If $\Delta = 0$, at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point. There is either a whole line of fixed points, as in Figure 5.1.5d, or a plane of fixed points, if $A = 0$.

Figure 5.2.8 shows that saddle points, nodes, and spirals are the major types of fixed points; they occur in large open regions of the (Δ,τ) plane. Centers, stars, degenerate nodes, and non-isolated fixed points are ***borderline cases*** that occur along curves in the (Δ,τ) plane. Of these borderline cases, centers are by far the most important. They occur very commonly in frictionless mechanical systems where energy is conserved.

EXAMPLE 5.2.6:

Classify the fixed point $\mathbf{x}^* = \mathbf{0}$ for the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution: The matrix has $\Delta = -2$; hence the fixed point is a saddle point. ■

EXAMPLE 5.2.7:

Redo Example 5.2.6 for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

Solution: Now $\Delta = 5$ and $\tau = 6$. Since $\Delta > 0$ and $\tau^2 - 4\Delta = 16 > 0$, the fixed point is a node. It is unstable, since $\tau > 0$. ■

5.3 Love Affairs

To arouse your interest in the classification of linear systems, we now discuss a simple model for the dynamics of love affairs (Strogatz 1988). The following story illustrates the idea.

Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let

$$\begin{aligned} R(t) &= \text{Romeo's love/hate for Juliet at time } t \\ J(t) &= \text{Juliet's love/hate for Romeo at time } t. \end{aligned}$$

Positive values of R, J signify love, negative values signify hate. Then a model for their star-crossed romance is

$$\dot{R} = aJ$$
$$\dot{J} = -bR$$

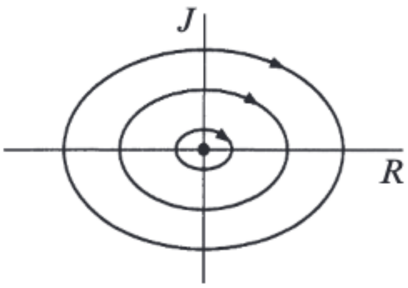


Figure 5.3.1

where the parameters a and b are positive, to be consistent with the story.

The sad outcome of their affair is, of course, a neverending cycle of love and hate; the governing system has a center at $(R, J) = (0, 0)$. At least they manage to achieve simultaneous love one-quarter of the time (Figure 5.3.1).

Now consider the forecast for lovers governed by the general linear system

$$\dot{R} = aR + bJ$$
$$\dot{J} = cR + dJ$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles. As named by one of my students, the choice $a > 0$, $b > 0$ means that Romeo is an "eager beaver"—he gets excited by Juliet's love for him, and is further spurred on by his own affectionate feelings for her. It's entertaining to name the other three romantic styles, and to predict the outcomes for the various pairings. For example, can a "cautious lover" ($a < 0$, $b > 0$) find true love with an eager beaver? These and other pressing questions will be considered in the exercises.

EXAMPLE 5.3.1:

What happens when two identically cautious lovers get together?

Solution: The system is

$$\dot{R} = aR + bJ$$
$$\dot{J} = bR + aJ$$

with $a < 0$, $b > 0$. Here a is a measure of cautiousness (they each try to avoid throwing themselves at the other) and b is a measure of responsiveness (they both get excited by the other's advances). We might suspect that the outcome depends on the relative size of a and b. Let's see what happens.

The corresponding matrix is

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

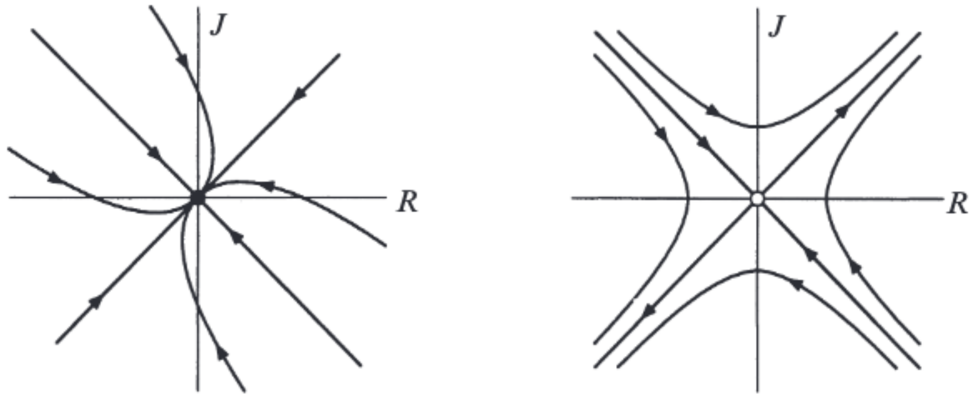
which has

$$\tau = 2a < 0\,, \qquad \Delta = a^2 - b^2\,, \qquad \tau^2 - 4\Delta = 4b^2 > 0\,.$$

Hence the fixed point $(R, J) = (0,0)$ is a saddle point if $a^2 < b^2$ and a stable node if $a^2 > b^2$. The eigenvalues and corresponding eigenvectors are

$$\lambda_1 = a + b\,, \qquad \mathbf{v}_1 = (1,1)\,, \qquad \lambda_2 = a - b\,, \qquad \mathbf{v}_2 = (1,-1).$$

Since $a + b > a - b$, the eigenvector $(1,1)$ spans the unstable manifold when the origin is a saddle point, and it spans the slow eigendirection when the origin is a stable node. Figure 5.3.2 shows the phase portrait for the two cases.



$a^2 > b^2$ $\qquad$ $a^2 < b^2$

Figure 5.3.2

If $a^2 > b^2$, the relationship always fizzles out to mutual indifference. The lesson seems to be that excessive caution can lead to apathy.

If $a^2 < b^2$, the lovers are more daring, or perhaps more sensitive to each other. Now the relationship is explosive. Depending on their feelings initially, their relationship either becomes a love fest or a war. In either case, all trajectories approach the line $R = J$, so their feelings are eventually mutual. ■