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# Solution 10

Release: 11.01.2021
Due: 18.01.2021

## Question 1: Programming Task

a) Implement the Heun's procedure to

$$
\begin{aligned}
y^*_{n+1} &= \tilde{y}_n + hf(t_n, \tilde{y}_n) \\
\tilde{y}_{n+1} &= \tilde{y}_n + \frac{h}{2}\left(f(t_n, \tilde{y}_n) + f(t_{n+1}, y^*_{n+1})\right)
\end{aligned}
$$

and solve the 2-body problem given by,

$$
\begin{aligned}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= -\frac{y_1}{(y_1^2 + y_2^2)^{3/2}} \\
\dot{y}_4 &= -\frac{y_2}{(y_1^2 + y_2^2)^{3/2}}
\end{aligned}
$$

with the initial values $y_1(0) = 0.5,\ y_2(0) = 0,\ y_3(0) = 0,\ y_4(0) = \sqrt(3)$. To do this, select $t_0 = 0, t_f = 8$, use the Heun's method with two steps $h = \frac{1}{10}, \frac{1}{100}$ and draw the paths $(\tilde{y}_1(t_n), \tilde{y}_2(t_n))$

b) Compare your results with the results of MATLAB's **ode23**

## Question 2: RK4

Consider the Linear Differential Equation system

$$
\dot{x} = \begin{pmatrix} -1001 & 999 \\ 999 & -1001 \end{pmatrix} x
$$

a) determine the general solution $x(t)$.

**Solution :**

$$
\begin{aligned}
\dot{x} &= Ax = TDT^{-1}x \\
\implies T^{-1}\dot{x} &= DT^{-1}x, \quad y = T^{-1}x \\
\dot{y} &= Dy \quad \text{where, } D = \text{diag}(\lambda_i)
\end{aligned}
$$

Eigen-values of $A$:

$$\lambda_1 = -2000; \quad v_1 = (-1, 1)^T$$
$$\lambda_2 = -2 \quad v_2 = (1, 1)^T$$

$$y(t) = \begin{pmatrix} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\lambda_2 t} \end{pmatrix}; \quad x(t) = Ty(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} y(t)$$

$$x(t) = \begin{pmatrix} -y_1(0)e^{\lambda_1 t} + y_2(0)e^{\lambda_2 t} \\ y_1(0)e^{\lambda_1 t} + y_2(0)e^{\lambda_2 t} \end{pmatrix}; \quad x(0) = \begin{pmatrix} -y_1(0) + y_2(0) \\ y_1(0) + y_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$\implies y_2(0) = \frac{1}{2}\left(x_1(0) + x_2(0)\right); \; y_1(0) = \frac{1}{2}\left(x_2(0) - x_1(0)\right).$

$$x(t) = \begin{pmatrix} \frac{1}{2}\left(x_1(0) - x_2(0)\right)e^{-2000t} + \frac{1}{2}\left(x_1(0) + x_2(0)\right)e^{-2t} \\ \\ \frac{1}{2}\left(x_2(0) - x_1(0)\right)e^{-2000t} + \frac{1}{2}\left(x_1(0) + x_2(0)\right)e^{-2t} \end{pmatrix}$$

## Question 3: Adaptive Step Size

Consider the joint Butcher table of the Dormand-Prince method (DOPRI5),

$$\begin{array}{c|ccccccc}
0 & & & & & & & \\
\frac{1}{5} & \frac{1}{5} & & & & & & \\
\frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & & \\
\frac{4}{5} & \frac{44}{45} & -\frac{56}{15} & \frac{32}{9} & & & & \\
\frac{8}{9} & \frac{19372}{6561} & -\frac{25360}{2187} & \frac{64448}{6561} & -\frac{212}{729} & & & \\
1 & \frac{9017}{3168} & -\frac{355}{33} & \frac{46732}{5247} & \frac{49}{176} & -\frac{5103}{18656} & & \\
1 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & \\
\hline
 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0 \\
\hline
 & \frac{5179}{57600} & 0 & \frac{7571}{16695} & \frac{393}{640} & -\frac{92097}{339200} & \frac{187}{2100} & \frac{1}{40}
\end{array}$$

a) Using the table, derive the expressions for computing each stage of the DOPRI5 method.

**Solution :**

$$
\begin{aligned}
k_1 &= f(t, x)\\
k_2 &= f\Big(t + \frac{h}{5}, x + \frac{h}{5}k_1\Big)\\
k_3 &= f\Big(t + \frac{3h}{10}, x + \frac{3h}{40}k_1 + \frac{9h}{40}k_2\Big)\\
k_4 &= f\Big(t + \frac{4h}{5}, x + \frac{44h}{45}k_1 - \frac{56h}{15}k_2 + \frac{32h}{9}k_3\Big)\\
k_5 &= f\Big(t + \frac{8h}{9}, x + \frac{19372h}{6561}k_1 - \frac{25360}{2187}k_2 + \frac{64448}{6561}k_3 - \frac{212}{729}k_4\Big)\\
k_6 &= f\Big(t + h, x + \frac{9017h}{3168}k_1 - \frac{355h}{33}k_2 + \frac{46732h}{5247}k_3 + \frac{49h}{176}k_4 - \frac{5103h}{18656}k_5\Big)\\
k_7 &= f\Big(t + h, x + \frac{35h}{384}k_1 + \frac{500h}{1113}k_3 + \frac{125h}{192}k_4 - \frac{2187h}{6784}k_5 + \frac{11h}{84}k_6\Big)
\end{aligned}
$$