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Solution 13

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Question 1: Solution of PDEs

Consider the PDE for advection equation $u_t + cu_x = 0$. Assuming that, we are only allowed to Fourier transform along x, i.e.

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\xi x} dx \quad (*)$$

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{i\xi x} dx \quad (**)$$

a) Formulate the analytical solution for $u(x, t)$ given the initial data for $\hat{u}(\xi, 0)$.

$$\begin{aligned}
u_t + cu_x = 0 \quad &\text{(Multiply by } e^{-i\xi x} \text{ and integrate from } -\infty \text{ to } \infty) \\
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u_t + cu_x) e^{-i\xi x} dx &= 0 \\
\hat{u}_t(\xi, t) + ci\xi \hat{u}(\xi, t) &= 0 \\
\hat{u}_t = -i\xi c\hat{u} \quad &\text{This is a ODE of the from } \dot{y} = \lambda y
\end{aligned}$$

The initial condition i.e.

$$\hat{u}(\xi, 0) = \hat{\eta}(\xi) = \int_{-\infty}^{\infty} \eta(x) e^{-i\xi x} dx$$

So, the analytical solution for $\hat{u}_t$ for each ξ and initial condition $\hat{u}(\xi, 0) = \hat{\eta}(\xi)$. is given by,

$$\hat{u}(\xi, t) = e^{-i\xi ct} \hat{\eta}(\xi)$$

Substituting for $\hat{u}(\xi, t)$ in (**), we get

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi ct} \hat{\eta}(\xi) e^{i\xi x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi(x-ct)} dx = \eta(x - ct)$$

This is equation for propagation of a wave solution.

b) Repeat the same analysis for diffusion equation $u_t = Du_{xx}$.

$$
\begin{aligned}
u_t - Du_{xx} &= 0 \ \ \text{(Multiply by } e^{-i\xi x} \text{ and integrate from } -\infty \text{ to } \infty) \\
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} &(u_t - Du_{xx}) e^{-i\xi x} dx = 0 \\
&\hat{u}_t(\xi, t) + D\xi^2 \hat{u}(\xi, t) = 0 \\
&\hat{u}_t(\xi, t) = -D\xi^2 \hat{u}(\xi, t)
\end{aligned}
$$

which has a solution of the form for initial condition $\hat{u}(\xi, 0) = \hat{\eta}(\xi)$

$$
\hat{u}(\xi, t) = e^{-D\xi^2 t} \hat{\eta}(\xi)
$$

Substituting for $\hat{u}(\xi, t)$ in (**) and using the result that Fourier transform of a gaussian function $\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$ is $e^{-D\xi^2 t}$, we get

$$
u(x, t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}} \eta(x) dx
$$

HINT : Take the fourier transform of the respective equations

Question 2: Semi-analytical solution

Given the PDE for heat conduction,

$$
u_t = u_{xx} - cos(2\pi x), \quad x \in \mathbb{R}
$$

with boundary conditions $u(t, 0) = u(t, 1) = 0$ and Initial conditions $u(0, x) = 0$. Approximate the solution by applying the Method of Lines with the implicit Euler procedure for time integration. Formulating the method and then calculate an example with $h = 0.01, t_f = 10$.

$$
\begin{aligned}
u_{xx} &= \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2} + \mathcal{O}(\Delta^2) \\
x_i &= ih; \quad u_i(t) = u(x_i, t) \\
\dot{u}_i(t) &= \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta x^2} - \cos(2\pi x_i); \quad i = 1, 2....., N - 1 \\
u_0(t) &= u_N(t) = 0, \ \text{ and } \ u_l(0) = 0 \ \text{ for all } \ l
\end{aligned}
$$

$$
\dot{u} = \frac{1}{h^2} \hat{A} u + b = Au + b \ \ \text{(System of linear-equations)}
$$

Implicit-Euler:

$$
\begin{aligned}
\hat{u}^{n+1} &= \hat{u}^n + \Delta t f(t + \Delta t, \hat{u}^{n+1}) \\
\hat{u}^{n+1} &= \hat{u}^n + \Delta t \left(A\hat{u}^{n+1} + b \right) = \hat{u}^j + \frac{\Delta t}{\Delta x^2} \hat{A} \hat{u}^{n+1} + b\Delta t \\
\left(I - \frac{\Delta t}{\Delta x^2} \hat{A} \right) \hat{u}^{n+1} &= \hat{u}^n + b\Delta t \\
\hat{u}^{n+1} &= \left(I - \frac{\Delta t}{\Delta x^2} \hat{A} \right)^{-1} (\hat{u}^n + b\Delta t) \, ; \quad u^0 = 0
\end{aligned}
$$

where,

$$A = \begin{pmatrix} -2 & 1 & 0 & \dots & & & \dots & \dots \\ 1 & -2 & 1 & \dots & & & & \\ 0 & \ddots & \ddots & \ddots & & & \dots & \vdots \\ & & \ddots & \ddots & \ddots & & & \\ & & & \dots & \ddots & -2 & 1 & 0 \\ & & & \dots & & 1 & -2 & 1 \\ \dots & & & \dots & & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N-1 \times N-1}$$

and $b = [-\cos(2\pi h), -\cos(4\pi h), -\cos(6\pi h), ..., -\cos(2(N-1)\pi h)]^T \in \mathbb{R}^{N-1}$

Question 3: Programming Task

Consider the poisson equation

$$\begin{aligned} \Delta u(x,y) &= 1 \quad (x,y) \in \Omega := (-1,1) \times (-1,1) \\ u &= 0 \quad \text{at } \partial\Omega \end{aligned}$$

a) Discretize the problem with 5-point FD stencil.

b) Solve the linear system and plot the solution.