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Solution 8

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Question 1: Trapezoidal method

Given $I = \int_a^b f(x)dx$ and $|f^{''}(x)| \leq M \quad \forall x \in [a, b]$
Show that the bound on the (absolute) error of the trapezoidal method can be written as

$$|I - T(H)| \leq \frac{M}{8}(b-a)h^2$$

The formula for trapezoidal rule

$$T = \frac{b-a}{2}(f(a) + f(b))$$

which can also be written as a integral of a linear polynomial $P_1(x)$

$$T = \int_a^b P_1(x)dx$$

Error in interpolation is given by

$$|f(x) - P_n(x)| = \frac{f^{n+1}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$$

$$|f(x) - P_1(x)| = \frac{f^{''}(\xi)}{(2)!}(x - x_0)(x - x_1) \qquad *$$

We are given a bound on $f^{''}(x) \leq M$, the bound on the polynomial $(x - x_0)(x - x_1) = (x-a)(x-b)$. The maximum occurs of this function occurs at $x = x_0 + (x_1 - x_0)/2 = a + h/2$. Substituting in the above equation, we get

$$|f(x) - P_1(x)| \leq \frac{Mh^2}{8} \tag{1}$$

For error in integral approximation,

$$|I - I(h)| = \int_a^b (f(x) - P_1(x))\, dx \leq \frac{Mh^2}{8}\int_a^b dx$$

$$\implies |I - I(h)| \leq \frac{Mh^2}{8}(b-a) = \frac{Mh^3}{8}$$

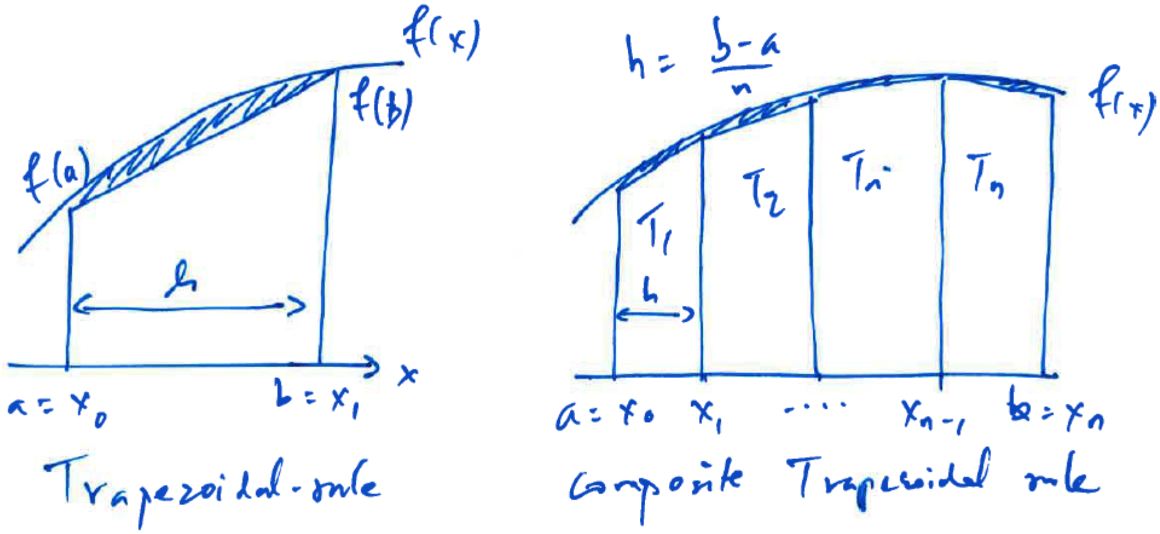

Figure 1: Trapezoidal rule

Error in composite trapezoidal rule

The interval is divided into b sub-intervals, i.e. $h = \frac{(b-a)}{n}$
Error in a sub-interval i

$$|I_i - T_i(h)| \leq \frac{M_i}{8} h^3$$

The maximum M for the entire interval $[a, b]$ is

$$M = \max_i M_i$$

The total error then can be bounded as the summation of errors in all intervals, i.e.,

$$|I - T(h)| = \sum_{i=1}^{n} |I_i - T_i(h)| \leq \sum_{i=1}^{n} \frac{M}{8} h^3 = \frac{M}{8} h^3 n = \frac{M}{8} \frac{h^3}{h} (b-a) = \frac{M}{8} h^2 (b-a)$$

$$|I - T(h)\ | \leq \frac{M}{8} (b-a) h^2$$

Question 2: Trapezoidal method

Use the trapezoidal method to calculate the approximation of the integral

$$\int_1^2 \frac{1}{x} dx = \ln 2$$

to a tolerance of 10^{-3} exactly.

$$|I - T(h)| \leq 10^{-3}$$

We know the error bound of the trapezoidal rule,

$$|I - T(h)| \leq \frac{M}{8} h^2 (b-a) \quad (**)$$

For the given function i.e. $f(x) = \frac{1}{x}, f^{''}(x) = \frac{2}{x^3}$. The function $f^{''}(x)$ is monotonically decreasing in the interval $[a, b] = [1, 2]$

$$\implies M = \max_{x \in [a,b]} |f^{''}(x)| = f^{''}(a) = 2$$

So substituting this in equation (**), we get

$$\frac{M}{8} h^2 (b-a) \leq 10^{-3} \implies h^2 \leq \frac{8 \times 10^{-3}}{M(b-a)} \implies h \leq 0.06324$$

We can chose any n, such that $h = \frac{(b-a)}{n} \leq 0.06324$. Let us choose $n = 16$, so that $h = \frac{(b-a)}{n} = 0.0625 < 0.06324$.

Applying the composite trapezoidal rule for $n = 16$ and $a = 1, b = 2$, we get

$$\begin{aligned} T(h) &= \frac{h}{2} (f_0 + 2f_1 + 2f_2 + ... + 2f_{n-1} + f_n) \\ T(h = \frac{1}{16}) &= \frac{1}{32} (f(1) + 2f(1+h) + 2f(1+2h) + ... + 2f(1+15h) + f(2)) \\ &= 0.69339 \end{aligned}$$

The exact value of the integral $\int_1^2 \frac{1}{x} dx = \ln 2 = 0.693147$, so the absolute error is given by $|0.693147 - 0.69339| = 2.4 \cdot 10^{-4} < 10^{-3}$

Question 3: Intergral approximation

a) Compute an approximation of the integral

$$\int_0^{\pi/2} \frac{\sin x}{1+x^2} dx = 0.526978557614\cdots,$$

by performing three steps of the trapezoidal method.

The formula for composite trapezoidal method for three steps reads
For 1 step $h = (b-a)$

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + f(b)\right]$$

Substituting for the above we get $\int_0^{\pi/2} \frac{\sin x}{1+x^2} dx \approx T_1 = 0.226509$. The error incurred is 0.3

For 2 steps $h = \frac{(b-a)}{2}$ and $x_i = a + ih$ and $i = 1$

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2f(x_1) + f(b)\right]$$

Substituting for the above we get $\int_0^{\pi/2} \frac{\sin x}{1+x^2} dx \approx T_2 = 0.45673745$. The error incurred is 0.0702411

For 3 steps $h = \frac{(b-a)}{4}$ and $x_i = a + ih$ and $i = 1, 2, 3$

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(b)\right]$$

Substituting for the above we get $\int_0^{\pi/2} \frac{\sin x}{1+x^2} dx \approx T_3 = 0.51050416$. The error incurred is 0.01647439.

b) Extrapolate according to Romberg-scheme the last two values to an improved value. Compare the result with the exact value.

Using Romberg-scheme, an higher order approximation can be created using richardson extrapolation formula

$$S_3 = \frac{4T_3 - T_2}{3} = 0.5284264$$

The error is 0.00144, and an order of magnitude improvement from T_3

Question 4: Gauss-formula

Given the Integral

$$I = \int_0^{\pi} \frac{1}{1+\cos^2 x} dx$$

Determine an approximation of I using the 5-point Gauss formula. This formula has the following nodes and weights (rounded to 8 digits):

$\pm x_i$	0	0.53846931	0.90617985
w_i	0.56888889	0.47862867	0.23692689

Change of interval formula for integration

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right) dt$$
$$\int_0^{\pi} \frac{dx}{1+\cos^2 x} = \int_{-1}^{1} \frac{\frac{\pi}{2}dt}{1+\cos^2\left(\frac{\pi}{2}(t+1)\right)}$$

Gaussian-quadrature expresses the integral as a convolution operation given as,

$$\int_{-1}^{1} f(x)W(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

is exact for polynomials of order $2n+1$, if x_i are the zeros of Legendre polynomials of order $n+1$ and $W(x) = 1$, then

$$\omega_i = \int_{-1}^{1} \prod_{k=0,k\neq i}^{k=n} \left(\frac{x - x_k}{x_i - x_k}\right) dx$$

For a 5-point Gauss formula, we have for $f(x_i) = \frac{\frac{\pi}{2}}{1+\cos^2\left(\frac{\pi}{2}(x_i+1)\right)}$

$$f_1 = 0.793958$$
$$f_2 = 1.006740$$
$$f_3 = 1.570796$$
$$f_4 = 1.006740$$
$$f_5 = 0.793958$$

$$Q_5 = \sum_{i=1}^{i=5} w_i f(x_i) = 2.233537$$

b) divide the interval $[0, \pi]$ into three equal partial intervals, and apply the 5-point Gauss formula to each partial interval.

Dividing the domain, into three equal parts i.e $[0, \frac{\pi}{3}], [\frac{\pi}{3}, \frac{2\pi}{3}]$ and $[\frac{2\pi}{3}, \pi]$. We can then use the gauss-quadrature rule to estimate the integral in each sub-interval and add them up.

For sub-interval $[0, \frac{\pi}{3}]$, using the formula for change of interval, we get $dx = \frac{\pi}{6}dt$ and $x = \frac{\pi}{6}(t+1)$

$$I = \int_{-1}^{1} \frac{\frac{\pi}{6}dt}{1+\cos^2\left(\frac{\pi}{6}(t+1)\right)}$$

$$f_1 = 0.262115$$
$$f_2 = 0.269517$$
$$f_3 = 0.299199$$
$$f_4 = 0.353816$$
$$f_5 = 0.404736$$

where $f(x_i) = \frac{\frac{\pi}{6}}{1+\cos^2\left(\frac{\pi}{6}(x_i+1)\right)}$

$$Q_5^1 = \sum_{i=1}^{i=5} w_i f(x_i) = 0.626551$$

For sub-interval $[\frac{\pi}{3}, \frac{2\pi}{3}]$, using the formula for change of interval, we get $dx = \frac{\pi}{6}dt$ and $x = \frac{\pi}{6}(t+3)$

$$I = \int_{-1}^{1} \frac{\frac{\pi}{6}dt}{1+\cos^2\left(\frac{\pi}{6}(t+3)\right)}$$

$$f_1 = 0.433180$$
$$f_2 = 0.485980$$
$$f_3 = 0.523598$$
$$f_4 = 0.485980$$
$$f_5 = 0.433180$$

where $f(x_i) = \frac{\frac{\pi}{6}}{1+\cos^2\left(\frac{\pi}{6}(x_i+3)\right)}$

$$Q_5^2 = \sum_{i=1}^{i=5} w_i f(x_i) = 0.968342$$

For sub-interval $[\frac{2\pi}{3}, \pi]$, using the formula for change of interval, we get $dx = \frac{\pi}{6}dt$ and $x = \frac{\pi}{6}(t+5)$

$$I = \int_{-1}^{1} \frac{\frac{\pi}{6}dt}{1+\cos^2\left(\frac{\pi}{6}(t+5)\right)}$$

$$f_1 = 0.404736$$
$$f_2 = 0.353815$$
$$f_3 = 0.299199$$
$$f_4 = 0.269517$$
$$f_5 = 0.262115$$

where $f(x_i) = \frac{\frac{\pi}{6}}{1+\cos^2\left(\frac{\pi}{6}(x_i+5)\right)}$

$$Q_5^3 = \sum_{i=1}^{i=5} w_i f(x_i) = 0.626551$$

$$Q_5^* = Q_5^1 + Q_5^2 + Q_5^3 = 2.221444$$

Difference between approximations $|Q_5 - Q_5^*| = 0.0121$

Question 5: Programming Task

a) Write a MATLAB program for integral evaluation using Romberg-scheme to some tolerance **tol**

b) Test your program with previous tasks

c) verify your results with the MATLAB command **quad**