2301631 Topology — Homework 12: Connectedness
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Throughout the homework sheet, IVT denotes the Intermediate Value Theorem.

Problem 1. *Let $f : X \to Y$ be a continuous function and X be connected. Prove that the graph of $f, \mathrm{Gf}(f) = \{(x, f(x)) | x \in X\}$, is connected in $X \times Y$.*

Proof. We Claim that $\tilde{f} : X \to X \times Y$ defined by $x \mapsto (x, f(x))$ is continuous. Consider the diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "\tilde{f}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \\
& X &
\end{tikzcd}$$

Since f and Id_X is continuous, $\tilde{f}$ is continuous by characteristic property of the product topology $X \times Y$. Clearly, $\tilde{f}(X) = \mathrm{Gf}(f)$. Thus $\mathrm{Gf}(f)$ is connected in $X \times Y$. $\square$

Problem 2. *Show that a connected metric space with more than one point must be uncountable. Hence, if S is a countable subset of a metric space, then S is totally disconnected. Hint: Use IVT.*

Proof. Let (X, d) be a connected metric space and $x, y \in X$ such that $x \neq y$. Then $d(x, y) > 0$. Define function $f : X \to \mathbb{R}$ by $f(z) = d(x, z)$ for all $z \in X$. By homework, f is continuous. Thus $f(X)$ is connected, so $f(X)$ is singleton or interval. Since $f(x) = 0$ and $f(y) > 0$, $f(X) = [a, b]$ for some $a, b \in \mathbb{R}$. Next, let $c \in [a, b]$, by IVT, there is $u \in X$ such that $f(u) = c$, that is f is surjective, so X is must be uncountable set. Note that every connected with more than one point is not totally disconnected. Now, we have if S is countable subset of a metric space, then S is totally disconnected as desired. $\square$

Problem 3. *Determine whether the following spaces are connected, totally disconnected or neither:*

i) $\mathbb{R}$ *with the cocountable topology;*

ii) $\mathbb{R}$ *with the lower limit topology;*

iii) $\{(x, y) \in \mathbb{R}^2 : x \text{ or } y \text{ is rational}\}$;

iv) $\{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ is rational}\}$;

v) $\{(x, y) \in \mathbb{R}^2 : x \text{ or } y \text{ but not both, is rational}\}$

Solution.

i) We consider $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$. Recall that F is closed in $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ if and only if F is countable or $F = \mathbb{R}$. Suppose to the contrary that X is disconnected. Then there is non-empty proper subset A of $\mathbb{R}$ such that A is closed and A is open. Since A is closed, we have A is countable. Since A is open $\mathbb{R} \setminus A$ is countable. This implies that $\mathbb{R} = A \cup (\mathbb{R} \setminus A)$ is also countable, which is a contradiction.Thus $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ is connected. Clearly, $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ is not totally disconnected.

ii) We consider $(\mathbb{R}, \mathcal{T}_{\text{lower limit}})$. Clearly, $[0, 1)$ is open in $(\mathbb{R}, \mathcal{T}_{\text{lower limit}})$. By Homework 8.2, $[0, 1)$ is closed. Thus $(\mathbb{R}, \mathcal{T}_{\text{lower limit}})$ is disconnected. Clearly, for all $x \in (\mathbb{R}, \mathcal{T}_{\text{lower limit}})$, $\{x\}$ is connected. Next, suppose A is any subset of $(\mathbb{R}, \mathcal{T}_{\text{lower limit}})$ that contains at least two point. Let a and b be distinct elements in A. WLOG we assume that $a < b$. Since $[a, b)$ is clopen in X, we have $[a, b) \cap A$ is also clopen in A. Thus A is disconnected. Now, we conclude that $(\mathbb{R}, \mathcal{T}_{\text{lower limit}})$ is totally disconnected.

iii) We consider $B := \{(x, y) : x \text{ or } y \text{ is rational}\}$. We show that for every $(x, y) \in B$ there is a path in B from (x, y) to $(0, 0)$. Let $(x, y) \in \mathbb{Q}$. WLOG suppose that $x \in \mathbb{Q}$. Since $\mathbb{R}$ is path-connected, there is a path $\alpha : [0, 1] \to \mathbb{R}$ such that $\alpha(0) = y$ and $\alpha(1) = 0$. Define $\tilde{\alpha} : [0, 1] \to B$ by $\tilde{\alpha}(t) = (x, \alpha(t))$ for all $t \in [0, 1]$. Since $x \in \mathbb{Q}$, $(x, \alpha(t)) \in B$ for all $t \in [0, 1]$. Clearly, $\tilde{\alpha}$ is continuous, $\tilde{\alpha}(0) = (x, y)$ and $\tilde{\alpha}(1) = (x, 0)$. Similarly, there is a path $\beta : [0, 1] \to \mathbb{R}$ such that $\beta(0) = x$ and $\beta(1) = 0$. Define $\tilde{\beta} : [0, 1] \to B$ by $\tilde{\beta}(t) = (\beta(t), 0)$ for all $t \in [0, 1]$, so $\tilde{\beta}$ is a path in B from $(x, 0)$ to $(0, 0)$. Thus $\tilde{\alpha} * \tilde{\beta}$ is a path in B from (x, y) to $(0, 0)$. Let $(v, w) \in B$. Then there is a path γ in B from (v, w) to $(0, 0)$. Let $\overline{\gamma}(t) = \gamma(1 - t)$. Then we have $\overline{\gamma}$ is a path in B from $(0, 0)$ to (v, w). Thus $(\tilde{\alpha} * \tilde{\beta}) * \overline{\gamma}$ is a path in B from (x, y) to (v, w), so B is path-connected.

iv) We consider $\{(x, y) \in \mathbb{Q}^2\}$. Observe that $\mathbb{Q}^2$ is disconnected by, say, $\{(x, y) \in \mathbb{Q}^2 : x > \pi\}$ and $\{(x, y) \in \mathbb{Q}^2 : x < \pi\}$. Next, suppose A is any subset of $\mathbb{Q}^2$ that contains at least two point. Let (x_1, y_1) and (x_2, y_2) be distinct elements in A. WLOG assume that $x_1 < x_2$. Then there is $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x_1 < \alpha < x_2$, it follows that $\{(x, y) \in A : x < \alpha\}$ and $\{(x, y) \in A : x > \alpha\}$ disconnect A, so no subset with more than one point is connected.Thus $\mathbb{Q}^2$ is totally disconnected.

v) We consider $\mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $q \in \mathbb{R} \setminus \mathbb{Q}$. Since $\{q\} \times \mathbb{R} \cong \mathbb{R} \cong \mathbb{R} \times \{q\}$, $\{q\} \times \mathbb{R}$ and $\mathbb{R} \times \{q\}$ are connected. Since $\{q\} \times \mathbb{R} \cap \mathbb{R} \times \{q\} = \{(q, q)\} \neq \varnothing$, we have $\{q\} \times \mathbb{R} \cup \mathbb{R} \times \{q\}$ is connected. For each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we define $A_\alpha := \{\alpha\} \times \mathbb{R} \cup \mathbb{R} \times \{\alpha\}$. It is easy to see that $A_\alpha \cap A_\beta = \{(\alpha, \beta), (\beta, \alpha)\} \neq \varnothing$ for all $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. Thus $\bigcup_{q \in \mathbb{R} \setminus \mathbb{Q}} A_q$ is connected, by Homework 4. Finally, we will show that $\mathbb{R}^2 \setminus \mathbb{Q}^2 = \bigcup_{q \in \mathbb{R} \setminus \mathbb{Q}} A_q$. Let $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$. Then $x \in \mathbb{R} \setminus \mathbb{Q}$ or $y \in \mathbb{R} \setminus \mathbb{Q}$. WLOG suppose that $x \in \mathbb{R} \setminus \mathbb{Q}$, so $(x, y) \in A_x \subseteq \bigcup_{q \in \mathbb{R} \setminus \mathbb{Q}} A_q$. Conversely, let $(x, y) \in \bigcup_{q \in \mathbb{R} \setminus \mathbb{Q}} A_q$, so $(x, y) \in A_x$ or $(x, y) \in A_y$. Thus $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$. Clearly, $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is not totally disconnected.

vi) We consider $H := (\mathbb{Q} \times \mathbb{R} \setminus \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{R} \setminus \mathbb{Q})$. Note that $x + y \in \mathbb{R} \setminus \mathbb{Q}$ if $(x, y) \in H$. For $q \in \mathbb{Q}$ let

$$G = \{(x, y) \in H : x + y > q\}$$

and

$$U = \{(x, y) \in H : x + y < q\}.$$

Thus $H = G \cup U$. Clearly G and U are disjoint open in H, so H is disconnected. Next, suppose A is any subset of H that contains at least two point. Let (x_1, y_1) and (x_2, y_2) be distinct elements in A. If $(x_1 + y_1) \neq (x_2 + y_2)$, WLOG $(x_1 + y_1) < (x_2 + y_2)$, then there is $r_1 \in \mathbb{Q}$ such that $\{(x, y) \in A : x + y > r_1\}$ and $\{(x, y) \in A : x + y < r_1\}$ disconnect A. If $x_1 + y_1 = x_2 + y_2$, then $\{(x, y) \in A : y - \frac{y_1+y_2}{2} > x - \frac{x_1+x_2}{2}\}$ and $\{(x, y) \in A : y - \frac{y_1+y_2}{2} < x - \frac{x_1+x_2}{2}\}$ disconnect A. Thus H is totally disconnected.

□

Problem 4. *Assume that $\{A_\alpha : \alpha \in \Lambda\}$ is a family of connected subsets of a topological space X such that $A_\alpha \cap A_\beta \neq \varnothing$ for all $\alpha, \beta \in \Lambda$. Prove that $\bigcup_{\alpha\in\Lambda} A_\alpha$ is connected.*

Proof. By assumption $A_\alpha \cap A_\beta \neq \varnothing$ for all $\alpha, \beta \in \Lambda$, we have $A_\alpha \neq \varnothing$ for all $\alpha \in \Lambda$. Fix $\lambda \in \Lambda$. Observe that $A_\lambda \cap A_\alpha \neq \varnothing$ for all $\alpha \in \Lambda$. It follows that $A_\lambda \cup A_\alpha$ is connected for all $\alpha \in \Lambda$, so $\{A_\lambda \cup A_\alpha : \alpha \in \Lambda\}$ is a family of connected subset of X. Since $\varnothing \neq A_\lambda \subseteq \bigcap_{\alpha\in\Lambda}(A_\lambda \cup A_\alpha)$, $\bigcup_{\alpha\in\Lambda}(A_\lambda \cup A_\alpha)$ is connected. Since $A_\lambda \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$, we have $\bigcup_{\alpha\in\Lambda}(A_\lambda \cup A_\alpha) = A_\lambda \cup \bigcup_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha\in\Lambda} A_\alpha$. We are done. □

Problem 5. *A point $x \in X$ is called a cut point if X is connected but $X - \{x\}$ is disconnected. Prove that if $f : X \to Y$ is a homeomorphism and x is a cut point of X, then $f(x)$ is a cut point of Y. Use this result to determine whether the following spaces are homeomorphic:*

(a) $(0, 1)$ *and* $[0, 1)$ *with the usual topology.*

(b) $\mathbb{R}$ *and* $\mathbb{R}^2$

Proof. Let $f : X \to Y$ is a homeomorphism and $x \in X$ is a cut point of X. Since X is connected, Y is also connected. We will show that $Y - \{f(x)\}$ is disconnected. Since $X - \{x\}$ is disconnected, there are two non-empty disjoint open subset of $X - \{x\}$, say A, B such that $A \cup B = X - \{x\}$.

- Since f is injective, $\varnothing = f(A \cap B) = f(A) \cap f(B)$ and $f(X - \{x\}) = f(X) - \{f(x)\}$.
- Since f is surjective, $f(X - \{x\}) = Y - \{f(x)\}$.
- Since restriction of an injective open map is an open map, $f(A)$ and $f(B)$ are open in $Y - \{f(x)\}$. We can conclude that $Y - \{f(x)\}$ is disconnected as desired.

a) Suppose to the contrary that there is a homeomorphism $h : (0, 1) \to [0, 1)$. Since h is surjective, there is $x \in (0, 1)$ such that $h(x) = 0$. Observe that $h(x)$ is not a cut point of $[0, 1)$. But $(0, 1) - \{x\} = (0, x) \cup (x, 1)$ is disconnected, so x is a cut point of $(0, 1)$. Thus f is not homeomorphism. It follows that $(0, 1) \ncong [0, 1)$.

b) Suppose to the contrary that there is a homeomorphism $g : \mathbb{R} \to \mathbb{R}^2$. Then there is $y \in \mathbb{R}$ such that $g(y) = (0, 0)$. Observe that for any $x_1, x_2 \in \mathbb{R}^2 - \{(0, 0)\}$ we can find path $\alpha : [0, 1] \to \mathbb{R}^2$ such that $\alpha(0) = x_1$ and $\alpha(x_2) = x_2$, that is $\mathbb{R}^2 - \{(0, 0)\}$ is path-connected. Thus $f(y)$ is not a cut point of $\mathbb{R}^2$. But $\mathbb{R} - \{y\}$ is a cut point of $\mathbb{R}$. Hence $\mathbb{R} \ncong \mathbb{R}^2$.

$\square$

Problem 6. *Let f be a $1-1$ continuous function from $\mathbb{R}$ into $\mathbb{R}$. Prove the following:*

i) For any $x < y$ in $\mathbb{R}$, if $f(x) < f(y)$, then for any $z \in (x, y), f(z) \in (f(x), f(y))$.

ii) For any $x < y$ in $\mathbb{R}$, if $f(x) < f(y)$, then for any $a < x, f(a) < f(x)$.

iii) f is monotone.

Proof. i) By iii), we are done.

ii) By iii), we are done.

iii) We will show that f is monotone. Suppose to the contrary that f is not monotone. WLOG assume that there exists $x < y < z \in \mathbb{R}$ such that $f(x) \leq f(y)$ and $f(y) \geq f(z)$. If $f(x) = f(y)$ or $f(y) = f(z)$, then f is not injective. Thus $f(x) < f(y)$ and $f(y) > f(z)$. WLOG assume that $f(x) < f(z) < f(y)$. By IVT, there is $c \in [x, y]$ such that $f(c) = f(z)$, but $z \notin [x, y]$, we have f is not injective, which is a contradiction.

$\square$

Problem 7. *Let X be a space, and define a relation $x \sim y$ if and only if there is a path in X from x to y. Show that $\sim$ is an equivalence relation whose equivalence classes are the path-components, i.e. the maximal path-connected subsets, of X.*

Proof. i) Reflexive: Clearly, constant path is continuous, i.e. $c : [0, 1] \to X$ such that $c(t) = x$ for all $t \in [0, 1]$. Thus $x \sim x$.

ii) Symmetric: Let $x \sim y$. Then there is a path $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Clearly, $\overline{\alpha}(t) = \alpha(1 - t)$ is a continuous function from $[0, 1]$ to X such that $\overline{\alpha}(0) = \alpha(1) = y$ and $\overline{\alpha}(1) = \alpha(0) = x$. Thus $y \sim x$

iii) Transitive: Let $y \sim z$. Then there is a path $\beta : [0, 1] \to X$ such that $\beta(0) = y$ and $\beta(1) = z$. Define $\alpha * \beta : [0, 1] \to X$ by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & \text{if } t \in [0, 1/2], \\ \beta(2t - 1), & \text{if } t \in [1/2, 1]. \end{cases}$$

The map $\alpha * \beta$ is well defined, since if $t = 1/2$, we have $\alpha(1) = \beta(0)$. Since $\alpha * \beta$ is continuous on the two closed subsets $[0, 1/2]$ and $[1/2, 1]$ of $[0, 1]$, it is continuous on $[0, 1]$, by the gluing lemma. It is easy to see that $\alpha * \beta(0) = x$ and $\alpha * \beta(1) = z$. Hence $x \sim z$.

$\square$

Problem 8. *Let X be a locally path-connected space. Prove that an open connected subset of X path-connected. In particular, a locally path-connected and connected space is path-connected.*
Hint: Let O be an open connected subset of X and fix $a \in O$. Define $U_\alpha = \{x \in O :$ there is a path in O joining a and $x\}$. Prove that U_α is both open and closed in O.

Proof. We show that U_α is both open and closed in O. Note that we can show that every open subset of a locally path-connected space is locally path-connected and every path component of locally path-connected are open as in the locally connected case. By the way U_α was constructed, we see that U_α is a path component of O, so it is open in O. Next we claim that every path component of a locally path-connected space are equal to its component. Let $p \in X$, and let A and B be the component and the path component containing p, respectively. Then we have $B \subseteq A$ and A can be written as a disjoint union of path components, each of which is open in X and thus in A. If B is not the only path component in A, then the sets B and $A \setminus B$ disconnected A, which is a contradiction because A is connected. This implies that $A = B$. Now, we have U_α is also a component of O, so it is closed in O. Since O is connected , $U_\alpha = O$. Thus O is path-connected. □

Problem 9. *Let A and B be subsets of a topological space X such that $A \cap \overline{B} \neq \varnothing$. Prove or disprove the following statements:*

a) If A and B are connected, then $A \cup B$ is connected.

b) If A and B are path-connected, then $A \cup B$ is path-connected.

Solution.

a) We claim that $A \cup B$ is connected. Suppose to the contrary that $A \cup B$ is disconnected. Then there is continuous surjective function $f : (A \cup B, \mathcal{T}_{A\cup B}) \to (\{a, b\}, \mathcal{T}_{\text{discrete}})$. Note that A and B are also connected in $A \cup B$ with the subspace topology. WLOG $f(B) = \{b\}$. Since f is continuous, $f^{-1}(\{b\})$ is closed set contain B, so $\overline{B} \subseteq f^{-1}(\{b\})$. But $A \cap \overline{B} \neq \varnothing$ and A is connected in $A \cup B$, $f(A) = \{b\}$. This implies that f is not surjective, which is a contradiction.

b) Choose $A = \{(0, 0)\} \subseteq \mathbb{R}^2$ and $B = \{(x, \sin(1/x)) : \ x > 0\} \subseteq \mathbb{R}^2$. Note that $A \cup B$ is the topologist's sine curve. Clearly, there is a contstant path from $[0, 1]$ to A, so A is path-connected. Let $(x, \sin(1/x), (y, \sin(1/y))) \in B$. Define $f : [0, 1] \to B$ by $f(t) : ((1-t)x + ty, \sin(1/((1-t)x + ty)))$. It is easy to check that f is continuous $f(0) = x$ and $f(1) = y$. This implies that B is path-connected. Observe that $A \subseteq A \cup B \subseteq \overline{B}$. Thus $A \cap \overline{B} \neq \varnothing$. But the topologist's sine curve is not path-connected.

□

Problem 10. *i) Prove that if X is totally disconnected and locally connected, then X is discrete.*

ii) Verify whether $\mathbb{R}$ with the lower limit topology is locally connected.

Proof. i) Let $x \in X$. Since X is totally disconnected, we have every singleton $\{x\}$ is component. Moreover, each component of a locally connected space X is open, so $\{x\}$ is open.

ii) By Problem 3 ii), we know that $\mathbb{R}$ with the lower limit topology is totally disconnected. If $\mathbb{R}$ with the lower limit topology is locally connected, it must be discrete, by Problem 10 i). Note that $[0, 2]$ is not open in $(\mathbb{R}, \mathcal{T}_{\text{lower limit}})$, by Homework 8.2 i). Hence $(\mathbb{R}, \mathcal{T}_{\text{lower limit}})$ is not locally connected.

□

Problem 11. *Prove that a locally connected compact space has finitely many components.*

Proof. Let X be a locally connected compact space. Then we have every component of X is open. Note that Components of X form a partition of X. Since X is compact, so finitely many of these components cover it, say $C_1, \ldots, C_n$. If G is component of X and let $g \in G$, then $g \in C_j$ for some $1 \leq j \leq n$, so $G \cup C_j$ is connected. Thus $G = C_j$. Now, we can conclude that X has finitely many components. $\square$

Problem 12. *Suppose that (X, d) is a compact metric space. Prove that X is locally connected if and only if for each $\varepsilon > 0$, there is a finite cover of X by compact connected sets of diameter less than ε.*

Proof. ($\Rightarrow$) Let $\varepsilon > 0$, $x \in X$ and $B_d(x : \varepsilon/4)$. Since X is locally connected, there is open connected nbhd V_x of x contained in $B_d(x : \varepsilon/4)$. By compactness of X, finitely many of these, say $\{V_{x_1}, \ldots, V_{x_n}\}$, cover X. Since X is compact, $\overline{V}_{x_j}$ is also compact for all $j \in \{1, \ldots, n\}$. Recall that $\operatorname{diam} \overline{V}_{x_j} = \operatorname{diam} V_{x_j} \leq \operatorname{diam} B_d(x_j : \varepsilon/4) \leq \varepsilon/2$.
($\Leftarrow$) Let $p \in X$ and U be an nbhd of p. Then there is $\varepsilon > 0$ such that $p \in B_d(p : \varepsilon) \subseteq U$. By assumption, there is a finite cover of X by compact connected sets of diameter less than ε, say $G_1, \ldots, G_m$.
Let $\mathcal{A}_1 = \{G_i : i \in \{1, \ldots, m\}$ and $x \in G_i\}$ and $\mathcal{A}_2 = \{G_i : i \in \{1, \ldots, m\}$ and $x \notin G_i\}$. Note that G_i is closed for all i, since it is a compact subset of Hausdorff space.
Clearly, $\mathcal{A}_1 \neq \varnothing$. If $\mathcal{A}_2 \neq \varnothing$, then $K = \bigcup_{A \in \mathcal{A}_2} A$ is a finite union of closed set does not containing x. Thus $X \setminus K$ is an open set containing x. Let $y \in X \setminus K$. Then we have $y \in A$ for some $A \in \mathcal{A}_1$, so $X \setminus K \subseteq \bigcup_{A \in \mathcal{A}_1} A$. Since $x \in A$ for all $A \in \mathcal{A}_1$ and $\bigcap_{A \in \mathcal{A}_1} A \neq \varnothing, \bigcup_{A \in \mathcal{A}_1} A$ is connected. Moreover, $A \subseteq B_d(x; \varepsilon)$ for all $A \in \mathcal{A}_1$, since $\operatorname{diam} A < \varepsilon$ and $x \in A$ for all $A \in \mathcal{A}_1$. Now, we have $\bigcup_{A \in \mathcal{A}_1} A$ is connected nbhd of x contained in U as desired. If $\mathcal{A}_2 = \varnothing$, we have X is a connected open set containing x such that $X \subseteq U$. $\square$

Problem 13. *Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f[\mathbb{Q}] \subseteq \mathbb{R} - \mathbb{Q}$ and $f[\mathbb{R} - \mathbb{Q}] \subseteq \mathbb{Q}$?*

Proof. Suppose to the contrary that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f[\mathbb{Q}] \subseteq \mathbb{R} - \mathbb{Q}$ and $f[\mathbb{R} - \mathbb{Q}] \subseteq \mathbb{Q}$. Since f is continuous, $f|_{[0,1]} : [0, 1] \to \mathbb{R}$ is also continuous. Note that $[0, 1]$ is compact and connected. Thus $f([0, 1])$ is compact and connected in $\mathbb{R}$, so it is closed bounded interval. Let $f([0, 1]) = [a, b]$. Next, define $g : x \mapsto \frac{f(x)-p}{q}$ where $p, q \in \mathbb{Q}$ such that $g([0, 1]) \subseteq [0, 1]$. Then we have $g : [0, 1] \to [0, 1]$. Observe that g is continuous, moreover $g(x)$ is rational if and only if f is rational. We can use the IVT to show that g has a fixed point, that is there is $x \in [0, 1]$ such that $g(x) = x$, so x is rational if and only if $x = g(x)$ is irrational, which is a contradiction. $\square$