



Dynamic Programming. Divide and Conquer. Backtracking

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Outline

Content

1. Dynamic Programming
 - DP Paradigm
 - Classical DP problems
 - Path reconstruction
 - Known types of DP problems
2. Divide and Conquer
 - Divide and Conquer paradigm
 - Problems
3. Backtracking
 - Recursive backtracking implementation
 - BKT tricks
4. Homework



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Dynamic Programming is probably the most frequent type of problems meet in algorithmic contests. Moreover, DP problems can also require a large other techniques such as: graphs, trees, data structures, optimizations, mathematics: combinatorics, geometry, game theory, number theory, backtracking and more.



DP Paradigm : algorithmic technique which is usually based on a recurrent formula / structure and one (or some) starting states. A sub-solution of the problem is constructed from previously found ones. DP solutions have a polynomial complexity. [[topcoder](#)].



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Sub-problems should be understood as sub-problem **instances** and they can *overlap*. The answer for a problem instance can be computed based on answers of other problem instances for which some atomic information is enough (like some number: minimum / sum / counter or other).



Classical DP problems. Fibonacci sequence.

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- Given n , find F_n



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- Given n , find F_n
- Most natural solution could be:

```
1  int fibo(int i) {  
2      if (i >= 3) {  
3          return fibo(i-1) + fibo(i-2);  
4      } else {  
5          return 1;  
6      }  
7  } // solution is fibo(n)
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- DP solution is:

```
1  int v[NMax];  
2  v[1] = v[2] = 1;  
3  for (int i = 3; i <= n; i++) {  
4      v[i] = v[i-1] + v[i-2];  
5  } // solution is v[n];
```



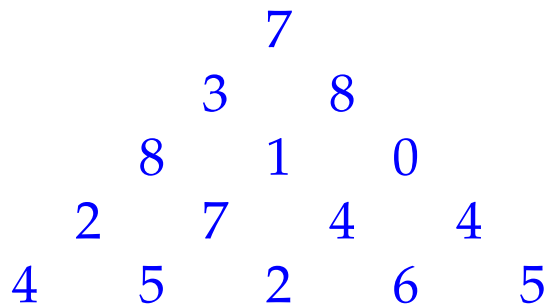
Classical DP problems. Triangle: IOI 1994.

- Given some numbers arranged like in the triangle below, find the maximum sum of elements on a path starting from the first number that reaches the bottom level and on each consecutive level goes either left or right.



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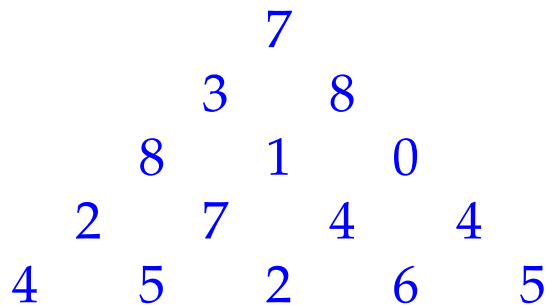
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- The solution is $30 = 7 + 3 + 8 + 7 + 5$.



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$$best_i = \max(best_j) + 1, \text{ for all } j \text{ such that } j < i \text{ and } v[j] < v[i]$$

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- $O(N \log N)$ solution?



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$$profit[i][j] = \max \begin{cases} profit[i-1][j] & \text{(don't add object i)} \\ profit[i-1][j-w_i] + p_i & \text{(add object i)} \end{cases}$$

for all i and j , avoiding any negative indices.



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In case of criteria like smallest lexicographic solutions, predecessors must be kept accordingly.



Known types of DP problems.



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- DP on trees: [asmax](#), [color2](#)



Known types of DP problems.

- DP on trees: [asmax](#), [color2](#)
- DP on configurations: [pavare](#), [false mirrors](#)
- DP and geometry: [pomi](#), [geometrie](#) (hard)



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Divide and Conquer



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It is different though because:

- sub-problems don't overlap
- the problem usually doesn't require finding a numeric value but to do some processing (sorting, searching, etc).



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A divide and conquer algorithm works by recursively breaking down a problem into two or more sub-problems of the same or related type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the original problem. [wikipedia](https://en.wikipedia.org/wiki/Divide_and_conquer).



Problems



Problems

- Fibonacci ...



Problems

- Fibonacci ...
- binary search ...



Problems

- Fibonacci ...
- binary search ...
- mergesort, quicksort



Problems

- Fibonacci ...
- binary search ...
- mergesort, quicksort
- **Maximum Sequence**



Problems

- Fibonacci ...
- binary search ...
- mergesort, quicksort
- Maximum Sequence
- Closest Points on a Plane



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Backtracking Backtracking is a general algorithm for finding all (or some) solutions to some computational problems, notably constraint satisfaction problems, that incrementally builds candidates to the solutions, and abandons a candidate ("backtracks") as soon as it determines that the candidate cannot possibly be completed to a valid solution [wikipedia](#).



Recursive backtracking implementation

Easier to implement recursively. Example: generating all permutations of elements [1..N].

```
1  int n, v[10], used[10];
2
3  void bkt(int k) {
4      if (k == n) {
5          for (int i = 0; i < n; i++) {
6              print(v[i]);
7          }
8      } else {
9          for (v[k] = 1; v[k] <= n; v[k]++) {
10             if (used[v[k]] == 0) {
11                 used[v[k]] = 1;
12                 bkt(k + 1);
13                 used[v[k]] = 0;
14             }
15         }
16     }
```

```
1  int main() {
2      read(n);
3      bkt(0);
4      return 0;
5  }
```



STL next permutation

```
1  for (int i = 1; i <= n; ++i)
2    v.push_back(i);
3  do {
4    for (int i = 0; i < n; ++i) print(v[i]);
5    print('\n');
6  } while (next_permutation(v.begin(), v.end()));
```



Use bitset to generate all subsets

To generate all subsets of a set of n elements.

```
1  for (int set = 0; set < (1 << n); set++) {
2    for (int i = 0; i < n; i++) {
3      if (set & (1 << i)) {
4        // included
5      } else {
6        // not included
7      }
8    }
9  }
```



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Dynamic Programming problems

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scmax

Divide and conquer problems

maximum sequence

Backtracking problems

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