*Prof. Dr. I. F. Sbalzarini*
*TU Dresden, 01187 Dresden, Germany*

# Solution 10

Release: 11.01.2021
Due: 18.01.2021

## Question 1: Programming Task

a) Implement the Heun's procedure to

$$\begin{aligned} y^*_{n+1} &= \tilde{y}_n + hf(t_n, \tilde{y}_n) \\ \tilde{y}_{n+1} &= \tilde{y}_n + \frac{h}{2}\left(f(t_n, \tilde{y}_n) + f(t_{n+1}, y^*_{n+1})\right) \end{aligned}$$

and solve the 2-body problem given by,

$$\begin{aligned} \dot{y}_1 &= y_3 \\ \dot{y}_2 &= y_4 \\ \dot{y}_3 &= -\frac{y_1}{(y_1^2 + y_2^2)^{3/2}} \\ \dot{y}_4 &= -\frac{y_2}{(y_1^2 + y_2^2)^{3/2}} \end{aligned}$$

with the initial values $y_1(0) = 0.5,\ y_2(0) = 0,\ y_3(0) = 0,\ y_4(0) = \sqrt{(3)}$. To do this, select $t_0 = 0, t_f = 8$, use the Heun's method with two steps $h = \frac{1}{10}, \frac{1}{100}$ and draw the paths $(\tilde{y}_1(t_n), \tilde{y}_2(t_n))$

b) Compare your results with the results of MATLAB's **ode23**

## Question 2: RK4

Consider the Linear Differential Equation system

$$\dot{x} = \begin{pmatrix} -1001 & 999 \\ 999 & -1001 \end{pmatrix} x$$

a) determine the general solution $x(t)$.

**Solution :**

$$\begin{aligned} \dot{x} &= Ax = TDT^{-1}x \\ \implies T^{-1}\dot{x} &= DT^{-1}x, \quad y = T^{-1}x \\ \dot{y} &= Dy \ \text{ where, } D = \text{diag}(\lambda_i) \end{aligned}$$

Eigen-values of $A$:

$$\lambda_1 = -2000; \quad v_1 = (-1, 1)^T$$
$$\lambda_2 = -2 \quad v_2 = (1, 1)^T$$

$$y(t) = \begin{pmatrix} y_1(0)e^{-\lambda_1 t} \\ y_2(0)e^{-\lambda_2 t} \end{pmatrix}; \quad x(t) = Ty(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$ y(t)

$$x(t) = \begin{pmatrix} -y_1(0)e^{-\lambda_1 t} + y_2(0)e^{-\lambda_2 t} \\ y_1(0)e^{-\lambda_1 t} + y_2(0)e^{-\lambda_2 t} \end{pmatrix}; \quad x(0) = \begin{pmatrix} -y_1(0) + y_2(0) \\ y_1(0) + y_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\implies y_2(0) = \tfrac{1}{2}\left(x_1(0) + x_2(0)\right); \; y_1(0) = \tfrac{1}{2}\left(x_2(0) - x_1(0)\right).$$

$$x(t) = \begin{pmatrix} \frac{1}{2}\left(x_1(0) - x_2(0)\right)e^{-2000t} + \frac{1}{2}\left(x_1(0) + x_2(0)\right)e^{-2t} \\ \\ \frac{1}{2}\left(x_2(0) - x_1(0)\right)e^{-2000t} + \frac{1}{2}\left(x_1(0) + x_2(0)\right)e^{-2t} \end{pmatrix}$$

## Question 3: Adaptive Step Size

Consider the joint Butcher table of the Dormand-Prince method (DOPRI5),

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $0$ | | | | | | | |
| $\frac{1}{5}$ | $\frac{1}{5}$ | | | | | | |
| $\frac{3}{10}$ | $\frac{3}{40}$ | $\frac{9}{40}$ | | | | | |
| $\frac{4}{5}$ | $\frac{44}{45}$ | $-\frac{56}{15}$ | $\frac{32}{9}$ | | | | |
| $\frac{8}{9}$ | $\frac{19372}{6561}$ | $-\frac{25360}{2187}$ | $\frac{64448}{6561}$ | $-\frac{212}{729}$ | | | |
| $1$ | $\frac{9017}{3168}$ | $-\frac{355}{33}$ | $\frac{46732}{5247}$ | $\frac{49}{176}$ | $-\frac{5103}{18656}$ | | |
| $1$ | $\frac{35}{384}$ | $0$ | $\frac{500}{1113}$ | $\frac{125}{192}$ | $-\frac{2187}{6784}$ | $\frac{11}{84}$ | |
| | $\frac{35}{384}$ | $0$ | $\frac{500}{1113}$ | $\frac{125}{192}$ | $-\frac{2187}{6784}$ | $\frac{11}{84}$ | $0$ |
| | $\frac{5179}{57600}$ | $0$ | $\frac{7571}{16695}$ | $\frac{393}{640}$ | $-\frac{92097}{339200}$ | $\frac{187}{2100}$ | $\frac{1}{40}$ |

a) Using the table, derive the expressions for computing each stage of the DOPRI5 method.

**Solution :**

$$
\begin{aligned}
k_1 &= f(t,x)\\
k_2 &= f\Big(t+\frac{h}{5}, x+\frac{h}{5}k_1\Big)\\
k_3 &= f\Big(t+\frac{3h}{10}, x+\frac{3h}{40}k_1+\frac{9h}{40}k_2\Big)\\
k_4 &= f\Big(t+\frac{4h}{5}, x+\frac{44h}{45}k_1-\frac{56h}{15}k_2+\frac{32h}{9}k_3\Big)\\
k_5 &= f\Big(t+\frac{8h}{9}, x+\frac{19372h}{6561}k_1-\frac{25360}{2187}k_2+\frac{64448}{6561}k_3-\frac{212}{729}k_4\Big)\\
k_6 &= f\Big(t+h, x+\frac{9017h}{3168}k_1-\frac{355h}{33}k_2+\frac{46732h}{5247}k_3+\frac{49h}{176}k_4-\frac{5103h}{18656}k_5\Big)\\
k_7 &= f\Big(t+h, x+\frac{35h}{384}k_1+\frac{500h}{1113}k_3+\frac{125h}{192}k_4-\frac{2187h}{6784}k_5+\frac{11h}{84}k_6\Big)
\end{aligned}
$$