Problem 1. *We say that a point $x \in X$ is an $\omega-$accumulation point of a set $A \subseteq X$ if every neighborhood of x contains infinitely many points of A. Show that a space X is countably compact if and only if every infinite subset of X has an $\omega-$accumulation point. Show further that if X is a T_1-space, then X is countably compact if and only if every infinite subset of X has an accumulation point.*

Proof. Let X be countably compact and any infinite subset K of X . We can choose an infinite sequence $k_1, \ldots$ of distinct points of K. Since X is countably compact, sequence $(k_n)_{n=1}^{\infty}$ has clusterpoint, says k by Theorem 15.3. Thus K has an $\omega-$ accumulation. Suppose that X is not countably compact. Then X has a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ which has no finite subcover, that is for each $n \in \mathbb{N}$, we can choose $x_n \notin \bigcup_{i=1}^{n} U_i$. Let $A = \{x_n : n \in \mathbb{N}\}$. If A is finite, then $A \subseteq \bigcup_{i=1}^{m} U_i$ for some $m \in \mathbb{N}$, but this is impossible because $x_m \in A$ but $x_m \notin \bigcup_{i=1}^{m} U_i$. Thus A is an infinite set. Next, let $x \in X$. Since $\mathcal{U}$ covers X, then $x \in U_n$ for some $n \in \mathbb{N}$. Let $F = U_n \cap A \setminus \{x\}$. Then we have $A \cap U_n \subseteq F \cup \{x\}$. Since $x_k \notin U_n$ for all $k \geq n$, we have F is a finite set. Thus there is an infinite subset A of X has no an $\omega-$ accumulation point. Moreover, we assume X is a T_1 space. Then we have F is closed set. It is easy to see that $x \notin F$. Let $V = U_n \cap (X \setminus F)$. Then V is a open nbhd of x in X. Furthermore,

$$V \cap (A \setminus \{x\}) = (U_n \cap (X \setminus F)) \cap A \setminus \{x\} = (U_n \cap A \setminus \{x\}) \cap X \setminus F = F \cap X \setminus F = \varnothing.$$

This implies that x is not a limit point of A. Thus X is not limit point compact.
Conversely, Suppose that X is not limit point compact. Then there is an infinite subset A of X that has no limit points. Let $B \subseteq A$ and $x \in X \setminus B$. Since A has no limit points, then there is a nbhd U of x such that $U \cap A \setminus \{x\} = \varnothing$, and thus $U \cap B = \varnothing$ (because $B \subseteq A \setminus \{x\}$). Thus, every point of $X \setminus B$ has a nbhd which is disjoint from B. Thus $X \setminus B$ is open, so B is closed. Next, we choose an infinite sequence $x_1, x_2, \ldots$ of distinct points of A. For any $n \in \mathbb{N}$, we define $U_n = X \setminus \{a_i : i \geq n\}$. Since $\{a_i : i \geq n\} \subseteq A$ for all $n \in \mathbb{N}$, it is closed. Thus U_n is open for all $n \in \mathbb{N}$. Let $z \in X$. Then $z \in U_1$ or $z \in \{a_i : i \geq 1\}$. If $z \in a_i : i \geq 1$, then $z = a_m$ for some $m \in \mathbb{N}$, and so $z \in U_{m+1}$. We can conclude that $\{U_n : n \in \mathbb{N}\}$ covers X. Since $a_n \notin U_i$ for all $i \in \{1, \ldots, n\}$, then no finite subcover of $\{U_n : n \in \mathbb{N}\}$ covers X. □

Problem 2. *Let $(X, \mathcal{T})$ be a Hausdorff space. Prove that the following statements are equivalent:*

a) X is locally compact.

b) For each $x \in X$ and each neighborhood W of x, there is an open set V such that $\overline{V}$ is compact and $x \in V \subseteq \overline{V} \subseteq W$.

c) For each compact set K in X and an open set W containing K, there is an open set V such that $\overline{V}$ is compact and $K \subseteq V \subseteq \overline{V} \subseteq W$.

d) $\mathcal{T}$ has a base consisting of open sets whose closures are compact.

Proof. Clearly, $(d) \Rightarrow (a)$ and $(b) \Rightarrow (a)$. Next, we show that $(a) \Rightarrow (d)$ Let $x \in X$. Then there is a compact nbhd K of x, so there is $U \in \mathcal{T}$ such that $x \in U \subseteq K$. Let $\mathcal{V}$ be the set of all open set $G \ni x$ contained in U, moreover $\mathcal{V}$ is a nbhd base at x. Since X is Hausdorff, K is closed in X. If $V \in \mathcal{V}$, then $\overline{V} \subseteq \overline{K} = K$, and so $\overline{V}$ is compact, because a closed subset of a compact set is compact. Thus $\mathcal{V}$ is nbhd base. Therefore $\mathcal{T}$ has a base consisting of open sets whose closures are compact as desired. Next, we show that $(c) \Rightarrow (b)$. Suppose (c). Let $x \in X$. and W be a nbhd of x. Note that $\{x\}$ is compact. Then there is an open set V such that $\overline{V}$ is compact and $x \in \{x\} \subseteq V \subseteq \overline{V} \subset W$ as desired. Finally, we show that $(a) \Rightarrow (c)$. Suppose (a), K is a compact subspace of X and an open set $W \supseteq K$. Since X is locally compact Hausdorff, the one-point compactification of X exist, says $\hat{X}$. Note that $\hat{X}$ is compact Hausdorff, so it is normal. Clearly, K is also compact in $\hat{X}$, and hence closed. Since X is open in $\hat{X}$ and W is open in X, W is also open in $\hat{X}$. We are done. □

Problem 3. *Let X be a locally compact Hausdorff space. We say that a continuous function $f : X \to \mathbb{R}$ vanishes at infinity if for any $\varepsilon > 0$, there is a compact subset K of X such that $|f(x)| < \varepsilon$ for any $x \in X \setminus K$. We denote the set of all continuous functions from X into $\mathbb{R}$ that vanish at infinity by $C_o(X)$.*

i) Prove that if $f, g \in C_o(X)$ and $\alpha \in \mathbb{R}$, then $f + g \in C_o(X)$ and $\alpha f \in C_o(X)$. (Hence, $C_o(X)$ is a vector space over $\mathbb{R}$.)

ii) Let $\hat{X}$ denote the one-point compactification of X. Let $f : X \to \mathbb{R}$ be a continuous function. Prove that f can be extended to a continuous function $\hat{f} : \hat{X} \to \mathbb{R}$ if and only if $f = g + c$ for some $g \in C_o(X)$ and some $c \in \mathbb{R}$.

Proof. i) Let $f, g \in C_0(X)$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Then there are compact subset K_1 , K_2 and K_3 of X such that $|f(x)| < \varepsilon/2$ for any $x \in X \setminus K_1$, $|g(x)| < \varepsilon/2$ for any $x \in X \setminus K_2$ and $|f(x)| < \frac{\varepsilon}{|\alpha|+1}$ for any $x \in X \setminus K_3$. Note that union of two compact sets is compact. Thus $|f(x) + g(x)| \leq |f(x)| + |g(x)| < \varepsilon$ for any $x \in X \setminus K_1 \cap X \setminus K_2 = X \setminus (K_1 \cup K_2)$, so $f + g \in C_0(X)$. It is easy to see that $|\alpha f(x)| = |\alpha||f(x)| < (|\alpha| + 1)|f| < \varepsilon$ for all $x \in X \setminus K_3$, so $\alpha f \in C_o(X)$.

ii) Since X is locally compact Hausdorff space, one-point compactification of X exist, says $\hat{X}$. $(\Rightarrow)$ Clearly, $\hat{f}(\infty) \in \mathbb{R}$. We will show that $f - \hat{f}(\infty) \in C_o(X)$. Let $\varepsilon > 0$. Since $\hat{f}$ is continuous at ∞, then there is an open subset $V \ni \infty$ of $\hat{X}$ such that $\hat{f}[V] \subseteq B_d(\hat{f}(\infty), \varepsilon)$. Since V is open containing ∞, $V = \hat{X} \setminus C$ where C is a compact subset of X. Thus for any $x \in X \setminus C \subseteq V$, we have $|f(x) - \hat{f}(\infty)| = |\hat{f}(x) - \hat{f}(\infty)| < \varepsilon$. Choose $c = \hat{f}(\infty) \in \mathbb{R}$ and $g = f - \hat{f}(\infty) \in C_o(X)$. We are done.
$(\Leftarrow)$ We define $\hat{f} : \hat{X} \to \mathbb{R}$ by $\hat{f}|_X = f$ and $\hat{f}(\infty) = c$. Note that $\hat{f}$ is continuous for all $x \in X$. It remains to show that $\hat{f}$ is continuous at ∞. Let $\varepsilon > 0$. Then there is compact subset K of X such that $|f(x) - \hat{f}(\infty)| = |g(x)| < \varepsilon$ for all $x \in X \setminus K$. Since K is compact in X, K is compact in $\hat{X}$, so K is closed in $\hat{X}$ (because $\hat{X}$ is Hausdorff). It follows that $\hat{X} \setminus K$ is an open set containing ∞. Let $y \in \hat{X} \setminus K$. If $y = \infty$, then $|\hat{f}(y) - \hat{f}(\infty)| = 0 < \varepsilon$. If $y \in X \setminus K$, then $|\hat{f}(y) - \hat{f}(\infty)| = |f(y) - \hat{f}(\infty)| < \varepsilon$. We conclude that $\hat{f}[\hat{X} \setminus K] \subseteq B_d(\hat{f}(\infty), \varepsilon)$, and thus $\hat{f}$ is continuous at ∞.

□

Problem 4. *Let E be a closed subset of a compact Hausdorff space X. Prove that the quotient space obtained from X by identifying E to a point is homeomorphic to the one-point compactification of $X \setminus E$.*

Proof. Let E be a closed subset of a compact Hausdorff X. Fix $e \in E$. We consider the set $S := (X \setminus E) \cup \{e\}$ and let $\pi : X \twoheadrightarrow S$ defined by

$$\pi(x) = \begin{cases} x, & \text{if } x \in X \setminus E; \\ e, & \text{if } x \in E. \end{cases}$$

Clearly, π is surjective. Now, we can gives quotient topology on S induced by π, say $\mathcal{T}_\pi$. First, we show that S is compact Hausdorff under quotient topology. Note that quotient map is continuous. Since X is compact , $\pi(X) = S$ is compact. Next, we show that the set $\mathcal{R} := \{(x_1, x_2) : \pi(x_1) = \pi(x_2)\}$ is closed in $X \times X$. Observe that $\mathcal{R} = E \times E \cup \triangle$. Since X is Hausdorff, we have $\triangle$ is closed in $X \times X$. By Homework 9.3 i), we have $\overline{E \times E} = \overline{E} \times \overline{E} = E \times E$. Thus $\mathcal{R}$ is closed in $X \times X$. Next, we show that $(S, \mathcal{T}_\pi)$ is Hausdorff. Let $y \in S$. If $y = e$, then $\pi^{-1}(\{y\}) = E$ is closed in X. If $y \in X \setminus E$, then $\pi^{-1}(\{y\}) = \{y\}$ is closed in X (because X is T_1). Now, let y_1 and y_2 be distinct points in S. Since X is normal, there are disjoint open subsets $U_1 \supseteq \pi^{-1}(\{y_1\})$ and $U_2 \supseteq \pi^{-1}(\{y_2\})$, and we define subsets $W_1, W_2 \subseteq Y$ by

$$W_i = \{y \in S : \pi^{-1}(\{y\}) \subseteq U_i\}.$$

Clearly, these are disjoint sets containing y_1 and y_2 respectively. Because π is a quotient map, W_i is open if and only if $X \setminus \pi^{-1}(W_i)$ is closed. From the definition of W_i, we have

$$\begin{aligned} X \setminus \pi^{-1}(W_i) &= \{x \in X : \textit{ there exists } x' \in X \setminus U_i \textit{ such that } \pi(x) = \pi(x')\} \\ &= p_1(\mathcal{R} \cap (X \times (X \setminus U_i))), \end{aligned}$$

where $p_1 : X \times X \to X$ is the projection on the first factor. Note that p_1 is continuous. Since $X \times X$ is compact and X is Hausdorff, p_1 is closed map. Since $\mathcal{R} \cap X \times X \setminus U_i$ is closed in $X \times X$, we have $X \setminus \pi^{-1}(W_i)$ is also closed in X, and therefore S is Hausdorff.

Since $X \setminus E$ is open, it is locally compact, so the one-point compactification of $X \setminus E$ exist, says $\sigma(X \setminus E)$. Finally, we show that $S \cong \sigma(X \setminus E)$. We define $h : S \to \sigma(X \setminus E)$ by

$$h(x) = \begin{cases} x, & \text{if } x \in X \setminus E \\ \infty, & \text{if } x = e \end{cases}.$$

Clearly, h is bijective. Let U be open in S. Case 1) if U does not containing e. Then $h(U) = U$. Since U is open in S, $\pi^{-1}(U) = U$ is open in X. Since $h(U)$ is open in $X \setminus E$ and $X \setminus E$ is open in $\sigma(X \setminus E)$, $h(U)$ is also open in $\sigma(X \setminus E)$. Case 2): Suppose that U contains e. Since $C = S \setminus U$ is closed in S, it is compact as a subspace of S. Since $h(C) = C$ contained in $X \setminus E$, it is a compact subspace of $X \setminus E$. Because $X \setminus E$ is a subspace of $\sigma(X \setminus E)$, the space $h(C)$ is also a compact subspace of $\sigma(X \setminus E)$. Since $\sigma(X \setminus E)$ is Hausdorff, $h(C)$ is closed in $\sigma(X \setminus E)$, so $h(U) = \sigma(X \setminus E) \setminus h(C)$ is open in $\sigma(X \setminus E)$. Thus h is an open map. By symmetry, we have h^{-1} is also an open map. Thus $S \cong \sigma(X \setminus E)$ as desired. $\square$

Problem 5. *Prove that the Stone-Čech compactification is unique up to homeomorphism.*

Proof. Let X be a complete regular space, $(\beta(X), e)$ and $(\beta'(X), e')$ are two Stone-Čech compactifications of X.

$$\begin{tikzcd} X \arrow[r, hook, "e"] \arrow[d, "e'"] & \beta(X) \arrow[dl, "f"] \\ \beta'(X) & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "e'"] \arrow[d, "e"] & \beta'(X) \arrow[dl, "g"] \\ \beta(X) & \end{tikzcd}$$

Consider the embedding mapping $e' : X \to \beta'(X)$. It is a continuous map of X into the compact Hausdorff space $\beta'(X)$. By universal property of Stone-Čech compactifications $\beta(X)$, there exists a unique continuous function $f : \beta(X) \to \beta'(X)$ such that $e' = f \circ e$. Similarly, there exsists a unique continuous function, $g : \beta'(X) \to \beta(X)$ such that $e = g \circ e'$. Next, we consider the composition $f \circ g : \beta'(X) \to \beta'(X)$, observe that $f \circ g \circ e' = e'$ for all $x \in X$. But $\mathrm{Id}_{\beta'(X)} \circ e' = e'$ for all $x \in X$, so $f \circ g = \mathrm{Id}_{\beta'(X)}$ by uniqueness of extension. Similarly, $g \circ f = \mathrm{Id}_{\beta(X)}$. By Homework 13.3 a), f is bijective. Because $f^{-1} = g$ is continuous, f is homeomorphism.

$$\begin{tikzcd} X \arrow[r, hook, "e"] \arrow[d, "e"] & \beta(X) \arrow[dl, "\mathrm{Id}_{\beta(X)}"] \\ \beta(X) & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "e'"] \arrow[d, "e'"] & \beta'(X) \arrow[dl, "\mathrm{Id}_{\beta'(X)}"] \\ \beta'(X) & \end{tikzcd}$$

$\square$

Problem 6. *The space $X \times Y$ is sequentially compact if and only if X and Y are sequentially compact.*

Proof. Note that sequential compactness is preserved under continuous surjection.
($\Rightarrow$) Since projection map $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous, X and Y are sequentially compact.
($\Leftarrow$) Let $((x_n, y_n))_{n=1}^{\infty}$ be a sequence in $X \times Y$. Since $(x_n)_{n=1}^{\infty}$ is a sequence in X, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \to x \in X$ as $k \to \infty$. We consider $(y_{n_k})_{k=1}^{\infty}$, there exists a subsequence $(y_{n_{k_l}})_{l=1}^{\infty}$ of $(y_{n_k})_{k=1}^{\infty}$ which $y_{n_{k_l}} \to y \in Y$ as $l \to \infty$ (because Y is sequentially compact). Clearly, $x_{n_{k_l}} \to x$ as $l \to \infty$. By Homework 9.4, we have $(x_{n_{k_l}}, y_{n_{k_l}}) \to (x, y)$ as $l \to \infty$. Therefore $X \times Y$ is sequentially compact. $\square$

Problem 7. *Show that the rationals $\mathbb{Q}$ are not locally compact.*

Proof. Suppose to the contrary that $\mathbb{Q}$ is locally compact. Choose $0 \in \mathbb{Q}$. Then there is a compact set K in $\mathbb{Q}$ and open set $(a, b) \cap \mathbb{Q}$ such that $0 \in (a, b) \cap \mathbb{Q} \subseteq K$. Since $\mathbb{Q}$ is Hausdorff, K is closed in $\mathbb{Q}$. It follows that $\overline{(a, b) \cap \mathbb{Q}}^{\mathbb{Q}}$ is compact in $\mathbb{Q}$ (because it is closed in K). Note that $\mathbb{Q}$ is dense in $\mathbb{R}$ and (a, b) is open in $\mathbb{R}$, so $\overline{(a, b) \cap \mathbb{Q}} = \overline{(a, b)} = [a, b]$. Observe that

$$\overline{(a, b) \cap \mathbb{Q}}^{\mathbb{Q}} = \overline{(a, b) \cap \mathbb{Q}} \cap \mathbb{Q} = [a, b] \cap \mathbb{Q}.$$

Now, we have $[a,b] \cap \mathbb{Q}$ is compact in $\mathbb{Q}$, so it is compact in $\mathbb{R}$ because inclusion map $\iota : \mathbb{Q} \hookrightarrow \mathbb{R}$ is continuous. By Heine-Borel Theorem, we have $[a,b] \cap \mathbb{Q}$ is closed and bounded. Choose $q \in [a,b] \cap \mathbb{R} \setminus \mathbb{Q}$. Then we can find sequence $(a_n)_{n=1}^{\infty}$ in $[a,b] \cap \mathbb{Q}$ which $a_n \to q$ as $n \to \infty$, so $q \in \overline{[a,b] \cap \mathbb{Q}}$. This implies that $\overline{[a,b] \cap \mathbb{Q}} \neq [a,b] \cap \mathbb{Q}$, so $[a,b] \cap \mathbb{Q}$ is not closed, which is a contradiction. $\square$