Prof. Dr. I. F. Sbalzarini
TU Dresden, 01187 Dresden, Germany

Solution 3

Release: 09.11.2020
Due: 16.11.2020

Question 1: Iterative schemes and Convergence

Given the matrices

$$A_1 = \begin{pmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{pmatrix}, A_3 = \begin{pmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{pmatrix}, A_4 = \begin{pmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{pmatrix}$$

a) For which of these matrices does the Jacobi and Gauss-Seidel methods converge ?

b) In the case when both Jacobi and Gauss-Seidel methods converge, which converges faster ?

The iteration matrix for Jacobi method and Gauss-siedel method are $T_J = -D^{-1}(L+U)$ and $T_{GS} = -(L+D)^{-1}U$, respectively. The spectral radius of iteratio matrix T is given by $\rho(T) = \max(|\text{eig}(T)|)$. So accordingly,

$$A_1(\rho(T_J)) = 1.125 \text{ and } A_1(\rho(T_{GS})) = 1.583$$

Both Jacobi and Gauss-Seidel methods fail to converge for A_1.

$$A_2(\rho(T_J)) = 0.813 \text{ and } A_2(\rho(T_{GS})) = 1.111$$

Jacobi is convergent and Gauss-Seidel methods fail to converge for A_2

$$A_3(\rho(T_J)) = 0.444 \text{ and } A_3(\rho(T_{GS})) = 0.019$$

Both Jacobi and Gauss-Seidel methods converge for A_1, but Gauss-Siedel converges faster than Jacobi, since $A_3(\rho(T_{GS})) < A_3(\rho(T_J))$

$$A_4(\rho(T_J)) = 0.641 \text{ and } A_4(\rho(T_{GS})) = 0.775$$

Both Jacobi and Gauss-Seidel methods converge for A_4, but Jacobi converges faster than Gauss-Seidel, since $A_4(\rho(T_J)) < A_4(\rho(T_{GS}))$

Question 2: Spectral radius and convergence rate

Given the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

of the linear system of equations $Ax = b$

a) For the iteration matrix T of Successive Over Relaxation(SOR) method, plot the spectral radius as a function of the parameter $\omega, 0 < \omega < 2$. What is the optimum value of ω for fast convergence ?

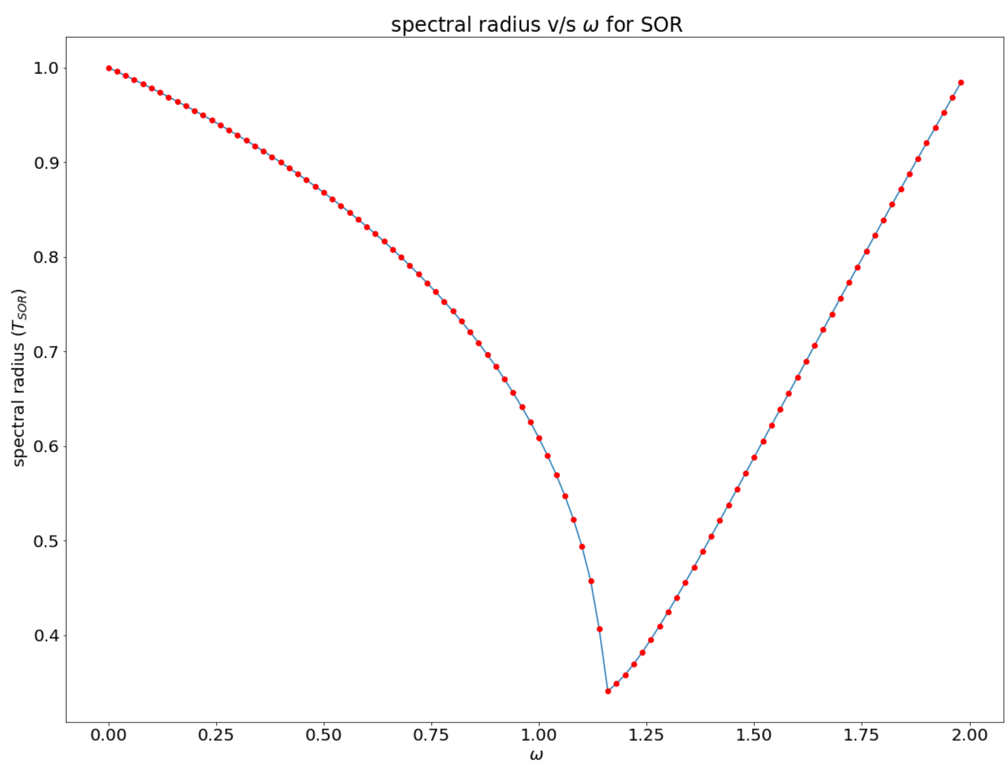
The iteration matrix for SOR is given by, $T_{SOR} = (D + \omega L)^{-1}\left[-\omega R + (1-\omega)D\right]$

$$T_{SOR} = \begin{pmatrix} 1-\omega & -\frac{2}{3}\omega & -\frac{1}{3}\omega \\ -\frac{2}{3}(\omega-\omega^2) & \frac{4}{9}\omega^2 - \omega + 1 & \frac{2}{9}\omega^2 - \frac{2}{3}\omega \\ -\frac{1}{9}(\omega^2-\omega)(4\omega-3) & -\frac{2}{27}\omega(3-2\omega)^2 & 1-\omega+\frac{5}{9}\omega^2 - \frac{4}{27}\omega^3 \end{pmatrix}$$

$$\rho(T_{SOR}) = \max(|\text{eig}(T_{SOR})|)$$

The fastest convergence occurs when the spectral radius $\rho(T_{SOR})$ is minimum. From the plot generated from **python** code we can infer that

$$\rho_{min} = 0.3387$$
$$\omega_{opt} = 1.152$$



b) How many iterations does the SOR method for $\omega = 0.4$ and 1.4 need to achieve the absolute tolerance of 10^{-10}.

$$\|e^k\| \leq \|T^k\|\|e^0\|$$
$$\text{we want} \;\; \|e^k\| \leq 10^{-10}$$

Let $\|e^0\| \approx 1$
For $\omega = 0.4$, $\rho(T_{SOR}) = 0.9 \implies k \geq 219$
For $\omega = 1.4$, $\rho(T_{SOR}) = 0.505 \implies k \geq 34$

Question 3: Programming task

a) Implement Jacobi, Gauss-Seidel and SOR-method in MATLAB/python.
(NOTE : Have a conditional check for spectral radius)

b) Given the linear system

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 4 & 11 & 5 \\ 7 & 8 & 16 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 8 \\ 20 \\ 31 \end{pmatrix}$$

start with the initial condition of $x^{[0]} = (0.5 \;\; 0.5 \;\; 0.5)^T$ and

i) Check if Jacobi and Gauss-Siedel methods converge ?

ii) Compute the number of steps required to reach a tolerance of 10^{-4} and also the CPU time.

iii) Check if SOR method improves the convergence rate.

Refer to the following link:
`https://crunchingnumbers.live/2017/07/09/iterative-methods-part-2/`