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Solution 2

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Question 1: LU decomposition and roundoff

Solve the Linear system Ax = y using LU decomposition

$$A = \begin{pmatrix} 0.005 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

a) Exactly

Row operation $R_2 = R_2 - 200 * R_1$ and storing the information('200') in the pivot column

$\begin{pmatrix} 0.005 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.005 & 1 \\ 200 & -199 \end{pmatrix}$

$L = \begin{pmatrix} 1 & 0 \\ 200 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 0.005 & 1 \\ 0 & -199 \end{pmatrix}$

Substitution $Lc = y$, leads to $c_1 = 0.5, c_2 = -99$. We can then back-substitute to find $x_1 = \frac{100}{199}$and $x_2 = \frac{99}{199}$

b) by rounding all the values of L and U to 2 decimal places after comma

After rounding to two decimal points L and U look like

$L = \begin{pmatrix} 1 & 0 \\ 200 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 0.01 & 1 \\ 0 & -199 \end{pmatrix}$ If we repeat the same process as in a) we get $c_1 = 0.5, c_2 = -99$ and by substitution $x_1 = 0, x_2 = 0.5$

c) by first swapping the rows of A and y and rounding them as in b)

By swapping rows of A and y and rounding them to 2 decimal points, we get

$A = \begin{pmatrix} 1 & 1 \\ 0.01 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$

If we continue the do LU decomposition of the above A and y, we get

$L = \begin{pmatrix} 1 & 0 \\ 0.01 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 1 \\ 0 & 0.99 \end{pmatrix}$

Solving for the coefficients, $c_1 = 1, c_2 = 0.49$ and by back-substitution $x_1 = 0.51, x_2 = 0.49$

The difference between the solution obtained in b) and the exact value obtained from a) is due to error accumulation during rounding in b).

Question 2: LU Decomposition

Determine the LU decomposition of the matrix

$$\text{A} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

and thus solve the linear equation system $Ax = b$ for $b = (1, 2, 3)^T$. Verify your results with MATLAB command **lu**

<u>Answer</u>:

Perform row operators in the following order

- $R_2 \to (1)R_1 + R_2$
- $R_3 \to (1/3)R_2 + R_3$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[\text{store}-1]{R_2 \to (1)R_1 + R_2} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 3 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[\text{store}-1/3]{R_3 \to (1/3)R_2 + R_3} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 3 \\ 0 & -1/3 & 2 \end{pmatrix}$$

We can then write L and U as follows,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1/3 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Substitute $Lc = y$, we obtain $c_1 = 1, c_2 = 3, c_3 = 4$ and by back-substitution we get $x_1 = -2, x_2 = -1$ and $x_3 = 2$

Question 3: Iterative schemes

a) Can you briefly explain the difference between Direct and Iterative methods for solution of the linear system $Ax = b$.

Direct method involve finding the solution using matrix decomposition for matrix inversion. The matrix here is assumed to be full rank and square.

Iterative methods start from an initial value and evolve until they converge.

b) For an iterative scheme given by $x^{k+1} = Tx^k + \text{c}$, where T is the iteration matrix, x^k is the solution at k iteration and c is a column vector

i) What is the sufficient condition for convergence of the iterative scheme ?

The norm of the iterative matrix should be less than 1.

$$\|T\| < 1$$

ii) What is the sufficient and necessary condition for convergence of the iterative scheme ?

The spectral radius of T is smaller than 1 i.e. $\max|\lambda(T)| < 1$. Where $\lambda(T)$ is the set of the eigen values of T

iii) See if matrix A in Problem 2) is diagonally dominant and also compute the spectral radius of the same.

Matrix A is not diagonally dominant. None of the rows of the matrix A obey the condition for diagonal dominant matrix.

The maximum eigen-value of matrix A is 2. So the spectral radius of $A = 2$

Question 4: Gaussian elimination in computer

Write a MATLAB/python code for solving the linear system $Ax = b$ using Gaussian elimination with partial pivoting. Test your code for linear system in Problem 2.

`https://martin-thoma.com/solving-linear-equations-with-gaussian-elimination/`