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Exercise 9

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Question 1: Derivative approximation

Given the function $f(x) = e^{-x^2}$, sample the function at N uniformly spaced grid points in the interval $[-1, 1]$. Compute the derivative $f^{'}(x_i)$ and the second derivative $f^{''}(x)$ of the function at sample point $x_i, i = 1, 2, ...N-1$ and spacing $h = \frac{2}{N}$ using

a) central difference $f^{'}(x) = \frac{f(x+h)-f(x-h)}{2h}$ for varying N

b) forward difference $f^{'}(x) = \frac{f(x+h)-f(x)}{h}$ for varying N

Compute the error of approximation w.r.t the analytical solution $f^{'}(x) = -2xe^{-x^2}$ for varying N for both a) and b). Plot the error for varying N

Question 2: Time integration

Given the initial value problem

$$\dot{y} = f(t, y(t)), \quad y(t_0) = y_0$$

a) The local truncation error of a time integration scheme is the numerical error made in a single step. Whereas, the global truncation error is the accumulated numerical error for computing the solution upto time t starting from t_0. Derive the expressions for the local and global truncation errors of the Euler method.

b) For the differential equation $\dot{y} = -100y$ and the initial condition $y(0) = 1$, evaluate the function at discrete times using explicit-Euler at $t = 0.01, 0.05, 0.1$ and compare the results with analytical solution. Compute both the local and global error of the time-stepping scheme for the same.

c) Compute the 4th order Central difference for a C^5 function f and prove that its truncation error is

$$E(f, h) = \frac{h^4 f^{(5)}(c)}{30}$$

Question 3: Programming Task

Consider the damped harmonic oscillator

$$\ddot{x}(t) + 0.5\dot{x}(t) + x(t) = 0$$

with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

a) Create a Matlab/Python subroutine to solve the given equation using explicit Euler method with spacing h as a parameter.

b) Using a) Plot the error for different step sizes h where the error is computed from the analytical solution given below

$$x(t) = \frac{\mathrm{e}^{-\frac{t}{4}}}{15} \left(15 \cos\left(\frac{\sqrt{15}\, t}{4} \right) + \sqrt{15} \sin\left(\frac{\sqrt{15}\, t}{4} \right) \right)$$