\ \tilde{x}_{j+1}\left(1 - \frac{1}{2}h\lambda\right) &= \tilde{x}_j\left(1 + \frac{1}{2}h\lambda\right) \end{aligned}$$

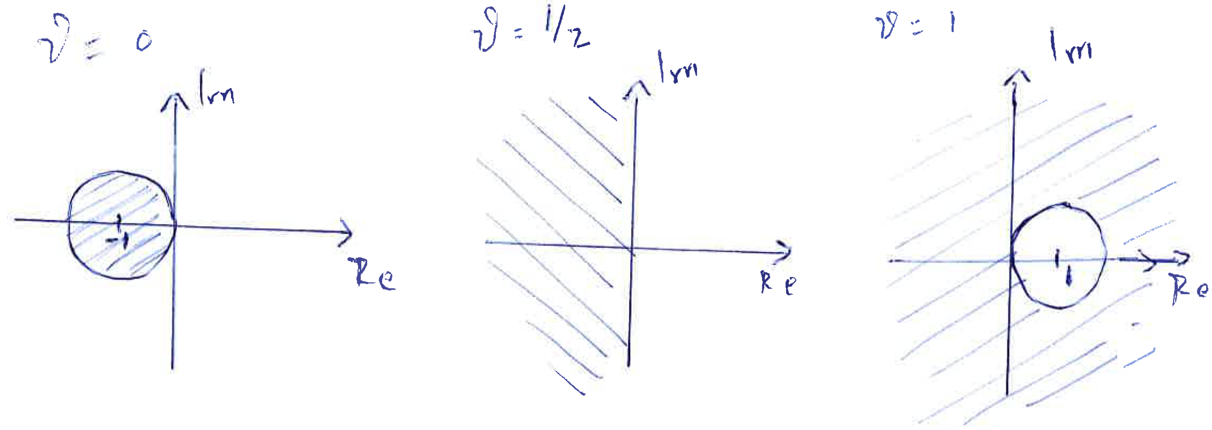
$\mu = h\lambda \implies R(\mu) = \frac{2+\mu}{2-\mu} \implies |R(\mu)|^2 = \frac{(2+u)^2+v^2}{(2-u)^2+v^2} = 1 \implies u = 0.$

For $\vartheta = 1$, the update rule is

$$\begin{aligned} \tilde{x}_{j+1} &= \tilde{x}_j + hf(t_j + h, \tilde{x}_{j+1}) \\ &= \tilde{x}_j + h\lambda\tilde{x}_{j+1} \\ \tilde{x}_{j+1}(1 - h\lambda) &= \tilde{x}_j \implies R(\mu) = \frac{1}{1-\mu} \end{aligned}$$

For $\mu = u + iv \implies |R(\mu)| = \frac{1}{(1-u)^2+v^2}$. Solving for $|R(\mu)|^2 = \frac{1}{(1-u)^2+v^2} < 1$, we get the stability regions.

Figure 1: stability regions for varying ϑ



Question 4: Programming Task

a) Implement the 3-step Adams-Bashforth procedure

$$\tilde{x}^{n+1} = \tilde{x}^n + \frac{h}{12}\left(23f^n - 16f^{n-1} + 5f^{n-2}\right)$$

Assume that the 3 starting values x^0, x^1, x^2 are known.

b) apply your program to

$$\begin{aligned} \dot{x}_1 &= bx_1 - cx_1x_2 \\ \dot{x}_2 &= -dx_2 + cx_1x_2 \end{aligned}$$

for $b = 1, d = 10, c = 1$, initial conditions $x_1(0) = \frac{1}{2}, x_2(0) = 1$ and $t_f = 10$. The starting values are, for $h = \frac{1}{100}$,

t	$x_1(t)$	$x_2(t)$
0.0	0.50000000000000	1.00000000000000
0.01	0.50023020652423	0.90937363770619
0.02	0.50089337004375	0.82696413439848

The above system of ODEs represents the predator-prey model, where x_1 is the prey and x_2 is the predator concentrations.

c) plot the two populations x_1, x_2 as a function of time t and also the trajectory $x_1(t), x_2(t)$ in the (x_1, x_2) plane (phase plane).