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Exercise 14

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Question 1: Von Neumann-stability analysis

$$u_t = \alpha u_{xx}$$

a) Show, using the Von Neumann-stability analysis, that the Crank-Nicolson method applied to the heat equation with central finite differences in space, is unconditionally stable

b) In a similar way, show that the Leap-frog method applied to above equation is unconditionally unstable.

Question 2: Hyperbolic equation

Consider the PDE for advection equation

$$u_t + cu_x = 0$$

Show that for the CTCS-method (Leapfrog?) the local truncation error is of the form

$$\text{error} = -\frac{1}{6}\Delta t^2 u_{ttt}|_i^n - \frac{c}{6}\Delta x^2 u_{xxx}|_i^n + \text{H.O.T in } \Delta t \text{ and } \Delta x$$

Question 3: Stability of hyperbolic PDEs

Work out the Von Neumann stability analysis for the wave equation with the CTCS scheme

$$u_{tt} = c^2 u_{xx}$$

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (CTCS)$$

Question 4: Mass conservation

Show that for the non-linear hyperbolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial [F(u)]}{\partial x} = 0$$

the following property holds

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx \ \ \forall t \geq 0$$

if we assume that $\lim_{x\to\pm\infty} F(u(x,t)) = 0, \ \forall t \geq 0$