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Solution 12

Release: 25.01.2021
Due: 01.02.2021

Question 1: Stability

Consider the Linear Differential Equation system

$$\dot{x} = \begin{pmatrix} -1001 & 999 \\ 999 & -1001 \end{pmatrix} x$$

Solution :

$$\begin{aligned} \dot{x} &= Ax = TDT^{-1}x \\ \implies T^{-1}\dot{x} &= DT^{-1}x, \quad y = T^{-1}x \\ \dot{y} &= Dy \quad \text{where, } D = \operatorname{diag}(\lambda_i) \end{aligned}$$

Eigen-values of A:

$$\begin{aligned} \lambda_1 &= -2000; \quad v_1 = (-1, 1)^T \\ \lambda_2 &= -2 \quad v_2 = (1, 1)^T \end{aligned}$$

$$y(t) = \begin{pmatrix} y_1(0)e^{-\lambda_1 t} \\ y_2(0)e^{-\lambda_2 t} \end{pmatrix}; \quad x(t) = Ty(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} -y_1(0)e^{-\lambda_1 t} + y_2(0)e^{-\lambda_2 t} \\ y_1(0)e^{-\lambda_1 t} + y_2(0)e^{-\lambda_2 t} \end{pmatrix}; \quad x(0) = \begin{pmatrix} -y_1(0) + y_2(0) \\ y_1(0) + y_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$\implies y_2(0) = \frac{1}{2}\left(x_1(0) + x_2(0)\right); \; y_1(0) = \frac{1}{2}\left(x_2(0) - x_1(0)\right).$

$$x(t) = \begin{pmatrix} \frac{1}{2}\left(x_1(0) - x_2(0)\right)e^{-2000t} + \frac{1}{2}\left(x_1(0) + x_2(0)\right)e^{-2t} \\ \\ \frac{1}{2}\left(x_2(0) - x_1(0)\right)e^{-2000t} + \frac{1}{2}\left(x_1(0) + x_2(0)\right)e^{-2t} \end{pmatrix}$$

a) we want to apply the explicit Euler procedure to this System. How big must the step size be for the numerical solution to the starting conditions

i) $x(0) = (-1, 1)^T$

the analytical solution is then,

$$x(t) = \begin{pmatrix} \frac{1}{2}(-2)\,e^{-2000t} \\ \\ \frac{1}{2}(2)\,e^{-2000t} \end{pmatrix}$$

So the only eigen-value left is $\lambda_1 = -2000$. For stability of explicit schemes, we need $h\lambda \in (-2, 0)$,

$$\begin{aligned} -2000h &\in (-2,0) \\ h &< 0.001 \end{aligned}$$

ii) $x(0) = (1,1)^T$

the analytical solution is then,

$$x(t) = \begin{pmatrix} \frac{1}{2}(2)\, e^{-2t} \\ \\ \frac{1}{2}(2)\, e^{-2t} \end{pmatrix}$$

So the only eigen-value left is $\lambda_2 = -2$. For stability of explicit schemes, we need $h\lambda \in (-2, 0)$,

$$\begin{aligned} -2h &\in (-2,0) \\ h &< 1 \end{aligned}$$

iii) $x(0) = (2,0)^T$

the analytical solution is then,

$$x(t) = \begin{pmatrix} \frac{1}{2}2e^{-2000t} + \frac{1}{2}2e^{-2t} \\ \\ -\frac{1}{2}2e^{-2000t} + \frac{1}{2}2e^{-2t} \end{pmatrix}$$

Here we choose the largest eigen-value $\lambda_1 = -2000$. For stability of explicit schemes, we need $h\lambda \in (-2, 0)$,

$$\begin{aligned} -2000h &\in (-2,0) \\ h &< 0.001 \end{aligned}$$

for stable qualitative behaviours .

Question 2: Stiff Differential Equations

Consider the IVP,

$$y'(t) = -15y(t)$$

with the initial condition $y(0) = 1$ and $t \geq 0$.

a) Find the exact analytical solution to the problem.

b) Use explicit Euler scheme to solve the problem with step size $h = \frac{1}{4}$ and $h = \frac{1}{8}$

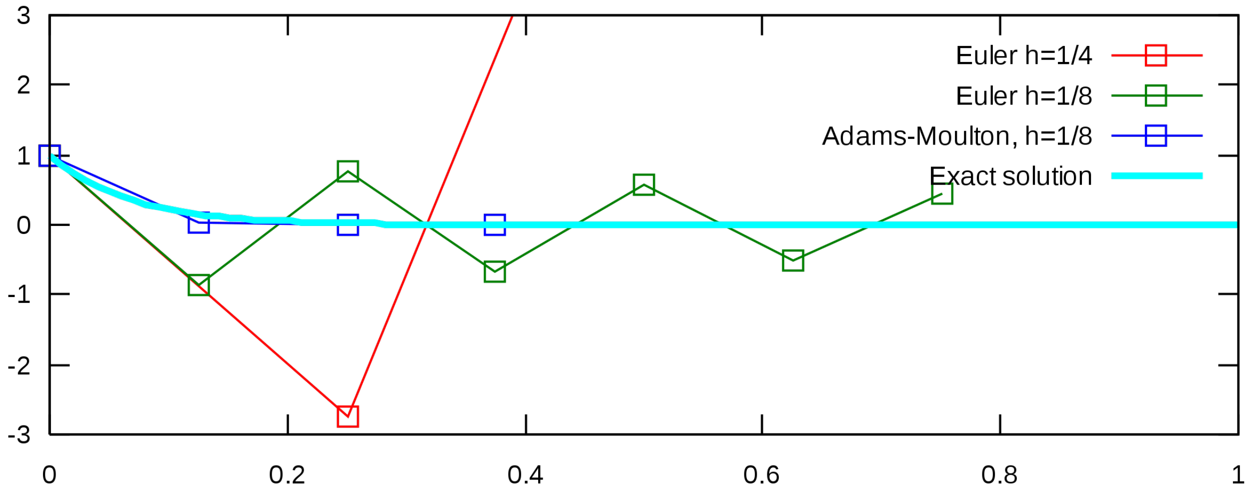
c) Use the implicit Trapezoidal scheme to solve the problem and plot a graph comparing explicit Euler ($h = \frac{1}{4}, \frac{1}{8}$), implicit Trapezoidal ($h = \frac{1}{8}$), and the analytical solution.

Solution :

The exact solution is

$$y(t) = e^{-15t} \text{ with } y(t) \to 0 \text{ as } t \to \infty$$

Figure 1: Numerical Simulation



Question 3: Classification of PDEs

Classify the following PDEs with respect to their order and type (parabolic, hyperbolic and ellitpic)

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = D\Delta u, \\
&\Delta u = 0 \\
&u_{tt} - c^2 u_{xx} = 0 \\
&u_{xx} + x u_{yy} = 0
\end{aligned}
$$

Solution :
The general case of second-order linear partial differential equation (PDE) with two independent variables is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

We base our classification on the sign of the quantity $B^2 - 4AC$.

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = D\Delta u, \quad \text{parabolic} \\
&\Delta u = 0 \ \ \text{elliptic} \\
&u_{tt} - c^2 u_{xx} = 0 \ \ \text{Hyperbolic}
\end{aligned}
$$

$$u_{xx} + x u_{yy} = 0$$
$$(B^2 - 4AC) = -4x \implies \text{for } x > 0 \text{ elliptic}, \ x = 0 \text{ parabolic}, \ x < 0 \text{ hyperbolic}$$