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1. Prove that if $\mathcal{F}$ is a finite subset of $C(X)$ then $\mathcal{F}$ is equicontinuous.

Proof Let $\{f_1, \dots, f_n\} = \mathcal{F} \subseteq C(X)$.

Let $x_0 \in X$ and $\varepsilon > 0$. Since f_i is continuous at x_0 for all $i \in \{1, \dots, n\}$, there exists $x_0 \in U_i \in \tau_X$ such that

$d(f_i(x), f_i(x_0)) < \varepsilon$ for all $x \in U_i$.

Choose $U = \bigcap_{i=1}^{n} U_i$. Clearly U is an open set containing x_0.

Then we have $d(f(x), f(x_0)) < \varepsilon$ for all $x \in U$ all $f \in \mathcal{F}$.

Hence $\mathcal{F}$ is equicontinuous.

2. Let $f_n, f \in C(X)$ for each $n \in \mathbb{N}$. Suppose that $(f_n(x))$ is decreasing and that $f_n(x) \to f(x)$ for each $x \in X$. Prove that $\{f_n\}$ is equicontinuous.

Proof. Since $f_n \to f$, $f_1(x) \geq f_2(x) \geq \dots \geq f(x)$ for all $x \in X$. It follows that

$f_1(x) - f(x) \geq f_2(x) - f(x) \geq \dots \geq 0$, and thus

$|f_1(x) - f(x)| \geq |f_2(x) - f(x)| \geq \dots \geq 0$.

Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \varepsilon/3$ for all $n \geq N$.

Next, we show that $f_m \in B_\rho(f_N : \varepsilon)$ for all $m \geq N$ where ρ is an uniform metric.

Observe that $|f_m(x) - f(x)| < |f_N(x) - f(x)| < \varepsilon/3$

It follows that $|f_m(x) - f_N(x)| < \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3$.

Thus $\sup_{x \in X} |f_m(x) - f_N(x)| \leq 2\varepsilon/3 < \varepsilon$.

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Hence $f_m \in B_\rho(f_N; \varepsilon)$.

Now, we see that for every $\varepsilon > 0$, there is a finite covering of $\{f_n\}$ by ε balls, say $B_\rho(f_1; \varepsilon), B_\rho(f_2; \varepsilon), \ldots B_\rho(f_N; \varepsilon)$.

Since $\{f_n\}$ is totally bounded under the uniform metric, then $\{f_n\}$ is equicontinuous under $|\cdot|$.

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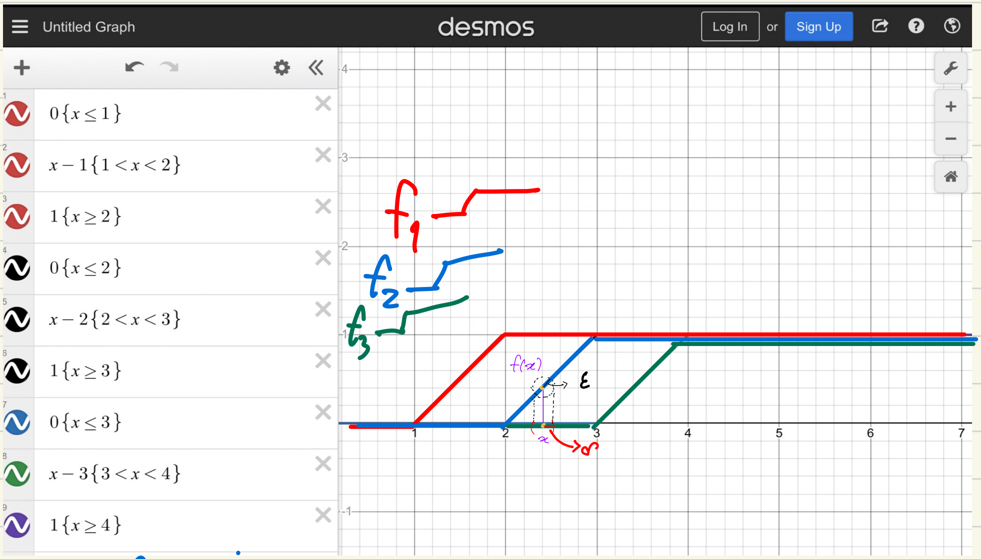
3. For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & \text{if } x \le n; \\ x - n, & \text{if } n < x < n+1; \\ 1, & \text{if } x \ge n+1, \end{cases}$$

Determine if the family $\{f_n\}$ is equicontinuous.

Sketch Proof.



Note that f_n is continuous for all $n \in \mathbb{N}$.

Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists unique $n \in \mathbb{N}$ such that $n \le x < n+1$. As we can seen that by the figure above we can find $\delta > 0$ such that

$|x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon$.

If $k < n$, then we have $|f_k(x) - f_k(y)| = |1-1| = 0 < \varepsilon$.

If $k > n$, then we have $|f_k(x) - f_k(y)| = |0-0| = 0 < \varepsilon$.

Hence $\{f_n\}$ is equicontinuous.

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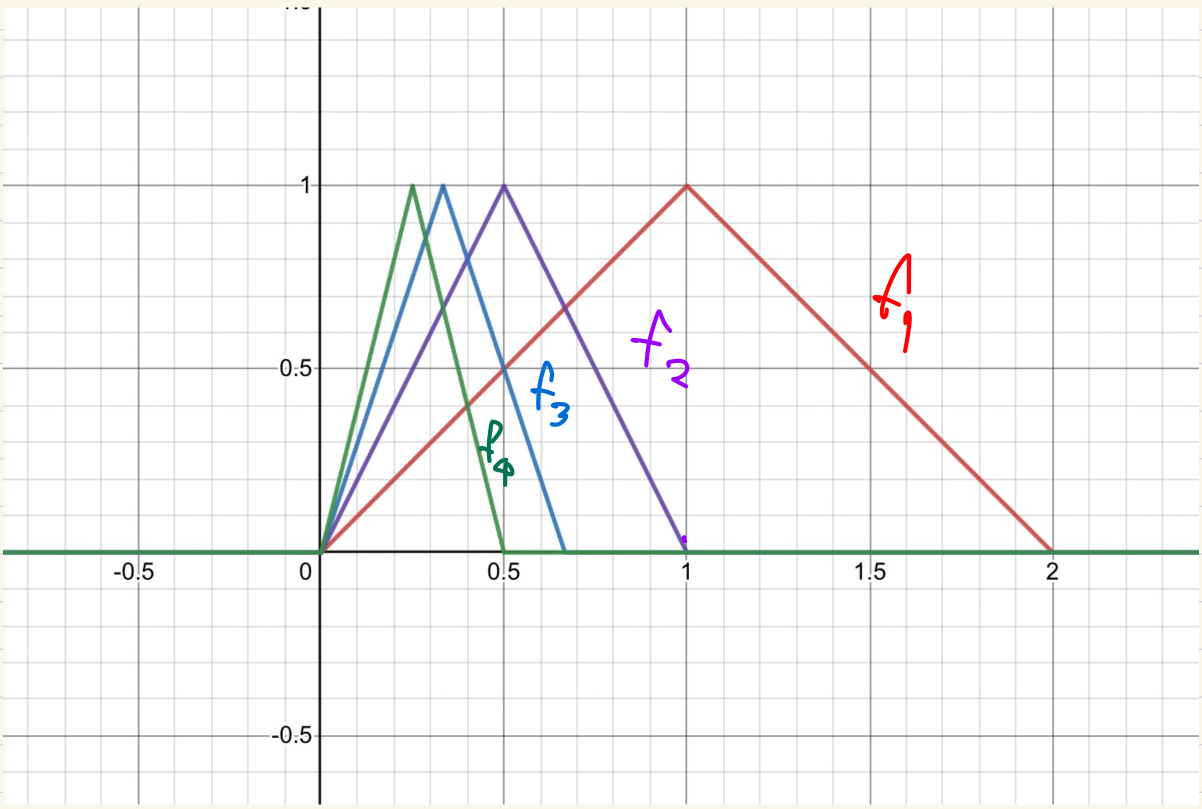
4. For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x > 2/n; \\ nx, & \text{if } 0 \le x \le 1/n; \\ 2 - nx, & \text{if } 1/n < x \le 2/n, \end{cases}$$

Determine if the family $\{f_n\}$ is equicontinuous.

Sketch Proof.



Choose $x_0 = 0$ and $\varepsilon = 1/4$. Let $\delta > 0$. Then there is $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$ by Archimedean Property. Let $x = \frac{1}{2k}$. Then we have $\left|\frac{1}{2k} - 0\right| < \frac{1}{k} < \delta$.

Consider $f_k \in \{f_n\}$, we see that

$$|f_k(x) - f_k(x_0)| = \left|\frac{1}{2k} \cdot k - 0k\right| = \frac{1}{2} \geq \frac{1}{4}.$$

Thus $\{f_n\}$ is not equicontinuous.

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$f_n \to f$ p.w.

5. If $\{f_n\}$ is a family of equicontinuous real-valued functions defined on a space X and (x_n) is a sequence in X converging to x, show that $(f(x_n))$ converges to $f(x)$.

Proof. Let $x_0 \in X$, $\varepsilon > 0$.

Since $\mathcal{F} = \{f_n\}$ is a family of equicontinuous real-valued functions, there is open set U in X containing x_0 such that for every $t \in U$, and for any $f_k \in \mathcal{F}$,

$$|f_k(t) - f_k(x_0)| < \frac{\varepsilon}{3}.$$

Since $f_n(x_0) \longrightarrow f(x_0)$, then there is $N_1 \in \mathbb{N}$ such that $|f_n(x_0) - f(x_0)| < \frac{\varepsilon}{3}$ for all $n \geq N_1$.

Let $y \in U$. Then there is $N_2 \in \mathbb{N}$, such that

$$|f_n(y) - f(y)| < \frac{\varepsilon}{3}.$$

Put $N = \max\{N_1, N_2\}$.

Now, we have

$$|f(x_0) - f(y)| \leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(y)| + |f_N(y) - f(y)|$$

$$< \frac{3\varepsilon}{3} = \varepsilon.$$

Hence f is continuous, and thus $f(x_n) \to f(x)$.

6. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be a continuously differentiable function such that $f_n(0) = 0$ and

$$\int_0^1 |f_n'(x)|^2 \, dx \le 1.$$

Show that the sequence (f_n) has a uniformly convergent subsequence.

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Proof We claim that $\int_0^1 |f_n'(x)| \, dx \le 1$.

By Cauchy Schwarz, we have $\int_0^1 |f_n'(x)| \cdot 1 \, dx \le \left(\int_0^1 |f_n'(x)|^2 dx\right)^{1/2} \left(\int_0^1 1 \, dx\right)^{1/2} \le 1$

Next, we show that $\{f_n\}$ is pointwise bounded.

Since for all $x \in [0,1]$ and all $n \in \mathbb{N}$ we have

$$|f_n(x)| = |f_n(x) - f_n(0)| = \left|\int_0^x f_n'(t) \, dt\right| \le \int_0^x |f_n'(t)| dt \le \int_0^x |f_n'(t)| dt + \underbrace{\int_x^1 |f_n'(t)| dt}_{\geq 0}$$

$$= \int_0^1 |f_n'(t)| dt \le 1.$$

This implies that $\{f_n\}$ is pointwise bounded.

Next, we show that $\{f_n\}$ is equicontinuous.

Let $\varepsilon > 0$ and $\delta = \varepsilon^2$ and $x, y \in [0,1]$ such that $|x-y| < \delta$.

Then we have

$$\underbrace{|f_n(x) - f_n(y)| = \left| \int_y^x f_n'(t)\,dt \right|}_{\text{By Fundamental theorem of Calculus.}} \le \int_y^x |f_n'(t)| \cdot 1 \, dt$$

$$\left.\begin{aligned} &\le \sqrt{\int_y^x |f_n'(t)|^2 dt} \sqrt{\int_y^x 1^2 dt} \end{aligned}\right\} \text{By Cauchy Schwarz inequality for integrals.}$$

$$= \sqrt{\int_y^x |f_n'(t)|^2 dt} \sqrt{|x-y|}$$

$$\le 1 \cdot \sqrt{\delta} = 1 \cdot \sqrt{\varepsilon^2} = \varepsilon$$

for all $n \in \mathbb{N}$. Hence $\{f_n\}$ is equicontinuous, and thus (f_n) has a uniformly convergent subsequence by Arzelà Theorem.