Problem 1. *Define a relation $\sim$ on $\mathbb{R}^2$ by*

$$(x_1, x_2) \sim (y_1, y_2) \iff (x_1 - y_1, x_2 - y_2) \in \mathbb{Z} \times \mathbb{Z}.$$

Show that this is an equivalence relation on $\mathbb{R}^2$ and determine its quotient space as a subset of $\mathbb{R}^3$

Proof. i) Reflexive: $(x_1, x_2) \sim (x_1, x_2)$, since $(x_1 - x_1, x_2 - x_2) = (0, 0) \in \mathbb{Z} \times \mathbb{Z}$.

ii) Symmetric: Let $(x_1, x_2) \sim (y_1, y_2)$. Then $x_1 - y_1 \in \mathbb{Z}$ and $x_2 - y_2 \in \mathbb{Z}$. It follows that $(y_1 - x_1) = -(x_1 - y_1) \in \mathbb{Z}$ and $(y_2 - x_2) = -(x_2 - y_2) \in \mathbb{Z}$. Thus $(y_1, y_2) \sim (x_1, x_2)$.

iii) Transitive: Let $(y_1, y_2) \sim (z_1, z_2)$. Then $y_1 - z_1 \in \mathbb{Z}$ and $y_2 - z_2 \in \mathbb{Z}$. Thus $x_1 - z_1 = (x_1 - y_1 + y_1 - z_1) \in \mathbb{Z}$ and $x_2 - z_2 = (x_2 - y_2 + y_2 - z_2) \in \mathbb{Z}$, and so $(x_1, x_2) \sim (z_1, z_2)$.

Consider the **doughnut surface**, which is the surface of revolution $D \subset \mathbb{R}^3$ obtained by starting with the circle $(x-2)^2 + z^2 = 1$ in the xz-plane, and revolving it around the $z-$axis. Define a map $f : \mathbb{R}^2 \to D$ by

$$f(u, v) = ((2 + \cos(2\pi u)) \cos(2\pi v), (2 + \cos(2\pi u)) \sin(2\pi v), \sin(2\pi u)). \tag{1}$$

Next, we show that the **torus** $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^4$ is homeomorphic to the doughnut surface D. For D the angles are $\varphi = 2\pi u$ and $\theta = 2\pi v$ as in (1); for $\mathbb{T}^2$, they are the angles in the two circles. Let $x_1 = \cos\theta, x_2 = \sin\theta, x_3 = \cos\varphi, x_4 = \sin\varphi$. Defining a map $G : \mathbb{T}^2 \to D$ by

$$G(x_1, x_2, x_3, x_4) = ((2 + x_3)x_1, (2 + x_3)x_2, x_4).$$

This is the restriction of a continuous map and is thus continuous, and G is bijective. To see that it is a homeomorphism, observe that its inverse is given by

$$G^{-1}(x, y, z) = (x/r, y/r, r - 2, z), \tag{2}$$

where $r = \sqrt{x^2 + y^2}$ is continuous. Thus $\mathbb{T}^2 \cong D$. Next, define $h : \mathbb{R}^2 \to \mathbb{T}^2$ by $h(x, y) = (e^{2\pi i x}, e^{2\pi i y})$. Clearly, h is a surjective continuous map. Observe that if $(l, k) \in \mathbb{Z} \times \mathbb{Z}$, then $h(x + l, y + k) = h(x, y)$. Let $h(x, y) = h(x', y')$. It follows that

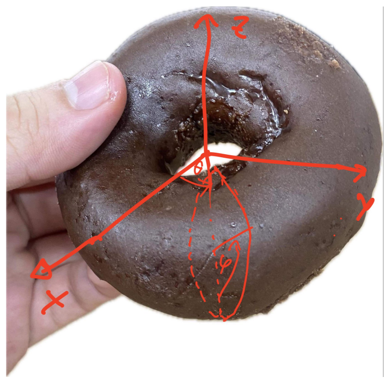
$$\cos 2\pi x - \cos 2\pi x' = 0 \tag{3}$$
$$\sin 2\pi x - \sin 2\pi x' = 0 \tag{4}$$
$$\cos 2\pi y - \cos 2\pi y' = 0 \tag{5}$$
$$\sin 2\pi y - \sin 2\pi y' = 0 \tag{6}$$

By (3) and (4), we have $x = \frac{1}{2}(2n_1 + 1)$ and $x' = \frac{1}{2}(2n_2 + 1)$ where $n_1, n_2 \in \mathbb{Z}$, so $x - x' \in \mathbb{Z}$. Similarly, $y - y' \in \mathbb{Z}$. Now, we have $h(x, y) = h(x', y')$ if and only if $(x, y) \sim (x', y')$. Next, we consider the canonical projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\sim$. Since π is continuous and

$[0,1]\times[0,1]$ is compact in $\mathbb{R}^2$, $\pi([0,1]\times[0,1]) = \mathbb{R}^2/\sim$ is also compact. Let $g : \mathbb{R}^2/\sim \longrightarrow \mathbb{T}^2$ by $g([x]) = h(x)$. Since h is continuous and surjective, g is also continuous and surjective, by universal property of quotient topology. Since $(x,y) \sim (x',y')$ if $h(x,y) = h(x',y')$, g is injective. Note that $\mathbb{T}^2 \subset \mathbb{R}^4$ is Hausdorff. Since g is a bijective continuous map from compact to Hausdorff, g is homeomorphism. Thus $\mathbb{R}^2/\sim \cong \mathbb{T}^2$.

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\pi"] \arrow[d, "h"] & \mathbb{R}^2/\sim \arrow[dl, "g"] \\
\mathbb{T}^2 &
\end{tikzcd}$$

Therefore $\mathbb{R}^2/\sim \cong D$. □



Problem 2. *Let X and Y be a topological spaces and $f : X \to Y$ a continuous surjective map. Let r be an equivalence relation on X. Let R be an equivalence relation on Y such that for all $a,b \in X$, if arb then $f(a)Rf(b)$. Define $g : X/r \to Y/R$ by $g([x]) = [f(x)]$. Prove that*

i) g is continuous;

ii) if f is a quotient map, then g is also a quotient map.

Proof. i) First, we how that g is well defined. Let $[x_1] = [x_2] \in X/r$. It follows that $x_1 r x_2$, so $f(x_1)Rf(x_2)$. Thus $g([x_1]) = [f(x_1)]) = [f(x_2)] = g([x_2])$. Let $x \overset{q_1}{\mapsto} [x]_r$ and $x \overset{q_2}{\mapsto} [x]_R$. Next, we consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "q_1"] \arrow[dr, "q_2\circ f"] & Y \arrow[d, two heads, "q_2"] \\
X/r \arrow[r, "g"'] & Y/R
\end{tikzcd}$$

Since f is continuous and q_2 is quotient map, $q_2 \circ f$ is continuous. Clearly, $g \circ q_1 = q_2 \circ f$. It foloows that g is continuous, by the characteristic property of the quotient topology X/r.

ii) Suppose that f is a quotient map. Note that any composition of quotient maps is a quotient map. This implies that $q_2 \circ f$ is a quotient map. Next, we consider the following diagram:

$$\begin{tikzcd} X \arrow[d, two heads, "q_1"] \arrow[dr, "q_2 \circ f"] & \\ X/r \arrow[r, "g"'] & Y/R \end{tikzcd}$$

Clearly, g is surjective. Let V be a subset of Y/R we show that V is open in Y/R if $g^{-1}(V)$ is open in X/r. Let $g^{-1}(V)$ be open in X/r. Since q_1 is continuous, $q^{-1}g^{-1}(V)$ is open in X. Since $q_2 \circ f$ is a quotient map and $f^{-1}q_2^{-1} = q_1^{-1}g^{-1}$, V is open in Y/R. We are done.

□

Problem 3. *Let X and Y be a topological spaces and $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, $g : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ be continuous maps such that $f \circ g$ is the identity map of Y onto itself. Prove tha following statements:*

a) f is surjective and g is injective.

b) Y has the quotient topology determined by f.

c) g maps Y homeomorphically onto a subspace of X (i.e. Y has the subspace topology determined by g).

d) If X is a Hausdorff space, then so is Y.

Proof. a) Let $y \in Y$. Then there is $y \in Y$ such that $f(g(y)) = \mathrm{Id}_Y(y) = y$. This implies that there is $g(y) \in X$ such that $f(g(y)) = y$, so f is surjective. Next, let y_1 and y_2 be distintc elements in Y which $g(y_1) = g(y_2)$. It is easy to see that $y_1 = \mathrm{Id}_Y(y_1) = f(g(y_1)) = f(g(y_2)) = \mathrm{Id}_Y(y_2) = y_2$. Thus g is injective.

b) Suppose that $\mathcal{T}$ is a topology on Y which make the map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T})$ is continuous. Next, we consider the following diagram:

$$\begin{tikzcd} & X \arrow[dl, shift left, "f"] \arrow[d, "f"] \\ (Y, \mathcal{T}_Y) \arrow[ur, shift left, "g"] \arrow[r, "\mathrm{Id}_Y"] & (Y, \mathcal{T}) \end{tikzcd}$$

Since $f \circ g$ is continuous, Id_Y is also continuous. Let $U \in \mathcal{T}$. Then $U = \mathrm{Id}_Y^{-1}(U) \in \mathcal{T}_Y$. Thus $\mathcal{T}_Y$ is the finest topology on Y for which f is continuous.

c) Let $g' : Y \to g(Y)$ be corestriction of g to $g(Y)$. Clearly, g' is bijective and it has an inverse function. Consider the following diagram:

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
Y \arrow[ur, "g"] \arrow[r, "g'"] & g(Y) \arrow[u, "\mathrm{Id}_{g(Y)}"] \arrow[r, "(g')^{-1}"] & Y
\end{tikzcd}$$

We applied the characteristic property to the subspace $g(Y)$ of X, then we have g' is continuous. Next, let $g(y) \in g(Y)$. Then we have

$$f \circ \mathrm{Id}_{g(Y)}(g(y)) = f(g(y)) = \mathrm{Id}_Y(y) = y = (g')^{-1}(g(y)) = (g')^{-1}(g'(y)) = y.$$

Since $g(Y)$ is a subspace topology of X, the inclusion map $\mathrm{Id}_{g(Y)}$ is continuous. Thus $(g')^{-1} = f \circ \mathrm{Id}_{g(Y)}$ is continuous. Therefore $Y \cong g(Y)$.

d) Let X be a Hausdorff space. Then we have $g(Y)$ is also Hausdorff. Let $y_1 \neq y_2 \in Y$. Since g' is injective, $g'(y_1) \neq g'(y_2)$. Since $g(Y)$ is Hausdorff, There exists disjoint open U, V in $g(Y)$ such that $g'(y_1) \in U$ and $g'(y_2) \in V$. So $\varnothing = (g')^{-1}(U \cap V) = (g')^{-1}(U) \cap (g')^{-1}(V)$. By continuity of g', we are done.

$\square$

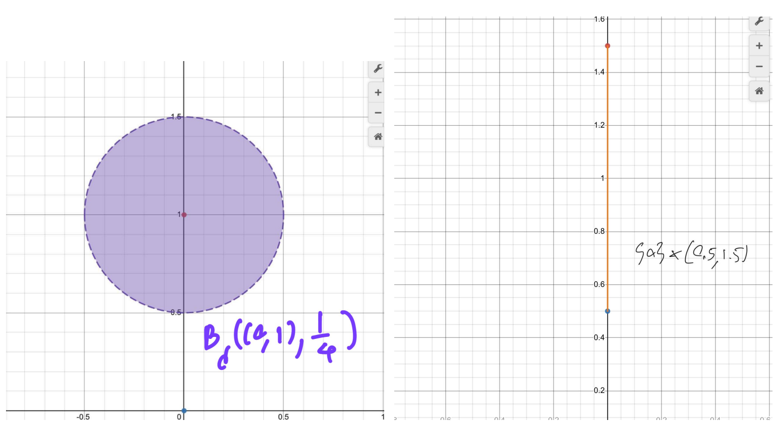
Problem 4. *Let $X = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}$. Let $f : \mathbb{R}^2 \to X$ be defined by*

$$f(x, y) = \begin{cases} (x, 0), & x \neq 0; \\ (0, y), & x = 0. \end{cases}$$

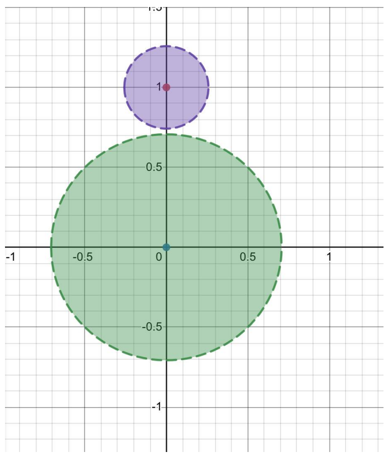
i) Consider X as a subspace of $\mathbb{R}^2$ with the usual topology. Is f is continuous ?

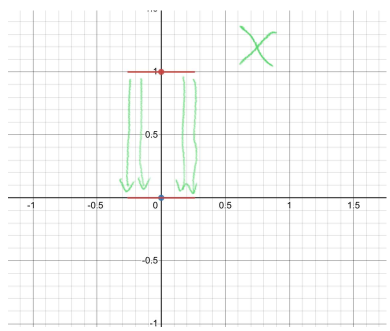
ii) Consider $(X, \mathcal{T}_f)$ where $\mathcal{T}_f$ is the quotient topology induced by f.
Is $(X, \mathcal{T}_f)$ a T_1- space ? Is $(X, \mathcal{T}_f)$ a T_2- space?

Proof. i) We will show that f is not continuous. Observe that $F := B_d((0, 1); 1/4) \cap X$ is open in X, but $f^{-1}(E) = \{0\} \times (0.5, 1.5)$ is not open in $\mathbb{R}^2$.



ii) We will show that $(X, \mathcal{T}_f)$ is a T_1- space. Let $(x, y) \in X$. If $x = 0$, we have $f^{-1}(X \setminus \{(x, y)\}) = \mathbb{R}^2 \setminus (x, y)$ is open in $\mathbb{R}^2$. Thus $\{(x, y)\}$ is closed in X. If $x \neq 0$, we have $f^{-1}(X \setminus \{(x, y)\}) = \mathbb{R}^2 \setminus \{(x', y') \in \mathbb{R}^2 : x' = x\}$ is open in $\mathbb{R}^2$. Thus $\{(x, y)\}$ is closed in X. Now, we conclude that $(X, \mathcal{T}_f)$ is a T_1-space. We will show that $(X, \mathcal{T}_f)$ is not a Hausdorff space. Choose $(0, 0), (0, 1) \in X$. Let U and V are open in X which containing $(0, 0)$ and $(0, 1)$, respectively. Thus $f^{-1}(U)$ is open in $\mathbb{R}^2$ and $f^{-1}(V)$ is open in $\mathbb{R}^2$. There are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $B_d((0, 0); \varepsilon_1) \subseteq f^{-1}(U)$ and $B_d((0, 1); \varepsilon_2) \subseteq f^{-1}(V)$. So $(0, \frac{1}{2}\min\{\varepsilon_1, \varepsilon_2\}) \in U \cap V$. It follows that $(X, \mathcal{T}_f)$ is not Hausdorff.





$\square$