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Solution 9

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Question 1: Derivative approximation

Given the function $f(x) = e^{-x^2}$, sample the function at N uniformly spaced grid points in the interval $[-1, 1]$. Compute the derivative $f^{'}(x_i)$ and the second derivative $f^{''}(x)$ of the function at sample point $x_i, i = 1, 2, ...N - 1$ and spacing $h = \frac{2}{N}$ using

a) central difference $f^{'}(x) = \frac{f(x+h)-f(x-h)}{2h}$ for varying N

b) forward difference $f^{'}(x) = \frac{f(x+h)-f(x)}{h}$ for varying N

Compute the error of approximation w.r.t the analytical solution $f^{'}(x) = -2xe^{-x^2}$ for varying N for both a) and b). Plot the error for varying N

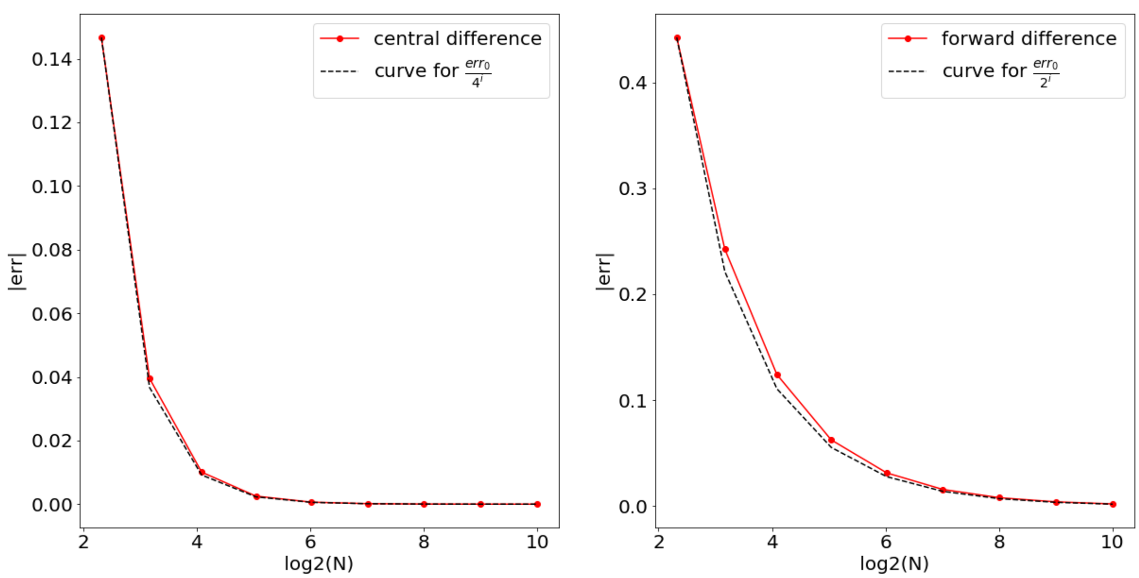


Figure 1: Convergence plot for both central difference(left) and forward difference(right). The plots demonstrate $\mathcal{O}(h^2)$ convergence for central difference and $\mathcal{O}(h)$ for forward difference

Question 2: Time integration

Given the initial value problem

$$\dot{y} = f(t, y(t)), \quad y(t_0) = y_0$$

a) Derive the expressions for the local and global truncation errors of the Euler method.

Local Truncation Error (LTE):
The solution obtained by Euler method in a single step is,

$$y_1 = y_0 + hf(t_0, y_0)$$

Using the Taylor's series expansion, the exact solution for that single step is,

$$y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{1}{2}h^2 y''(t_0) + O(h^3).$$

Hence the local truncation error is the difference of the two solutions.

$$\text{LTE} = E_0 = y(t_0 + h) - y_1 = \frac{1}{2}h^2 y''(t_0) + O(h^3).$$

or using the remainder term of the Taylor series,

$$\text{LTE} = y(t_0 + h) - y_1 = \frac{1}{2}h^2 y''(\xi_0).$$

where $\xi_0 \in [t_0, t_0 + h]$.

Global Truncation Error (GTE):

$$\text{GTE} := y(t_n) - y_n = |E_{n+1}| \leq (1 + h\,|f_y\,(t_n, \eta)|)\,|E_n| + \frac{|y''(\tau)|}{2}h^2$$

The Global Truncation Error would be the due the accumulation of local truncation errors.
Let

$$K = \max_{(t,y)\in R} |f_y(t, y)| < \infty, \quad M = \max_{(t,y)\in R} |(y''))\,(t, y)| < \infty$$

then

$$|E_{n+1}| \leq (1 + Kh)|En| + \frac{Mh^2}{2}$$

Putting the value of E_n in the recursion, we get

$$|E_n| \leq \frac{Mh^2}{2}S(h), S(h) = \sum_{i=0}^{n-1}(1 + Kh)^i$$

The equation on the right is a geometric series, therefore the sum simplifies to

$$|E_n| \leq \frac{Mh}{2K}\left[(1 + Kh)^{t_n/h} - 1\right]$$

As $h \to 0$, $\lim_{h\to 0}(1 + Kh)^{t_n/h} = e^{Kt_n}$. Hence,

$$\text{GTE} = |E_n| \leq \frac{Mh}{2K}\left(e^{Kt_n} - 1\right)$$

b) Evaluate the function at discrete times using explicit-Euler at $t = 0.01, 0.05, 0.1$ and compare the results with analytical solution. Compute both the local and global error of the time-stepping scheme.

$$\dot{y} = -100y$$

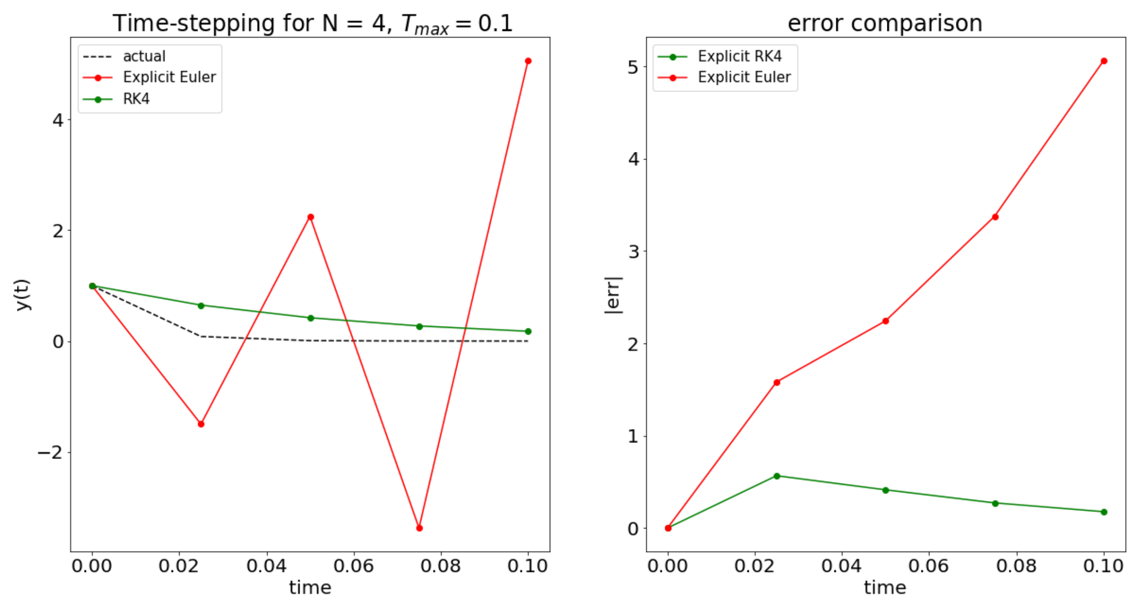
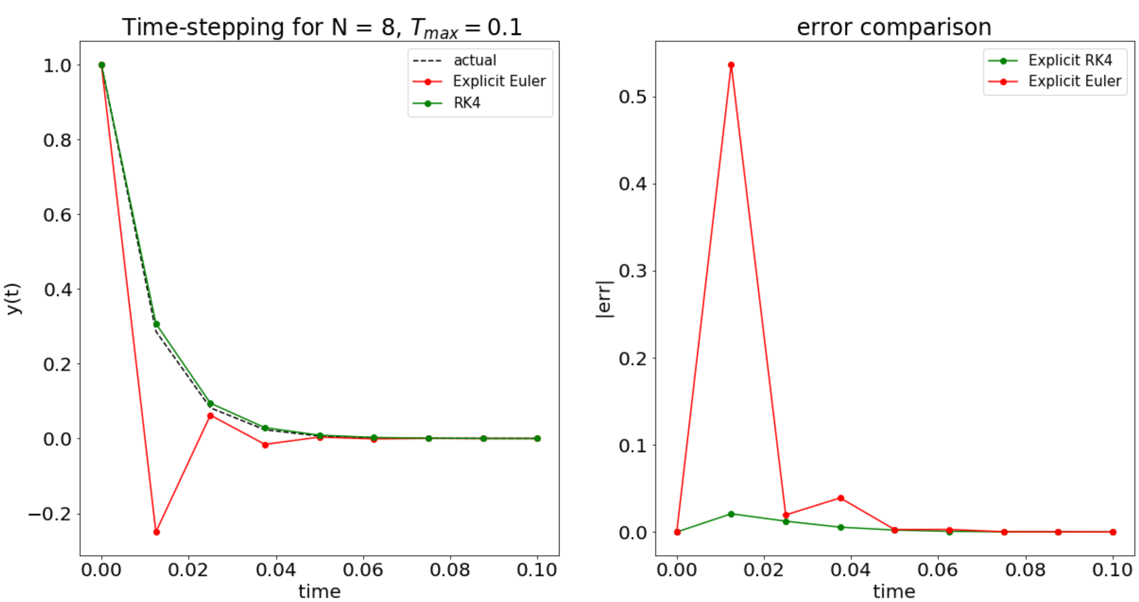
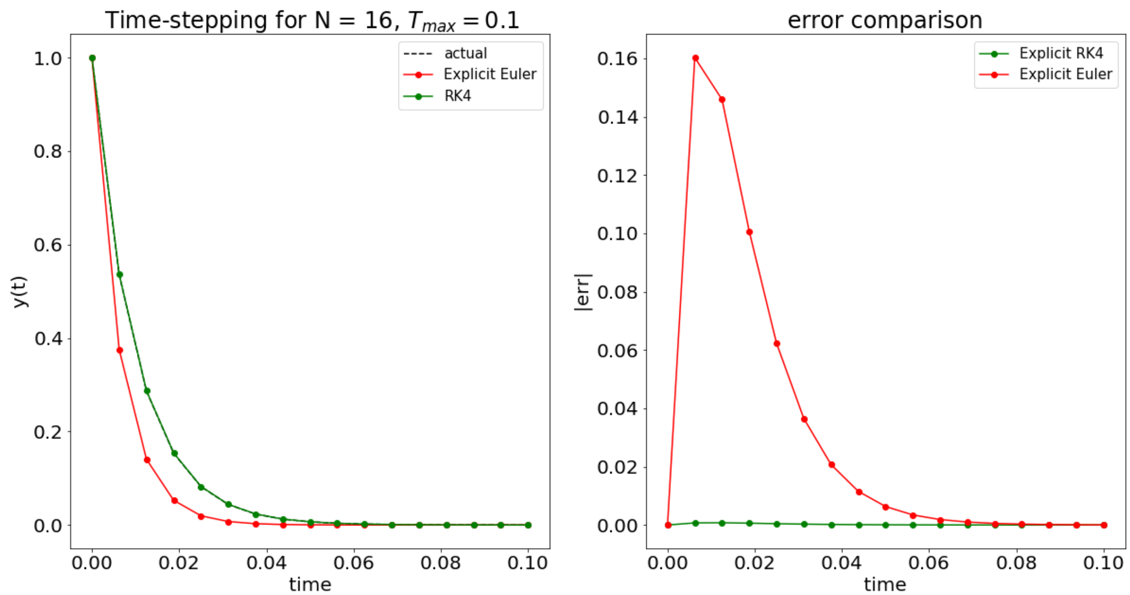


Figure 2: $\Delta t = \frac{T_{max}}{N} = 0.025$, Explict Euler is unstable in this regime





c) Compute the 4th order Central difference for a C^5 function f and prove that its truncation error is

$$E(f,h) = \frac{h^4 f^{(5)}(c)}{30}$$

Starting with the 4th degree expansion of the Taylor's series,

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{2f^{(3)}(x)h^3}{3!} + \frac{2f^{(5)}(c_1)\,h^5}{5!}$$

Then using the step size 2h,

$$f(x+2h) - f(x-2h) = 4f'(x)h + \frac{16f^{(3)}(x)h^3}{3!} + \frac{64f^{(5)}(c_2)\,h^5}{5!}$$

Now multiplying the first equation by 8 and subtracting it from the above equation, we get

$$\begin{aligned} -f(x+2h) + 8 \quad & f(x+h) - 8f(x-h) + f(x-2h) \\ & = 12f'(x)h + \frac{\left(16f^{(5)}(c_1) - 64f^{(5)}(c_2)\right)h^5}{120} \end{aligned}$$

Given f a C^5 function, we can find value c such that $c \in [x-2h, x+2h]$ and,

$$16f^{(5)}(c_1) - 64f^{(5)}(c_2) = -48f^{(5)}(c)$$

Therefore after substituting it back,

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{f^{(5)}(c)h^4}{30}.$$