Make a forward pass before the backward pass

Backpropagation: Understanding the implications of the chain rule

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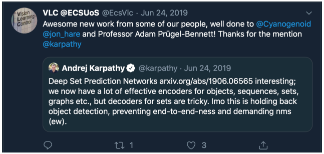
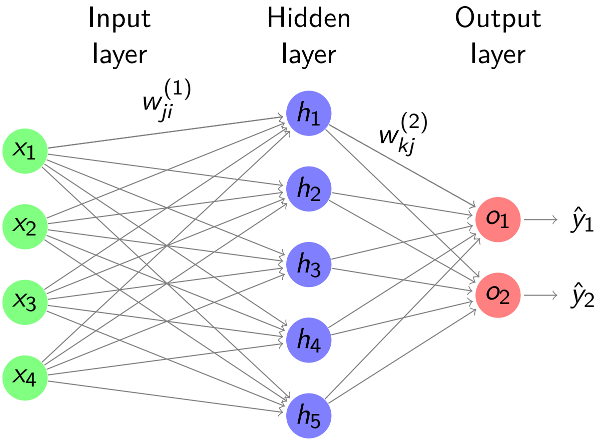
A lot of the ideas in this lecture come from Andrej Karpathy's blog post on backprop (https://medium.com/@karpathy/yes-you-should-understand-backprop-e2f06eab496b) and his CS231n Lecture Notes (http://cs231n.github.io/optimization-2/)



Topics

- A quick look at an MLP again
- The chain rule (again)
- Uninititive gradient effects
- A closer look at basic stochastic gradient descent algorithms

The unbiased Multilayer Perceptron (again)...



Without loss of generality, we can write the above as:

$$\hat{\boldsymbol{y}} = g(f(\boldsymbol{x}; \boldsymbol{W}^{(1)}); \boldsymbol{W}^{(2)}) = g(\boldsymbol{W}^{(2)} f(\boldsymbol{W}^{(1)} \boldsymbol{x}))$$

where f and g are activation functions.

Gradients of our simple unbiased MLP

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- (But we're not that crazy!)

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so $\nabla_{[x,y,z]} f = [z, z, q]$

A computational graph perspective

$$f(x, y, z) = (x + y)z$$

An intuition of the chain rule

- Notice how every operation in the computational graph given its inputs can immediately compute two things:
  1. its output value
  2. the *local* gradient of its inputs with respect to its output value
- The chain rule tells us literally that each operation should take its local gradients and multiply them by the gradient that *flows* backwards into it

This is backpropagation

- The backprop algorithm is just the idea that you can perform the forward pass (computing and caching the local gradients as you go),
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- Backprop is just another name for 'Reverse Mode Automatic Differentiation'...

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  - **Hence you need to always pay attention to data normalisation!**

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    - What's the implication of this in a deep network with sigmoid activations?

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- These are dead ReLUs - ones that never fire for all training data
    - Sometimes you can find that you have a large fraction of these
    - if you get them from the beginning, check weight initialisation and data normalisation
    - if they're appearing during training, maybe η is too big?

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- Same thing happens in the backward pass of an RNN (although with matrices rather than scalars, so the reasoning applies to the largest eigenvalue)