

Approximate Functions

Going Deep

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- No free lunch and universal approximation
- Why go deep?
- Problems of going deep
- Some fixes:
 - Improving gradient flow with skip connections
 - Regularising with Dropout

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 - **No machine learning algorithm is universally better than any other!**
 - Fortunately, in the real world, data is generated by a small subset of generating distributions...

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\implies simple neural networks can represent a wide variety of interesting functions when given appropriate parameters.

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 - The training algorithm might just choose the wrong solution as a result of overfitting.
 - *There is no known universal procedure for examining a set of examples and choosing a function that will generalise to points out of the training set.*

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Then Why Go Deep?

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- Alternatively, one could just consider that a deep architecture just expresses that the function we wish to learn is a program made of multiple steps where each step makes use of the previous steps outputs.
- **Empirically, deeper networks just seem to generalise better!**

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 - Overfitting

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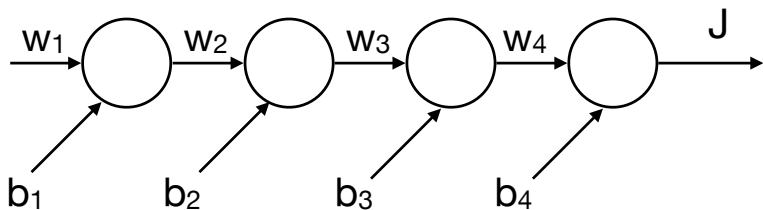
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- In principle, optimisers that rescale the gradients of each weight should be able to deal with this issue (as long as numeric precision doesn't become problematic).

Issues with Going Deep



Residual Connections

- One of the most effective ways to resolve diminishing gradients is with residual neural networks (ResNets)³.

³K. He, X. Zhang, S. Ren and J. Sun, "Deep Residual Learning for Image Recognition," CVPR, Las Vegas, NV, 2016, pp. 770-778.

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- Is this the full story though? Skip connections also break symmetries, which could be much more important...

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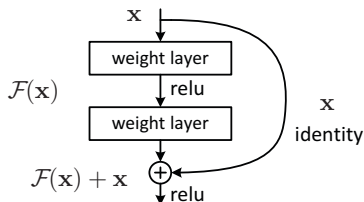


Figure 2. Residual learning: a building block.

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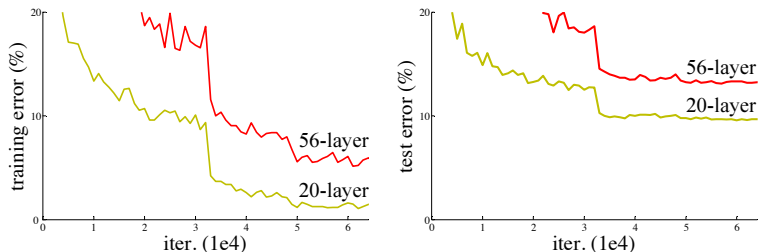


Figure 1. Training error (left) and test error (right) on CIFAR-10 with 20-layer and 56-layer “plain” networks. The deeper network has higher training error, and thus test error. Similar phenomena on ImageNet is presented in Fig. 4.

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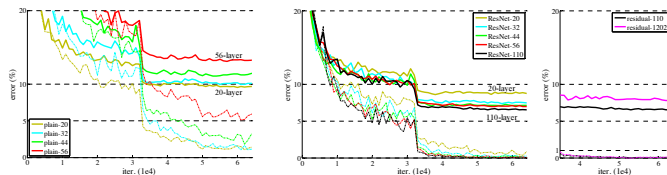


Figure 6. Training on **CIFAR-10**. Dashed lines denote training error, and bold lines denote testing error. **Left:** plain networks. The error of plain-110 is higher than 60% and not displayed. **Middle:** ResNets. **Right:** ResNets with 110 and 1202 layers.

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 - Regularise by smoothing the optimisation landscape (e.g. Batch Normalisation)

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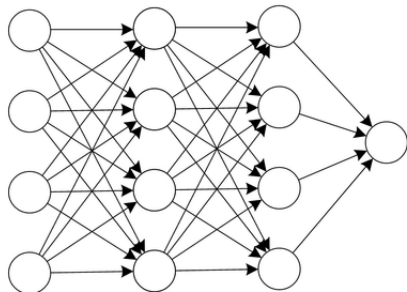
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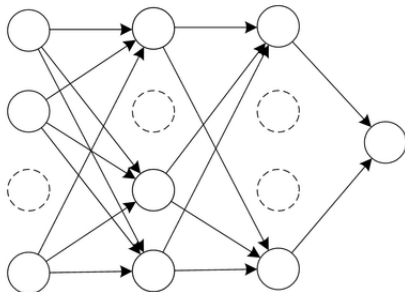
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- Motivation: the best way to regularise a fixed size model is to average predictions over all possible parameter settings, weighting each setting by the posterior probability given the training data.
 - Clearly this isn't actually tractable - dropout is an approximation of this idea.
 - The idea of averaging predictions to resolve the bias-variance dilemma is called ensembling.

Dropout



(a) Standard Neural Network



(b) Network after Dropout

Image from: https://www.researchgate.net/figure/Dropout-neural-network-model-a-is-a-standard-neural-network-b-is-the-same-network_fig3_309206911

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- For each batch we then randomly decide whether or not a given neuron in a given layer is removed.

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$$\delta^l = ((w^{l+1})^T \delta^{l+1}) \odot \sigma'(z^l) \odot m^{(l)} \quad (4)$$

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- By breaking co-adaptation, each unit will ultimately find more general features