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Problem 1. *Prove that a one-point compactification is unique up to homeomorphism.*

Proof. Let X be a locally compact Hausdorff space, then X has a one-point compactification. Suppose that $Y_1 = X \cup \omega$ and $Y_2 = X \cup \infty$ are a one-point compactification of X. Define $h : Y_1 \to Y_2$ by

$$h(x) = \begin{cases} x, \ x \in X; \\ \infty, \ x = \omega. \end{cases}$$

Clearly, h is bijective. We show that h is an open map. Let U be an open subset of Y_1. Case 1): If $\omega \notin U$, then we have $h(U) = U$, and U is open in X. Because X is open in Y_2, $h(U) = U$ is also open in Y_2. Case 2): Suppose that $\omega \in U$. Then $U = G \cup \{\omega\}$ where G is open in X and $X \setminus G$ is compact. It follows that $X \setminus G$ is compact in Y_1, and hence closed in Y_1. So $X \setminus G$ is also compact in Y_2. Since Y_2 is Hausdorff, $X \setminus G$ is closed in Y_2. Thus $h(U) = Y_2 \setminus X \setminus G$ is open in Y_2. By symmetry, we can show that h^{-1} is an open map. Now, we conclude that $Y_1 \cong Y_2$ □

Problem 2 (Urysohn's Lemma, LCH version)**.** *Let X be a locally compact Hausdorff space. Let K be a compact and G open subset of X such that $K \subseteq G$. Prove that there exists a continuous function $f : X \to [0, 1]$ such that $f[K] = \{1\}$ and $f \equiv 0$ outside a compact subset of G.*

Proof. Since X is locally compact Hausdorff, X has a one-point compactification, says $\hat{X}$. Since K is compact in X, K is also compact in $\hat{X}$, and hence closed in $\hat{X}$. Because G is open in X and X is open in $\hat{X}$, then G is open in $\hat{X}$. It follows that $\hat{X} \setminus G$ is closed. Observe that $K \cap \hat{X} \setminus G = \varnothing$. Since $\hat{X}$ is normal, there is a continuous function $\hat{f} : \hat{X} \to [0, 1]$ such that $\hat{f}[K] = \{1\}$ and $\hat{f}[\hat{H} \setminus G] = \{0\}$. Next, $f := \hat{f}|_X : X \to [0, 1]$ is also continuous such that $f[K] = \{1\}$ and $f[X \setminus G] = \{0\}$ as desired. □