Approximate Functions

VLC = $\iiint$ Vision Learning Control $dV dL dC$

Going Deep

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Overview

- No free lunch and universal approximation
- Why go deep?
- Problems of going deep
- Some fixes:
  - Improving gradient flow with skip connections
  - Regularising with Dropout

No Free Lunch

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  - **No machine learning algorithm is universally better than any other!**
  - Fortunately, in the real world, data is generated by a small subset of generating distributions...

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$\implies$ simple neural networks can represent a wide variety of interesting functions when given appropriate parameters.

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  - We've not said anything about learnability...
    - The optimiser might not find a good solution[2].
    - The training algorithm might just choose the wrong solution as a result of overfitting.
    - *There is no known universal proceedure for examining a set of examples and choosing a function that will generalise to points out of the training set.*

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Then Why Go Deep?

- There are functions you can compute with a deep neural network that shallow networks require exponentially more hidden units to compute.
- The following function is more efficient to implement using a deep neural network: $y = x_1 \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_n$

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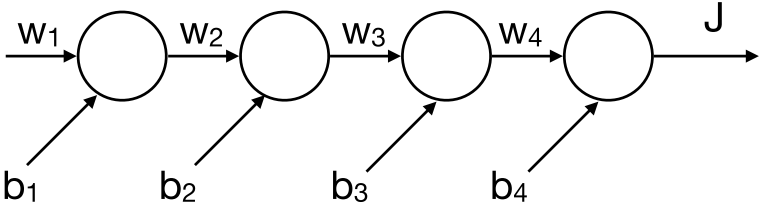
- This leads to the neural network not being able to train.
- This issue affects many-layered networks (feed-forward), as well as recurrent networks.

Issues with Going Deep



Residual Connections

- One of the most effective ways to resolve the vanishing gradient problem is with residual neural networks (ResNets)[3].

[3] K. He, X. Zhang, S. Ren and J. Sun, "Deep Residual Learning for Image Recognition," CVPR, Las Vegas, NV, 2016, pp. 770-778.

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- ResNets are artificial neural networks that use *skip connections* to jump over layers.
- The vanishing gradient problem is mitigated in ResNets by reusing activations from a previous layer.

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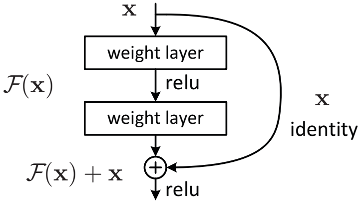

Figure 2. Residual learning: a building block.[4]

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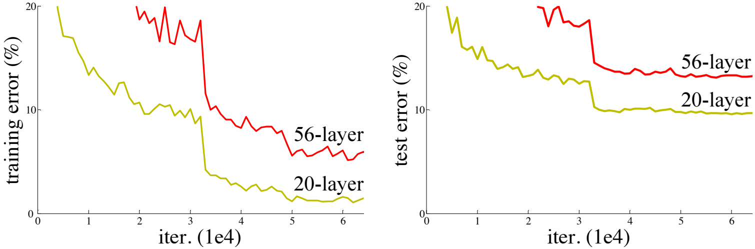

Figure 1. Training error (left) and test error (right) on CIFAR-10 with 20-layer and 56-layer "plain" networks. The deeper network has higher training error, and thus test error. Similar phenomena on ImageNet is presented in Fig. 4. [5]

[5] K. He, X. Zhang, S. Ren and J. Sun, "Deep Residual Learning for Image Recognition," CVPR, Las Vegas, NV, 2016, pp. 770-778.

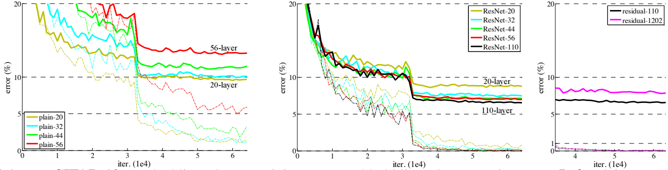

Figure 6. Training on **CIFAR-10**. Dashed lines denote training error, and bold lines denote testing error. **Left**: plain networks. The error of plain-110 is higher than 60% and not displayed. **Middle**: ResNets. **Right**: ResNets with 110 and 1202 layers. [6]

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- Dropout is a form of regularization
- The key idea in dropout is to randomly drop neurons, including all of the connections, from the neural network during training.

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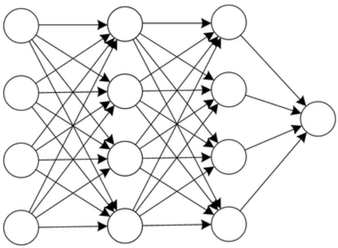
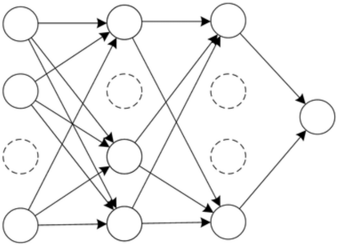

(a) Standard Neural Network


(b) Network after Dropout [7]

[7] Image from: https://www.researchgate.net/figure/Dropout-neural-network-model-a-is-a-standard-neural-network-b-is-the-same-fig3_309206911

How Does Dropout Work in Practice?

- In the learning phase, we stochastically remove hidden units by setting a dropout probability for each layer in the network. We then randomly decide wether or not a neuron in a given layer is removed stochastically.

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$$\delta^l = ((w^{l+1})^T \delta^{l+1}) \odot \sigma'(z^l) \odot m^{(l)} \tag{4}$$

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- By breaking co-adaptation, each unit will ultimately find more general features