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Solution 6

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Question 1: Fixed-Point Theorem

For the function

$$f(x) = 2e^{-x/2}$$

a) Graphically show that the function has one fixed point.

b) Show that in the closed interval $I = [0.8, 1.4]$, the requirements for Banach's fixed-point theorem are fulfilled.

Solution:

From the plot we can see that $f(x) = 2e^{-x/2}$ is a monotonic decreasing function for $x \in I$

$$\begin{aligned} f(0.8) &= 1.341 \in I \\ f(1.4) &= 0.986 \in I \\ \implies f(I) &\subset I \end{aligned}$$

A function f is a contraction mapping on I, if

$$\forall x, x' \in I : |f(x') - f(x) \leq c|x' - x||$$

for some real number $c < 1$

For the interval c, we can identify the maximum c, i.e. $c = \max_{x \in I} |f'(x)| = \max_{x \in I} \left(e^{-x/2}\right)$. Since $f'(x) = e^{-x/2}$ is a monotonic decreasing function, $\max_{x \in I} \left(e^{-x/2}\right) = f'(0.8) = 0.67032 < 1$. So the function f is a contraction mapping in interval I.

c) Find an upper-limit for the number of iterations

$$x_{j+1} = 2e^{-x_j/2}$$

such that, $|x_n - x^*| \leq 10^{-6}$, where $x_0 \in I$ is any starting point in I.

Solution:

Banach fixed-point theorem says: $|x_n - x^*| \leq \frac{c^n}{1-c}|x_1 - x_0| = 10^{-6}$
For $|x_1 - x_0| = |1.4 - 0.8| = 0.6$ and $c = 0.67032$, we get $n \leq 37$

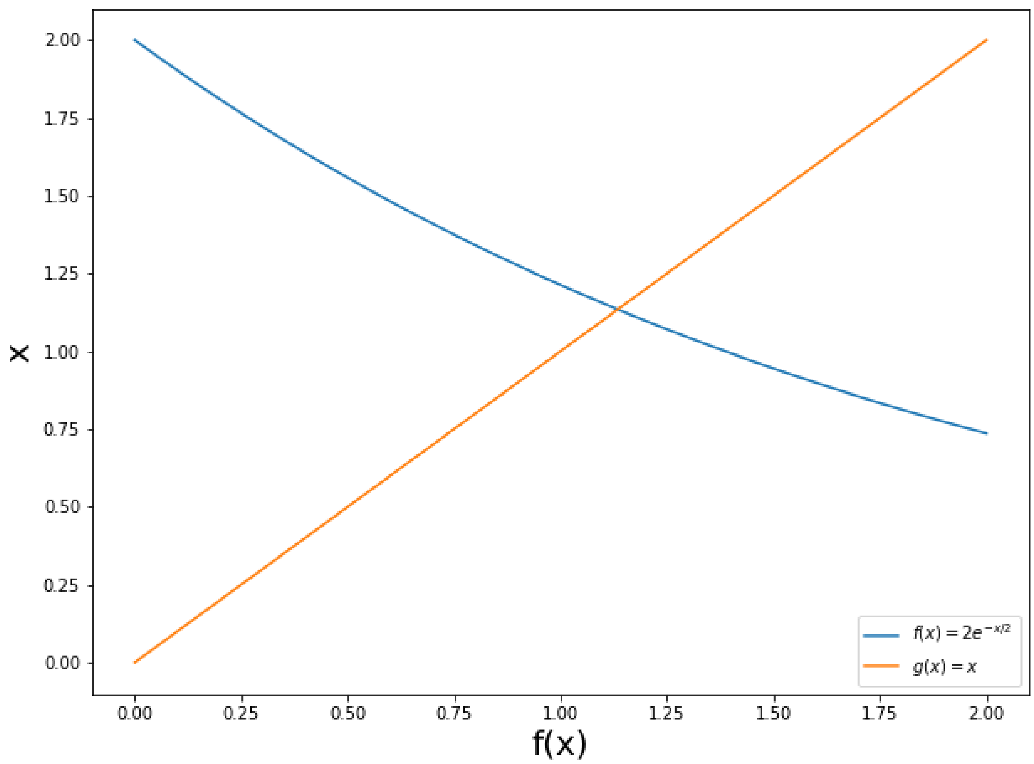

Figure 1: The function $f_1(x) = 2e^{-x/2}$ has one intersection point with $f_2(x) = x$

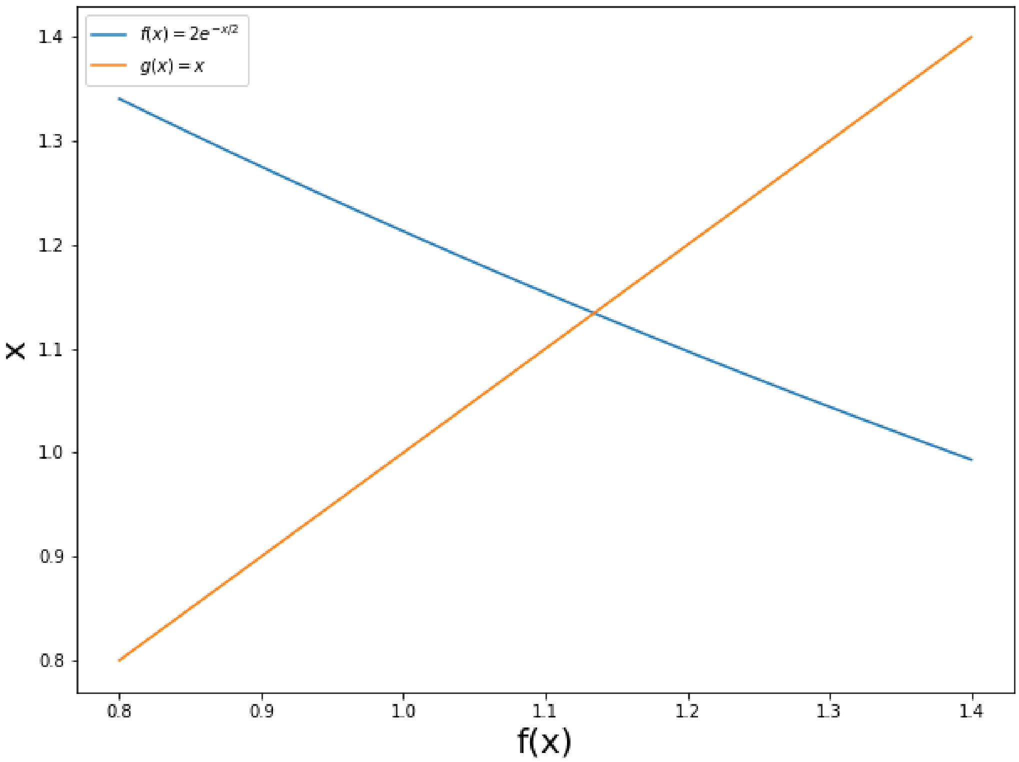

Figure 2: close-up for the interval I=[0.8,1.4]

Question 2: Newton method

The Kepler's equation given by,

$$x - e\sin x = t$$

with parameters e and t should be solved for x using Newton method.

a) Formulate the Newton's iteration for the solution of Kepler's equation.

Solution:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$f(x) = x - e\sin x - t \ , \ f^{'}(x) = 1 - e\cos x$$

$$x_{k+1} = x_k - \frac{x_k - e\sin x_k - t}{1 - e\cos x_k}$$

$$x_{k+1} = \frac{e\sin x_k - x_k\, e\cos x_k + t}{1 - e\ \mathrm{cos} x_k}$$

b) Solve the Newton's iteration numerically for $e = 0.4$ and $t = 0.4$ with starting value $x_0 = 0.7$ $x_0 = 0.7, x_1 = 0.6390$

Question 3: System of non-linear equations

Solve the non-linear system of equations,

$$
\begin{aligned}
e^{xy} + x^2 + y - 1.4 &= 0 \\
x^2 + y^2 + x - 0.46 &= 0
\end{aligned}
$$

with the help of the Newton procedure. Select $(x_0, y_0) = (0.5, 0.4)$. Formulate the procedure and carry out one Newton step by hand.

Solution:

$$
J(x,y) = \begin{pmatrix} ye^{xy} + 2x & xe^{xy} + 1 \\ 2x + 1 & 2y \end{pmatrix}
$$

$(x_0, y_0) = (0.5, 0.4)$ Start Newton iteration with $(x_0, y_0) = (0.5, 0.4)$ and **tol**.

$$
\begin{aligned}
J(x^k, y^k)\Delta^k &= -f(x^k, y^k) \\
x^{k+1} &= x^k + \Delta_x^k \\
y^{k+1} &= y^k + \Delta_y^k
\end{aligned}
$$

For $k = 0$,

$$
J(x^0, y^0) = \begin{pmatrix} 1.4885 & 1.6107 \\ 2.0 & 0.8 \end{pmatrix} \text{ and } f(x^0, y^0) = \begin{pmatrix} -0.4714 \\ -0.45 \end{pmatrix}
$$

$\Delta_x^0 = -0.17123, \;\; x^1 = x^0 + \Delta_x^0 = 0.3287$
$\Delta_y^0 = -0.1344, \;\; y^1 = y^0 + \Delta_y^0 = 0.2655$

Repeat this process for $k = 1, 2, 3...$ until convergence.

Question 4: Programming task

a) Implement a robust method for calculating the zeros of a scalar function. The algorithm should work like this :

- Starting from an interval $[a_0, b_0]$, apply bisection-method until the zero is known with an accuracy of $\mathbf{tol}_0$, i.e. $|b_j - a_j| < \mathbf{tol}_0$
- Use Newton-method with starting value $x = \frac{1}{2}(a_j + b_j)$ to determine the zero point with a relative accuracy of **tol**
- STOP, if the Newton procedure exits the interval $[a, b]$

b) Repeat a) with Secant method instead of Newton's procedure

c) Compute the zero(s) of the functions

$$f(x) = x - 2 + \ln x, \quad x > 0,$$
$$g(x) = (x-1)(x+1)(x+2)$$

using the two methods you have implemented in a) and b). First select an appropriate interval $[a_0, b_0]$ and use $\mathbf{tol}_0 = 10, \mathbf{tol} = 10^{-10}$. Now apply the procedures only to $g(x)$ with the parameters $a_0 = -0.9, b_0 = 1.1, \mathbf{tol}_0 = 3, \mathbf{tol} = 10^{-10}$. Explain the result.